HAMILTONIAN FLOWS ON STATIONARY MANIFOLDS

Allan P. Fordy and Simon D. Harris

Dedicated to Martin Kruskal to celebrate his 70th birthday.

ABSTRACT. We consider the important question of the relationship between an "integrable" nonlinear evolution equation, a partial differential equation (PDE), and its stationary reduction, an ordinary differential equation (ODE). We are mainly concerned with "transporting" algebraic machinery from the PDE to the ODE. In particular, given a Hamiltonian representation of the original PDE, we would like to *derive* the Poisson brackets, Hamiltonians, etc. for the stationary flow.

In this paper, we give a *systematic* derivation of these Hamiltonian properties, using an $x - t$ reversed form of the spectral problem for the PDE. We obtain, in this way, the Hamiltonian structures, etc. of the PDE, but written in terms of the stationary manifold coordinates. These structures naturally reduce to their finite dimensional counterparts for the stationary flow.

We illustrate our results with several examples, including the Korteweg–de Vries, dispersive water wave, and Ito hierarchies.

1. Introduction

When Martin Kruskal and his collaborators discovered the soliton and introduced the Inverse Scattering Method for solving the Korteweg–de Vries (KdV) equation, they opened up a whole new field which exploded into numerous subfields having repercussions far beyond the subject of nonlinear wave equations. One of the early discoveries was that the KdV equation can be regarded as a completely integrable, infinite dimensional Hamiltonian system [14, 23]. This gave a whole new impetus to the general theory of completely integrable Hamiltonian systems, (both infinite and finite dimensional). During the 19th Century, there had been great interest in such systems, but the subject fell from grace when Bruns and Poincaré showed the non-integrability of the 3-body problem. However, following the discovery of the KdV's Hamiltonian properties, many new examples were found, such as the Toda lattice, Calogero-Moser system, and numerous partial differential equations (PDEs). At the end of the 1970's, two very important discoveries were made.

1. Bogoyavlenskii and Novikov [7] showed that each of the *stationary reductions* of the KdV hierarchy constitutes a completely integrable (finite dimensional) Hamiltonian system.
2. Magri [18] and Gelfand and Dorfman [15] independently developed the theory of "bi-Hamiltonian" systems, giving the KdV equation as a prime example.

The first of these is concerned with *stationary flows*, meaning that we consider solutions $u(x, t)$ (for example) of a PDE (generally a system) which are *independent of* t. Such solutions satisfy an ordinary differential equation (ODE) (generally a system). In simple cases, such as the KdV equation, this is obviously a Hamiltonian system (even Newtonian). Generally, this is highly non-trivial and follows (in our context) from the Hamiltonian property of the PDE. As noted in (1) above, this Hamiltonian system of ODEs can actually be completely integrable.

The "second" (local) Hamiltonian structure of the KdV equation (noted in (2) above) is perhaps best understood in the context of Miura's "Remarkable explicit nonlinear transformation" [19], which is, in fact, a Poisson map, under which the second Poisson bracket of the KdV equation is just the image of the single (local) Poisson bracket of the modified KdV (MKdV) equation. These ideas were generalised in the 1980s, being developed for general Lax and many other classes of equations [11, 12, 17, 3].

In [5], it was shown that the stationary flows of the KdV and MKdV hierarchies are also bi-Hamiltonian. Thus, we have examples of bi-Hamiltonian PDEs whose stationary flows are also bi-Hamiltonian. However, as has often been pointed out (see, for example, [8]), the situation is unsatisfactory, in that the *relationship* between structures (such as Poisson brackets) associated with a PDE and those associated with its stationary reduction is rather obscure. Ideally, we would like to "transport" *known* machinery from an infinite dimensional system, directly to its finite dimensional reductions. However, the original approach of Bogoyavlenskii and Novikov required going out of the Lax pair/zero curvature framework and using a Lagrangian representation to start afresh. Whilst the Lagrangian itself is derived from a PDE Hamiltonian density, the resulting Poisson bracket bears no resemblance to that of the PDE. The situation was improved by Antonowicz and Blaszak [1], who used $x - t$ reversal (see the discussion leading to equation (24) below) together with the Miura map (following the approach of [5]).

In this paper, we give a systematic treatment of this problem, by starting with a zero curvature formulation of the given PDE in $x - t$ reversed coordinates. Using techniques employed in [3], we systematically derive (from a knowledge of the zero-curvature representation) a Hamiltonian formulation of the PDE in $x - t$ reversed coordinates (our examples are, in fact, bi- or tri-Hamiltonian). The resulting infinite dimensional Poisson brackets naturally reduce to their finite dimensional counterparts for the stationary flows. The basic approach is to rearrange the Lax or zero curvature equation in an appropriate way. In the case of a "generic" spectral problem, this is straightforward, but for our stationary flows, which are deep reductions of the generic case, the calculation is highly non-trivial.

In Section 2, we briefly describe how to build Hamiltonian operators from a zero curvature representation, using the approach of [3, 4]. We first describe the generic case, illustrating these ideas with the KdV hierarchy. For the benefit of the non-specialist, we use the latter as a vehicle for introducing a number of "well-known" ideas and facts, such as Miura maps and the construction of conserved densities. This section also contains some important background information on stationary flows and the idea of $x - t$ reversal. We refer the reader to [4] for a more formal and complete exposition of all these ideas.

The main results of this paper are in Section 3, where we show how to construct the Hamiltonian structure of a PDE in terms of any coordinate system on the stationary manifold. The main difficulty here stems from the fact that the zero curvature representation is a deep reduction of the generic case. After some general discussion, we consider the case of the stationary KdV equation. Whilst this example is suitable for introducing a number of ideas, it is too special. The stationary 5th-order KdV equation is much more representative, so it is studied next in this section.

In Section 4, we consider some *systems* of PDEs associated with "energy dependent" Schrödinger operators. In particular, we discuss the dispersive water wave (DWW) and Ito hierarchies. In these cases, it is possible to use the *recursion operator* of the PDE to derive a third Hamiltonian structure. Whilst this *construction* would be invalid in the stationary reduction, the resulting Poisson brackets *do* reduce correctly.

2. The zero curvature representation in Hamiltonian form

Since the early discoveries of Gardner, Greene, Kruskal, and Miura, there has been interest in equations with a so-called *Lax* or *zero curvature* representation. We consider *nonlinear* evolution equations which can be represented as the integrability conditions of a pair of *linear* equations:

$$\Psi_x = U\Psi, \tag{1}$$

$$\Psi_t = V\Psi \tag{2}$$

where U and V are $N \times N$ matrices, dependent both upon the functions entering the nonlinear equations and upon a spectral parameter λ. The integrability conditions are

$$U_t - V_x + [U, V] = 0. \tag{3}$$

This can be interpreted as the zero curvature condition for the Lie algebra-valued connection defined by (1) and (2). With $\lambda_t = 0$, this is said to define an *isospectral flow* of the *spectral problem* (1). For a given U, the matrix V can be determined algorithmically for simple classes of λ−dependence. We write (3) in the form

$$U_t = (\partial - \operatorname{ad} U)V \quad \text{with } \partial \equiv \partial_x \tag{4}$$

where $(\operatorname{ad} U)V = [U, V]$. These can be represented with respect to some basis, with $U = u_i e_i$:

$$\sum_i u_{it} e_i = -\left(\sum_{i,j} B_{ij} e_i \otimes e_j\right)\left(\sum_k v_k^* e_k^*\right) = -\sum_{i,j}\left(B_{ij} v_j^*\right) e_i,$$

with e_i^* the dual basis of e_i , so that $B_{ij} e_i \otimes e_j$ is a $(2,0)$ tensor whose components are (at most) first-order differential operators. In components,

$$u_{it} = -B_{ij} v_j^*, \quad v_i^* = g_{ij} v_j, \quad g_{ij} = tr(e_i e_j). \tag{5}$$

Here the matrix **B** (components B_{ij}) is the *Hamiltonian operator* and the vector $(v_1^*, \ldots, v_N^*)^T$ is the gradient of some Hamiltonian functional. The simplest way of calculating this is to check the trace form $tr(U_t V)$ since

$$tr(U_t V) = \mathbf{u}_t \cdot \mathbf{v}^*.$$

We are mainly interested in the *hierarchy* of isospectral flows of a spectral problem with U dependent upon λ and the "potential functions" in some specified way. We

often have polynomial dependence upon λ, generating a sequence of *polynomial flows*, labeled by "times" t_m:

$$U_{t_m} - V_{(m)x} + [U, V_{(m)}] = 0. \tag{6}$$

All the examples of this paper involve second-order spectral problems, so we just present the case $N = 2$ here (see [16] for some higher order examples).

Example 2.1 (2×2 matrices). We have

$$U = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad V = \begin{pmatrix} \gamma & \alpha \\ \beta & -\gamma \end{pmatrix},$$

and define

$$\mathbf{u} = (b, c, a)^T, \mathbf{v} = (\alpha, \beta, \gamma)^T, \quad \text{giving} \quad \mathbf{v}^* = (\beta, \alpha, 2\gamma)^T = -\delta_u h.$$

The above formula gives

$$\mathbf{B} = \begin{pmatrix} 0 & -\partial + 2a & -b \\ -\partial - 2a & 0 & c \\ b & -c & -\frac{1}{2}\partial \end{pmatrix}. \tag{7}$$

With some λ-dependence of a, b, and c, these can form blocks within larger Hamiltonian structures. In [9], this is used to generate multi-Hamiltonian structures. When U is linear in λ, we get a bi-Hamiltonian formulation.

Example 2.2 (The linear pencil: $w = \lambda + w_0$, $q = q_0$, $r = r_0$). This is the case of the generalised Zakharov-Shabat (ZS) spectral problem, which immediately gives the bi-Hamiltonian formulation

$$u_t = (\mathbf{B}_1 - \lambda \mathbf{B}_0)\delta h, \quad \delta h \equiv (\delta_{q_0} h, \delta_{r_0} h, \delta_{w_0} h)^T,$$

written explicitly as

$$\begin{pmatrix} q_0 \\ r_0 \\ w_0 \end{pmatrix}_t = \left[\begin{pmatrix} 0 & -\partial + 2w_0 & -q_0 \\ -\partial - 2w_0 & 0 & r_0 \\ q_0 & -r_0 & -\frac{1}{2}\partial \end{pmatrix} - \lambda \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \beta \\ \alpha \\ 2\gamma \end{pmatrix}.$$

This example is deceptively simple as a consequence of the simple dependence upon λ. The spectral problems associated with stationary flows are deep reductions of the generic case and have complicated λ-dependence.

2.1. The KdV hierarchy. If, in example (2.1), we put $a = 0$, $b = 1$, $c = \frac{1}{4}\lambda - u$, corresponding to the Schrödinger equation $\psi_{1xx} + u\psi_1 = \frac{1}{4}\lambda\psi_1$, then we find

$$\gamma = -\frac{1}{2}\alpha_x, \quad \beta = -\frac{1}{2}\alpha_{xx} - (u - \frac{1}{4}\lambda)\alpha, \tag{8}$$

and

$$u_t = (\mathbf{B}_1 - \lambda \mathbf{B}_0)\,\alpha,$$

where

$$\mathbf{B}_0 = \frac{1}{2}\partial, \quad \mathbf{B}_1 = \frac{1}{2}\partial^3 + 2u\partial + u_x. \tag{9}$$

In a simple way, this has generated the bi-Hamiltonian formulation of the KdV hierarchy, with $\alpha = \delta_u \mathcal{H}$. Taking α to be a polynomial expansion in λ gives an element of the usual KdV hierarchy. The first three polynomial flows are $u_{t_0} = u_x$ and

$$u_{t_1} = u_{xxx} + 6uu_x, \tag{10}$$

$$u_{t_2} = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x, \tag{11}$$

which correspond, respectively, to $\alpha_{(0)} = 1$, $\alpha_{(1)} = \lambda + 2u$, and $\alpha_{(2)} = \lambda^2 + 2\lambda u + 2u_{xx} + 6u^2$.

The Miura map. The second Hamiltonian structure $\mathbf{B}_1$ is the image of a much simpler structure through the Miura map. By adding a parameter, we get Gardner's generalisation, which enables us to obtain *both* Hamiltonian structures and an infinite number of conservation laws. Explicitly, this takes the form

$$u = M[v] \equiv -v_x - v^2 + \frac{1}{4}\lambda, \tag{12}$$

The Jacobian (Fréchet derivative) of this map is defined by $M'[v]w = \frac{d}{d\varepsilon}M[v+\varepsilon w]\big|_{\varepsilon=0}$, giving $M' = -\partial - 2v$. As in the finite dimensional case, the Jacobian is used to "transport" Poisson brackets (now infinite dimensional). Generally, with

$$v_t = \tilde{\mathbf{B}}\delta_v\tilde{\mathcal{H}}$$

where $\tilde{\mathbf{B}}$ is some Hamiltonian operator, and $u = M[v]$, we have

$$u_t = M'v_t \quad \text{and} \quad \delta_v\tilde{\mathcal{H}} = (M')^\dagger \delta_u \mathcal{H}$$

where $\tilde{\mathcal{H}}[v]$ and $\mathcal{H}[u]$ are related by $\tilde{\mathcal{H}} = \mathcal{H} \circ M \pmod{\text{Im}\partial}$ and $(M')^\dagger$ is the formal adjoint of M'. Piecing these formulae together, we get

$$u_t = M'\tilde{\mathbf{B}}(M')^\dagger \delta_u \mathcal{H}.$$

In this way, the single (local) Hamiltonian structure of the MKdV hierarchy, $-\frac{1}{2}\partial$, is transported to give

$$M'\left(-\frac{1}{2}\partial\right)(M')^\dagger = \frac{1}{2}\partial^3 + 2u\partial + u_x - \frac{1}{2}\lambda\partial \equiv \mathbf{B}_1 - \lambda\mathbf{B}_0.$$

Each Hamiltonian functional, $\mathcal{H}[u]$, is pulled back to a functional (an equivalence class of functionals) of v:

$$\tilde{\mathcal{H}}[v] = \mathcal{H}[M[v]],$$

and we have the mapping of Hamiltonian vector fields

$$v_t = -\frac{1}{2}\partial\delta_v\tilde{\mathcal{H}} \quad \mapsto \quad u_t = (\mathbf{B}_1 - \lambda\mathbf{B}_0)\delta_u\mathcal{H}.$$

The Miura map can be used to build an infinite family of commuting Hamiltonians. We introduce the asymptotic expansion

$$v = -\frac{1}{2}k + uk^{-1} + \sum_{i=2}^{\infty} v_i k^{-i}, \quad \lambda = k^2, \tag{13}$$

which starts as

$$v_2 = u_x, \qquad\qquad v_3 = u^2 + u_{xx},$$
$$v_4 = \left(2u^2 + u_{xx}\right)_x, \quad v_5 = 2u^3 - u_x^2 + (u_{xxx} + 6uu_x)_x,$$

and satisfies (for $i \geq 1$)

$$v_{i+1} = v_{ix} + \sum_{j=0}^{i} v_j v_{i-j},$$

so, at each level, v_{i+1} can be solved in terms of a *local* functional of $v_1, \ldots, v_i$. It is easy to prove that the even coefficients v_{2i} are exact derivatives, so that

$$\delta_u v = k^{-1} \left(1 + 2u\lambda^{-1} + \left(2u_{xx} + 6u^2\right)\lambda^{-2} + \cdots\right).$$

If we define $\mathcal{H}_n = v_{2n+1}$, $n = 0, 1, \ldots$, we get $\delta_u v = k^{-1} \sum_{n=0}^{\infty} \lambda^{-n} \delta_u \mathcal{H}_n$. Since

$$(\mathbf{B}_1 - \lambda \mathbf{B}_0)\, \delta_u v = -\frac{1}{2} M' \partial \delta_v v = 0,$$

we may define $\alpha = k\delta_u v = \sum_{n=0}^{\infty} \lambda^{-n} \delta_u \mathcal{H}_n$, and the mth flow by

$$u_{t_m} = (\mathbf{B}_1 - \lambda \mathbf{B}_0)\, \alpha_{(m)}, \quad \alpha_{(m)} = (\lambda^m \alpha)_+$$

where the subscript "+" means taking only the non-negative powers of λ. This gives the bi-Hamiltonian representation

$$u_{t_m} = \mathbf{B}_1 \delta_u \mathcal{H}_m = \mathbf{B}_0 \delta_u \mathcal{H}_{m+1}. \tag{14}$$

This bi-Hamiltonian ladder property guarantees that the Hamiltonians $\mathcal{H}_n$ are in involution with respect to *both* Poisson brackets. The flow (14) is the image of the vector field

$$v_{t_m} = -\frac{1}{2} \partial\, \beta_{(m)}, \quad \beta_{(m)} = (M')^\dagger \alpha_{(m)}\big|_{u=M[v]}. \tag{15}$$

The first two flows and their "modifications" are

$$\alpha_{(0)} = 1, \quad \beta_{(0)} = -2v \qquad \Rightarrow \qquad u_{t_0} = u_x, \quad v_{t_0} = v_x,$$

$$\left.\begin{array}{l} \alpha_{(1)} = \lambda + 2u, \\[4pt] \beta_{(1)} = -3\lambda v + 4v^3 - 2v_{xx} \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} u_{t_1} = u_{xxx} + 6uu_x, \\[6pt] v_{t_1} = v_{xxx} - 6v^2 v_x + \dfrac{3}{2}\lambda v_x. \end{array}\right. \tag{16}$$

For each flow (15), $-\frac{1}{2}\beta_{(m)}$ is an expression in v and its x-derivatives. By substituting (13) into (15) and equating coefficients of k, we obtain an infinite sequence of conservation laws, obeyed by each member of the hierarchy

$$\mathcal{H}_{n t_m} = \mathcal{F}_{nm\,x}.$$

The *fluxes* $\mathcal{F}_{nm}$ play the role of conserved densities when we consider the $x-t_m$ reversal in later sections. The above discussion of the Miura map and its uses is very important for what follows. These ideas and methods have a wide range of applicability in the theory of integrable systems (see [4] for a review).

Stationary reductions. Bogoyavlenskii and Novikov [7] used a generalised Legendre transformation to write the stationary flows of the KdV hierarchy in Hamiltonian form and then proved complete integrability. In [5], the Miura map was used to construct a second Hamiltonian structure, and in [1], the equations were viewed with x and t reversing their roles.

The mth-stationary flow in the KdV hierarchy is defined by

$$u_{t_m} = \frac{1}{2}\partial\delta_u\mathcal{H}_{m+1} = 0, \quad m \geq 1,$$

which is in the form of a generalised Lagrangian equation (see the Appendix)

$$\delta_u\mathcal{L}_{m+1} = 0, \quad \mathcal{L}_{m+1} = \mathcal{H}_{m+1} - c_m u.$$

Remark 2.1. Not all operators have such a simple kernel. For the KdV hierarchy, Ker $\mathbf{B}_0 = \delta_u(cu)$, but Ker $\mathbf{B}_1$ is not so simple. For $\mathbf{B}_1$, however, it is possible to extend the space of variables in order to achieve this [6]. Generally, constants such as c above arise from the kernel of the relevant Hamiltonian operator $\mathbf{B}$ (see the examples of Section 4).

It is possible to write these Lagrangian equations as an m-degrees of freedom canonical Hamiltonian system. If we take c_m as a dynamical variable, then the Poisson bracket is degenerate, with c_m as Casimir:

$$\{f,g\} = \frac{\partial f}{\partial x_i}J_{ij}\frac{\partial g}{\partial x_j} = (\nabla f)^T J(\nabla g), \quad J = \begin{pmatrix} 0 & I_m & 0 \\ -I_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where I_m is the $m \times m$ unit matrix. Bogoyavlenskii and Novikov [7] showed that this Hamiltonian system is completely integrable with first integrals given by the fluxes of the Hamiltonians $\mathcal{H}_1, \ldots, \mathcal{H}_m$:

$$0 = \mathcal{H}_{0t_m} = \mathcal{F}_{0mx},$$
$$0 = \mathcal{H}_{kt_m} = \mathcal{F}_{kmx} \quad k = 1, \ldots, m,$$

where the first of these is just the equation, which is an ODE of order $2m$. This is used to eliminate higher derivatives from the fluxes $\mathcal{F}_{km}$, which then can be written as functions of the variables (q_i, p_i, c_m): $\mathcal{F}_{km} = h_k(q_i, p_i, c_m)$, $k = 1, \ldots, m$. The later fluxes become functionally dependent upon $h_1, \ldots, h_m$. Since $\{H_i, H_m\} = 0$ for all elements of the KdV hierarchy, the stationary manifold is invariant under the action of the flows. The first components of each of the commuting Hamiltonian flows of the stationary flow are precisely

$$u_{t_k} = \frac{1}{2}\partial\delta\mathcal{H}_{k+1}, \quad k = 1, \ldots, m-1,$$

when written in terms of the stationary manifold coordinates, since $q_1 = u$.

The Miura map between the KdV and MKdV equations induces a diffeomorphism between corresponding stationary manifolds. The fluxes/Hamiltonians for the stationary flows are calculated directly from

$$v_{t_m} = -\frac{1}{2}\tilde{c}_{mx}$$

with $\tilde{c}_m$ given as a differential polynomial in v. Thus, substituting the expansion for v into $\tilde{c}_m$, we directly construct the fluxes as coefficients in the k-expansion of $\tilde{c}_m$.

Furthermore, it is possible to use the Miura map to construct the second Hamiltonian structure for the stationary flows.

Example 2.3 (The stationary KdV equation). The stationary KdV equation (which defines c_1) is

$$2u_{xx} + 6u^2 = c_1, \tag{17}$$

with Lagrangian $\mathcal{L}$ given by $\mathcal{L} = 2u^3 - u_x^2 - c_1 u$ and canonical coordinates

$$q = u, \quad p = -2u_x, \quad c_1, \tag{18}$$

c_1 being the Casimir of the degenerate canonical bracket. The Legendre transformation gives

$$h_{\mathcal{L}} = c_1 q - 2q^3 - \frac{1}{4}p^2,$$

which is the flux of $\mathcal{H}_1 = u^2$.

The Miura map induces the diffeomorphism M given by

$$\begin{aligned}
q &= \frac{1}{4}\lambda - \frac{1}{2}\tilde{p} - \tilde{q}^2, \\
p &= -\tilde{c}_1 + 4\tilde{q}^3 + 2\tilde{q}\tilde{p} - 3\lambda\tilde{q}, \\
c_1 &= \frac{1}{2}\tilde{p}^2 - 2\tilde{q}^4 + 2\tilde{q}\tilde{c}_1 + 3\lambda\tilde{q}^2 + \frac{3}{8}\lambda^2,
\end{aligned} \tag{19}$$

relating (17) to the stationary MKdV equation

$$-2v_{xx} + 4v^3 - 3\lambda v = \tilde{c}_1, \tag{20}$$

which has Lagrangian $\mathcal{L} = v_x^2 + v^4 - \tilde{c}_1 v - \frac{3}{2}\lambda v^2$ and canonical coordinates

$$\tilde{q} = v, \quad \tilde{p} = 2v_x, \quad \tilde{c}_1.$$

The canonical Poisson brackets between the modified variables

$$\{\tilde{q}, \tilde{p}\} = 1, \quad \{\tilde{q}, \tilde{c}_1\} = \{\tilde{p}, \tilde{c}_1\} = 0$$

induce

$$\{q, p\} = -2q - \lambda, \quad \{q, c_1\} = -p, \quad \{p, c_1\} = -2c_1 + 12q^2. \tag{21}$$

This is a non-canonical Poisson bracket for the stationary KdV equation. In fact, it is a one-parameter family of Poisson brackets; the coefficient of λ is just the canonical bracket, while the remaining part is the *second* bracket [5]. This proves not only that each coefficient is independently a Poisson bracket, but also that they are compatible.

The variable c_1 is a Casimir for the degenerate canonical bracket with its level surfaces being symplectic leaves. The symplectic leaves of the second Poisson bracket are different, being the level surfaces of $h_{\mathcal{L}}$, while c_1 now generates the stationary KdV equation.

Reversing the role of x and t. The KdV equation (10) defines a flow on the infinite dimensional space with coordinates $(u, u_x, u_{xx}, \dots)$ and *evolution parameter t*. Since we are interested in the stationary flow, we consider the KdV equation (a PDE) as a special flow on the infinite dimensional space with coordinates $(q, p, c_1, q_t, p_t, c_{1t}, \dots)$ where q, p, and c_1 are defined by (17)–(18), and with *evolution parameter x*. We now must give the first *three* components of this flow:

$$\begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2}p \\ 6q^2 - c_1 \\ 2q_t \end{pmatrix}.$$

In [1], this flow was shown to be Hamiltonian, with a 3×3 matrix Poisson bracket, *which reduces to the stationary case when the potential functions are independent of t*:

$$\begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2\partial_t \end{pmatrix} \begin{pmatrix} c_1 - 6q^2 \\ -\frac{1}{2}p \\ q \end{pmatrix} \tag{22}$$

where this last vector is just $\delta_s(c_1 q - 2q^3 - \frac{1}{4}p^2)$ where $\delta_s h = (\delta_q h, \delta_p h, \delta_{c_1} h)^T$ denotes the gradient of h with respect to the stationary coordinates. This could have been done in *any* coordinate system, generally producing a non-canonical Poisson bracket. However, with the same canonical variables as before, the equation took "canonical" form.

Similarly, the MKdV equation (see (16)) can be written

$$\begin{pmatrix} \tilde{q} \\ \tilde{p} \\ \tilde{c}_1 \end{pmatrix}_x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2\partial_t \end{pmatrix} \begin{pmatrix} \tilde{c}_1 - 4\tilde{q}^3 + 3\lambda\tilde{q} \\ \frac{1}{2}\tilde{p} \\ \tilde{q} \end{pmatrix} = \hat{\mathbf{B}}\delta_{\tilde{s}}\tilde{h} \tag{23}$$

where $\tilde{h} = \tilde{q}\tilde{c}_1 - \tilde{q}^4 + \frac{1}{4}\tilde{p}^2 + \frac{3}{2}\lambda\tilde{q}^2$, and $\delta_{\tilde{s}}$ denotes the gradient with respect to the modified stationary coordinates.

The Miura map (19) is only changed by $-2\tilde{q}_t$ in the definition of c_1. As shown in [1], the above bracket for the MKdV equation then is transformed to

$$\begin{pmatrix} 0 & -2q & \partial_t - p \\ 2q & -2\partial_t & 12q^2 - 2c_1 \\ \partial_t + p & 2c_1 - 12q^2 & 4(q\partial_t + \partial_t q) \end{pmatrix} - \lambda \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2\partial_t \end{pmatrix}. \tag{24}$$

These define Poisson brackets for the *full* KdV equation written in terms of the stationary manifold coordinates and clearly reduce to those (see (21)) of the stationary equations when the potential functions are independent of t.

Such $x - t$ reversed flows and Poisson brackets also have been considered in [13, 20].

3. The $x - t$ reversed spectral problem

The above derivation of (24) was ad hoc, suited only to simple examples. We now fit our stationary flows into the scheme described in Section 2 and thus treat them as isospectral flows. In so doing, we give a systematic construction of Hamiltonian structures, Hamiltonians, and time evolutions of the "wave functions" (of the associated spectral problem). For the latter, we give a formula in terms of gradients of the corresponding Hamiltonians. In this section, we mainly present this in the context of

the KdV hierarchy, but start with some general statements. Other examples are dealt with in the next section and in [16].

Consider an equation in some hierarchy with zero curvature representation (6). For simplicity of notation, we restrict attention to scalar equations, but multi-component examples present no difficulty (see Section 4). In order to consider the t_m stationary flow, we just reverse the roles of U (which equals $V_{(0)}$) and $V_{(m)}$ (which now is playing the role of the spectral problem). We write the zero curvature equations as

$$V_{(m)x} = U_\xi - [V_{(m)}, U]$$

where $\xi = t_m$. It should be noted (see the examples) that $V_{(m)}$ depends upon u and some (but not too many) of its x-derivatives. Following the argument which led to (22), we write this in terms of some set of coordinates on the stationary manifold. For comparison of results, we choose the canonical coordinates.

We write $V_{(m)}$, when written in terms of the stationary manifold coordinates, as $\mathcal{U}$ and consider a general evolution (labeled by τ) with zero curvature representation

$$\mathcal{U}_\tau = \mathcal{V}_\xi - [\mathcal{U}, \mathcal{V}]. \tag{25}$$

The particular choice $\mathcal{V} = U = V_{(0)}$ corresponds to our original equation in $x - t$ reversed form (in terms of the stationary manifold coordinates). All the examples of this paper are associated with 2×2 matrix problems, so we now restrict to this case (see [16] for other cases). We have

$$\mathcal{U} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} \gamma & \alpha \\ \beta & -\gamma \end{pmatrix} \tag{26}$$

where a, b, and c are functions of q_i, p_i, c_i, and λ. Except for very simple cases (such as the KdV equation), there are more than 3 coordinates, so (25) does not give the full evolution $\mathbf{q}_\tau$ where $\mathbf{q} = (q_1, \ldots, q_n, p_1, \ldots, p_n, c_1, \ldots, c_k)^T$. Indeed, if we represent $\mathcal{U}$ by the vector $\mathbf{b} = (b, c, a)^T$, we obtain

$$\mathbf{b}_\tau = \tilde{\mathbf{B}} \delta_b \tilde{h} \tag{27}$$

where $\tilde{\mathbf{B}}$ is given by (7) (with $\partial \equiv \frac{\partial}{\partial \xi}$) and $\delta_b \tilde{h}$ is shorthand for

$$\left(\frac{\delta \tilde{h}}{\delta b}, \frac{\delta \tilde{h}}{\delta c}, \frac{\delta \tilde{h}}{\delta a} \right)^T = -(\beta, \alpha, 2\gamma)^T.$$

If $\mathcal{J}_\varphi$ denotes the Fréchet derivative of the map $\varphi : \mathbf{q} \mapsto \mathbf{b}$, then

$$\mathbf{b}_\tau = \mathcal{J}_\varphi \mathbf{q}_\tau \quad \text{and} \quad \delta_q h = \mathcal{J}_\varphi^\dagger \cdot \delta_b \tilde{h} \tag{28}$$

where $h = \tilde{h} \circ \varphi$. Our task is to resolve (27) and (28) into the form

$$\mathbf{q}_\tau = \mathbf{B} \delta_q h,$$

even though $\mathcal{J}_\varphi$ is a $3 \times (2n + k)$ matrix, with right kernel of dimension $2n + k - 3$.

This construction deals with each stationary manifold separately.

3.1. The KdV equation. In this case, we have example 2.1 with (8):

$$\mathcal{U} \equiv V_{(1)} = \begin{pmatrix} \frac{1}{2}p & \lambda + 2q \\ \frac{1}{4}\lambda^2 - \frac{1}{2}\lambda q + q^2 - \frac{1}{2}c_1 & -\frac{1}{2}p \end{pmatrix}$$

where coordinates (q, p, c_1) are defined by (17)–(18). The map φ then has Jacobian

$$\mathcal{J}_\varphi = \begin{pmatrix} 2 & 0 & 0 \\ 2q - \frac{1}{2}\lambda & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

In this case, it is possible to invert $\mathcal{J}_\varphi$ to obtain

$$\mathbf{q}_\tau = \mathcal{J}_\varphi^{-1} \tilde{\mathbf{B}} \left(\mathcal{J}_\varphi^{-1} \right)^\dagger \delta_q h = \mathbf{B} \, \delta_q h \tag{29}$$

where

$$\mathbf{B} = \begin{pmatrix} 0 & -2q - \lambda & -p + \partial \\ 2q + \lambda & -2\partial & 12q^2 - 2c_1 \\ p + \partial & -12q^2 + 2c_1 & (4q - \lambda)\partial + \partial(4q - \lambda) \end{pmatrix}. \tag{30}$$

As previously mentioned, this is related to the simple Hamiltonian structure of (23) through the Miura map. This is an alternative way of proving that $\mathbf{B}$ is indeed Hamiltonian.

To calculate the hierarchy of Hamiltonians, we repeat the argument used with the KdV hierarchy in the usual coordinates. Since $\tilde{c}_1$ is a *flux* in the usual MKdV language, it is now a conserved density under τ-evolutions. Furthermore, it is a Casimir function of the Poisson bracket given in (23), so it generates the bi-Hamiltonian ladder (24)

$$\hat{\mathbf{B}}\delta_{\tilde{q}}\tilde{c}_1 = 0 \quad \Rightarrow \quad \mathbf{B}\delta_q\tilde{c}_1 = M'\hat{\mathbf{B}}(M')^\dagger\delta_q\tilde{c}_1 = M'\hat{\mathbf{B}}\delta_{\tilde{q}}\tilde{c}_1 = 0,$$

where $\mathbf{B}$ is given by (30). The asymptotic series for $\tilde{c}_1$ can be derived from the *known* series (13) by substitution into the formula (20), replacing x-derivatives of u by q, p, c_1, and their ξ-derivatives. Explicitly, we have

$$\tilde{q} = -\frac{1}{2}k + qk^{-1} - \frac{1}{2}pk^{-2} + (\frac{1}{2}c_1 - 2q^2)k^{-3} + (q_\xi + qp)k^{-4} + \cdots,$$

$$\tilde{c}_1 = -2v_{xx} + 4v^3 - 3\lambda v$$

$$= k^3 - c_1 k^{-1} - 2q_\xi k^{-2} + \left(\frac{1}{2}p^2 + 4q^3 - 2c_1 q + p_\xi\right) k^{-3} + \cdots.$$

The Hamiltonians

$$h_{-1} = \frac{1}{2}c_1, \quad h_0 = -\frac{1}{4}p^2 + qc_1 - 2q^3, \quad h_1 = \frac{1}{4}c_1^2 + q_\xi p,$$

and second Hamiltonian structure, generate

$$\begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_{\tau_{-1}} = \begin{pmatrix} -\frac{1}{2}p \\ 6q^2 - c_1 \\ 2q_\xi \end{pmatrix}, \quad \begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_{\tau_0} = \begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_\xi,$$

$$\begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_{\tau_1} = \begin{pmatrix} \frac{1}{2}c_{1\xi} - 2qq_\xi - \frac{1}{2}pc_1 \\ -2q_{\xi\xi} - 2qp_\xi + 6q^2c_1 - c_1^2 \\ \left(4c_1q - 4q^3 - \frac{1}{2}p^2 - p_\xi\right)_\xi \end{pmatrix}.$$

In general, we have

$$\mathbf{q}_{\tau_r} = (\mathbf{B}_1 - \lambda\mathbf{B}_0)\delta_q h_{(r)} = \mathbf{B}_1\delta_q h_r, \quad h_{(r)} = \lambda^{r+1}h_{-1} + \cdots + h_r.$$

The time evolution Ψ_{τ_r} is given in terms of $\delta_q h_{(r)}$:

$$\gamma = -\frac{\delta h_{(r)}}{\delta p}, \quad \alpha = 2\frac{\delta h_{(r)}}{\delta c_1}, \quad \beta = -\frac{1}{2}\frac{\delta h_{(r)}}{\delta q} + \left(\frac{1}{2}\lambda - 2q\right)\frac{\delta h_{(r)}}{\delta c_1}. \tag{31}$$

Thus h_{-1} generates the KdV equation and h_0 the "translational flow", telling us that $\tau_0 = \xi$. We can use the τ_{-1} flow to write p, c_1 and q_ξ in terms of q and its τ_{-1} derivatives after which

$$q_{\tau_1} = q_{xxxxx} + 10qq_{xxx} + 20q_xq_{xx} + 30q^2q_x,$$

where we have written $\tau_{-1} = x$.

Thus, the hierarchy is just the KdV hierarchy in disguise. However, without the τ_{-1} flow, this hierarchy shows no hint of being reducible from the (q, p, c_1) space to the q space. In these coordinates, the reduction to the stationary flow $q_\xi = 0$ is very natural, with (30) reducing to (21) and the bi-Hamiltonian ladder becoming finite.

3.2. The fifth-order KdV equation. Here

$$u_{t_2} = \frac{1}{2}\partial\delta\mathcal{H}_3 = 0, \quad \mathcal{H}_3 = u_{xx}^2 - 10uu_x^2 + 5u^4.$$

The Lagrangian $\mathcal{L}_3 = u_{xx}^2 - 10uu_x^2 + 5u^4 - c_2u$ gives the canonical variables

$$q_1 = u, \quad q_2 = u_x, \quad p_2 = 2u_{xx}, \quad p_1 = -2u_{xxx} - 20uu_x,$$
$$c_2 = 2u_{xxxx} + 20uu_{xx} + 10u_x^2 + 20u^3, \tag{32}$$

with c_2 being a Casimir function of the degenerate canonical bracket. The spectral problem is given by $V_{(2)}$ of the KdV hierarchy (example 2.1 with (8) and $\alpha = \alpha_{(2)}$), with (32):

$$\Psi_\xi = \mathcal{U}\Psi, \quad \text{where} \quad \mathcal{U} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{33}$$

with a, b, c given by

$$a = -\lambda q_2 + \frac{1}{2}p_1 + 4q_1 q_2,$$

$$b = \lambda^2 + 2\lambda q_1 + p_2 + 6q_1^2,$$

$$c = \frac{1}{4}\lambda^3 - \frac{1}{2}\lambda^2 q_1 - \frac{1}{4}\lambda(p_2 + 2q_1^2) - \frac{1}{2}c_2 + q_1 p_2 + 4q_1^3 - q_2^2.$$

We denote this mapping by $\varphi : \mathbf{q} \mapsto \mathbf{b}$. As before, we consider an evolution $\mathcal{V}$ (see (26)) where α, β, and γ are functions of (q_i, p_i, c_2), their ξ-derivatives, and λ. The integrability conditions are again (25).

The Jacobian $\mathcal{J}_\varphi$ is the 3×5 matrix

$$\mathcal{J}_\varphi = \begin{pmatrix} 12q_1 + 2\lambda & 0 & 0 & 1 & 0 \\ 12q_1^2 + p_2 - \lambda q_1 - \frac{1}{2}\lambda^2 & -2q_2 & 0 & q_1 - \frac{1}{4}\lambda & -\frac{1}{2} \\ 4q_2 & 4q_1 - \lambda & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

which does not, therefore, have a unique inverse. Thus the equation

$$\mathbf{b}_\tau = \mathcal{J}_\varphi \, \mathbf{q}_\tau \tag{34}$$

is underdetermined for the components of $\mathbf{q}_\tau$, whilst the equation

$$\delta_q h = \mathcal{J}_\varphi^\dagger \, \delta_b \tilde{h} \tag{35}$$

gives an overdetermined system of equations for the components of $\delta_b \tilde{h}$, which we would like to solve in terms of $\delta_q h$. Thus $\delta_q h$ must be orthogonal to $\mathbf{w} \in Ker \, \mathcal{J}_\varphi$

$$\mathbf{w}^\dagger \delta_q \, h = (\mathcal{J}_\varphi \mathbf{w})^\dagger \delta_b \tilde{h} \Rightarrow \mathbf{w}^\dagger \delta_q h = 0,$$

apparently giving rise to some constraints on $\delta_q h$. In fact, these "constraints" can be identified with part of the recursion relation of the bi-Hamiltonian ladder, so do not constitute a restriction. The (right) kernel of $\mathcal{J}_\varphi$ is important in what follows, and is spanned by

$$\mathbf{v}_1 = (1, 0, -8q_2, -12q_1 - 2\lambda, 2p_2)^T,$$

$$\mathbf{v}_2 = (0, 1, -8q_1 + 2\lambda, 0, -4q_2)^T.$$

The left kernel of $\mathcal{J}_\varphi$ is $\mathbf{0}$. We then can solve (uniquely) for $\delta_b \tilde{h}$:

$$\delta_a \tilde{h} = 2\delta_{p_1} h, \quad \delta_b \tilde{h} = \delta_{p_2} h + (2q_1 - \frac{1}{2}\lambda)\delta_{c_2} h, \quad \delta_c \tilde{h} = -2\delta_{c_2} h, \tag{36}$$

with "constraints"

$$\lambda \delta_{p_1} h = -\frac{1}{2}\delta_{q_2} h + 4q_1 \delta_{p_1} h + 2q_2 \delta_{c_2} h,$$

$$\lambda \delta_{p_2} h = \frac{1}{2}\delta_{q_1} h - 4q_2 \delta_{p_1} h - 6q_1 \delta_{p_2} h + p_2 \delta_{c_2} h. \tag{37}$$

Since we do not have a left inverse for $\mathcal{J}_\varphi$, we cannot use (29). We write (25) as

$$\mathcal{J}_\varphi \, \mathbf{q}_\tau = \tilde{\mathbf{B}}\delta_b \tilde{h}, \tag{38}$$

where $\tilde{\mathbf{B}}$ is the 3×3 Hamiltonian structure given in (27). We write (36) as $\delta_b \tilde{h} = D\delta_q h$ where

$$D = \begin{pmatrix} 0 & 0 & 0 & 1 & 2q_1 - \frac{1}{2}\lambda \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix},$$

but are free to add the transpose of an element of the right kernel of $\mathcal{J}_\varphi$

$$\delta_b \tilde{h} = \Big(D + \sum_{i=1}^{2} \mathbf{f}_i \mathbf{v}_i^\dagger\Big)\delta_q h \equiv \hat{D}\delta_q h. \qquad (39)$$

Indeed,

$$\mathcal{J}_\varphi^\dagger \delta_b \tilde{h} = \delta_q h \quad \Rightarrow \quad \hat{D}\mathcal{J}_\varphi^\dagger \delta_b \tilde{h} = \hat{D}\delta_q h = \delta_b \tilde{h} \quad \Rightarrow \quad \hat{D}\mathcal{J}_\varphi^\dagger = I_3$$

where I_3 is the 3×3 identity matrix. Thus, for *any* choice of $\mathbf{f}_i$, $\hat{D}$ is a left inverse of $\mathcal{J}_\varphi^\dagger$.

Whilst the $\mathbf{f}_i$ make no difference to the formula (36) for $\delta_b \tilde{h}$, they *can* change the right-hand side of

$$\mathcal{J}_\varphi \mathbf{q}_{\tau_r} = \tilde{\mathbf{B}}\Big(D + \sum_{i=1}^{2} \mathbf{f}_i \mathbf{v}_i^\dagger\Big)\delta_q h_{(r)}, \qquad (40)$$

which involves the *truncated* Hamiltonian $h_{(r)}$, which no longer satisfies the infinite recursion relation represented by the "constraint" (37).

We now use our knowledge of the standard KdV hierarchy. The Hamiltonian $h_{(r)} = (\lambda^{r+2}h)_+$ is given in terms of the fluxes previously calculated:

$$h_{-2} = \frac{1}{2}c_2, \quad h_{-1} = -5q_1^4 + 10q_1 q_2^2 + q_1 c_2 + q_2 p_1 + \frac{1}{4}p_2^2.$$

We have the following correspondence between the usual KdV t_k and τ_m:

$$t_0 = x, t_1, \qquad t_2 = \xi, \quad t_3, \dots$$
$$\tau_{-2} = x, \tau_{-1}, \quad \tau_0 = \xi, \quad \tau_1, \dots.$$

The τ_{-2} flow is just the $x - t_2$ reversed fifth-order KdV equation

$$\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ c_2 \end{pmatrix}_{\tau_{-2}} = \begin{pmatrix} q_2 \\ \frac{1}{2}p_2 \\ 20q_1^3 - 10q_2^2 - c_2 \\ -20q_1 q_2 - p_1 \\ 2q_{1\xi} \end{pmatrix},$$

whilst the τ_{-1} flow is the KdV equation for q_1 and its prolongation to the other variables (but written in these coordinates)

$$
\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ c_2 \end{pmatrix}_{\tau_{-1}} = \begin{pmatrix} -4q_1q_2 - \frac{1}{2}p_1 \\ -10q_1^3 - 2q_1p_2 + q_2^2 + \frac{1}{2}c_2 \\ 120q_1^4 + 30q_1^2p_2 + 20q_1q_2^2 - 6q_1c_2 + 4q_2p_1 + p_2^2 - 2q_{2\xi} \\ 20q_1^2q_2 + 4q_1p_1 - 2q_2p_2 + 2q_{1\xi} \\ 12q_1q_{1\xi} + p_{2\xi} \end{pmatrix}.
$$

Substituting these equations into (40) determines $\mathbf{f}_i$ to be zero. Thus, for this example, the correct choice of $\hat{D}$ is D.

Remark 3.1. Whilst for the KdV hierarchy and several other examples, we find $\mathbf{f}_i = 0$, this is *not* always true (in this paper, see Ito's equation).

We wish to find a 5×5 matrix $\mathbf{B}$ such that

$$
\tilde{\mathbf{B}} = \mathcal{J}_\varphi \mathbf{B} \mathcal{J}_\varphi^\dagger. \tag{41}
$$

Using $\hat{D}$ ($= D$), we can find a particular solution $\bar{\mathbf{B}}$ of (41) satisfying

$$
\bar{\mathbf{B}} = D^\dagger \tilde{\mathbf{B}} D \quad \text{so that} \quad \mathcal{J}_\varphi \bar{\mathbf{B}} \mathcal{J}_\varphi^\dagger = (\mathcal{J}_\varphi D^\dagger) \tilde{\mathbf{B}} (D \mathcal{J}_\varphi^\dagger) = \tilde{\mathbf{B}}.
$$

This $\bar{\mathbf{B}}$ explicitly takes the form

$$
\bar{\mathbf{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\partial & \Gamma_3 & \Gamma_1 \\ 0 & 0 & -\Gamma_3 & 0 & 2\partial + \Gamma_2 \\ 0 & 0 & -\Gamma_1 & 2\partial - \Gamma_2 & 4(\partial q_1 + q_1\partial) - 2\lambda\partial \end{pmatrix}
$$

where

$$
\begin{aligned}
\Gamma_1 &= 40q_1^3 + 8q_1p_2 - 4q_2^2 - 2c_2 - 2\lambda p_2, \\
\Gamma_2 &= -16q_1q_2 - 2p_1 + 4\lambda q_2, \\
\Gamma_3 &= 12q_1^2 + 2p_2 + 4\lambda q_1 + 2\lambda^2.
\end{aligned}
$$

To this we can add any skew-symmetric matrix which is in the kernel of $M \mapsto \mathcal{J}_\varphi M \mathcal{J}_\varphi^\dagger$:

$$
\mathbf{B} = \bar{\mathbf{B}} + \sum_{i,j=1}^{2} \left(\mathbf{a}_i \mathbf{v}_i^\dagger - \mathbf{v}_i \mathbf{a}_i^\dagger \right).
$$

We wish to write the evolution equations for $\mathbf{q}$ as

$$\mathbf{q}_{\tau_r} = \mathbf{B}\delta_q h_{(r)} = \Big(\bar{\mathbf{B}} + \sum_{i,j=1}^{2} \big(\mathbf{a}_i \mathbf{v}_i^\dagger - \mathbf{v}_i \mathbf{a}_i^\dagger\big)\Big)\delta_q h_{(r)}, \qquad (42)$$

but also have (38) after replacing $\tilde{\mathbf{B}}$:

$$\mathcal{J}_\varphi \big(\mathbf{q}_{\tau_r} - \bar{\mathbf{B}}\delta_q h_{(r)}\big) = 0. \qquad (43)$$

To be consistent, we require

$$\mathcal{J}_\varphi \sum_{i,j=1}^{2} \big(\mathbf{a}_i \mathbf{v}_i^\dagger - \mathbf{v}_i \mathbf{a}_i^\dagger\big)\delta_q h_{(r)} = 0.$$

Since we do *not* have $\mathbf{v}_i^\dagger \delta_q h_{(r)} = 0$, we need $\mathcal{J}_\varphi \mathbf{a}_i = 0$, so that $\mathbf{a}_i = \sum_{j=1}^{2} \mathbf{v}_j d_{ji}$, and

$$\mathbf{B} = \bar{\mathbf{B}} + \sum_{i,j=1}^{2} \mathbf{v}_i b_{ij} \mathbf{v}_j^\dagger$$

where $b_{ij} = (\mathbf{b})_{ij}$ with $\mathbf{b}$ a skew-adjoint matrix differential operator (in the present case, a simple skew-symmetric matrix). Equation (43) implies $\mathbf{q}_{\tau_r} - \bar{\mathbf{B}}\delta_q h_{(r)} \in Ker \, \mathcal{J}_\varphi$. From the forms of $\bar{\mathbf{B}}$ and $\mathbf{v}_i$, we have explicitly

$$\mathbf{q}_{\tau_r} = \bar{\mathbf{B}}\delta_q h_{(r)} + \mathbf{v}_1 q_{1\tau_r} + \mathbf{v}_2 q_{2\tau_r}. \qquad (44)$$

Matching (42) and (44), we have

$$q_{i\tau_r} = \sum_{j=1}^{2} b_{ij} \mathbf{v}_j^\dagger \delta_q h_{(r)}, \quad i = 1, 2.$$

The coordinates (32) define the fifth-order KdV equation as an $x - t$ reversed flow, with $\tau_{-2} = x$, from which we find

$$q_{1\tau_{-2}} = q_2, \quad q_{2\tau_{-2}} = \frac{1}{2}p_2.$$

In our Hamiltonian formulation, this should be generated by the function $h_{-2} = \frac{1}{2}c_2$. Substituting these into (44) gives

$$b_{11} = b_{22} = 0 \quad , \quad b_{12} = -b_{21} = -\frac{1}{2}.$$

The Hamiltonian operator $\mathbf{B}$ then takes the form

$$\mathbf{B} = \mathbf{B}_1 - \lambda \mathbf{B}_0$$

where

$$\mathbf{B}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\partial \end{pmatrix},$$

$$\mathbf{B}_1 = \begin{pmatrix} 0 & -\frac{1}{2} & 4q_1 & 0 & 2q_2 \\ \frac{1}{2} & 0 & -4q_2 & -6q_1 & p_2 \\ -4q_1 & 4q_2 & -2\partial & 60q_1^2 + 2p_2 & \Omega_1 \\ 0 & 6q_1 & -60q_1^2 - 2p_2 & 0 & 2\partial + \Omega_2 \\ -2q_2 & -p_2 & -\Omega_1 & 2\partial - \Omega_2 & 4(\partial q_1 + q_1\partial) \end{pmatrix},$$

and $\Omega_1 = 40q_1^3 - 20q_2^2 - 2c_2$, $\Omega_2 = -40q_1q_2 - 2p_1$.

The modified fifth-order hierarchy has a *single* local Hamiltonian structure (almost identical to $\mathbf{B}_0$ above). This simple operator is transformed onto $\mathbf{B}_1 - \lambda\mathbf{B}_0$ by the Miura map (12), when extended to this 5-dimensional space. Once again, this is the simplest proof of the fact that $\mathbf{B}_0$ and $\mathbf{B}_1$ are both Hamiltonian *and* compatible. The modified fifth-order KdV equation is

$$v_{t_2} = -\frac{1}{2}\tilde{c}_{2x}, \quad \tilde{c}_2 = -2v_{xxxx} + 20v^2v_{xx} + 20vv_x^2 - 12v^5 - 5k^2(v_{xx} - 2v^3) - \frac{15}{4}k^4v.$$

Comparing with (15), we have

$$\tilde{c}_2 = \alpha_{(2)x} - 2v\alpha_{(2)}.$$

Using the expansion (13), the coordinates (32), and their ξ-derivatives, we obtain an infinite expansion for $\tilde{c}_2$. This is our h, from which $h_{(r)} = (\lambda^{r+2}h)_+$ is calculated.

The time evolution (26) is calculated from $h_{(r)}$ by (39) (compare with (31)):

$$\gamma = \delta_{p_1}h_{(r)}, \quad \alpha = -2\delta_{c_2}h_{(r)}, \quad \beta = \delta_{p_2}h_{(r)} + (2q_1 - \frac{1}{2}\lambda)\delta_{c_2}h_{(r)}.$$

When ξ-derivatives of the phase space variables are zero, then $\mathbf{B}_0$ and $\mathbf{B}_1$ reduce to the previously known [5] Poisson matrices of the stationary fifth-order KdV equation. We then have just three independent Hamiltonians: c_2, h_{-2}, h_{-1}. The integral c_2 is the Casimir of $\mathbf{B}_0$, and h_{-1} is the Casimir of $\mathbf{B}_1$ in this reduction.

4. Some two-component examples

We have illustrated our procedure in the context of the KdV hierarchy. The KdV equation itself is a special case, but the fifth-order member of the hierarchy already includes many of the general features. The details of the seventh-order KdV equation and its modification are presented in [16].

In this section, we present some interesting examples which illustrate certain additional features. For instance, Ito's equation requires non-zero $\mathbf{f}_i$ (see Remark (3.1)).

4.1. The DWW hierarchy. This hierarchy is isospectral to a particular case of the energy dependent Schrödinger operator [3]. Since the KdV hierarchy is also a special case of this, the zero curvature representations are very similar:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ \lambda^2 - \lambda u_1 - u_0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv U\Psi,$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_{t_m} = \begin{pmatrix} -\frac{1}{2}\alpha_{(m)x} & \alpha_{(m)} \\ C_{(m)} & \frac{1}{2}\alpha_{(m)x} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv V_{(m)}\Psi \tag{45}$$

where $C_{(m)} = -\frac{1}{2}\alpha_{(m)xx} - (u_0 + u_1\lambda - \lambda^2)\alpha_{(m)}$ and $\alpha_{(m)} = \sum_{i=0}^{m} \alpha_{m-i}\lambda^i$. The isospectral flows are tri-Hamiltonian, with

$$\mathbf{u}_{t_m} = \mathbf{B}_2\delta_u\mathcal{H}_m = \mathbf{B}_1\delta_u\mathcal{H}_{m+1} = \mathbf{B}_0\delta_u\mathcal{H}_{m+2}$$

where $\mathbf{u} = (u_0, u_1)^T$, and

$$\mathbf{B}_0 = \begin{pmatrix} -J_1 & -J_2 \\ -J_2 & 0 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} J_0 & 0 \\ 0 & -J_2 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix}, \tag{46}$$

with $J_0 = \frac{1}{2}\partial^3 + 2u_0\partial + u_{0x}$, $J_1 = 2u_1\partial + u_{1x}$, $J_2 = -2\partial$. The first few Hamiltonians are

$$\mathcal{H}_0 = u_1, \quad \mathcal{H}_1 = u_0 + \frac{1}{4}u_1^2, \quad \mathcal{H}_2 = \frac{1}{2}u_0u_1 + \frac{1}{8}u_1^3,$$

$$\mathcal{H}_3 = -\frac{1}{16}u_{1x}^2 + \frac{1}{4}u_0^2 + \frac{3}{8}u_0u_1^2 + \frac{5}{64}u_1^4,$$

$$\mathcal{H}_4 = -\frac{1}{8}u_{0x}u_{1x} - \frac{5}{32}u_1u_{1x}^2 + \frac{3}{8}u_0^2u_1 + \frac{5}{16}u_0u_1^3 + \frac{7}{128}u_1^5.$$

The t_1 and t_2 flows are

$$u_{0t_1} = \frac{1}{4}(u_{1xxx} + 4u_0u_{1x} + 2u_{0x}u_1),$$

$$u_{1t_1} = (u_0 + \frac{3}{4}u_1^2)_x, \tag{47}$$

$$u_{0t_2} = \frac{1}{4}(u_{0xxx} + 6u_0u_{0x}) + \frac{3}{8}(u_1u_{1xxx} + 3u_{1x}u_{1xx} + u_{0x}u_1^2 + 4u_0u_1u_{1x}),$$

$$u_{1t_2} = \frac{1}{4}(u_{1xx} + 6u_0u_1 + \frac{5}{2}u_1^3)_x. \tag{48}$$

The first of these is known as the dispersive water wave (DWW) equation (after a simple change of coordinates) whilst the second is a coupled KdV equation. Half the flows reduce to those of the KdV hierarchy when $u_1 = 0$.

Example 4.1 (The DWW equation). The stationary solutions of the DWW equation (47) have Lagrangian

$$\mathcal{L} = \mathcal{H}_3 - u_0c_1 - u_1c_2 - \frac{1}{4}u_1^2c_1$$

$$= -\frac{1}{16}u_{1x}^2 + \frac{1}{4}u_0^2 + \frac{3}{8}u_0u_1^2 + \frac{5}{64}u_1^4 - u_0c_1 - u_1c_2 - \frac{1}{4}u_1^2c_1$$

where the terms involving c_i form the Casimirs of $\mathbf{B}_0$. As a function of u_0 and u_1, this is degenerate, but Lagrange's equations

$$c_1 = \frac{1}{2}u_0 + \frac{3}{8}u_1^2, \quad c_2 = \frac{1}{8}u_{1xx} + \frac{3}{4}u_0 u_1 + \frac{5}{16}u_1^3 - \frac{1}{2}u_1 c_1, \qquad (49)$$

can be used to eliminate u_0, resulting in a one degree of freedom system with Lagrangian

$$\hat{\mathcal{L}} = -\frac{1}{8}\left(\frac{1}{2}u_{1x}^2 + \frac{1}{2}u_1^4 - 4c_1 u_1^2 + 8c_2 u_1\right).$$

Convenient coordinates for the stationary manifold of the DWW equation thus are

$$q = u_1, \quad p = -\frac{1}{8}u_{1x}, \quad c_1, \quad c_2.$$

The DWW equation corresponds to $V_{(1)}$ (of (45)) with $\alpha_{(1)} = \lambda + \frac{1}{2}u_1$, and with u_0 and u_{1xx} eliminated using (49), so the $x - t$ reversed spectral problem is (33) with $\varphi : \mathbf{q} \mapsto \mathbf{b}$ given by

$$a = 2p, \quad b = \lambda + \frac{1}{2}q, \quad c = \lambda^3 - \frac{1}{2}\lambda^2 q + \lambda\left(\frac{1}{4}q^2 - 2c_1\right) - 2c_2 + qc_1 - \frac{1}{8}q^3,$$

with Jacobian

$$\mathcal{J}_\varphi = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \gamma & 0 & q - 2\lambda & -2 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

where $\gamma = -\frac{3}{8}q^2 + c_1 + \frac{1}{2}\lambda q - \frac{1}{2}\lambda^2$, whose right kernel is spanned by

$$\mathbf{v}_1 = (0, 0, 1, \tfrac{1}{2}q - \lambda)^T.$$

Thus, with time evolution (45), we once again have the underdetermined system (34) for $\mathbf{q}_\tau$ and overdetermined system (35) for $\delta_b\tilde{h}$. One solution for $\delta_b\tilde{h}$ is given by

$$\delta_b\tilde{h} = D\,\delta_q h \quad \text{with} \quad D\,\mathcal{J}_\varphi^\dagger = I_3$$

where

$$D = \begin{pmatrix} 2 & 0 & 0 & \gamma \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Once again the $\mathbf{f}_1$ of (39) is zero, so $\hat{D} = D$. Our aim is to find a 4×4 matrix operator $\mathbf{B}$ satisfying (41). We use D and $\bar{\mathbf{B}}$ of (27) to obtain a particular solution $\bar{\mathbf{B}} = D^\dagger \tilde{\mathbf{B}} D$

$$\bar{\mathbf{B}} = \begin{pmatrix} 0 & -\frac{1}{2}q - \lambda & 0 & -4p + \partial \\ \frac{1}{2}q + \lambda & -\frac{1}{8}\partial & 0 & -\Gamma \\ 0 & 0 & 0 & 0 \\ 4p + \partial & \Gamma & 0 & \chi \end{pmatrix}$$

where

$$\Gamma = \frac{1}{8}q^3 - \frac{1}{2}qc_1 + \frac{1}{2}c_2,$$

$$\chi = \partial\left(-\frac{3}{16}q^2 + \frac{1}{2}c_1 + \frac{1}{4}\lambda q\right) + \left(-\frac{3}{16}q^2 + \frac{1}{2}c_1 + \frac{1}{4}\lambda q\right)\partial - \frac{1}{2}\lambda^2\partial.$$

To this we can add an extra term:

$$\mathbf{B} = \bar{\mathbf{B}} + \mathbf{v}_1 b_{11} \mathbf{v}_1^{\dagger}$$

where b_{11} is determined by comparison with the τ_{-1} flow: $b_{11} = \frac{1}{2}\partial$. We find

$$\mathbf{B} = \mathbf{B}_1 - \lambda\mathbf{B}_0$$

with

$$\mathbf{B}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\partial \\ 0 & 0 & \frac{1}{2}\partial & 0 \end{pmatrix},$$

$$\mathbf{B}_1 = \begin{pmatrix} 0 & -\frac{1}{2}q & 0 & -4p+\partial \\ \frac{1}{2}q & -\frac{1}{8}\partial & 0 & -\Gamma \\ 0 & 0 & \frac{1}{2}\partial & \frac{1}{4}\partial q \\ 4p+\partial & \Gamma & \frac{1}{4}q\partial & \partial\rho+\rho\partial \end{pmatrix},$$

where $\rho = -\frac{1}{8}q^2 + \frac{1}{2}c_1$.

The DWW equation is tri-Hamiltonian in *any* coordinate system. We form the recursion operator

$$\mathbf{R} = \mathbf{B}_1\mathbf{B}_0^{-1} = \begin{pmatrix} -\frac{1}{2}q & 0 & -8p\partial^{-1}+2 & 0 \\ -\frac{1}{8}\partial & -\frac{1}{2}q & -2\Gamma\partial^{-1} & 0 \\ 0 & 0 & \frac{1}{2}\partial q\partial^{-1} & 1 \\ \Gamma & -4p-\partial & 2\rho+2\partial\rho\partial^{-1} & \frac{1}{2}q \end{pmatrix},$$

and then define $\mathbf{B}_2 = \mathbf{R}\mathbf{B}_1$ to give

$$\mathbf{B}_2 = \begin{pmatrix} 0 & \frac{1}{4}q^2 & -4p+\partial & \frac{1}{2}(\partial q - q\partial) \\ -\frac{1}{4}q^2 & \frac{1}{16}(\partial q + q\partial) & -\Gamma & \frac{1}{2}\partial p - \frac{1}{8}\partial^2 \\ 4p+\partial & \Gamma & \frac{1}{4}(\partial q + q\partial) & -\frac{1}{8}q^2\partial + \frac{1}{2}(\partial c_1 + c_1\partial) \\ \frac{1}{2}(q\partial - \partial q) & \frac{1}{2}p\partial + \frac{1}{8}\partial^2 & -\frac{1}{8}\partial q^2 + \frac{1}{2}(\partial c_1 + c_1\partial) & \frac{1}{2}(\partial c_2 + c_2\partial) \end{pmatrix}.$$

Remark 4.1. This *construction* is only possible on the PDE level since $\mathbf{B}_0$ becomes degenerate (and thus cannot be inverted) when reduced to the stationary case. Nevertheless, the reduced $\mathbf{B}_2$ is still valid.

The hierarchy is tri-Hamiltonian with

$$\mathbf{q}_{\tau_r} = \mathbf{B}_2 \delta h_{r-1} = \mathbf{B}_1 \delta h_r = \mathbf{B}_0 \delta h_{r+1}.$$

The integrals $h_{(r)}$ can be calculated as the fluxes of the DWW integrals in the usual (u_0, u_1) coordinates, or can be calculated with a parameter dependent Miura map as previously explained. Parameter dependent Miura maps also enable us to prove that the operators $\mathbf{B}_k$ are Hamiltonian and compatible. The time evolution (45) is given in terms of $\delta h_{(r)}$ in a similar way to (31). The details can be found in [16]. Here we just give the parameter independent Miura maps and the modified Hamiltonian structures, together with the connecting diagram. The Miura maps,

$$M_1 \begin{cases} q = 2\tilde{q}, \quad p = -\frac{1}{2}\tilde{q}\tilde{p} - \frac{1}{4}\tilde{c}_1, \quad c_1 = \frac{1}{2}\tilde{c}_2, \\ c_2 = -\frac{1}{2}\tilde{q}^3 + \frac{1}{2}\tilde{q}\tilde{p}^2 + \frac{1}{2}\tilde{q}\tilde{c}_2 + \frac{1}{2}\tilde{p}\tilde{c}_1 - \frac{1}{2}\tilde{p}_\xi, \end{cases}$$

$$M_2 \begin{cases} \tilde{q} = 2\hat{p}, \quad \tilde{p} = \hat{q}\hat{p} - \frac{1}{2}\hat{c}_1, \quad \tilde{c}_1 = -2\hat{c}_2, \\ \tilde{c}_2 = -\hat{q}^2\hat{p}^2 + \hat{q}\hat{p}\hat{c}_1 + \hat{q}\hat{c}_2 + 4\hat{p}^2 - \frac{1}{4}\hat{c}_1^2 + \frac{1}{2}\hat{q}_\xi, \end{cases}$$

transport Hamiltonian structures as shown in the figure below:

$$
\begin{array}{ccc}
q & \longleftarrow \tilde{q} & \longleftarrow \hat{q} \\
\mathbf{B}_2 & \longleftarrow \tilde{\mathbf{B}}_2 & \longleftarrow \hat{\mathbf{B}}_2 \\
\mathbf{B}_1 & \longleftarrow \tilde{\mathbf{B}}_1 & \\
\mathbf{B}_0 & &
\end{array}
$$

where

$$\tilde{\mathbf{B}}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -2\partial & 0 \\ 0 & 0 & 0 & 2\partial \end{pmatrix},$$

$$\tilde{\mathbf{B}}_2 = \begin{pmatrix} 0 & -\tilde{q} & 0 & 2\tilde{q}\tilde{p} + \tilde{c}_1 + \partial \\ \tilde{q} & 0 & \partial & 3\tilde{q}^2 - \tilde{p}^2 - \tilde{c}_2 \\ 0 & \partial & 0 & -2\partial\tilde{p} \\ -2\tilde{q}\tilde{p} - \tilde{c}_1 + \partial & -3\tilde{q}^2 + \tilde{p}^2 + \tilde{c}_2 & -2\tilde{p}\partial & 2\partial\tilde{q} + 2\tilde{q}\partial \end{pmatrix},$$

$$\hat{\mathbf{B}}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial \\ 0 & 0 & \partial & 0 \end{pmatrix}.$$

Remark 4.2. The other members of the DWW hierarchy are very similar, but the matrix operators are larger (6×6 in the case of the third-order DWW equation (48)).

4.2. The Ito hierarchy. We consider the isospectral flows of

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ \lambda - u_1 - u_0\lambda^{-1} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

The general time evolution is given by $V_{(m)}$ of (45), but with $C_{(m)} = -\frac{1}{2}\alpha_{(m)xx} - (u_0\lambda^{-1} + u_1 - \lambda)\alpha_{(m)}$. The t_1 flow, corresponding to $\alpha_{(1)} = \lambda + \frac{1}{2}u_1$, is the coupled KdV system

$$u_{0t_1} = \frac{1}{2}u_{0x}u_1 + u_0u_{1x}, \qquad u_{1t_1} = \frac{1}{4}(u_{1xxx} + 6u_1u_{1x} + 4u_{0x}). \tag{50}$$

In the coordinates $q = u_1$, $r = 2u_0^{1/2}$, this takes the form of Ito's equation. This system is tri-Hamiltonian [2]

$$\mathbf{u}_{t_m} = \mathbf{B}_2\delta_u\mathcal{H}_m = \mathbf{B}_1\delta_u\mathcal{H}_{m+1} = \mathbf{B}_0\delta_u\mathcal{H}_{m+2},$$

where $\mathbf{B}_k$ takes the same form as (46), but with

$$J_0 = 2u_0\partial + u_{0x}, \quad J_1 = \frac{1}{2}\partial^3 + 2u_1\partial + u_{1x}, \quad J_2 = -2\partial.$$

The first few Hamiltonians are

$$\mathcal{H}_0 = u_1, \quad \mathcal{H}_1 = u_0 + \frac{1}{4}u_1^2, \quad \mathcal{H}_2 = -\frac{1}{16}u_{1x}^2 + \frac{1}{2}u_0u_1 + \frac{1}{8}u_1^3,$$

$$\mathcal{H}_3 = \frac{1}{8}u_0u_{1xx} + \frac{1}{64}u_{1xx}^2 - \frac{5}{32}u_1u_{1x}^2 + \frac{1}{4}u_0^2 + \frac{3}{8}u_0u_1^2 + \frac{5}{64}u_1^4.$$

Example 4.2 (Ito's equation). Ito's equation (50) corresponds to $\mathbf{B}_0\delta\mathcal{H}_3$. For the stationary flow, we need the kernel of $\mathbf{B}_0$, which is $(c_1, \frac{1}{2}u_1c_1 + c_2)^T$ where c_i are constants. Allowing c_i to be non-constant, this is still a good representation for the gradient, even in the non-stationary case. Putting $\delta\mathcal{H}_3 = (c_1, \frac{1}{2}u_1c_1 + c_2)^T$,

$$\frac{1}{8}\left(u_{1xx} + 3u_1^2 + 4u_0\right) = c_1, \tag{51}$$

$$\frac{1}{32}\left(u_{1xxxx} + 4u_{0xx}\right) + \frac{5}{32}\left(u_{1x}^2 + 2u_1u_{1xx} + 2u_1^3\right) + \frac{3}{4}u_0u_1 = c_2 + \frac{1}{2}u_1c_1,$$

the second of which can be simplified to give

$$c_2 = \frac{1}{4}c_{1xx} - \frac{1}{32}u_{1x}^2 + \frac{1}{4}u_0u_1 - \frac{1}{16}u_1^3 + \frac{1}{2}u_1c_1. \tag{52}$$

The stationary flow of Ito's equation (50) is reduced to one degree of freedom by eliminating u_0 and is Lagrangian with

$$\mathcal{L} = \frac{1}{32}u_1u_{1x}^2 - \frac{1}{16}u_1^4 + \frac{1}{2}u_1^2c_1 - u_1c_2$$

where the c_i are constants. This, up to an exact derivative, corresponds, after eliminating u_0, to $\mathcal{H}_3 - u_0c_1 - u_1c_2 - \frac{1}{4}u_1^2c_1$. The term $u_0c_1 + u_1c_2 + \frac{1}{4}u_1^2c_1$ is the general Casimir of $\mathbf{B}_0$. Convenient coordinates for the stationary manifold of Ito's equation are thus

$$q = u_1, \quad p = \frac{1}{16}u_1u_{1x}, \quad c_1, \quad c_2.$$

In the definition of $\mathcal{U} = V_{(1)}$, we thus can use (51), (52) to eliminate u_0 and u_{1xx} to give the $x - t$ reversed spectral problem as (33) with $\varphi : \mathbf{q} \mapsto \mathbf{b}$ given by

$$a = -4pq^{-1}, \qquad b = \lambda + \frac{1}{2}q,$$

$$c = \lambda^2 - \frac{1}{2}\lambda q + \frac{1}{4}q^2 - 2c_1 + \left(qc_1 - 2c_2 - \frac{1}{8}q^3 - 16p^2q^{-2} + 4(pq^{-1})_\xi\right)\lambda^{-1},$$

where the term $4(pq^{-1})_\xi$ stems from $c_{1xx} = \frac{1}{2}u_{1\xi x} = 8(pq^{-1})_\xi$.

The Jacobian $\mathcal{J}_\varphi$ is the Fréchet derivative of this map:

$$\mathcal{J}_\varphi = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \sigma & \lambda^{-1}(-32pq^{-2} + 4\partial q^{-1}) & \lambda^{-1}q - 2 & -2\lambda^{-1} \\ 4pq^{-2} & -4q^{-1} & 0 & 0 \end{pmatrix},$$

where

$$\sigma = \lambda^{-1}(-\frac{3}{8}q^2 + c_1 + 32p^2q^{-3} - 4\partial pq^{-2}) + \frac{1}{2}q - \frac{1}{2}\lambda.$$

The right kernel of $\mathcal{J}_\varphi$ is spanned by

$$\mathbf{v}_1 = (0, 0, 1, \frac{1}{2}q - \lambda)^T.$$

Thus, with this time evolution, we once again have the underdetermined system (34) for $\mathbf{q}_\tau$ and overdetermined system (35) for $\delta_b \tilde{h}$. One solution for $\delta_b \tilde{h}$ is given by

$$\delta_b \tilde{h} = D\delta_q h \quad \text{with} \quad D\mathcal{J}_\varphi^\dagger = I_3$$

where

$$D = \begin{pmatrix} 2 & 2pq^{-1} & 0 & -\frac{3}{8}q^2 + c_1 + \frac{1}{2}\lambda q - \frac{1}{2}\lambda^2 \\ 0 & 0 & 0 & -\frac{1}{2}\lambda \\ 0 & -\frac{1}{4}q & 0 & 4pq^{-1} + \frac{1}{2}\partial \end{pmatrix}.$$

To this D we are free to add the transpose of an element of the right kernel of $\mathcal{J}_\varphi$, since this does not change the relationship (35):

$$\delta_b \tilde{h} = (D + \mathbf{f}_1\mathbf{v}_1^\dagger)\delta_q h \equiv \hat{D}\delta_q h.$$

However, in this case, $\mathbf{f}_1 \neq 0$ (see Remark (3.1)), as we find by substituting the τ_{-1} flow (stationary Ito's equation) into

$$\mathcal{J}_\varphi\mathbf{q}_{\tau_r} = \tilde{\mathbf{B}}\hat{D}\delta_q h_{(r)}.$$

We find

$$\mathbf{f}_1 = (\Upsilon_1, -\frac{1}{2}, 0)^T$$

where

$$\Upsilon_1 = \lambda^{-1}\theta + \frac{1}{2}q - \frac{1}{2}\lambda, \quad \text{with} \quad \theta = \frac{1}{8}q^2 - c_1 + 2c_2q^{-1} + 16p^2q^{-3} - 4q^{-1}(pq^{-1})_\xi,$$

after which

$$\hat{D} = \begin{pmatrix} 2 & 2pq^{-1} & \Upsilon_1 & \Upsilon_2 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{4}q \\ 0 & -\frac{1}{4}q & 0 & 4pq^{-1} + \frac{1}{2}\partial \end{pmatrix}$$

where

$$\Upsilon_2 = \left(\frac{1}{2}\lambda^{-1}q - 1 \right)\theta - \frac{1}{8}q(q + 2\lambda) + c_1.$$

We then can calculate $\bar{\mathbf{B}} = D^\dagger \tilde{\mathbf{B}} D$:

$$\bar{\mathbf{B}} = \begin{pmatrix} 0 & \frac{1}{4}q^2 + \frac{1}{2}\lambda q & 8pq^{-1} + \partial & \frac{1}{2}\partial q - \frac{1}{2}q\partial - 8\lambda pq^{-1} - \lambda\partial \\ -\frac{1}{4}q^2 - \frac{1}{2}\lambda q & -\frac{1}{32}q\partial q & \Omega_1 + \partial pq^{-1} & \frac{1}{16}q\partial^2 + \frac{1}{2}\partial p - \lambda\Omega_1 - \lambda\partial pq^{-1} \\ \partial - 8pq^{-1} & pq^{-1}\partial - \Omega_1 & \Gamma_5 & \Omega_2(\frac{1}{2}q - \lambda) + \rho_1\partial + \partial\rho_2 \\ \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 \end{pmatrix}$$

where

$$\Gamma_1 = -\frac{1}{2}\partial q + \frac{1}{2}q\partial + 8\lambda pq^{-1} - \lambda\partial, \quad \Gamma_2 = -\frac{1}{16}\partial^2 q + \frac{1}{2}p\partial + \lambda\Omega_1 - \lambda pq^{-1}\partial,$$

$$\Gamma_3 = (\frac{1}{2}q - \lambda)\Omega_2 + \partial\rho_1 + \rho_2\partial, \quad \Gamma_4 = (\frac{1}{2}q - \lambda)\Omega_2(\frac{1}{2}q - \lambda) + \partial\rho_3 + \rho_3\partial + \frac{1}{8}\partial^3,$$

$$\Gamma_5 = \Omega_2 + \frac{1}{4}(\partial q + q\partial) - \frac{1}{2}\lambda\partial,$$

$$\rho_1 = -\frac{1}{8}q^2 + \frac{1}{2}c_1, \quad \rho_2 = \frac{1}{2}c_1 - \frac{1}{4}\lambda q, \quad \rho_3 = \frac{1}{2}c_2 + \frac{1}{8}\lambda q^2 - \frac{1}{2}\lambda c_1,$$

$$\Omega_1 = q\rho_1 - \frac{1}{2}c_2 + 4p^2q^{-2}, \quad \Omega_2 = \frac{1}{2}\lambda^{-1}(\partial\theta + \theta\partial).$$

To this, we add the extra term

$$\mathbf{B} = \bar{\mathbf{B}} + \mathbf{v}_1 b_{11} \mathbf{v}_1^\dagger$$

where b_{11} is determined by comparison with the τ_{-1} flow: $b_{11} = -\Omega_2$. We find $\mathbf{B} = \mathbf{B}_2 - \lambda\mathbf{B}_1$ where

$$\mathbf{B}_1 = \begin{pmatrix} 0 & -\frac{1}{2}q & 0 & 8pq^{-1} + \partial \\ \frac{1}{2}q & 0 & 0 & \Omega_1 + \partial pq^{-1} \\ 0 & 0 & \frac{1}{2}\partial & \frac{1}{4}\partial q \\ -8pq^{-1} + \partial & -\Omega_1 + pq^{-1}\partial & \frac{1}{4}q\partial & \partial\rho_1 + \rho_1\partial \end{pmatrix},$$

$$\mathbf{B}_2 = \begin{pmatrix} 0 & \frac{1}{4}q^2 & 8pq^{-1} + \partial & \frac{1}{2}\partial q - \frac{1}{2}q\partial \\ -\frac{1}{4}q^2 & -\frac{1}{32}q\partial q & \Omega_1 + \partial pq^{-1} & \frac{1}{16}q\partial^2 + \frac{1}{2}\partial p \\ -8pq^{-1} + \partial & -\Omega_1 + pq^{-1}\partial & \frac{1}{4}(\partial q + q\partial) & \rho_1\partial + \frac{1}{2}\partial c_1 \\ -\frac{1}{2}\partial q + \frac{1}{2}q\partial & -\frac{1}{16}\partial^2 q + \frac{1}{2}p\partial & \partial\rho_1 + \frac{1}{2}c_1\partial & \frac{1}{2}(\partial c_2 + c_2\partial) + \frac{1}{8}\partial^3 \end{pmatrix}.$$

The notation anticipates the same phenomenon as in the (u_0, u_1) coordinates. That is, the *third* Hamiltonian structure is actually the first! We define the recursion operators $\mathbf{R} = \mathbf{B}_2 \mathbf{B}_1^{-1}$

$$\mathbf{R} = \begin{pmatrix} -\frac{1}{2}q & 0 & 16pq^{-1}\partial^{-1} + 2 & 0 \\ \frac{1}{16}q\partial & -\frac{1}{2}q & (2\Omega_1 + 2\partial pq^{-1})\partial^{-1} & 0 \\ 0 & 0 & \frac{1}{2}\partial q\partial^{-1} & 1 \\ -\Omega_1 + \frac{1}{8}\partial^2 & 8pq^{-1} - \partial & 2\rho_1 + 2\partial\rho_1\partial^{-1} & \frac{1}{2}q \end{pmatrix},$$

which is invertible. The Hamiltonian operator $\mathbf{B}_0$ then is constructed by $\mathbf{B}_0 = \mathbf{R}^{-1}\mathbf{B}_1$

$$\mathbf{B}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & \frac{1}{8}\partial & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\partial \\ 0 & 0 & \frac{1}{2}\partial & 0 \end{pmatrix}.$$

Remark 4.3. Once again, since $\mathbf{B}_1$ degenerates upon reduction to the stationary flow, this *construction* is only valid on the PDE level. However, $\mathbf{B}_0$ is still valid upon this reduction.

In the usual way, we can construct an infinite sequence of Hamiltonians satisfying the bi-Hamiltonian ladder, Miura maps and modified systems, and all the other formulae, such as (31), in a systematic way [16].

5. Conclusions

In this paper, we have been interested in the relationship between an integrable nonlinear evolution equation (PDE) and its stationary flow. We were particularly interested in the reduction of the infinite dimensional Hamiltonian structures to their finite dimensional counterparts. This reduction is most transparent when the PDE is written as a flow in a larger space whose coordinates are those of the stationary manifold, together with their t-derivatives.

Starting from a zero-curvature representation (reversing the roles of U and V), we gave a systematic construction of the isospectral flows and their Hamiltonian structures. We have adopted the approach presented in [3, 4], which *simultaneously* constructs the isospectral flows, time evolutions of the wave functions, the Hamiltonians, and Hamiltonian structures. This close relationship between the zero curvature representation and the Hamiltonian formulation of the equations perhaps is best seen by the formula (31).

An unsatisfactory feature of our approach is that each stationary manifold has to be treated separately. Thus we construct 1×1, 3×3, 5×5, etc. Hamiltonian representations of the KdV equation, but know of no way of reducing from higher to lower order representations. This deficiency perhaps could be overcome by giving some Lie algebraic description of stationary reductions (see [21] for the general approach). However, since the appropriate spectral problems are such deep reductions of the generic case, we can expect the appropriate algebra to be rather complicated.

Whilst the importance of our results is mainly in the realm of the stationary reductions, we have, in passing, answered a number of the questions raised in [13], where

they believed they had found a *new* hierarchy of equations. We have seen that, in fact, this hierarchy is just the KdV hierarchy in disguise.

In this paper, we presented a selection of examples, including the KdV, DWW, and Ito hierarchies. Further examples from the DWW and Boussinesq hierarchies can be found in [16] and will be presented elsewhere.

Appendix: Generalised Lagrangians

We consider a (generalised) Lagrangian:

$$\mathcal{L}(q^{(0)}, q^{(1)}, \ldots, q^{(n)}), \qquad q^{(i)} = \frac{d^i q}{dx^i}, \quad n \geq 1,$$

where $\mathcal{L}$ is non-degenerate ($\frac{\partial^2 \mathcal{L}}{\partial q^{(n)2}} \neq 0$). The corresponding Euler-Lagrange equations are

$$\sum_{i=0}^{n} (-\partial)^i \frac{\partial \mathcal{L}}{\partial q^{(i)}} = 0.$$

Canonical coordinates can be defined as

$$q_i = q^{(i-1)}, \qquad i = 1, \ldots, n,$$

$$p_n = \frac{\partial \mathcal{L}}{\partial q^{(n)}}, \quad p_i = \frac{\partial \mathcal{L}}{\partial q^{(i)}} - \dot{p}_{i+1}, \quad i = 1, \ldots, n-1.$$

The Euler-Lagrange equations then take canonical Hamiltonian form with

$$h = \sum_{i=1}^{n-1} q_{i+1} p_i + q_{nx} p_n - \mathcal{L}(q_1, \ldots, q_n, q_{nx}),$$

where q_{nx} is removed by inverting the non-degenerate Legendre transformation [22]. The generalisation to more than one component q_i is straightforward.

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DEPARTMENT OF APPLIED MATHEMATICAL STUDIES AND CENTRE FOR NONLINEAR STUDIES, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, UK

E-mail: `allan@amsta.leeds.ac.uk`