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HYPERASYMPTOTIC SOLUTIONS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH A SINGULARITY OF ARBITRARY INTEGER RANK

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ABSTRACT. We develop a hyperasymptotic expansion for solutions of the general second-order homogeneous differential equation with a singularity of arbitrary integer Poincaré rank at infinity. This expansion is in terms of certain integrals which are generalisations of the hyperterminant integrals developed in other papers for the rank one case.

1. Introduction

The general linear homogeneous differential equation of the second-order is given by

$$\frac{d^2W}{dz^2} + f(z)\frac{dW}{dz} + g(z)W = 0. \tag{1.1}$$

The problem we shall study is that of an irregular singularity of rank r at infinity. In this case, the functions f and g can be expanded in power series about infinity of the form

$$f(z) = z^{r-1}\sum_{s=0}^{\infty}\frac{f_s}{z^s}, \qquad g(z) = z^{2r-2}\sum_{s=0}^{\infty}\frac{g_s}{z^s},$$

which converge in an open annulus $|z| > a$. At least one of the coefficients f_0, g_0, g_1 is non-zero; otherwise the singularity would have lower rank.

This equation (1.1) is studied in detail for the case $r = 1$ in [2] where a method of rigorous re-expansion of the remainder terms in the asymptotic expansion of the solution is developed. The re-expansions are in terms of certain multiple integrals, the so-called *hyperterminant integrals.* This equation also is studied in [3] for the case of arbitrary r, and a method for the calculation of Stokes' multipliers is derived. Using these results, we have developed the *hyperasymptotic expansions* (see [1, 2] for references) for the differential equation (1.1) for the general case of the second-order linear differential equation of arbitrary rank r.

2. Setting up the problem

By making the transformation

$$w(z) = \exp\left(\frac{1}{2}\int^z f(t)dt\right)W(z), \tag{2.1}$$

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the differential equation (1.1) is transformed to the equation

$$\frac{d^2w}{dz^2} = \phi(z)w \tag{2.2}$$

where

$$\phi(z) = \frac{1}{4}f^2(z) + \frac{1}{2}f'(z) - g(z).$$

We may assume without loss of generality that $\frac{1}{4}f_0^2 - g_0$ is non-zero[1] so that the square root of ϕ can be expanded in the form

$$\{\phi(z)\}^{\frac{1}{2}} = z^{r-1}\sum_{s=0}^{\infty}\frac{\phi_s}{z^s}.$$

We now define sectors

$$S_k = \left\{z \ : \ \frac{(-k - \frac{1}{2} - \sigma)}{r}\pi \leq \operatorname{ph} z \leq \frac{(-k + \frac{1}{2} - \sigma)}{r}\pi\right\}$$

where $\sigma = \operatorname{ph}\phi_0$.

If we define $\hat{S}_k$ to be any closed sector properly interior to $S_{k-1} \cup S_k \cup S_{k+1}$, then the differential equation (2.2) has unique solutions w_k defined by

$$w_k(z) \sim e^{-\xi(z)}z^{\mu_1}\sum_{s=0}^{\infty}\frac{a_{s,1}}{z^s}, \qquad z \to \infty \text{ in } \hat{S}_k \tag{2.3}$$

for k even, and

$$w_k(z) \sim e^{\xi(z)}z^{\mu_2}\sum_{s=0}^{\infty}\frac{a_{s,2}}{z^s}, \qquad z \to \infty \text{ in } \hat{S}_k \tag{2.4}$$

for k odd (see [3, 4]). These sectors differ slightly from those in [3], and because of this so do the solutions $w_k(z)$. The order r polynomial ξ is given by

$$\xi(z) = z^r\sum_{s=0}^{r-1}\frac{\phi_s}{(r-s)z^s},$$

and the coefficients μ_1, μ_2, $a_{s,1}$ and $a_{s,2}$ can be calculated using a recursion relation derived by substituting the expressions (2.3) and (2.4) into the differential equation (2.2) (see [3]).

We now would like to define a new variable x such that

$$x^r = 2\xi(z),$$

so that the polynomial ξ in the exponent of the asymptotic form of the solutions (2.3), (2.4) becomes simply x^r. We can do this by writing

$$z = x\sum_{s=0}^{\infty}\frac{c_s}{x^s}$$

[1] The case when $\frac{1}{4}f_0^2 - g_0$ is zero is dealt with by using the transformation of Fabry; see [4].

and calculating the coefficients c_s by reversion of power series. Performing a full reversion of power series is laborious, but if we now truncate this series and make the change of variables

$$z = x\sum_{s=0}^{r-1}\frac{c_s}{x^s},$$

we then can transform equation (1.1) to write down a new differential equation for W in the variable x:

$$\frac{d^2W}{dx^2} + \hat{f}(x)\frac{dW}{dx} + \hat{g}(x)W = 0. \tag{2.5}$$

We can apply the transform (2.1) to this differential equation, and this yields an equation with solutions in the form (2.3), (2.4) with the polynomial ξ having the simple form $x^r/2$ in the new variable x. Without loss of generality, we now can assume that the equation (1.1) is in the correct form initially, so that when we derive (2.2) from it, there are two solutions which have the following behavior

$$w_k(z) \sim e^{-z^r/2}z^{\mu_1}\sum_{s=0}^{\infty}\frac{a_{s,1}}{z^s}, \qquad z\to\infty \text{ in } \hat{S}_k \tag{2.6}$$

for k even, and

$$w_k(z) \sim e^{z^r/2}z^{\mu_2}\sum_{s=0}^{\infty}\frac{a_{s,2}}{z^s}, \qquad z\to\infty \text{ in } \hat{S}_k \tag{2.7}$$

for k odd.

There are only two linearly independent solutions to the second-order differential equation (2.2), so there must be a linear relationship between any three solutions; a *connection formula.* In particular, we can write

$$w_{k+2}(z) = C_{k+1}w_{k+1}(z) + w_k(z). \tag{2.8}$$

The coefficient of $w_k(z)$ is unity because $w_{k+2}(z)$ and $w_k(z)$ have the same dominant asymptotic form in their common sector of validity.

We now define functions

$$u_k(z) = e^{z^r/2}z^{-\mu_1}e^{\mu_1 k\pi\iota/r}w_k(ze^{-k\pi\iota/r})$$

for k even and

$$u_k(z) = e^{z^r/2}z^{-\mu_2}e^{\mu_2 k\pi\iota/r}w_k(ze^{-k\pi\iota/r})$$

for k odd. These functions have the asymptotic form

$$u_k(z) \sim \sum_{s=0}^{\infty}\frac{a_{s,1}}{(ze^{-k\pi\iota/r})^s}, \qquad z\to\infty \text{ in } \hat{S}_0 \tag{2.9}$$

for k even, and

$$u_k(z) \sim \sum_{s=0}^{\infty}\frac{a_{s,2}}{(ze^{-k\pi\iota/r})^s}, \qquad z\to\infty \text{ in } \hat{S}_0 \tag{2.10}$$

for k odd. Using the connection formula (2.8) for w, we now can define connection formulae for u

$$u_{k+2}(ze^{2\pi\iota/r}) = C_{k+1}e^{z^r}z^{\omega}e^{-k\pi\iota\omega/r}u_{k+1}(ze^{\pi\iota/r}) + u_k(z) \tag{2.11}$$

for k even, and

$$u_{k+2}(ze^{2\pi\iota/r}) = C_{k+1}e^{z^r}z^{-\omega}e^{k\pi\iota\omega/r}u_{k+1}(ze^{\pi\iota/r}) + u_k(z) \tag{2.12}$$

for k odd. The number $\omega = \mu_2 - \mu_1$. Note that $u_{k+2r}(z) = u_k(z)$.

In a similar manner to [3], we now can write down a Stieltjes integral representation for each of the functions u_k. The representation has a slightly different form depending on whether k is an even or odd integer.

Lemma 1. *For even k,*

$$\begin{aligned} u_k(z) = -\frac{z}{2\pi\iota}\Bigg[\sum_{j=0}^{r-1}\int_{\rho e^{(-2j+k-1)\pi\iota/r}}^{\rho e^{(-2j+k+1)\pi\iota/r}} \frac{u_{2j}(te^{(2j-k)\pi\iota/r})}{t(t-z)}dt \\ + e^{-k\pi\iota\omega/r}\sum_{j=0}^{r-1} C_{2j+1}\int_{\rho e^{-(2j-k+1)\pi\iota/r}}^{\infty e^{-(2j-k+1)\pi\iota/r}} \frac{e^{t^r}t^{\omega}u_{2j+1}(te^{(2j-k+1)\pi\iota/r})}{t(t-z)}dt\Bigg]. \end{aligned} \tag{2.13}$$

For odd k,

$$\begin{aligned} u_k(z) = -\frac{z}{2\pi\iota}\Bigg[\sum_{j=0}^{r-1}\int_{\rho e^{(-2j+k-2)\pi\iota/r}}^{\rho e^{(-2j+k)\pi\iota/r}} \frac{u_{2j+1}(te^{(2j-k+1)\pi\iota/r})}{t(t-z)}dt \\ + e^{k\pi\iota\omega/r}\sum_{j=0}^{r-1} C_{2j}\int_{\rho e^{-(2j-k)\pi\iota/r}}^{\infty e^{-(2j-k)\pi\iota/r}} \frac{e^{t^r}t^{-\omega}u_{2j}(te^{(2j-k)\pi\iota/r})}{t(t-z)}dt\Bigg]. \end{aligned} \tag{2.14}$$

We use these integral representations for u_k to derive an integral representation for the remainder after truncation of its asymptotic series. This is done by expanding the term $(t-z)^{-1}$ as a finite geometric series. The details of the proof are similar to [3] and are omitted. As a byproduct of this process, we also find an integral representation for the coefficients of the asymptotic expansion for u_k. These can be used to develop asymptotic expansions for the coefficients (see [2, 3]).

Theorem 1.

$$u_k(z) = \sum_{s=0}^{n-1}\frac{a_{s,1}}{(ze^{-k\pi\iota/r})^s} + R_k^0(z,n) \tag{2.15}$$

where the coefficients $a_{s,1}$ in the expansion are given by

$$\begin{aligned} a_{s,1} = \frac{1}{2\pi\iota}\Bigg[\sum_{j=0}^{r-1} e^{-2js\pi\iota/r}\int_{\rho e^{-\pi\iota/r}}^{\rho e^{\pi\iota/r}} u_{2j}(t)t^{s-1}dt \\ + \sum_{j=0}^{r-1} C_{2j+1}e^{-(2j+1)(s+\omega)\pi\iota/r}\int_{\rho}^{\infty} e^{-t^r}t^{s+\omega-1}u_{2j+1}(t)dt\Bigg] \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} R_k^0(z,n) = -\frac{1}{2\pi\iota z^{n-1}}\epsilon_k^0(z;\rho,n) \\ -\frac{1}{2\pi\iota z^{n-1}}e^{kn\pi\iota/r}\sum_{j=0}^{r-1} C_{2j+1}e^{-(2j+1)(n+\omega)\pi\iota/r}\int_{\rho}^{\infty}\frac{e^{-t^r}t^{n+\omega-1}u_{2j+1}(t)}{te^{-(2j-k+1)\pi\iota/r}-z}dt \end{aligned} \tag{2.17}$$

for k even, and

$$u_k(z) = \sum_{s=0}^{n-1} \frac{a_{s,2}}{(ze^{-k\pi\iota/r})^s} + R_k^0(z,n) \tag{2.18}$$

where

$$a_{s,2} = \frac{1}{2\pi\iota}\Bigg[\sum_{j=0}^{r-1} e^{-(2j+1)s\pi\iota/r} \int_{\rho e^{-\pi\iota/r}}^{\rho e^{\pi\iota/r}} u_{2j+1}(t)t^{s-1}dt + \sum_{j=0}^{r-1} C_{2j}e^{-2j(s-\omega)\pi\iota/r} \int_\rho^\infty e^{-t^r}t^{s-\omega-1}u_{2j}(t)dt\Bigg] \tag{2.19}$$

and

$$R_k^0(z,n) = -\frac{1}{2\pi\iota z^{n-1}}\epsilon_k^0(z;\rho,n) - \frac{1}{2\pi\iota z^{n-1}}e^{kn\pi\iota/r}\sum_{j=0}^{r-1} C_{2j}e^{-2j(n-\omega)\pi\iota/r}\int_\rho^\infty \frac{e^{-t^r}t^{n-\omega-1}u_{2j}(t)}{te^{-(2j-k)\pi\iota/r}-z}dt \tag{2.20}$$

for k odd.

The definition of ϵ_k^0 is given by

$$\epsilon_k^0(z;\rho,n) = \sum_{j=0}^{r-1} e^{n(-2j+k)\pi\iota/r}\int_{\rho e^{-\pi\iota/r}}^{\rho e^{\pi\iota/r}} \frac{u_{2j}(t)t^{n-1}}{te^{(-2j+k)\pi\iota/r}-z}dt$$

for k even and by

$$\epsilon_k^0(z;\rho,n) = \sum_{j=0}^{r-1} e^{n(-2j+k-1)\pi\iota/r}\int_{\rho e^{-\pi\iota/r}}^{\rho e^{\pi\iota/r}} \frac{u_{2j+1}(t)t^{n-1}}{te^{(-2j+k-1)\pi\iota/r}-z}dt$$

for k odd.

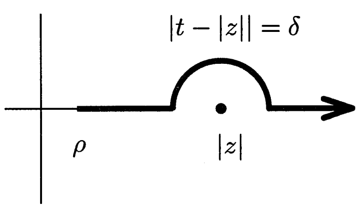

FIGURE 1. $\mathcal{P}$ for $0 \le \operatorname{ph} z \le \pi/r$

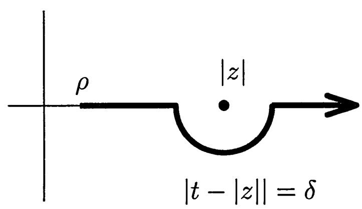

FIGURE 2. $\mathcal{P}$ for $-\pi/r \le \operatorname{ph} z \le 0$

In the integrals in (2.17) and (2.20), z has been restricted to the phase range $|\operatorname{ph} z| \le \pi/r - \delta$. We would like now to include the phases $\pm\pi/r$. We do this by analytically continuing the integrals. This is performed in the standard way, i.e., indenting the straight line contours from $t = \rho$ to $t = \infty$ in a semi-circle $\big||z| - t\big| = \delta$ where the indentation goes to the left (resp. to the right) of z when $0 \le \operatorname{ph} z \le \pi/r$ (resp. $-\pi/r \le \operatorname{ph} z \le 0$) (see Figures 1,2). We shall call this contour $\mathcal{P}$. With this extension and the continuation formulae, we can get a representation for any solution of the differential equation for any z.

3. Optimal expansion at level zero (Poincaré asymptotics)

We now wish to minimise the remainder in (2.15) and (2.18) to determine the optimal number of terms of these expansions to use with the single standard Poincaré asymptotic series. We will consider in detail the case for even k; the calculations are similar for odd k.

Let $n = N_0$ in (2.15). The remainder term is given in (2.17). We deal with the two terms separately. The first term is estimated by

$$\frac{1}{2\pi\iota z^{N_0-1}}\epsilon_k^0(z;\rho;N_0) = O(\rho^{N_0}z^{-N_0}). \tag{3.1}$$

In the second term of (2.17), we can see that the dominant contribution to the bound occurs when $-\pi/r \leq \operatorname{ph} z \leq 0$ and arises from the integral for which $2j-k=0$.[2] The path along which we integrate is indented at $|z|$ to pass to the right of $|z|$. To derive sharp error bounds, we need to perform the analytic continuation of the previous section in a different way.

Starting with the dominant integral (assume $0 \geq \operatorname{ph} z > -\pi/r$)

$$\int_\rho^\infty \frac{e^{-t^r}t^{N_0+\omega-1}u_{k+1}(t)}{te^{-\pi\iota/r}-z}dt,$$

we replace z by $z\exp(-\pi\iota/r)$ and make the substitution $t = v^{1/r}$ (taking the principal branch) to give

$$\frac{e^{\pi\iota/r}}{r}\int_{\rho^r}^\infty \frac{e^{-v}v^{(N_0+\omega)/r-1}u_{k+1}(v^{\frac{1}{r}})}{v^{1/r}-z}dv.$$

Now we perform the analytic continuation allowing z to be real and indenting the contour in a semi-circle centered on and to the right of z of radius δ_1 in the v plane.

We bound this integral in two parts. For the integrals

$$\frac{e^{\pi\iota/r}}{r}\left\{\int_{\rho^r}^{|z|^r-\delta_1}+\int_{|z|^r+\delta_1}^\infty\right\}\frac{e^{-v}v^{(N_0+\omega)/r-1}u_{k+1}(v^{\frac{1}{r}})}{v^{1/r}-z}dv,$$

we can say that $|u^{1/r}-z| = O(1)$, u_k is $O(1)$ uniformly on the region $|z| \geq \rho$, so that the sum of integrals is

$$c\,\Gamma((N_0+\mathcal{R}\omega)/r) \tag{3.2}$$

where $c = O(1)$ is a generic constant.

For the part of the integral around the semicircle we have $\left|v-|z|^r\right| = \delta_1$ so that

$$\frac{e^{\pi\iota/r}}{r}\int_{|v-|z|^r|=\delta_1}\frac{e^{-v}v^{(N_0+\omega)/r-1}u_{k+1}(v^{\frac{1}{r}})}{v^{1/r}-z}dv = e^{-|z|^r}|z^r+\delta_1|^{(N_0+\mathcal{R}\omega)/r-1}\,O(1) \tag{3.3}$$

uniformly in the region of validity for z.

Now we assume that

$$N_0 = \beta_0|z|^r+\alpha_0 \tag{3.4}$$

[2]Symmetrically when $0 \leq \operatorname{ph} z \leq \pi/r$ and $2j-k=-2$.

where $\beta_0 > 0$ is $O(1)$ and α_0 is bounded. Now using this form for N_0, comparing (3.3) to (3.2), and using Stirling's asymptotic estimate for the gamma function, we have

$$\begin{aligned}\frac{e^{-|z|^r}\left(|z|^r+\delta_1\right)^{(N_0+\mathcal{R}\omega)/r-1}}{\Gamma((N_0+\mathcal{R}\omega)/r)} &= ce^{-|z|^r}|z|^{\beta_0|z|^r+\alpha_0+\mathcal{R}\omega-r}e^{(\beta_0|z|^r+\alpha_0+\mathcal{R}\omega)/r}\\ &\quad\times\left[(\beta_0|z|^r+\alpha_0+\mathcal{R}\omega)/r\right]^{-(\beta_0|z|^r+\alpha_0+\mathcal{R}\omega)/r+\frac{1}{2}}\\ &= ce^{-|z|^r}|z|^{\beta_0|z|^r+\alpha_0+\mathcal{R}\omega-r}e^{\beta_0|z|^r/r}\\ &\quad\times(\beta_0/r)^{-(\beta_0/r)|z|^r}|z|^{-(\beta_0|z|^r+\alpha_0+\mathcal{R}\omega)+\frac{r}{2}}\\ &= ce^{-|z|^r}|z|^{-r/2}\left(e^{\beta_0/r}(\beta_0/r)^{-\beta_0/r}\right)^{|z|^r} = O(|z|^{-r/2})\end{aligned}$$

The last step is due to the fact that $e^{\beta_0/r}(\beta_0/r)^{-\beta_0/r}$ has its maximum at $\beta_0 = r$. This estimate for the semicircular indentation of the integral therefore can be absorbed in the estimate for the straight-line part of the integral (3.2), and this is our final estimate for the integral. The remainder term in (2.17) then is estimated by

$$O(\Gamma((N_0+\mathcal{R}\omega)/r)z^{-N_0+1}). \tag{3.5}$$

Using the value of N_0 in (3.4) and Stirling's formula, we can minimize (3.5) with respect to β_0. We find that

$$z^{-N_0+1}\Gamma((N_0+\mathcal{R}\omega)/r) = c\left[e^{-(\beta_0/r)}(\beta_0/r)^{(\beta_0/r)}\right]^{|z|^r}|z|^{1+\mathcal{R}\omega-r/2}$$

and $\beta_0 = r$ for the remainder to be minimal. Substituting this value of β_0 in (3.5), we find that the minimal remainder estimate for k even is

$$O(z^{1-r/2+\mathcal{R}\omega}e^{-|z|^r}).$$

The corresponding estimate for k odd may be shown in a similar manner to be

$$O(z^{1-r/2-\mathcal{R}\omega}e^{-|z|^r}).$$

4. Optimal expansion at level one

To construct the first level of hyperasymptotic expansions, we re-expand the remainder terms in (2.17) and (2.20). The calculations for even and odd k are similar, so only the even k calculations will be shown. Substituting the expressions (2.18) into (2.17), we find that

$$\begin{aligned}R_k^0(z,N_0) = -\frac{z^{1-N_0}}{2\pi\iota}e^{kN_0\pi\iota/r}\sum_{s=0}^{N_1-1}a_{s,2}\sum_{j=0}^{r-1}C_{2j+1}e^{-(2j+1)(N_0+\omega-s)\pi\iota/r}\\ \times G_{2j-k+1}^{1,\omega}(z;N_0-s)+R_k^1(z;N_0,N_1)\end{aligned}$$

where

$$G_k^{1,\omega}(z,N_0) = \int_0^\infty \frac{e^{-t^r}t^{N_0+\omega-1}}{te^{-k\pi\iota/r}-z}dt, \tag{4.1}$$

and

$$
\begin{aligned}
R_k^1(z;N_0,N_1) = &-\frac{z^{1-N_0}}{2\pi\iota}\epsilon_k^0(z;\rho;N_0)\\
&+\frac{z^{1-N_0}}{2\pi\iota}e^{kN_0\pi\iota/r}\sum_{s=0}^{N_1-1}a_{s,2}\sum_{j=0}^{r-1}C_{2j+1}e^{-(2j+1)(N_0+\omega-s)\pi\iota/r}\\
&\times\int_0^\rho\frac{e^{-t^r}t^{N_0+\omega-1-s}}{te^{-(2j-k+1)\pi\iota/r}-z}dt\\
&-\frac{z^{1-N_0}}{2\pi\iota}e^{kN_0\pi\iota/r}\sum_{j=0}^{r-1}C_{2j+1}e^{-(2j+1)(N_0+\omega)\pi\iota/r}\\
&\times\int_{\mathcal{P}}\frac{e^{-t^r}t^{N_0+\omega-1}R_{2j+1}^0(t,N_1)}{te^{-(2j-k+1)\pi\iota/r}-z}dt. \qquad (4.2)
\end{aligned}
$$

Now we estimate the remainder as in the previous section and then proceed to minimize it. The first term on the right-hand side of (4.2) is $\epsilon_k^0(z;\rho;N_0)$, which is estimated as before in (3.1) to be $O(z^{-N_0}\rho^{N_0})$.

In the second term in (4.2), we have that $|te^{-(2j-k+1)\pi\iota/r}-z|\geq|z|-\rho$ in the worst case, so that

$$
\int_0^\rho\frac{e^{-t^r}t^{N_0+\omega-1-s}}{te^{-(2j-k+1)\pi\iota/r}-z}dt=O\big(z^{-1}\gamma((N_0+\mathcal{R}\omega-s)/r,\rho^r)\big)=O\big(z^{-1}\rho^{N_0}/N_0\big).
$$

From [3] we have that

$$
a_{s,2}=O(\Gamma((s-\mathcal{R}\omega)/r))\text{ as }s\to\infty,
$$

so that the whole second term is estimated by

$$
O(\Gamma((N_1-\omega)/r)z^{-N_0}\rho^{N_0}/N_0).
$$

In the third term, taking the expression for $R_{2j+1}^0(t,N_1)$ from (2.20) and replacing all occurrences of ρ by $\rho-\delta$ (to ensure convergence of the integrals),

$$
\begin{aligned}
\int_{\mathcal{P}}\frac{e^{-t^r}t^{N_0+\omega-1}R_{2j+1}^0(t,N_1)}{te^{-(2j-k+1)\pi\iota/r}-z}dt = &\frac{1}{2\pi\iota}\int_{\mathcal{P}}\frac{e^{-t^r}t^{N_0-N_1+\omega}}{te^{-(2j-k+1)\pi\iota/r}-z}dt\\
&\times\Bigg[\epsilon_{2j+1}^0(t;\rho-\delta;N_1)+e^{(2j+1)N_1\pi\iota/r}\\
&\times\sum_{l=0}^{r-1}C_{2l}e^{-(2l)(N_1-\omega)\pi\iota/r}\int_{\rho-\delta}^{\infty}\frac{e^{-t_1^r}t_1^{N_1-\omega-1}u_{2l}(t_1)}{t_1e^{-(2l-2j-1)\pi\iota/r}-t}dt_1\Bigg].
\end{aligned}
$$

The estimate for the integral of the ϵ term is given by (3.1), so that we have

$$
\int_{\mathcal{P}}\frac{e^{-t^r}t^{N_0-N_1+\omega}\epsilon_{2j+1}^0(t;\rho-\delta;N_1)}{te^{-(2j-k+1)\pi\iota/r}-z}dt=O\big((\rho-\delta)^{N_1}\Gamma((N_0-N_1+\mathcal{R}\omega)/r)\big).
$$

The double integral written out in full is

$$\int_{\mathcal{P}}\int_{\rho-\delta}^{\infty} \frac{e^{-t^r} t^{N_0-N_1+\omega} e^{-t_1^r} t_1^{N_1-\omega-1} u_{2l}(t_1)}{(te^{-(2j-k+1)\pi\iota/r}-z)(t_1 e^{-(2l-2j-1)\pi\iota/r}-t)} dt_1 dt. \tag{4.3}$$

We estimate this by noting that

$$|te^{-(2j-k+1)\pi\iota/r}-z| \geq \delta \quad \text{and} \quad |t_1 e^{-(2l-2j-1)\pi\iota/r}-t| \geq |t|c.$$

Then (4.3) splits into the product of two single integrals and is estimated by

$$O\big(\Gamma((N_0-N_1+\mathcal{R}\omega)/r)\Gamma((N_1-\mathcal{R}\omega)/r)\big).$$

All of the other terms can be absorbed into this estimate, so that we have our final estimate

$$R_k^1(z;N_0,N_1) = O(z^{-N_0+1}\Gamma((N_0-N_1+\mathcal{R}\omega)/r)\Gamma((N_1-\mathcal{R}\omega)/r)). \tag{4.4}$$

Following (3.4), now assume the standard form for N_0 and N_1:

$$N_0 = \beta_0|z|^r + \alpha_0,$$
$$N_1 = \beta_1|z|^r + \alpha_1.$$

Here $\beta_0 > \beta_1 > 0$ are $O(1)$, and α_0, α_1 are bounded.

Using Stirling's formula to give an asymptotic estimate for the gamma functions in (4.4), we find that

$$R_k^1(z;N_0,N_1) = O(z^{-\beta_0|z|^r-\alpha_0+1})|z|^{(\beta_0-\beta_1)|z|^r+\alpha_0-\alpha_1+\mathcal{R}\omega-r/2}$$
$$\times\left(e^{-\frac{\beta_0-\beta_1}{r}}\left[\frac{\beta_0-\beta_1}{r}\right]^{\frac{\beta_0-\beta_1}{r}}\right)^{|z|^r} |z|^{\beta_1|z|^r+\alpha_1-\mathcal{R}\omega-r/2}\left(e^{-\frac{\beta_1}{r}}\left[\frac{\beta_1}{r}\right]^{\frac{\beta_1}{r}}\right)^{|z|^r}. \tag{4.5}$$

We find that the estimate (4.5) is minimized when $\beta_0-\beta_1 = r$ and $\beta_1 = r$, so that $\beta_0 = 2r$, and the optimal estimate for the remainder at level 1 is

$$R_k^1(z;N_0,N_1) = z^{1-r}e^{-2|z|^r}.$$

5. General levels

The complete expansion for $u_k(z)$ can now be written down and proved by induction. The number ν_k in Theorem 2 is 1 when k is even and 0 when k is odd.

Theorem 2. *Assume $-\pi/r \leq \operatorname{ph} z \leq \pi/r$. For k even*

$$u_k(z) = \sum_{s=0}^{N_0-1}\frac{a_{s,1}}{(ze^{-k\pi\iota/r})^s} + z^{1-N_0}e^{kN_0\pi\iota/r}\sum_{n=1}^{p}(-)^n\sum_{s=0}^{N_n-1}a_{s,2-\nu_n}$$
$$\times\prod_{l=0}^{n-2}\left[\sum_{j_l=0}^{r-1}\frac{C_{2j_l+\nu_l}}{2\pi\iota}e^{-(2j_l+\nu_l)(N_l-N_{l+1}+(-)^l\omega)\pi\iota/r}\right]$$
$$\times\sum_{j_{n-1}=0}^{r-1}\frac{C_{2j_{n-1}+1-\nu_n}}{2\pi\iota}e^{-(2j_{n-1}+1-\nu_n)(N_{n-1}-s-(-)^n\omega)\pi\iota/r}$$
$$\times G^{n,\omega}_{2j_0-k+1,2j_1-2j_0-1,\ldots,2(j_{n-1}-j_{n-2})-(-1)^n}(z;N_0-N_1,\ldots,N_{n-2}-N_{n-1},N_{n-1}-s)$$
$$+R_k^p(z;N_0,N_1,\ldots,N_p)$$

where the remainder is estimated by

$$R_k^p(z;N_0,N_1,\dots,N_p) = O(z^{-N_0+1}\Gamma((N_0-N_1+\Re\omega)/r)\Gamma((N_1-N_2-\Re\omega)/r)\times\cdots \\ \times\Gamma((N_{p-1}-N_p+(-)^{p-1}\Re\omega)/r)\Gamma((N_p+(-)^p\Re\omega)/r)).$$

For k odd

$$u_k(z) = \sum_{s=0}^{N_0-1}\frac{a_{s,2}}{(ze^{-k\pi\iota/r})^s} + z^{1-N_0}e^{kN_0\pi\iota/r}\sum_{n=1}^{p}(-)^n\sum_{s=0}^{N_n-1}a_{s,1+\nu_n} \\ \times\prod_{l=0}^{n-2}\left[\sum_{j_l=0}^{r-1}\frac{C_{2j_l+1-\nu_l}}{2\pi\iota}e^{-(2j_l+1-\nu_l)(N_l-N_{l+1}-(-)^l\omega)\pi\iota/r}\right] \\ \times\sum_{j_{n-1}=0}^{r-1}\frac{C_{2j_{n-1}+\nu_n}}{2\pi\iota}e^{-(2j_{n-1}+\nu_n)(N_{n-1}-s+(-)^n\omega)\pi\iota/r} \\ \times G^{n,-\omega}_{2j_0-k,2j_1-2j_0+1,\dots,2(j_{n-1}-j_{n-2})+(-1)^n}(z;N_0-N_1,\dots,N_{n-2}-N_{n-1},N_{n-1}-s) \\ + R_k^p(z;N_0,N_1,\dots,N_p)$$

where the remainder is estimated by

$$R_k^p(z;N_0,N_1,\dots,N_p) = O(z^{-N_0+1}\Gamma((N_0-N_1-\Re\omega)/r)\Gamma((N_1-N_2+\Re\omega)/r)\times\cdots \\ \times\Gamma((N_{p-1}-N_p+(-)^{p-2}\Re\omega)/r)\Gamma((N_p+(-)^{p-1}\Re\omega)/r)).$$

In the case where the expansions are optimally truncated after n series, the number of terms in the final re-expansion is $N_p = r|z|^r+\alpha_p$; then in each previous expansion, the number of terms increases approximately by this amount, i.e., $N_{p-i} = (i+1)r|z|^r+\alpha_{p-i}$, etc. The optimal error term in this case is $R_k^p = O(z^{1-p\frac{r}{2}}e^{-p|z|^r})$.

The general integral G appearing in the expansions above is given by

$$G^{n+1,\omega}_{k,k_1,\dots,k_n}(z;M_0,M_1,\dots,M_n) = \\ \int_0^\infty\cdots\int_0^\infty\frac{e^{-t^r-t_1^r\cdots-t_n^r}t^{M_0+\omega}t_1^{M_1-\omega}\cdots t_{n-1}^{M_{n-1}+(-)^{n-1}\omega}t_n^{M_n+(-)^n\omega-1}}{(te^{-k\pi\iota/r}-z)(t_1e^{-k_1\pi\iota/r}-t)\cdots(t_ne^{-k_n\pi\iota/r}-t_{n-1})}dt_n\cdots dt_1\,dt \tag{5.1}$$

for $n\geq 1$ and for $n=0$, it is given by (4.1).

6. On calculation of the integrals G appearing in expansions

The integrals (5.1) and (4.1) (often called terminant integrals) can be calculated numerically by writing them in terms of the integrals in [2]. We then can use some results in [1] which express these simpler integrals as a convergent infinite series of confluent hypergeometric functions.

To recast our integrals in terms of those in [1], we first substitute $t=v^{1/r}$, $t_j=v_j^{1/r}$, $j=1,\dots,n$ into (5.1) taking the principal branch in all cases. We then use the results

$$\frac{1}{(ve^{-k\pi\iota})^{1/r}-z} = \frac{1}{ve^{-k\pi\iota}-z^r}\sum_{j_0=0}^{r-1}(ve^{-k\pi\iota})^{j_0/r}z^{r-1-j_0}$$

and

$$\frac{1}{(v_n e^{-k_n\pi\iota})^{1/r} - v_{n-1}^{1/r}} = \frac{1}{v_n e^{-k_n\pi\iota} - v_{n-1}} \sum_{j_n=0}^{r-1} (v_n e^{-k_n\pi\iota})^{j_n/r} v_{n-1}^{(r-1-j_n)/r}$$

to show that (5.1) can be expressed as the $n+1$-fold sum of $n+1$-fold integrals F defined in [1] in the case where $k_1, \dots, k_n$ are all odd integers:

$$G_{k,k_1,\dots,k_n}^{n+1,\omega}(z; M_0, M_1, \dots, M_n) =$$

$$\frac{1}{(-r)^{n+1}} \sum_{j_0=0}^{r-1} z^{r-1-j_0} \sum_{j_1=0}^{r-1} \cdots \sum_{j_n=0}^{r-1} e^{-(kj_0+k_1j_1+\cdots+k_nj_n)\pi\iota/r}$$

$$\times F^{n+1}\Big(z^r; \frac{M_0+j_0-j_1+\omega}{r+1}, \frac{M_1+j_1-j_2-\omega}{r+1}; \dots, \frac{M_n+j_n+(-1)^n\omega}{r}\Big).$$

To calculate with Theorem 2, we then use the results of [1]. These calculations for various examples confirm numerically the theoretical error estimates of Theorem 2.

7. Conclusions

In Theorem 2, we have obtained in general form a hyperasymptotic expansion at all levels for solutions of a second-order homogeneous linear ordinary differential equation which has an irregular singularity at infinity of arbitrary rank r.

As in [2], this expansion for sufficiently large level p is numerically unstable. This instability is due to the fact that in general we sum a divergent series past the point where the last term added is of order one, leading to severe cancellation. This may be dealt with in the same manner as in [2]. The optimal numerically stable scheme will use fewer terms than the corresponding optimal series from Theorem 2, but it will not be as accurate for the same level p.

We also may extend the region of validity of the exponentially improved expansions in a manner similar to [2, Section 10], with a corresponding weakening of the error estimates in the expanded sectors. This is really of only theoretical interest, however; in practice, the high accuracy results in this paper can be used to generate approximations to any solution anywhere in the complex plane by direct use of the connection formulae (2.11) and (2.12).

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