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SOME BERNSTEIN-DURRMEYER-TYPE OPERATORS

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To Professor Lee Lorch on his 80^{th} *Birthday*

ABSTRACT. Three Bernstien-Durrmeyer-type operators are introduced in this paper. These operators are based on the linear positive operators defined by A. Meir and A. Sharma that generalize the Bernstein and Szász operators. Some approximation properties of the operators are investigated. In each case, we obtain a Voronovskaja-type theorem.

1. Introduction

Thirty years ago A. Meir and A. Sharma [8] defined two linear positive operators which generalize the Bernstein and Szász operators. The first one is denoted by $B_n^{\lambda,\alpha}$ ($\lambda < 0$, $\alpha > -1$) and is based on the Laguerre polynomial $L_n^{(\alpha)}(x)$, $\alpha > -1$. For any $f \in C[0,1]$, they set

$$B_n^{\lambda,\alpha}(f;x) := \frac{1}{L_n^{(\alpha)}(\lambda)} \sum_{\nu=0}^{n} \binom{n+\alpha}{\nu+\alpha} L_\nu^{(\alpha)}\left(\frac{\lambda}{x}\right) x^\nu (1-x)^{n-\nu} f\left(\frac{\nu}{n}\right). \tag{1.1}$$

It was shown in [2] that if $f \in C[0,1]$, then $B_n^{\lambda,\alpha}(f;x)$ converges uniformly to $f(x)$ in $[0,1]$. For $\lambda = 0$, $B_n^{\lambda,\alpha}(f;x)$ becomes the Bernstein polynomial since

$$L_n^{(\alpha)}(\lambda) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \frac{(-\lambda)^\nu}{\nu!}. \tag{1.2}$$

Their second operator S_n^λ (λ real) uses the Hermite polynomial $H_\nu(x)$ of degree ν and is based on the identity

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_{2k}(i\lambda)}{(2k)!} (nx)^k = e^{nx} \cosh\, 2\lambda\, \sqrt{nx}\,. \tag{1.3}$$

For any $f \in C[0,\infty)$, they set

$$S_n^\lambda(f;x) := \sum_{k=0}^{\infty} \alpha_{nk}^\lambda(x) f\left(\frac{k}{n}\right), \quad \lambda \text{ real}, \tag{1.4}$$

where

$$\alpha_{nk}^\lambda(x) := e^{-nx}\, \operatorname{sech}(2\lambda\, \sqrt{nx})\, (-1)^k \frac{H_{2k}(i\lambda)}{(2k)!} (nx)^k. \tag{1.5}$$

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Our objective is to define two Durrmeyer-type operators based on the operators $B_n^{\lambda,\alpha}$ and S_n^{λ}. We denote our operators by $M_n^{(\lambda,\alpha)}$ and $\tilde{S}_n^{\lambda}$, respectively. The precise definition of $M_n^{(\lambda,\alpha)}$ is given in Section 2. The operator $P_n^{(\lambda,\alpha)}$ is the subject of Section 3. In Section 4, we treat the operator $\tilde{S}_n^{\lambda}$. In each case, we obtain a Voronovskaja-type theorem.

The Bernstein-Durrmeyer operator was introduced by Durrmeyer [5] in 1967 as a certain modification of the Bernstein operator. Some approximation properties of this operator were investigated by Derriennic [4]. This operator has some very nice properties, and the strong converse inequality of this operator was discussed in [1]. Similar modifications of Szász operators were introduced and studied by Mazhar and Totik [7]. In [3], Chui, He, and Hsu examined the asymptotic properties of certain positive summation-integral operators. The operator $M_n^{(\lambda,\alpha)}(f;x)$ can be considered as a special case of the summation-integral operators in [3].

2. The operator $M_n^{(\lambda,\alpha)}$

For any $f \in C[0,1]$ and for any $\lambda \le 0$ and $\alpha > -1$, the operator $M_n^{(\lambda,\alpha)}$ maps $C[0,1]$ into π_n and is given precisely by

$$M_n^{(\lambda,\alpha)}(f;x) := \frac{n+1}{L_n^{(\alpha)}(\lambda)} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} L_k^{(\alpha)}\left(\frac{\lambda}{x}\right) x^k (1-x)^{n-k} \int_0^1 p_{n,k}(t) f(t)\, dt \tag{2.1}$$

where $p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$ and $L_n^{(\alpha)}(\lambda)$ defined by (1.2) is the Laguerre polynomial of degree n. Since $\lambda \le 0$, $L_n^{(\alpha)}(\lambda) > 0$ and $M_n^{(\lambda,\alpha)}$ is a linear positive operator. For $\lambda = 0$, $M_n^{(\lambda,\alpha)}(f;x)$ is $M_n(f;x)$, the Durrmeyer operator. We shall prove

Theorem 1. *If $f \in C[0,1]$, then $M_n^{(\lambda,\alpha)}(f;x)$ converges uniformly to $f(x)$ in $[0,1]$ as $n \to \infty$.*

The proof of this theorem will be based on the following two lemmas.

Lemma 1 ([8]). *For a given $\lambda \le 0$, we have*

$$\frac{1}{\sqrt{n}} \frac{L_{n-1}^{(\alpha+1)}(\lambda)}{L_n^{(\alpha)}(\lambda)} = O(1) \qquad \text{as } n \to \infty \tag{2.2}$$

$$\frac{L_{n-1}^{(\alpha)}(\lambda)}{L_n^{(\alpha)}(\lambda)} = 1 + O\left(\frac{1}{\sqrt{n}}\right), \qquad \text{as } n \to \infty. \tag{2.3}$$

For a proof, see Meir and Sharma [8].

Lemma 2. $M_n^{(\lambda,\alpha)}$ *is a contraction operator and satisfies the properties:*

$$M_n^{(\lambda,\alpha)}(1;x) = 1, \tag{2.4}$$

$$M_n^{(\lambda,\alpha)}(t;x) = x - \frac{\lambda(1-x)}{n+2}\frac{L_{n-1}^{(\alpha+1)}(\lambda)}{L_n^{(\alpha)}(\lambda)} + \frac{1-2x}{n+2}, \tag{2.5}$$

$$\begin{aligned} M_n^{(\lambda,\alpha)}(t^2;x) = x^2 &+ \frac{2\lambda[(n-1)x^2-(n-3)x-2]}{(n+2)(n+3)}\frac{L_{n-1}^{(\alpha+1)}(\lambda)}{L_n^{(\alpha)}(\lambda)} \\ &- \frac{6(n+1)x^2-4nx-2}{(n+2)(n+3)} + \frac{\lambda^2(1-x)^2}{(n+2)(n+3)}\frac{L_{n-2}^{(\alpha+2)}(\lambda)}{L_n^{(\alpha)}(\lambda)}. \end{aligned} \tag{2.6}$$

The proof depends on (1.2), the integral

$$\int_0^1 p_{n,k}(t)t^\ell dt = \frac{(k+1)\cdots(k+\ell)}{(n+1)(n+2)\cdots(n+\ell+1)},$$

and elementary calculations.

Using Lemmas 1 and 2 and the Korovkin Theorem about linear positive operators, we get Theorem 1.

To obtain a result stronger than Theorem 1, we need to calculate $M_n^{(\lambda,\alpha)}((\cdot - x)^i;x)$ for $i = 3,4$. The following lemma is useful in the computation and is easy to prove.

Lemma 3. *For* $j \in \mathbb{N}$, *we have*

$$\sum_{k=0}^{n}\binom{n}{k} k(k-1)\cdots(k-j)x^k(1-x)^{n-k} = n(n-1)\cdots(n-j)x^{j+1}, \tag{2.7}$$

$$\sum_{k=0}^{n}\frac{(-\lambda)^k}{k!}\binom{n+\alpha}{n-k} k(k-1)\cdots(k-j) = (-\lambda)^{j+1}L_{n-j-1}^{(\alpha+j+1)}(\lambda). \tag{2.8}$$

From (2.5) and (2.6), we have $M_n^{(\lambda,\alpha)}((\cdot - x)^2;x) = o(\frac{1}{n})$. Using Lemma 3, we can obtain an explicit expression for $M_n^{(\lambda,\alpha)}(t^3;x)$ and $M_n^{(\lambda,\alpha)}(t^4;x)$ after some cumbersome calculations. We thus obtain

$$M_n^{(\lambda,\alpha)}(t^3;x) = \frac{1}{(n+2)(n+3)(n+4)}\sum_{j=0}^{3}\lambda^j A_{n,j}(x)\frac{L_{n-j}^{(\alpha+j)}(\lambda)}{L_n^{(\alpha)}(\lambda)} \tag{2.9}$$

where $A_{n,j}(x)$ are cubic polynomials in x and of degree $3-j$ in n. Similarly, we can obtain

$$M_n^{(\lambda,\alpha)}(t^4;x) = \frac{1}{(n+2)(n+3)(n+4)(n+5)}\sum_{j=0}^{4}\lambda^j B_{n,j}(x)\frac{L_{n-j}^{(\alpha+j)}(\lambda)}{L_n^{(\alpha)}(\lambda)} \tag{2.10}$$

where $B_{n,j}(x)$ are polynomials in x of degree 4 and of degree $4-j$ in n. It is easy to see from the above that

$$M_n^{(\lambda,\alpha)}((\cdot - x)^3;x) = \frac{1}{(n+2)(n+3)(n+4)}\sum_{j=0}^{3}c_{n,j}(x)\lambda^j\frac{L_{n-j}^{(\alpha+j)}(\lambda)}{L_n^{(\alpha)}(\lambda)} \tag{2.11}$$

and

$$M_n^{(\lambda,\alpha)}\big((\cdot - x)^4; x\big) = \frac{1}{(n+2)\cdots(n+5)} \sum_{j=0}^{4} d_{n,j}(x)\lambda^j \, \frac{L_{n-j}^{(\alpha+j)}(\lambda)}{L_n^{(\alpha)}(\lambda)} \tag{2.12}$$

where $c_{n,j}(x)$ are cubic polynomials in x and $d_{n,j}(x)$ are quartic polynomials in x.

From these, we can obtain

$$M_n^{(\lambda,\alpha)}\big((\cdot - x)^j; x\big) = O(n^{-\frac{j}{2}}), \quad j = 3, 4. \tag{2.12a}$$

We shall prove

Theorem 2. *If $f \in C^3[0,1]$ and $M_n^{(\lambda,\alpha)}(f;x)$ is given by (2.1), then we have*

$$\left\| M_n^{(\lambda,\alpha)}(f) - f + \frac{\lambda(1-x)L_{n-1}^{(\alpha+1)}(\lambda)}{(n+2)L_n^{(\alpha)}(\lambda)}\, f' - \frac{1}{n+2}\, P(D)f \right\|_{L_\infty[0,1]}$$
$$\leq C_1 \left(\frac{\lambda^2}{n^2} \frac{L_{n-2}^{(\alpha+2)}(\lambda)}{L_n^{(\alpha)}(\lambda)} + \frac{|\lambda|}{n^2} \frac{L_{n-1}^{(\alpha+1)}(\lambda)}{L_n^{(\alpha)}(\lambda)} + \frac{1}{n^2} \right) \|f''\|_{L_\infty[0,1]} + \frac{C_2}{n^{3/2}} \, \|f'''\|_{L_\infty[0,1]} \tag{2.13}$$

where C_1, C_2 are constants independent of n and f and $P(D) = \frac{d}{dx}\big(x(1-x)\big)\frac{d}{dx}$.

For $\lambda = 0$, this gives Voronovskaja's Theorem for the Durrmeyer operator.

Proof. Applying the operator $M_n^{(\lambda,\alpha)}$ to both sides of Taylor's expansions of $f(t) - f(x)$ yields

$$M_n^{(\lambda,\alpha)}(f;x) - f(x) = M_n^{(\lambda,\alpha)}\big((\cdot - x); x\big) f'(x) + \tfrac{1}{2} M_n^{(\lambda,\alpha)}\big((\cdot - x)^2; x\big) f''(x) + I_n(f;x)$$

where

$$I_n(f;x) = M_n^{(\lambda,\alpha)}\big(R_x(\cdot); x\big), \quad R_x(t) := \tfrac{1}{2} \int_x^t (t-v)^2 f'''(v)\, dv. \tag{2.14}$$

From (2.5) and (2.6) in Lemma 2, we have upon simplifying

$$M_n^{(\lambda,\alpha)}\big((\cdot - x); x\big) f'(x) + \frac{1}{2}\, M_n^{(\lambda,\alpha)}\big((\cdot - x)^2; x\big) f''(x)$$
$$= \frac{P(D)f}{(n+2)} - \frac{\lambda(1-x)L_{n-1}^{(\alpha+1)}(\lambda)}{L_n^{(\alpha)}(\lambda)} f'(x) + f''(x) \left[\frac{1 - 6x(1-x)}{(n+2)(n+3)} - \frac{2\lambda(2x-1)(x-1)}{(n+2)(n+3)} \frac{L_{n-1}^{(\alpha+1)}(\lambda)}{L_n^{(\alpha)}(\lambda)} + \frac{\lambda^2(1-x)^2 L_{n-2}^{(\alpha+2)}(\lambda)}{2(n+2)(n+3)L_n^{(\alpha)}(\lambda)} \right].$$

It remains to estimate $|I_n(f;x)|$. From (2.14), we have

$$|I_n(f;x)| \leq \frac{\|f'''\|(n+1)}{2L_n^{(\alpha)}(\lambda)} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} L_k^{(\alpha)}\left(\frac{\lambda}{x}\right) x^k (1-x)^{n-k} \int_0^1 p_{n,k}(t)\, \frac{|t-x|^3}{3}\, dt.$$

Using the Schwarz inequality first on the integral and then on the sum and using (2.12a), we obtain

$$|I_n(f;x)| \leq \frac{\|f'''\|}{6} \left[M_n^{(\lambda,\alpha)}\big((\cdot - x)^2; x\big) M_n^{(\lambda,\alpha)}\big((\cdot - x)^4; x\big) \right]^{1/2} \leq \frac{\|f'''\|}{6}\, O\left(\frac{1}{n^{3/2}}\right).$$

This completes the proof. □

3. The operator $P_n^{(\lambda,\alpha)}$

We now define the operator $P_n^{(\lambda,\alpha)}$ which maps polynomials of degree m into polynomials of degree $m+1$ except when $m=0$. For any $f \in C[0,\infty)$, we set

$$P_n^{(\lambda,\alpha)}(f;x) := n(1+x)^{-\alpha-1}\exp(\lambda x)\sum_{\nu=0}^{\infty} L_\nu^{(\alpha)}(\lambda)\left(\frac{x}{1+x}\right)^{\nu}\int_0^\infty e^{-nt}\,\frac{(nt)^\nu}{\nu!}\,f(t)\,dt \tag{3.1}$$

where $\lambda \le 0$ and $L_\nu^{(\alpha)}(\lambda)$ is the Laguerre polynomial of degree ν. Our definition is based on the following identity:

$$\sum_{\nu=0}^{\infty} L_\nu^{(\alpha)}(\lambda)\left(\frac{x}{1+x}\right)^{\nu} = (1+x)^{\alpha+1}e^{-\lambda x}, \tag{3.2}$$

(Szegö [9, p. 100, (5.1.9)]). It is clear from (3.2) and $n\int_0^\infty e^{-nt}\frac{(nt)^\nu}{\nu!}\,dt = 1$ that $P_n^{(\lambda,\alpha)}(1;x)=1$. Differentiating (3.2) with respect to x and simplifying, we have

$$\sum_{\nu=0}^{\infty} \nu L_\nu^{(\alpha)}(\lambda)\left(\frac{x}{1+x}\right)^{\nu} = e^{-\lambda x}(1+x)^{\alpha+1}[x(\alpha+1)-\lambda x(1+x)]. \tag{3.3}$$

This shows that

$$P_n^{(\lambda,\alpha)}(t;x) = \frac{\alpha x}{n} + \frac{(1+x)(1-\lambda x)}{n} \to x \quad \text{as } n\to\infty \text{ if } \frac{\alpha}{n}\to 1. \tag{3.4}$$

From now on, we shall take $\alpha = n$ and write $P_n^{(\lambda,n)}$ as P_n^λ. Again, we can evaluate $P_n^\lambda(t^2;x)$ in a similar way. Indeed, we obtain

$$\begin{aligned} P_n^\lambda(t^2;x) &= \frac{1}{n^2}\Big\{(n+1)x-\lambda x(1+x)\Big\}^2 \\ &\quad + \frac{1}{n^2}\Big\{2+x(x+4)(n+1)-2\lambda x(x+1)(x+2)\Big\}. \end{aligned} \tag{3.5}$$

This shows that $P_n^\lambda(t^2;x)\to x^2$ as $n\to\infty$. By Korovkin's theorem, it follows that if $f\in C[0,\infty)$, then $P_n^\lambda(f;x)$ converges uniformly to $f(x)$ on any compact subset of $[0,\infty)$.

It is easy to see that P_n^λ is a contraction operator. We now shall prove the following Voronovskaja-type result for the operator P_n^λ.

Theorem 3. *If $[a,b]$ is any closed interval in $[0,\infty)$ and if $f\in C^2[0,\infty)$, then*

$$\left\|P_n^\lambda(f;x)-f(x)-\frac{(1+x)(1-\lambda x)}{n}\,f'(x)\right\|_{L_\infty[a,b]} \le \frac{C}{n}\,\|f''\|_{L_\infty[0,\infty)}.$$

Proof. Using (3.3) and (3.4) and the Taylor expansion, we now can see that

$$\begin{aligned} &\left|P_n^\lambda(f;x)-f(x)-\frac{(1+x)(1-\lambda x)}{n}f'(x)\right| \\ &\quad = \left|n(1+x)^{-n-1}e^{\lambda x}\sum_{\nu=0}^{\infty}L_\nu^{(n)}(\lambda)\left(\frac{x}{1+x}\right)^{\nu}\int_0^\infty e^{-nt}\frac{(nt)^\nu}{\nu!}\int_x^t (t-v)f''(v)\,dv\,dt\right| \\ &\quad \le \frac{1}{2}\|f''\|_{L_\infty[0,\infty)}P_n^\lambda\big((\cdot-x)^2;x\big). \end{aligned}$$

Using (3.2), (3.3), and (3.5), we obtain $\|P_n^\lambda\big((\cdot-x)^2;x\big)\|_{L_\infty[a,b]} \le \frac{C}{n}$. □

Remark. An operator analogous to P_n^λ is obtained when $\lambda = 0$, on setting

$$\tilde{P}_n^0(f;x) := n(1+x)^{-n-1}\sum_{\mu=0}^{\infty}\binom{n+\mu}{\mu}\left(\frac{x}{1+x}\right)^{\mu} \times \int_0^\infty \binom{n+\mu}{\mu} t^\mu (1+t)^{-n-\mu-1} f(t)\,dt.$$

The above operator is very close to the operator defined by Gupta in [6]. It can be easily seen that

$$\tilde{P}_n^0(1;x) = 1, \quad \tilde{P}_n^0(t;x) = x + \frac{2x+1}{n-1}, \quad \tilde{P}_n^0(t^2;x) = x^2 + \frac{6nx^2+4(n+1)x+2}{(n-1)(n-2)}.$$

From these formulae, one can easily prove the following Voronovskaja-type result for the operator $\tilde{P}_n^0$ for a closed interval $[a,b]$ in $[0,\infty)$:

$$\left\|\tilde{P}_n^0(f) - f - \frac{2x+1}{n-1}f'\right\|_{L_\infty[a,b]} \le \frac{C}{n}\|f''\|_{L_\infty[0,\infty)}.$$

4. The operator $\tilde{S}_n^\lambda$

For a fixed $\beta \ge 0$, we shall denote by $L_{\infty,\beta}[0,\infty)$ the set of all functions satisfying

$$\|f\|_{L_{\infty,\beta}[0,\infty)} = \operatorname{esssup}_{t\in[0,\infty)}\left|f(t)e^{-\beta t}\right| < \infty.$$

If $f(t) \in L_{\infty,\beta}[0,\infty)$ and $\alpha_{n,k}^\lambda(x)$ is given by (1.5), we define

$$\tilde{S}_n^\lambda(f;x) := n\sum_{k=0}^{\infty}\alpha_{n,k}^\lambda(x)\int_0^\infty e^{-nt}\frac{(nt)^k}{k!}\,f(t)\,dt. \tag{4.1}$$

It is easy to see that $\tilde{S}_n^\lambda$ is a positive linear operator. We prove below some properties of $\tilde{S}_n^\lambda$.

Lemma 4. *The following relations hold for $\tilde{S}_n^\lambda(f;x)$:*

$$\tilde{S}_n^\lambda(1;x) = 1, \qquad \tilde{S}_n^\lambda(t;x) = x + \lambda\sqrt{\frac{x}{n}}\tanh(2\lambda\sqrt{nx}) + \frac{1}{n}, \tag{4.2}$$

$$\tilde{S}_n^\lambda(t^2;x) = x^2 + \frac{(\lambda^2+4)x}{n} + \frac{2}{n^2} + \left(2\lambda x + \frac{7\lambda}{2n}\right)\sqrt{\frac{x}{n}}\tanh(2\lambda\sqrt{nx}), \tag{4.3}$$

$$\tilde{S}_n^\lambda(t^3;x) = x^3 + \frac{(3\lambda^2+9)x^2}{n} + \frac{(15\lambda^2+24)x}{2n^2} + \frac{6}{n^3} + \tanh(2\lambda\sqrt{nx})\left(\frac{3\lambda x^{5/2}}{\sqrt{n}} + \frac{(2\lambda^3+33\lambda)x^{3/2}}{2n\sqrt{n}} + \frac{33\lambda}{4}\frac{x^{1/2}}{n^2\sqrt{n}}\right), \tag{4.4}$$

$$\tilde{S}_n^\lambda(t^4;x) = x^4 + \frac{6\lambda^2+16x^3}{n} + \frac{(\lambda^4+42\lambda^2+72)x^2}{n^2} + \left(\frac{207}{4}\lambda^2+96\right)\frac{x}{n^3} + \frac{24}{n^4} + \tanh(2\lambda\sqrt{nx})\left[\frac{4\lambda x^{7/2}}{\sqrt{n}} + \frac{(4\lambda^3+45\lambda)x^{5/2}}{n\sqrt{n}} + \frac{(13\lambda^3+123\lambda)x^{3/2}}{n^2\sqrt{n}} + \frac{57\lambda\sqrt{x}}{n^3\sqrt{n}}\right]. \tag{4.5}$$

Proof. From the identity (1.3), we get $\sum_{k=0}^{\infty} \alpha_{n,k}^{\lambda}(x) = 1$ which implies (4.2). Differentiating (1.3) with respect to x, we get on simplifying

$$\sum_{k=0}^{\infty} k\alpha_{n,k}^{\lambda}(x) = nx + \lambda\sqrt{nx}\tanh(2\lambda\sqrt{nx}). \tag{4.6}$$

Since $\tilde{S}_n^{\lambda}(t;x) = n\sum_{k=0}^{\infty} \alpha_{nk}^{\lambda}(x)\frac{(k+1)}{n^2} = \frac{1}{n} + \frac{1}{n}\sum_{k=0}^{\infty} k\alpha_{n,k}^{\lambda}(x)$, we obtain the second formula in (4.2). Differentiating (1.3) twice and simplifying, we obtain (4.3). Similarly we get (4.4) and (4.5). □

Lemma 4 leads easily to

Lemma 5. *For $\tilde{S}_n^{\lambda}(f;x)$ given by (4.1), we have*

$$\tilde{S}_n^{\lambda}((\cdot - x);x) = \lambda\sqrt{\frac{x}{n}}\tanh(2\lambda\sqrt{nx}) + \frac{1}{n}, \tag{4.7}$$

$$\tilde{S}_n^{\lambda}((\cdot - x)^2;x) = \frac{\lambda^2+2}{n}x + \frac{7\lambda\sqrt{x}}{2n\sqrt{n}}\tanh(2\lambda\sqrt{nx}) + \frac{2}{n^2}, \tag{4.8}$$

$$\begin{aligned}\tilde{S}_n^{\lambda}((\cdot - x)^3;x) &= \frac{(\lambda^3+6\lambda)x^{3/2}}{n^{3/2}}\tanh(2\lambda\sqrt{nx})\\ &\quad + \frac{(15\lambda^2+12)x}{2n^2} + \frac{33\lambda x^{1/2}}{4n^2\sqrt{n}}\tanh(2\lambda\sqrt{nx}) + \frac{6}{n^3},\end{aligned} \tag{4.9}$$

$$\begin{aligned}\tilde{S}_n^{\lambda}((\cdot - x)^4;x) &= \frac{(\lambda^4+12\lambda^2+12)x^2}{n^2} + \frac{(13\lambda^3+90\lambda)x^{3/2}}{n^{5/2}}\tanh(2\lambda\sqrt{nx})\\ &\quad + \frac{9(23\lambda^2+32)x}{4n^3} + \frac{57\lambda}{n^3}\sqrt{\frac{x}{n}}\tanh(2\lambda\sqrt{nx}) + \frac{24}{n^4}.\end{aligned} \tag{4.10}$$

Remark. The operator $\tilde{S}_n^{\lambda}$ is not a bounded operator from $L_{\infty,\beta}[0,\infty)$ to itself. To see this, we consider the case when $f(x) = e^{\beta x}$ which belongs to $L_{\infty,\beta}[0,\infty)$. Then using the relation (1.3), we have

$$\begin{aligned}\tilde{S}_n^{\lambda}(f;x) &= \frac{n}{n-\beta}\sum_{k=0}^{\infty} \alpha_{n,k}^{\lambda}(x)\left(\frac{n}{n-\beta}\right)^k\\ &= \frac{n}{n-\beta}\exp\left(\frac{n\beta x}{n-\beta}\right)\frac{\cosh\left(2\lambda\sqrt{\frac{n^2x}{n-\beta}}\right)}{\cosh(2\lambda\sqrt{nx})}.\end{aligned} \tag{4.11}$$

Thus we have

$$\tilde{S}_n^{\lambda}(f;x)e^{-\beta x} = \frac{n}{n-\beta}\exp\left(\frac{\beta^2 x}{n-\beta} + \frac{2\lambda\beta\sqrt{\frac{nx}{n-\beta}}}{\sqrt{n}+\sqrt{n-\beta}}\right)\frac{1+\exp\left(-4\lambda\sqrt{\frac{n^2x}{n-\beta}}\right)}{1+e^{-4\lambda\sqrt{nx}}} =: \Lambda(x). \tag{4.12}$$

Since the right side is bounded for fixed x, but is not so when $x \in [0,\infty)$, it follows that $\tilde{S}_n^{\lambda}(f;x) \notin L_{\infty,\beta}[0,\infty)$.

Lemma 6. *If $\tilde{S}_n^{\lambda}(f;x)$ is given by (4.1) and if $f \in L_{\infty,\beta}[0,\infty)$, then we have for any $\varepsilon > 0$*

$$\|\tilde{S}_n^{\lambda}(f;x)\|_{L_{\infty,\beta+\varepsilon}(0,\infty)} \le C(\lambda,\beta)\|f\|_{L_{\infty,\beta}[0,\infty)} \tag{4.13}$$

where $C(\lambda,\beta)$ is independent of n and f.

Proof. From (4.1), it follows that

$$\begin{aligned}|\tilde{S}_n^\lambda(f;x)e^{-(\beta+\varepsilon)x}| &\le \|f\|_{L_{\infty,\beta}[0,\infty)}e^{-(\beta+\varepsilon)x}n\sum_{k=0}^{\infty}\alpha_{nk}^\lambda(x)\int_0^\infty e^{-nt}\frac{(nt)^k}{k!}e^{\beta t}dt\\ &\le C(\lambda,\beta)\|f\|_{L_{\infty,\beta}[0,\infty)} \qquad (4.14)\end{aligned}$$

where $C(\lambda,\beta)$ is the bound on the half real line of $e^{-\varepsilon x}\Lambda(x)$, $\Lambda(x)$ being given by (4.12). □

We now shall prove a Voronovskaja-type theorem for $\tilde{S}_n^\lambda(f;x)$.

Theorem 4. *If $[a,b]$ is a closed interval in $[0,\infty)$ and if $f\in C^3[0,\infty)$, then for $\tilde{S}_n^\lambda(f;x)$ given by (4.1), we have*

$$\begin{aligned}&\left\|\tilde{S}_n^\lambda(f)-f-\frac{f'}{n}-\frac{(\lambda^2+2)x}{2n}f''-\lambda\sqrt{\frac{x}{n}}\tanh(2\lambda\sqrt{nx})\left(f'+\frac{7f''}{4n}\right)\right\|_{L_\infty[a,b]}\\ &\qquad\le\frac{1}{n^2}\|f''\|_{L_\infty[0,\infty)}+\frac{C}{n^{3/2}}\|f'''\|_{L_\infty[0,\infty)}. \qquad (4.15)\end{aligned}$$

Proof. Applying $\tilde{S}_n^\lambda$ to the Taylor expansion of f, we obtain

$$\tilde{S}_n^\lambda(f;x)-f(x)=\tilde{S}_n^\lambda((\cdot-x);x)f'(x)+\frac{1}{2}\tilde{S}_n^\lambda((\cdot-x)^2;x)f''(x)+I_n(f;x) \qquad (4.16)$$

where

$$I_n(f;x):=n\sum_{k=0}^{\infty}\alpha_{n,k}^\lambda(x)\int_0^\infty e^{-nt}\frac{(nt)^k}{k!}\int_x^t\frac{(t-v)^2}{2}f'''(v)\,dvdt. \qquad (4.17)$$

Clearly $|I_n(f;x)|$ is bounded from above by

$$\begin{aligned}&\frac{n}{6}\|f'''\|_{L_\infty[0,\infty)}\sum_{k=0}^{\infty}\alpha_{n,k}^\lambda(x)\int_0^\infty e^{-nt}\frac{(nt)^k}{k!}|t-x|^3dt\\ &\le\frac{1}{6}\|f'''\|_{L_\infty[0,\infty)}\left\{\tilde{S}_n^\lambda((\cdot-x)^2;x)\right\}^{1/2}\left\{\tilde{S}_n^\lambda((\cdot-x)^4;x)\right\}^{1/2}\\ &\le C\|f'''\|_{L_\infty[0,\infty)}\frac{1}{n^{3/2}}.\end{aligned}$$

The result (4.14) follows on using (4.8) and (4.10). □

For convergence, on $[0,\infty)$ we shall use the weighted $L_{\infty,\beta}[0,\infty)$ norm. We shall prove

Theorem 5. *If $\beta>0$ and $f''\in L_{\infty,\beta}[0,\infty)$, then for any $\varepsilon>0$*

$$\left\|\tilde{S}_n^{(\lambda)}(f)-f-\left(\frac{\lambda\sqrt{x}}{\sqrt{n}}\tanh(2\lambda\sqrt{nx})+\frac{1}{n}\right)f'\right\|_{L_{\infty,\beta+\varepsilon}[0,\infty)}\le\frac{C(\lambda,\beta)}{n}\|f''\|_{L_{\infty,\beta}[0,\infty)}$$

where $C(\lambda,\beta)$ is independent of n and f.

Proof. Using Taylor's formula and the formula (4.7), we have

$$\tilde{S}_n^{(\lambda)}(f;x)-f(x)-\left(\frac{1}{n}+\lambda\sqrt{\frac{x}{n}}\tanh\big(2\lambda\sqrt{nx}\big)\right)f'(x)=J_n(f;x)$$

where

$$J_n(f;x) := n\sum_{k=0}^{\infty}\alpha_{n,k}^{\lambda}(x)\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}\int_x^t (t-v)f''(v)\,dvdt.$$

Note that $|J_n(f;x)|\exp(-\beta-\varepsilon)x$ is bounded above by

$$\|f''\|_{L_{\infty,\beta}[0,\infty)}e^{-(\beta+\varepsilon)x}n\sum_{k=0}^{\infty}\alpha_{n,k}^{\lambda}(x)\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}\left|\int_x^t |t-v|e^{\beta v}dv\right|\,dt. \tag{4.18}$$

Since

$$\left|\int_x^t |t-v|e^{\beta v}dv\right| = \frac{1}{\beta}e^{\beta x}\left(x-\frac{1}{\beta}\right)-\frac{t}{\beta}e^{\beta x}+\frac{1}{\beta^2}e^{\beta t}$$

and $\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}e^{\beta t}dt = \frac{n^k}{(n-\beta)^{k+1}}$, it follows that

$$\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}\left|\int_x^t |t-v|e^{\beta v}dv\right|\,dt = \frac{e^{\beta x}}{n\beta}\left(x-\frac{1}{\beta}\right)-\frac{e^{\beta x}}{\beta}\frac{k+1}{n^2}+\frac{1}{\beta^2}\frac{n^k}{(n-\beta)^{k+1}}.$$

Now from (4.6), we have

$$\frac{1}{n}\sum_{k=0}^{\infty}\alpha_{n,k}^{\lambda}(x)(k+1) = x+\lambda\sqrt{\frac{x}{n}}\tanh\big(2\lambda\sqrt{nx}\big)+\frac{1}{n}$$

and

$$\begin{aligned} n\sum_{k=0}^{\infty}\alpha_{n,k}^{\lambda}(x)\frac{n^k}{(n-\beta)^{k+1}} &= \frac{n}{n-\beta}e^{-nx}\operatorname{sech}\big(2\lambda\sqrt{nx}\big)\sum_{k=0}^{\infty}(-1)^k\frac{H_{2k}(i\lambda)}{(2k)!}\left(\frac{n^2x}{n-\beta}\right)^k \\ &= \frac{n}{n-\beta}\exp\left(\frac{n\beta x}{n-\beta}\right)\frac{\cosh\big(2\lambda\sqrt{\frac{n^2x}{n-\beta}}\big)}{\cosh\big(2\lambda\sqrt{nx}\big)}. \end{aligned}$$

Therefore, from (4.18), we get after some simplification

$$\big|J_n(f;x)e^{-(\beta+\varepsilon)x}\big| \le \|f''\|_{L_{\infty}[0,\infty)}e^{-\varepsilon x}\left[\frac{1}{\beta n}+\frac{1}{\beta^2}|K_{n,1}(x)|+|K_{n,2}(x)|\right]$$

where

$$K_{n,1}(x) = \frac{\beta}{n-\beta}e^{\frac{\beta^2 x}{n-\beta}}\frac{\cosh\Big(2\lambda\sqrt{\frac{n^2x}{n-\beta}}\Big)}{\cosh(2\lambda\sqrt{nx})} \tag{4.19}$$

and

$$K_{n,2}(x) = e^{\frac{\beta^2 x}{n-\beta}}\frac{\cosh\Big(2\lambda\sqrt{\frac{n^2x}{n-\beta}}\Big)}{\cosh(2\lambda\sqrt{nx})}-1-\lambda\beta\sqrt{\frac{x}{n}}\frac{\sinh(2\lambda\sqrt{nx})}{\cosh(2\lambda\sqrt{nx})}. \tag{4.20}$$

In the following lemma, we shall obtain estimates for $|K_{n,1}(x)|e^{-\varepsilon x}$ and $|K_{n,2}(x)|e^{-\varepsilon x}$, which yield our result. □

Lemma 7. *For $K_{n,1}(x)$ and $K_{n,2}(x)$ given by* (4.19) *and* (4.20), *we have*

$$|K_{n,j}(x)|e^{-\varepsilon x} \le \frac{C(\lambda,\beta)}{n}, \qquad j=1,2. \tag{4.21}$$

Proof. Writing the hyperbolic cosines in terms of exponentials and simplifying, we get

$$K_{n,1}(x) = \frac{\beta}{n}\Lambda(x)$$

where $\Lambda(x)$ is given by (4.12) which yields (4.21) for $j = 1$. For $K_{n,2}(x)$, we rewrite

$$K_{n,2}(x) = \frac{e^{2\lambda\sqrt{nx}}}{2\cosh(2\lambda\sqrt{nx})}\left\{S_1(x) + S_2(x) + \lambda\beta\sqrt{\frac{x}{n}}e^{-4\lambda\sqrt{nx}}\right\} \tag{4.22}$$

where we set

$$S_1(x) := \exp\left[\frac{\beta^2 x}{n-\beta} + \frac{2\lambda\beta\sqrt{nx/(n-\beta)}}{\sqrt{n}+\sqrt{n-\beta}}\right] - 1 - \lambda\beta\sqrt{\frac{x}{n}},$$

$$S_2(x) := \exp\left[\frac{\beta^2 x}{n-\beta} - 2\lambda\sqrt{\frac{nx}{n-\beta}}(\sqrt{n}+\sqrt{n-\beta})\right] - e^{-4\lambda\sqrt{nx}}.$$

Putting $a := \frac{\beta^2 x}{n-\beta} + \frac{2\lambda\beta\sqrt{nx/(n-\beta)}}{\sqrt{n}+\sqrt{n-\beta}}$ and using the inequality $0 \le e^a - 1 - a \le a^2 e^a/2$ for $a \ge 0$, we see that

$$|S_1(x)| \le \frac{1}{2}a^2 e^a + \left|a - \lambda\beta\sqrt{\frac{x}{n}}\right|.$$

On the other hand, we can write

$$\begin{aligned}
|S_2(x)| &= \left|e^{-4\lambda\sqrt{nx}}\left[\exp\left(\frac{\beta^2 x}{n-\beta} - \frac{2\lambda\sqrt{nx/(n-\beta)}}{\sqrt{n}+\sqrt{n-\beta}}\right) - 1\right]\right| \\
&\le \frac{1}{1+4\lambda\sqrt{nx}}\left|\frac{\beta^2 x}{n-\beta} - \frac{2\lambda\beta\sqrt{nx/(n-\beta)}}{\sqrt{n}+\sqrt{n-\beta}}\right| \exp\left[\frac{\beta^2 x}{n-\beta} - \frac{2\lambda\beta\sqrt{nx/(n-\beta)}}{\sqrt{n}+\sqrt{n-\beta}}\right] \\
&\le C(\lambda,\beta)\left(\frac{\sqrt{x}}{n^{3/2}} + \frac{1}{n}\right)\exp\left[\frac{\beta^2 x}{n-\beta} - \frac{2\lambda\beta\sqrt{nx/(n-\beta)}}{\sqrt{n}+\sqrt{n-\beta}}\right]
\end{aligned}$$

on using inequalities $e^a - 1 \le ae^a$ for all a and $e^{-a} \le (1+a)^{-1}$ for $a \ge 0$. Similarly, we have

$$\left|\beta\lambda\sqrt{\frac{x}{n}}e^{-4\lambda\sqrt{nx}}\right| \le \frac{\beta\lambda}{1+4\lambda\sqrt{nx}}\sqrt{\frac{x}{n}} \le \frac{C(\lambda,\beta)}{n}.$$

Now notice that

$$e^{-\varepsilon x}\frac{e^{2\lambda\sqrt{nx}}}{\cosh(2\lambda\sqrt{nx})}x^b \exp\left(\frac{\beta^2 x}{n-\beta} \pm \frac{2\lambda\beta\sqrt{nx/(n-\beta)}}{\sqrt{n}+\sqrt{n-\beta}}\right) \le C(\lambda,\beta)$$

and for any $b \ge 0$, we have

$$e^{-\varepsilon x}\frac{e^{2\lambda\sqrt{nx}}}{\cosh(2\lambda\sqrt{nx})}x^b \le C(\lambda,\beta),$$

$$e^{-\varepsilon x}\frac{e^{2\lambda\sqrt{nx}}}{2\cosh(2\lambda\sqrt{nx})}|S_j(x)| \le \frac{C(\lambda,\beta)}{n}, \qquad j = 1, 2,$$

from which (4.21) follows for $j = 2$. □

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