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THE CLASSICAL, DIRECT, AND NONCLASSICAL METHODS FOR SYMMETRY REDUCTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Martin D. Kruskal on the occasion of his seventieth birthday.

ABSTRACT. In this paper we discuss the derivation of symmetry reductions and exact solutions of nonlinear partial differential equations using the classical Lie method of infinitesimal transformations, the direct method due to Clarkson and Kruskal [22], and the nonclassical method due to Bluman and Cole [11]. In particular, we compare and contrast the application of these three methods and discuss the relationships among the methods.

1. Introduction

Nonlinear phenomena have many important applications in several aspects of physics as well as other natural and applied sciences. Essentially all the fundamental equations of physics are nonlinear, and, in general, such nonlinear equations often are very difficult to solve explicitly. Consequently perturbation, asymptotic, and numerical methods often are used, with much success, to obtain *approximate* solutions of these equations; however, there also is much current interest in obtaining *exact* analytical solutions of nonlinear equations. Symmetry group techniques provide one method for obtaining such solutions of partial differential equations. These have many mathematical and physical applications and are usually obtained either by seeking a solution in a special form or, more generally, by exploiting symmetries of the equation. This provides a method for obtaining *exact* and *special* solutions of a given equation in terms of solutions of lower dimensional equations, in particular, ordinary differential equations. Furthermore, the methods do not depend upon whether or not the equation is "integrable" (in any sense of the word).

The classical method for finding symmetry reductions of partial differential equations is the Lie group method [12, 58]. Suppose $(x,t) \in \mathbb{R}^2$ are the independent variables, $u \in \mathbb{R}$ the dependent variable, and $\mathbf{u}^{(\ell)}(x,t)$ denotes the set of all the partial derivatives of order ℓ of u. To apply the classical method to the general Nth-order partial differential equations

$$\Delta = \Delta\big(x,t,u,\mathbf{u}^{(1)}(x,t),\ldots,\mathbf{u}^{(N)}(x,t)\big) = 0, \tag{1.1}$$

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we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$\tilde{x} = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \tag{1.2a}$$

$$\tilde{t} = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \tag{1.2b}$$

$$\tilde{u} = u + \varepsilon\phi(x,t,u) + O(\varepsilon^2) \tag{1.2c}$$

where ε is the group parameter. Requiring that (1.1) is invariant under this transformation yields an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$, and $\phi(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u \tag{1.3}$$

where $\partial_x \equiv \partial/\partial x$, etc. Though this method is entirely algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which can become virtually unmanageable if attempted manually, and so symbolic manipulation programs have been developed, for example, in MACSYMA, MAPLE, MATHEMATICA, MUMATH, and REDUCE, to facilitate the calculations; an excellent survey of the different packages presently available and a discussion of their strengths and applications is given by Hereman [40].

There have been several generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov [64] developed the method of partially invariant solutions; recently Ondich [63] has shown that this method can be considered as a special case of the method of differential constraints introduced by Yanenko [80] and Olver and Rosenau [60,61]. Bluman and Cole [11], in their study of symmetry reductions of the linear heat equation, proposed the so-called nonclassical method of group-invariant solutions; this technique also is known as the "method of conditional symmetries" [46] and the "method of partial symmetries of the first type" [75]. In this method, the original partial differential equation (1.1) is augmented with the invariant surface condition

$$\psi \equiv \xi(x,t,u)u_x + \tau(x,t,u)u_t - \phi(x,t,u) = 0, \tag{1.4}$$

which is associated with the vector field (1.3). By requiring that the set of simultaneous solutions of (1.1) and (1.4) are invariant under the transformation (1.2), one obtains an overdetermined, *nonlinear* system of equations, as opposed to a linear system in the classical case, for the infinitesimals ξ, τ, and ϕ, which appear in both the transformations (1.2) and the supplementary condition (1.4). The number of determining equations arising in the nonclassical method is smaller than for the classical method since there are fewer linearly independent expressions in the derivatives. Since all solutions of the classical determining equations necessarily satisfy the nonclassical determining equations, the solution set may be larger in the nonclassical case. For some equations, such as the Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.5}$$

which is the prototypical soliton equation solved by Gardner, Greene, Kruskal, and Miura [38] using the inverse scattering method, the infinitesimals arising from the classical and nonclassical methods coincide. It should be emphasized that the vector fields associated with the nonclassical method do not form a vector space, still less a

Lie algebra, since the invariant surface condition (1.4) depends upon the particular reduction. For example, the sum of two nonclassical symmetry operators is not, in general, a symmetry operator at all; similarly, the commutator of two nonclassical symmetry operators, or the sum of a classical symmetry operator and a nonclassical symmetry operator are not, in general, symmetry operators.

Subsequently, these methods were generalized further by Olver and Rosenau [60,61] to include "weak symmetries" and, even more generally, "side conditions" or "differential constraints" (see also Yanenko [80]). However, their framework appears to be too general to be practical, and they concluded that:

> "the unifying theme behind finding special solutions of partial differential equations is not, as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of partial differential equations."

Motivated by the fact that symmetry reductions of the Boussinesq equation (see equation (2.1) below) were known that are not obtainable using the classical Lie group method [56, 60, 61, 69, 70], Clarkson and Kruskal [22] developed a direct, algorithmic method for finding symmetry reductions (in the following referred to as the *direct method*), which they used to obtain previously unknown reductions of the Boussinesq equation (see §2 for details). The basic idea of the direct method is to seek a solution of a partial differential equation such as (1.1) in the form

$$u(x,t) = F\big(x,t,w(z(x,t))\big) \tag{1.6}$$

and require that $w(z)$ satisfy an ordinary differential equation. This imposes conditions upon $F(x,t,w)$, $z(x,t)$, and their derivatives in the form of an overdetermined system of equations whose solution yields the desired reductions. Levi and Winternitz [46] subsequently gave a group theoretical explanation of these results by showing that all the new reductions of the Boussinesq equation could be obtained using the nonclassical method of Bluman and Cole [11]. The novel characteristic about the direct method, in comparison to the others mentioned above, is that it involves no use of group theory. We remark that the direct method has certain resemblances to the so-called "method of free parameter analysis" [39]; although in the latter method the boundary conditions are crucially used in the determination of the reduction whereas they are not used in the direct method. Additionally ansatz-based methods for determining reductions and exact solutions of partial differential equations have been used by Fushchych and co-workers (see [34–36] and the references therein).

The nonclassical method lay dormant until the papers by Olver and Rosenau [60,61]. However, following the development of the direct method, there has been renewed interest in the nonclassical method. Recently both methods have been used to generate many new symmetry reductions and exact solutions for several physically significant partial differential equations, which represents significant and important progress (see [20, 34, 36] and the references therein). At the time of writing, according to BIDS[1], there have been 117 citations of the paper by Bluman and Cole [11], 18 before 1988 and 99 since 1989.

[1]Bath Information and Data Services ISI database, Science Citation Index

Recent generalizations of the direct method include those due to Burdé [16, 17], Galaktionov [37], and Hood [44]. Generalizations of the nonclassical method are discussed by Bluman and Shtelen [13], Burdé [18], and Olver and Vorob'ev [62].

In §2 of this paper, we discuss the application of the classical, direct, and nonclassical methods to the Boussinesq equation. In §3 we apply the nonclassical method to five variants of a shallow water wave equation in particular, comparing the complexity of the associated calculations. In §4 we discuss the relationship between the classical, direct, and nonclassical methods.

2. The Boussinesq equation

In this section, we discuss symmetry reductions of the Boussinesq equation

$$\Delta \equiv u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0, \tag{2.1}$$

which is also a soliton equation solvable by inverse scattering [1, 81]. The Boussinesq equation arises in several physical applications: propagation of long waves in shallow water [14, 15, 74, 78], one-dimensional nonlinear lattice-waves [73, 82], vibrations in a nonlinear string [81], and ion sound waves in a plasma [71].

2.1. Classical Lie method. To apply the classical Lie method to the Boussinesq (BQ) equation (2.1), we require that the set $\mathcal{S} := \{u(x,t) : \Delta(u) = 0\}$ of solutions of (2.1) is invariant under the transformation (1.2). This yields the *determining equations*, a system of *linear*, *homogeneous* PDEs for ξ, τ, and ϕ, and that is accomplished by requiring that $\mathrm{pr}^{(4)}\mathbf{v}(\Delta)|_{\Delta=0} = 0$ where $\mathrm{pr}^{(4)}\mathbf{v}$ is the fourth prolongation of the vector field (1.3) [12, 58].

Hence, we obtain twelve determining equations for the infinitesimals, which have the general solution

$$\xi = \alpha x + \beta, \qquad \tau = 2\alpha t + \gamma, \qquad \phi = -2\alpha u \tag{2.2}$$

where α, β, and γ are arbitrary constants [56, 70]. Consequently, there are two canonical (classical) symmetry reductions (see [20, 22] for further details).

Case 1. $\alpha = 0$. In this case, we set $\gamma = 1$ and obtain the traveling wave reduction

$$u(x,t) = w(z), \qquad z = x - \beta t \tag{2.3}$$

where $w(z)$ satisfies

$$w'''' + ww'' + (w')^2 + \beta^2 w'' = 0 \tag{2.4}$$

with $' := d/dz$, which is solvable in terms of the first Painlevé equation or elliptic functions, depending upon the choice of constants of integration.

Case 2. $\alpha = 1$. In this case, we set $\beta = \gamma = 0$ and obtain the scaling reduction

$$u(x,t) = t^{-1} w(z), \qquad z = x/t^{1/2} \tag{2.5}$$

where $w(z)$ satisfies

$$w'''' + ww'' + (w')^2 + \frac{1}{4}z^2 w'' + \frac{7}{4} z w' + 2w = 0, \tag{2.6}$$

which is solvable in terms of the fourth Painlevé equation.

However, as noted by several authors [56, 60, 61, 69, 70], the BQ equation (2.1) also possesses the accelerating wave solution

$$u(x,t) = w(z) - 4\mu^2 t^2, \qquad z = x + \mu t^2 \tag{2.7}$$

where μ is an arbitrary constant and $w(z)$ satisfies

$$w''' + ww' + 2\mu w = 8\mu^2 z + A$$

with A an arbitrary constant, which is solvable in terms of the second Painlevé equation. Associated infinitesimals for this reduction are $\xi = 2\mu t$, $\tau = -1$, $\phi = 8\mu^2 t$, which are clearly *not* a special case of (2.2), with associated vector field $\mathbf{v} = 2\mu t\partial_x - \partial_t + 8\mu^2 t\partial_u$.

2.2. Direct method. Clarkson and Kruskal [22] developed the direct method in an attempt to understand the symmetry reduction (2.7) and derive it systematically (the previous derivations had been by seemingly *ad hoc* techniques). For the BQ equation (2.1), Clarkson and Kruskal [22] showed that it is sufficient to seek a solution in the linear form

$$u(x,t) = \beta(x,t)w(z(x,t)) + \alpha(x,t), \tag{2.8}$$

rather than the more general form (1.6). There are two cases to consider, $z_x \not\equiv 0$ and $z_x \equiv 0$, though we shall consider only the generic case when $z_x \not\equiv 0$; the case $z_x \equiv 0$ is discussed in [19, 47].

In the generic case when $z_x \not\equiv 0$, substituting (2.8) into the BQ equation (2.1) yields

$$\begin{aligned}
&\beta z_x^4 w'''' + (6\beta z_x^2 z_{xx} + 4\beta_x z_x^3)w''' \\
&\quad + \left[\alpha\beta z_x^2 + \beta(z_t^2 + 3z_{xx}^2 + 4z_x z_{xxx}) + 12\beta_x z_x z_{xx} + 6\beta_{xx} z_x^2\right]w'' \\
&\quad + \left[\beta z_{xxxx} + 4\beta_x z_{xxx} + 6\beta_{xx} z_{xx} + 4\beta_{xxx} z_x + 2\alpha_x\beta z_x + 2\alpha\beta_x z_x \right. \\
&\qquad \left. + \alpha\beta z_{xx} + 2\beta_t z_t + \beta z_{tt}\right]w' \\
&\quad + \left[\beta_{xxxx} + 2\alpha_x\beta_x + \alpha\beta_{xx} + \alpha_{xx}\beta + \beta_{tt}\right]w + \beta^2 z_x^2 ww'' + \beta(4\beta_x z_x + \beta z_{xx})ww' \\
&\quad + ({\beta_x}^2 + \beta\beta_{xx})w^2 + \beta^2 z_x^2 (w')^2 + \alpha_{tt} + \alpha\alpha_{xx} + \alpha_x^2 + \alpha_{xxxx} = 0.
\end{aligned} \tag{2.9}$$

For this to be an ordinary differential equation for $w(z)$, the coefficients must be of the form $\beta z_x^4 \Gamma(z)$ (using the coefficient of w'''' as the normalizing coefficient). This requirement generates an overdetermined system of equations for $\alpha(x,t)$, $\beta(x,t)$, and $z(x,t)$. Solving this yields the generic symmetry reduction of the BQ equation (2.1) given by

$$u(x,t) = \theta^2(t)w(z) - \frac{1}{\theta^2(t)}\left(x\frac{d\theta}{dt} + \frac{d\phi}{dt}\right)^2, \qquad z(x,t) = x\theta(t) + \phi(t) \tag{2.10}$$

where $\theta(t)$ and $\phi(t)$ are any solutions of

$$\frac{d^2\theta}{dt^2} = A\theta^5, \qquad \frac{d^2\phi}{dt^2} = (A\phi + B)\theta^4, \tag{2.11}$$

A and B are arbitrary constants, and $w(z)$ satisfies

$$w'''' + ww'' + (w')^2 + (Az + B)w' + 2Aw = 2(Az + B)^2. \tag{2.12}$$

Depending upon the choice of the constants, this equation is solvable in terms of the first, second, and fourth Painlevé equations [22].

Solving (2.11) yields six canonical types of symmetry reductions:

$$u(x,t) = w_1(z), \qquad z = x + \mu_1 t, \tag{2.13a}$$

$$u(x,t) = t^2 w_2(z) - x^2/t^2, \qquad z = xt, \tag{2.13b}$$

$$u(x,t) = w_3(z) - 4\mu_3^2 t^2, \qquad z = x + \mu_3 t^2, \tag{2.13c}$$

$$u(x,t) = t^2 w_4(z) - (x + 6\mu_4 t^5)^2/t^2, \qquad z = xt + \mu_4 t^6, \tag{2.13d}$$

$$u(x,t) = t^{-1} w_5(z) - (x - 3\mu_5 t^2)^2/(4t^2), \qquad z = xt^{-1/2} + \mu_5 t^{3/2}, \tag{2.13e}$$

$$u(x,t) = \frac{1}{\wp(t)}\left\{ w_6(z) - \left[\frac{1}{2} z \frac{d\wp}{dt} + \mu_6 \wp^{3/2}(t)\right]^2 \right\}, \qquad z = \wp^{-1/2}(t)[x + \mu_6 \zeta(t)] \tag{2.13f}$$

where μ_1, $\mu_3, \ldots, \mu_6$ are arbitrary constants, $\wp(t) = \wp(t + t_0; 0, g_3)$ is the Weierstrass elliptic function, $\zeta(t) = \zeta(t + t_0; 0, g_3)$ is the Weierstrass zeta function, $w_1(z)$ and $w_2(z)$ satisfy an equation equivalent to the first Painlevé equation, $w_3(z)$ and $w_4(z)$ satisfy an equation equivalent to the second Painlevé equation and $w_5(z)$, and $w_6(z)$ satisfy an equation equivalent to the fourth Painlevé equation.

2.3. Nonclassical method. In the concluding discussion of [22], Clarkson and Kruskal expressed the hope "that a group theoretical explanation of the [direct] method will be possible in due course". Levi and Winternitz [46] subsequently gave such an explanation of Clarkson and Kruskal's results by showing that all their new reductions of the Boussinesq equation could be obtained using the nonclassical method of Bluman and Cole [11], as we show in this subsection. Recently Olver [59] has provided a proof of the precise relationship between the direct methods which we discuss in §4 below (see also [9, 68]).

In the nonclassical method, it is required that the infinitesimal transformation (1.2) leaves invariant the set of simultaneous solutions of the BQ equation (2.1) and the surface condition (1.4) where ξ, τ, and ϕ are the same as in the transformation (1.2). That is, we require that the subset of $\mathcal{S}$ given by $\mathcal{S}_\psi = \{\, u(x,t) : \Delta(u) = 0, \psi(u) = 0 \,\}$ be invariant under the transformation (1.2). Thus "nonclassical symmetries", or "conditional symmetries", of a partial differential equation Δ are transformations that leave only the subset S_ψ of the solution set S of the system invariant. Other solutions of Δ that are not in the subset S_ψ are *not* necessarily transformed to the set S.

The usual method of applying the nonclassical method (e.g., as described in [46]), to the BQ equation (2.1) involves applying the prolongation $\mathrm{pr}^{(4)}\mathbf{v}$ to the system of equations given by (2.1) and the invariant surface condition (1.4) and requiring that the resulting expressions vanish for $u \in \mathcal{S}_\psi$, i.e.,

$$\mathrm{pr}^{(4)}\mathbf{v}(\Delta)\Big|_{\Delta=0,\psi=0} = 0, \qquad \mathrm{pr}^{(1)}\mathbf{v}(\psi)\Big|_{\Delta=0,\psi=0} = 0. \tag{2.14}$$

It is easily shown that $\mathrm{pr}^{(1)}\mathbf{v}(\psi) = -(\xi_u u_x + \tau_u u_t - \phi_u)\psi$, which vanishes identically when $\psi = 0$, without imposing any conditions upon ξ, τ, and ϕ. However, as shown by Clarkson and Mansfield [24], this procedure for applying the nonclassical method can create difficulties, in particular, in the implementation of symbolic manipulation programs. These difficulties often arise for equations such as (2.1) which require the use of differential consequences of the invariant surface condition (1.4). In [24], Clarkson and Mansfield proposed an algorithm for calculating the determining equations associated with the nonclassical method which avoids many of the difficulties commonly encountered, and we use this algorithm here.

There are two cases to consider: (i), $\tau \neq 0$; and (ii), $\tau = 0$ and $\xi \neq 0$. We only shall consider the generic case when $\tau \neq 0$.

In the case $\tau \not\equiv 0$, we set $\tau = 1$, without loss of generality, and then we use the invariant surface condition (1.4) to eliminate u_{tt} in (2.1), yielding

$$\begin{aligned}\phi_t + \phi_u(\phi - \xi u_x) - \xi_t u_x - \xi_u u_x(\phi - \xi u_x) \\ - \xi\left[\phi_x + \phi_u u_x - (\xi_x u_x + \xi_u u_x^2 + \xi u_{xx})\right] \\ + u u_{xx} + u_x^2 + u_{xxxx} = 0,\end{aligned} \tag{2.15}$$

which essentially is an ordinary differential equation for $u(x)$ with t a parameter since only x-derivatives of u arise. Now we apply the classical Lie algorithm to this equation, i.e., we require that it be invariant under the transformation (1.2) with $\tau = 1$, and then we use (2.15) to eliminate u_{xxxx}. This yields a system of seven determining equations which have solution

$$\xi = xf(t) + g(t), \tag{2.16a}$$

$$\begin{aligned}\phi = -\Bigg\{2f(t)u + 2x^2 f(t)\left[\frac{df}{dt} + 2f^2(t)\right] \\ + 2x\left[\frac{df}{dt}g(t) + f(t)\frac{dg}{dt} + 4f^2(t)g(t)\right] + 2g(t)\left[\frac{dg}{dt} + 2f(t)g(t)\right]\Bigg\}\end{aligned} \tag{2.16b}$$

where

$$f(t) = \frac{1}{2p(t)}\frac{dp}{dt}, \qquad g(t) = \frac{\kappa_1}{2p(t)}\frac{dp}{dt} + \frac{\kappa_0}{p(t)}\frac{dp}{dt}\int^t \frac{p(s)}{[p'(s)]^2}\,ds,$$

and $p(t)$ satisfies

$$\left(\frac{dp}{dt}\right)^2 = \kappa_3 p^3 + \kappa_2 \tag{2.17}$$

where κ_3, κ_2, κ_1, and κ_0 are arbitrary constants (see [20] for details). Equation (2.17) is solvable in terms of the Weierstrass elliptic function $\wp(t; 0, g_3)$ if $\kappa_3\kappa_2 \neq 0$ and elementary functions otherwise. Solving (2.17) for $p(t)$ yields the six canonical symmetry reductions (2.13) of the BQ equation (2.1) that were derived by Clarkson and Kruskal [22] using the direct method, as discussed in the previous section.

3. Shallow water wave equations

The shallow water wave (SWW) equation,

$$v_{xxt} + \alpha v v_t - \beta v_x \partial_x^{-1} v_t - v_t - v_x = 0 \tag{3.1}$$

where $(\partial_x^{-1} f)(x) = \int_x^\infty f(y)\,dy$ and α and β are arbitrary nonzero constants, can be derived from the classical shallow water theory in the Boussinesq approximation [31]. Two special cases of (3.1) have attracted some attention in the literature, namely the cases when $\alpha = 2\beta$ and $\alpha = \beta$. In their seminal paper on soliton theory, Ablowitz, Kaup, Newell, and Segur [2] showed that

$$v_{xxt} + 2\beta v v_t - \beta v_x \partial_x^{-1} v_t - v_t - v_x = 0 \tag{3.2}$$

(which is (3.1) with $\alpha = 2\beta$) is solvable by inverse scattering. Further, they remark that (3.2) reduces in the long wave, small amplitude limit to the KdV equation (1.5), and they also comment that (3.2) has the desirable properties of the regularized long wave (RLW) equation [10, 65]

$$v_{xxt} + v v_x - v_t - v_x = 0, \tag{3.3}$$

sometimes known as the Benjamin-Bona-Mahoney equation, in that it responds feebly to short waves. We note that (3.2) and (3.3) have the same linear dispersion relation $\omega(k) = -k/(1+k^2)$ for the complex exponential $v(x,t) \sim \exp\{i[kx+\omega(k)t]\}$. However, in contrast to (3.2), the RLW equation (3.3) is thought *not* to be solvable by inverse scattering [53]. Subsequently, Hirota and Satsuma [43] studied both (3.1) and (3.2) with $\alpha = \beta$, i.e.,

$$v_{xxt} + \beta v v_t - \beta v_x \partial_x^{-1} v_t - v_t - v_x = 0, \tag{3.4}$$

using Hirota's bi-linear method [42] and obtained N-soliton solutions for both equations.

The SWW equation (3.1) also was discussed by Hietarinta [41] who showed that it can be expressed in Hirota's bi-linear form [42] if and only if either (i) $\alpha = \beta$, when it reduces to (3.4), or (ii) $\alpha = 2\beta$, when it reduces to (3.2). Further, the SWW equation (3.1) satisfies the necessary conditions of the Painlevé tests due to Ablowitz et al. [3,4] and Weiss et al. [77] to be completely integrable if and only if either $\alpha = \beta$ or $\alpha = 2\beta$ (see [25]). These results strongly suggest that the SWW equation (3.1) is completely integrable if and only if it has one of the two special forms (3.2) or (3.4), which are both known to be solvable by inverse scattering (see [2] and [30], respectively).

Here we are interested in symmetry reductions and exact solutions of five variants of SWW equation (3.1). Since (3.1) contains a nonlocal term, in order to undertake symmetry analysis we need to write (3.1) as an analytic equation or system. These five variants are

(i) the scalar equation

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - u_{xx} = 0, \tag{3.5}$$

(ii) the system with two dependent variables

$$v_{xxt} + \alpha v v_t + \beta w v_x - v_t - v_x = 0, \tag{3.6a}$$

$$w_x = v_t, \tag{3.6b}$$

(iii) the system with three dependent variables

$$v = u_x, \tag{3.7a}$$

$$w = u_t, \tag{3.7b}$$

$$v_{xxt} + \alpha v v_t + \beta w v_x - v_t - v_x = 0, \tag{3.7c}$$

(iv) the system derived from a conservation law

$$\psi_x = u_{xxx} + \frac{1}{2}(\alpha - \beta)u_x^2 - u_x, \tag{3.8a}$$

$$\psi_t = u_x - \beta u_t u_x, \tag{3.8b}$$

and (v) a system with two dependent variables only slightly different from the second system, but including the dependent variable u,

$$v_{xxt} + \alpha v v_t + \beta u_t v_x - v_t - v_x = 0, \tag{3.9a}$$

$$v = u_x. \tag{3.9b}$$

3.1. Classical symmetries. The number of determining equations obtained by applying the classical method to these five systems is given in the following table.

Equation	No. determining equations
(3.5)	14 linear equations
(3.6)	20 linear equations
(3.7)	24 linear equations
(3.8)	17 linear equations
(3.9)	17 linear equations

Solving the associated determining equations in the five cases yields the infinitesimals

$$\xi = \mu_1 x + \mu_2, \qquad \tau = f(t),$$

$$\phi_1 = -\mu_1 u + \frac{f(t) + \mu_1 t}{\beta} + \frac{2\mu_1 x}{\alpha} + \mu_3, \qquad \phi_3 = -\left(\frac{df}{dt} + \mu_1\right)\left(w - \frac{1}{\beta}\right),$$

$$\phi_2 = -2\mu_1\left(v - \frac{1}{\alpha}\right), \qquad \phi_4 = -3\mu_1\psi + \frac{2\mu_1(t - x - \beta u)}{\alpha} + \mu_4$$

where μ_1, μ_2, μ_3, and μ_4 are arbitrary constants and $f(t)$ is an arbitrary function. Hence there are two canonical reductions.

Case 1. $\mu_1 \neq 0$. In this case, we set $\mu_1 = 1$, $\mu_2 = \mu_3 = \mu_4 = 0$, and $\frac{1}{f(t)} = -\frac{d}{dt}[\ln g(t)]$, without loss of generality. Hence, we obtain the reduction

$$u(x,t) = g(t)U(z) + \frac{t}{\beta} + \frac{x}{\alpha}, \qquad w(x,t) = \frac{dg}{dt}W(z) + \frac{1}{\beta},$$

$$v(x,t) = g^2(t)V(z) + \frac{1}{\alpha}, \qquad \psi(x,t) = g^3(t)\Phi(z) - \frac{\beta}{\alpha}U(z)g(t) - \frac{(\alpha+\beta)x}{2\alpha^2}$$

where $z = xg(t)$, $U(z)$ satisfies

$$zU'''' + 4U''' + 2\alpha\,(U')^2 + \beta UU'' + (\alpha+\beta)zU'U'' = 0, \tag{3.10}$$

and $V(z)$, $W(z)$, and $\Phi(z)$ are given by

$$V = U', \qquad W = U + zU', \qquad \Phi = -\frac{1}{3}zU''' - \frac{1}{6}(\alpha+\beta)z\,(U')^2 - \frac{1}{3}\beta UU'.$$

It is straightforward to show using the algorithm of Ablowitz et al. [4] that (3.10) is of Painlevé-type, i.e., its solutions have no movable singularities other than poles, only if either (i) $\alpha = \beta$ or (ii) $\alpha = 2\beta$. These two special cases of (3.10) are solvable in terms of solutions of the third Painlevé equation [25].

Case 2. $\mu_1 = 0$. In this case, we set $\mu_2 = 1$ and $\frac{1}{f(t)} = \frac{dg}{dt}$, without loss of generality. Hence, we obtain the reduction

$$u(x,t) = U(z) + \mu_3 g(t) + \frac{t}{\beta}, \qquad w(x,t) = \frac{dg}{dt} W(z) + \frac{1}{\beta},$$
$$v(x,t) = V(z), \qquad \psi(x,t) = \Phi(z) + \mu_4 g(t)$$

where $z = x - g(t)$, $U(z)$ satisfies

$$U''' + \frac{1}{2}(\alpha + \beta)\left(U'\right)^2 - (1 + \beta\mu_3)U' = A, \tag{3.11}$$

with A an arbitrary constant, and $V(z)$, $W(z)$, and $\Phi(z)$ are given by

$$V = U', \qquad W = \mu_3 - U', \qquad \Phi' = \mu_4 + \beta\mu_3 U' - \beta\left(U'\right)^2.$$

If $\alpha + \beta \neq 0$, then (3.11) is equivalent to the Weierstrass elliptic function equation, otherwise it is a linear equation.

3.2. Nonclassical symmetries. Next we apply the nonclassical method to the five systems (3.5)–(3.9) in the canonical case when $\tau \not\equiv 0$; we set $\tau = 1$ without loss of generality. The number of (nonclassical) determining equations and lines of output for these five systems is given in the following table.

Equation	Output	No. determining equations	
(3.5)	67 lines	8 equations	[3 linear, 5 nonlinear]
(3.6)	583 lines	11 equations	[all nonlinear]
(3.7)	1136 lines	13 equations	[all nonlinear]
(3.8)	181 lines	15 equations	[4 linear, 11 nonlinear]
(3.9)	166 lines	9 equations	[5 linear, 4 nonlinear]

3.2.1. *Equation* (3.5). In this case, we obtain a system of 8 determining equations, 3 linear and 5 nonlinear. It is straightforward to show that there are three cases to consider: (i) $\xi_x = 0$, with $\alpha = \beta$, (ii) $\xi_x = 0$, with $\alpha = -\beta$, and (iii) $\xi_x \neq 0$, which yields the classical reductions (see [25] for details).

Case (i). $\alpha = \beta$; $\xi_x = 0$. In this case, we obtain the infinitesimals

$$\xi = \frac{df}{dt}, \qquad \phi_1 = 2P(\zeta)\frac{df}{dt} + \frac{1}{\alpha} \tag{3.12}$$

where $\zeta = x + f(t)$, $f(t)$ is an arbitrary function and $P(\zeta)$ satisfies

$$\frac{d^2P}{d\zeta^2} + \alpha P^2 - P = \lambda\zeta + \mu_1 \tag{3.13}$$

with λ and μ_1 arbitrary constants (see [25] for details). If $\lambda \neq 0$, then this equation is equivalent to the first Painlevé equation; otherwise it is equivalent to the Weierstrass elliptic function equation. Thus, solving the characteristic equations yields the nonclassical reduction

$$u(x,t) = p(\zeta) + q(z) + t/\alpha \tag{3.14}$$

where $\zeta = x + f(t)$, $z = x - f(t)$, $f(t)$ is an arbitrary function, $P(\zeta) = p_\zeta$ satisfies (3.13), and $Q(z) = q'(z)$ satisfies

$$Q'' + \alpha Q^2 - Q = \lambda z + \mu_2 \tag{3.15}$$

where μ_2 is an arbitrary constant; λ is (effectively) a "separation" constant.

In particular, if $\lambda = \mu_1 = \mu_2 = 0$ (we set $\alpha = 1$ without loss of generality), then equations (3.13) and (3.15) possess the special solutions $P(\zeta) = \frac{3}{2}\operatorname{sech}^2\left(\frac{1}{2}\zeta\right)$ and $Q(z) = \frac{3}{2}\operatorname{sech}^2\left(\frac{1}{2}z\right)$, respectively. Hence we obtain the exact solution of (3.5) with $\alpha = \beta = 1$ given by

$$u(x,t) = 3\tanh\left\{\frac{1}{2}\left[x + f(t)\right]\right\} + 3\tanh\left\{\frac{1}{2}\left[x - f(t)\right]\right\} + t.$$

This is one of the simplest, nontrivial family of solutions of (3.5) with $\alpha = \beta = 1$ obtainable using this reduction and has a rich variety of qualitative behaviours. This is due to the freedom in the choice of the arbitrary function $f(t)$. One can choose $f_1(t)$ and $f_2(t)$ such that $|f_1(t) - f_2(t)|$ is exponentially small as $t \to -\infty$, yet $f_1(t)$ and $f_2(t)$ are quite different as $t \to +\infty$. By a judicious choice of $f(t)$, one can obtain a plethora of different solutions (see the figures in [25]). We believe these results suggest that solving (3.5) with $\alpha = \beta = 1$ numerically could pose some fundamental difficulties. An exponentially small change in the initial data yields a fundamentally different solution as $t \to \infty$. How can any numerical scheme in current use cope with such behavior?

Case (ii). $\alpha = -\beta$, $\xi_x = 0$. In this case, we obtain the infinitesimals

$$\xi = \frac{df}{dt}, \qquad \phi_1 = \frac{df}{dt}\eta(z) - \frac{1}{\alpha}$$

where $z = x - f(t)$ and $\eta(z)$ satisfies

$$\eta'''' - \eta'' + \alpha\left[\eta\eta'' - (\eta')^2\right] = 0, \tag{3.16}$$

which is not of Painlevé type. Then, solving the characteristic equations yields the nonclassical reduction

$$u(x,t) = U(z) + f(t)\eta(z) - t/\alpha \tag{3.17}$$

where $U(z)$ satisfies the linear equation

$$U'''' + (\alpha\eta - 1)U'' - \alpha\eta' U' = \eta''' - \eta'. \tag{3.18}$$

3.2.2. *System* (3.6). Applying the nonclassical method to (3.6) yields a system of eleven determining equations, all of which are nonlinear, and the biggest equation has 166 summands! Solving these equations could be highly intractable manually; however using symbolic manipulation programs, in particular `diffgrob2` in MAPLE [49], the problem becomes more tractable though still non-trivial. We find both classical reductions and two nonclassical reductions (which themselves have interesting special cases). These two nonclassical reductions arise when $\beta = \pm\alpha$ from infinitesimals which give the following invariant surface conditions

$$\frac{df}{dt}v_x + v_t = \frac{df}{dt}\Phi(\theta,\zeta), \tag{3.19a}$$

$$\frac{df}{dt}w_x + w_t = \frac{\alpha}{\beta}\left(\frac{df}{dt}\right)^2\Phi(\theta,\zeta) + \left(w - \frac{1}{\beta}\right)\frac{d}{dt}\left(\ln\frac{df}{dt}\right) \tag{3.19b}$$

where

$$\theta = v + \left(w - \frac{1}{\beta}\right)\Big/\frac{df}{dt}, \qquad \zeta = \frac{\beta x}{\alpha} + f(t),$$

and $\Phi(\theta,\zeta)$ satisfies

$$\begin{aligned}&\alpha\theta\Phi_\zeta \pm \alpha\theta\Phi\Phi_\theta + \Phi_{\zeta\zeta\zeta} \pm \Phi_\theta\Phi_{\zeta\zeta} + 3\Phi\Phi_\zeta\Phi_{\theta\theta}\\ &\quad \pm 3\Phi_{\theta\zeta}\Phi_\zeta + \Phi_\theta^2\Phi_\zeta - \Phi_\zeta \pm \Phi^3\Phi_{\theta\theta\theta} + 3\Phi^2\Phi_{\theta\theta\zeta}\\ &\quad \pm 4\Phi^2\Phi_\theta\Phi_{\theta\theta} \pm 3\Phi\Phi_{\theta\zeta\zeta} + 5\Phi\Phi_\theta\Phi_{\theta\zeta} \pm \Phi\Phi_\theta^3 \mp \Phi\Phi_\theta + \alpha\Phi^2 = 0. \end{aligned} \tag{3.20$^\pm$}$$

At first glance, this equation appears to be difficult to solve in full generality; indeed it is more complex than the original equation. However, by using the associated invariant surface conditions, we can make progress. Remember that in (3.19) the variables v, w are dependent rather than independent variables. If we define

$$\Theta(x,t) = \left(w + \frac{df}{dt}v - \frac{1}{\beta}\right)\Big/\frac{df}{dt},$$

then (3.19) yield

$$\frac{df}{dt}\Theta_x + \Theta_t = \left(1 + \frac{\alpha}{\beta}\right)\frac{df}{dt}\Phi(\theta,\zeta). \tag{3.21}$$

Case (i). $\alpha = -\beta$. In this case, equation (3.21) is easily solved to give

$$\Theta(x,t) = p(z), \qquad z = x - f(t).$$

Our two earlier variables, θ and ζ, now are equivalent to the new variable z defined above, so we let $\Phi(\theta,\zeta) = \eta'(z)$ (for convenience). The invariant surface conditions (3.19) are now in a form that can be solved and give the following reduction

$$v(x,t) = V(z) + f(t)\eta', \qquad w(x,t) = \frac{df}{dt}W(z) - f(t)\frac{df}{dt}\eta' - \frac{1}{\alpha}$$

where $z = x - f(t)$, $\eta(z)$ satisfies (3.16), $W(z) = \eta(z) - V(z)$, and $V(z)$ satisfies the linear equation

$$V''' + (\alpha\eta - 1)V' - \alpha\eta' V = \eta' - \eta'. \tag{3.22}$$

Whilst the reduction holds, $\eta(z)$ should be pre-determined by the infinitesimals or should, in some sense, satisfy (3.20$^-$). To show that this is the case, notice that

$\Theta(x,t)$ has the same form as θ, though the former is a dependent and the latter an independent variable. We apply the hodograph transformation

$$\Phi(\theta,\zeta) = \Omega_s(s,\zeta), \qquad \theta = \Omega(s,\zeta), \tag{3.23}$$

to equation (3.20^-), which implements this role reversal to leave a (large) equation for $\Omega(s,\zeta)$. Since $\Phi(\theta,\zeta)$ becomes a function of one variable only, we let $\zeta = s$, so $\Omega = \Omega(s)$. The large equation for Ω then simplifies enormously to give

$$\frac{d^4\Omega}{ds^4} + \alpha\Omega\frac{d^2\Omega}{ds^2} - \frac{d^2\Omega}{ds^s} - \alpha\left(\frac{d\Omega}{ds}\right)^2 = 0. \tag{3.24}$$

We note that since $s = \zeta = z = x - f(t)$, it is not difficult to show that $d\eta/dz \equiv d\Omega/ds$, as required.

Case (ii). $\alpha = \beta$. In this case, the right-hand side of (3.21) is no longer zero but yields

$$\frac{df}{dt}\Theta_x + \Theta_t = 2\frac{df}{dt}\Phi(\theta,\zeta). \tag{3.25}$$

First, we consider the simplified case when $\Phi_\zeta = 0$. Again, we apply a hodograph-type transformation

$$\Phi(\theta) = 2\frac{d\Omega}{ds}, \qquad \theta = 2\Omega(s) + c_6. \tag{3.26}$$

Thus, (3.20^+) becomes

$$\frac{d^4\Omega}{ds^4} + 2\alpha\Omega\frac{d^2\Omega}{ds^2} + (\alpha c_6 - 1)\frac{d^2\Omega}{ds^2} + 2\alpha\left(\frac{d\Omega}{ds}\right)^2 = 0, \tag{3.27}$$

which can be integrated twice to yield either the first Painlevé equation or the Weierstrass elliptic function equation. However, knowing this doesn't appear to make (3.25) much easier to solve. To progress, we note, as before, that $\Theta(x,t)$ has the same form as θ. Therefore, from (3.25), the chain rule, and using (3.26), we have

$$\frac{d\Omega}{ds}\left(\frac{df}{dt}s_x + s_t - 2\frac{df}{dt}\right) = 0. \tag{3.28}$$

If $d\Omega/ds = 0$, we obtain a classical reduction, whilst assuming that $d\Omega/ds \neq 0$ gives

$$s = 2f(t) + G(z), \qquad z = x - f(t). \tag{3.29}$$

Now we are able to solve the invariant surface conditions (3.19) since our z in (3.29) is a characteristic direction in both of equations (3.19). These yield

$$v(x,t) = \Omega(s) + Q(z), \qquad w(x,t) = \frac{df}{dt}[\Omega(s) + H(z)] + \frac{1}{\alpha} \tag{3.30}$$

where $z = x - f(t)$. Substituting this into (3.6b) gives

$$H'(z) + Q'(z) = 2\left(1 - G'(z)\right)\frac{d\Omega}{ds}. \tag{3.31}$$

There are two possibilities. First, if $G'(z) \neq 1$ we divide by $1 - G'(z)$ to obtain

$$\frac{H'(z) + Q'(z)}{1 - G'(z)} = 2\frac{d\Omega}{ds} = \lambda \tag{3.32}$$

where λ is a separation constant, since now s and z are independent. Putting $d\Omega/ds = \frac{1}{2}\lambda$ into (3.27) gives $\Omega(s) = 0$, a classical reduction. Second, if $G'(z) = 1$ then

$s = x + f(t)$ and $H'(z) + Q'(z) = 0$. Integrating this last expression and substituting our values of v, w into (3.6) give the reduction

$$v(x,t) = P(\zeta) + Q(z), \qquad w(x,t) = \frac{df}{dt}[P(\zeta) - Q(z)] + \frac{1}{\alpha} \tag{3.33}$$

where $z = x - f(t)$, $\zeta = x + f(t)$, and $P(\zeta)$ and $Q(z)$ satisfy (3.13) and (3.15), respectively. This is the analogue of reduction (3.14).

The solution of the general case $\Phi = \Phi(\theta, \zeta)$, with $\Phi_\zeta \neq 0$, follows a path similar to the special case we have just considered. We make a slightly different hodograph transformation than previously, namely

$$\Phi(\theta,\zeta) = 2[\Omega_s + \Omega_\zeta], \qquad \theta = 2\Omega(s,\zeta), \tag{3.34}$$

which, when using the chain rule, transforms (3.25) to the equation,

$$[\Omega_s + \Omega_\zeta]\left(\frac{df}{dt}s + x + s_t - 2\frac{df}{dt}\right) = 0. \tag{3.35}$$

If $\Omega_s + \Omega_\zeta = 0$, then we obtain a classical reduction, whilst if $\Omega_s + \Omega_\zeta \neq 0$, then, as previously, it can be shown that s is given by (3.29). We now may solve the invariant surface conditions (3.19) so that $v(x,t)$ and $w(x,t)$ are given by (3.30), which when substituted into (3.6b) give (3.31); though note that now Ω is a function of s and ζ, so that we've differentiated partially with respect to s. If $dG/dz = 1$, then $s = \zeta = x + f(t)$; hence Ω is a function of s only, and so this simplifies to the special case discussed above. If $dG/dz \neq 1$, then we obtain (3.32) since s and ζ are independent of z (again Ω is a function of s and ζ). Hence $\Omega(s,\zeta) = \frac{1}{2}\lambda s + M(\zeta)$. Substituting this into the transformed (3.12+) does not help much. However, substituting (3.30) with $\Omega(s,\zeta) = \frac{1}{2}\lambda s + M(\zeta)$ into (3.6a) and requiring that the resulting equation be an ordinary differential equation yields that either $v(x,t)$ is a constant or that we can obtain the same reduction as found in the special case above.

3.2.3. *System* (3.7). Applying the nonclassical method to (3.7) yields a system of 13 determining equations, all of which are nonlinear. From these one can obtain the classical reductions and also two nonclassical reductions. We note that if we apply the nonclassical method to equations (3.7a) and (3.7b) individually, we are able to obtain explicit expressions for ϕ_2 and ϕ_3 in terms of ξ and ϕ_1 and, significantly, to find that ξ and ϕ_1 are independent of v and w. With this information the nonclassical method generates a system of determining equations in ξ and ϕ_1 which are equivalent and of similar complexity to those of the scalar equation (3.5).

3.2.4. *System* (3.8). Applying the nonclassical method to (3.8) yields a system of 15 determining equations, of which 4 are linear and 11 are nonlinear. Solving these yields both classical reductions but only one of the nonclassical reductions. The associated infinitesimals yield the nonclassical reduction for (3.8) without recourse to a hodograph transformation.

In the case when $\alpha = \beta$, the infinitesimals are

$$\xi = \frac{df}{dt}, \qquad \phi_1 = 2\frac{df}{dt}\frac{dp}{d\zeta} + \frac{1}{\alpha}, \qquad \phi_4 = \frac{df}{dt}\left(2\frac{d^3p}{d\zeta^3} - 2\frac{dp}{d\zeta} + \mu_2 - \mu_1 - 22\lambda f(t)\right)$$

where λ, μ_1, and μ_2 are arbitrary constants, $f(t)$ is an arbitrary function, $\zeta = x + f(t)$, and $P(\zeta) = \frac{dp}{d\zeta}$ satisfies (3.13). Hence, we obtain the following nonclassical reduction

$$u(x,t) = p(\zeta) + q(z) + \frac{t}{\alpha}, \quad \psi(x,t) = \Phi(z) + \frac{d^2p}{d\zeta^2} - p(\zeta) + (\mu_2 - \mu_1)f(t) - \lambda f^2(t)$$

where $z = x - f(t)$, $Q(z) = q'(z)$ satisfies (3.15), and $\Phi(z) = q''(z) - q(z)$. This is the analogue of reduction (3.14).

3.2.5. *System* (3.9). Applying the nonclassical method to (3.9) yields a system of 9 determining equations, of which 5 are linear and 4 are nonlinear. Solving these yields both classical reductions and two nonclassical reductions. The infinitesimals associated with both the $\alpha = \beta$ and $\alpha = -\beta$ nonclassical reductions for the system (3.9) are obtained without recourse to a hodograph transformation. Further, the complexity of the nonclassical determining equations is similar to that for the scalar equation (3.5).

Case (i). $\alpha = \beta$. In this case, the infinitesimals are

$$\xi = \frac{df}{dt}, \qquad \phi_1 = 2\frac{df}{dt}\frac{dp}{d\zeta} + \frac{1}{\alpha}, \qquad \phi_2 = 2\frac{df}{dt}\frac{d^2p}{d\zeta^2}$$

where $\zeta = x + f(t)$, $f(t)$ is an arbitrary function, and $P(\zeta) = \frac{dp}{d\zeta}$ satisfies (3.13). Hence, we obtain the nonclassical reduction

$$u(x,t) = p(\zeta) + q(z) + t/\alpha, \qquad v(x,t) = V(z) + P(\zeta)$$

where $z = x - f(t)$, $Q(z) = q'(z)$ satisfies (3.15), and $V(z) = Q(z)$. This is the analogue of reduction (3.14).

Case (ii). $\alpha = -\beta$. In this case, the infinitesimals are

$$\xi = \frac{df}{dt}, \qquad \phi_1 = \frac{df}{dt}\eta(z) - \frac{1}{\alpha}, \qquad \phi_2 = \frac{df}{dt}\eta'$$

where $\eta(z)$ satisfies (3.16). Hence, we obtain the following nonclassical reduction

$$u(x,t) = U(z) + f(t)\eta(z) - t/\alpha, \qquad v(x,t) = V(z) + f(t)\eta'$$

where $z = x - f(t)$, $U(z)$ satisfies (3.18), and $V(z) = U'$. This is the analogue of reduction (3.17).

3.3. Discussion. In this section, we have discussed symmetry reductions using the classical and nonclassical methods for five variants of a shallow water wave equation, namely the scalar equation (3.5) and the systems (3.6)–(3.9). Both methods give the same reductions when applied to the system (3.6) as when applied to the scalar counterpart (3.5). What is unusual about the calculation for the system (3.6) is the large increase in complexity in moving from a scalar equation to this system. Whilst for both the system and the scalar equation the determining equations for the classical method are of similar complexity (and are all linear), the nonclassical method paints quite a different picture. For the system, there are 11 determining equations, all nonlinear, which constitute 583 lines of computer generated output. Even when we look at the subcase $\xi_w = \xi_v = 0$, this only reduces to 8 nonlinear equations and 117 lines of output. In comparison, the scalar equation (3.5) has only

8 determining equations, 3 linear and 5 nonlinear which produce 67 lines of output, greatly simplifying the problem at hand.

Another difficulty we observed was that the reductions obtained using the nonclassical method arise in a very unusual manner, and one has to use a hodograph transformation. Indeed we can establish a set of infinitesimals which would give the reduction (3.33) more naturally by working the method of characteristics backwards, namely,

$$\xi = \frac{df}{dt}, \quad \phi_2 = 2\frac{df}{dt}\frac{dP}{d\zeta}, \quad \phi_3 = \left(w - \frac{1}{\alpha}\right)\frac{d}{dt}\left(\ln\frac{df}{dt}\right) + 2\left(\frac{df}{dt}\right)^2\frac{dP}{d\zeta} \tag{3.36}$$

where $\zeta = x + f(t)$, $f(t)$ is an arbitrary function, and $P(y)$ satisfies

$$\frac{d^3P}{d\zeta^3} + 2\alpha P\frac{dP}{d\zeta} - \frac{dP}{d\zeta} = c_1 \tag{3.37}$$

with c_1 an arbitrary constant. It is straightforward to show that the infinitesimals (3.36) satisfy the determining equations arising from the nonclassical method for (3.6) if and only if $P(\zeta) = \kappa_0$, a constant. Consequently, we assert that the infinitesimals arising from the nonclassical method which give rise to the nonclassical reductions are "unnatural."

The system (3.7) admits both "unnatural" and "natural" infinitesimals for *both* nonclassical reductions; however these could be difficult to find if they were not known *a priori*.

The system (3.8) admits (natural) infinitesimals which give rise only to one of the nonclassical reductions, in the case when $\alpha = \beta$.

The system (3.9) appears to be the simplest representation of the shallow water wave system, and the associated calculations are similar in complexity to the scalar equation (3.5).

We also have applied the direct method due to Clarkson and Kruskal [22] to the five equations (3.5)–(3.9) and obtained the same results as with the nonclassical method. The direct method is not entirely straightforward, especially for the systems (3.6)–(3.9), though the direct method seems to be much easier to implement; details of the application of the direct method to (3.6) are given in [67].

This raises an important open question as to how one determines *a priori* the most suitable representation, from the point of view of symmetry calculations, of a nonlocal equation such as (3.1).

4. Relationship between classical, direct, and nonclassical methods

In §2 it was shown that applying the nonclassical method yields all the new symmetry reductions of the Boussinesq equation (2.1) that were derived by Clarkson and Kruskal [22] using the direct method (see also [46]). Clarkson and Winternitz [29] obtained a similar result for the Kadomtsev-Petviashvili (KP) equation

$$(u_t + uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0, \qquad \sigma^2 = \pm 1, \tag{4.1}$$

showing that the direct and nonclassical methods yield the same symmetry reductions for (4.1).

The results in [29, 46] suggested that the direct and nonclassical methods were equivalent, i.e., they yield the same reductions. Indeed Clarkson and Kruskal [22] posed the question on the relationship between these two methods in the conclusion of their original paper on the direct method.

This question was investigated by Nucci and Clarkson [57] who applied both the direct and nonclassical methods to the Fitzhugh-Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-a) \tag{4.2}$$

where a is an arbitrary parameter, which arises in population genetics [7,8] and models the transmission of nerve impulses [32, 55].

Example 4.1. *The Fitzhugh-Nagumo Equation*

Applying the classical Lie method to the Fitzhugh-Nagumo equation (4.2) yields the traveling wave solution

$$u(x,t) = w(z), \qquad z = \mu x - \lambda t. \tag{4.3}$$

Applying the direct method to the Fitzhugh-Nagumo equation (4.2) yields, in addition to the traveling wave solution (4.3), the exact solutions of the Fitzhugh-Nagumo equation (4.2) expressed in terms of Jacobi elliptic functions for $a=-1$, $a=\frac{1}{2}$, and $a=2$. For example, for $a=-1$,

$$u(x,t) = z_x ds\Big(z; \frac{1}{2}\sqrt{2}\Big), \tag{4.4a}$$

$$z(x,t) = c_1 \exp\Big[\frac{1}{2}(\sqrt{2}\,x+3t)\Big] + c_2 \exp\Big[\frac{1}{2}(-\sqrt{2}\,x+3t)\Big] + c_3 \tag{4.4b}$$

where $ds(z;k)$ is the Jacobi elliptic function satisfying

$$(\eta')^2 = \eta^4 + (2k^2-1)\eta^2 + k^2(k^2-1).$$

Applying the nonclassical method to the Fitzhugh-Nagumo equation (4.2) yields, in addition to the traveling wave solution (4.3) and the elliptic functions solutions (4.4), the following exact solution of the Fitzhugh-Nagumo equation (4.2), for $a \neq 0$, $a \neq 1$;

$$u(x,t) = \frac{ac_1 \exp\left\{\frac{1}{2}\left(\pm\sqrt{2}\,ax + a^2t\right)\right\} + c_2 \exp\left\{\frac{1}{2}\left(\pm\sqrt{2}\,x + t\right)\right\}}{c_1 \exp\left\{\frac{1}{2}\left(\pm\sqrt{2}\,ax + a^2t\right)\right\} + c_2 \exp\left\{\frac{1}{2}\left(\pm\sqrt{2}\,x + t\right)\right\} + c_3 \exp\left(at\right)} \tag{4.5}$$

where c_1, c_2, and c_3 are arbitrary constants. If $a=0$ or $a=1$, then similar solutions are obtained.

These results pose the following important open question: "for which partial differential equations does the nonclassical method yield more symmetry reductions than the direct method?" Furthermore, it remains an open question to determine *a priori* which partial differential equations possess symmetry reductions that are not obtainable using the classical Lie group approach.

The ansatz $u(x,t) = F(x,t,w(z))$ with $z = z(x,t)$ used in the direct method assumes that the symmetry variable z does not depend upon u. Consequently it is implicitly assumed that the ratio of infinitesimals ξ/τ is independent of u. For the exact solution (4.5), this ratio of infinitesimals is dependent upon u. However, even if the ratio is dependent upon u, this does not guarantee that the associated symmetry solution is not obtainable using the direct method [20].

Recently Olver [59] (see also [9, 21, 68]) has proved the precise relationship between the direct and nonclassical methods. The general Nth-order partial differential equation (1.1) admits a *direct reduction* if there exist functions $z = z(x,t)$ and

$u = U(x,t,w)$ such that the Clarkson-Kruskal ansatz

$$u(x,t) = U(x,t,w(z)) \tag{4.6}$$

reduces (1.1) to a single ordinary differential equation for $w(z)$. (Note that U is not uniquely determined since we can incorporate any functions of the similarity variable z into w.) Olver [59] proved the following two theorems.

Theorem 4.2. *There is a one-to-one correspondence between the ansätze of the direct method* (4.6) *with* $U_w \neq 0$ *and the quasi-linear first-order differential constraint*

$$\mathbf{v}(u) \equiv \xi(x,t)u_x + \tau(x,t)u_t = \phi(x,t,u). \tag{4.7}$$

Theorem 4.3. *The ansatz* (4.6) *will reduce the partial differential equation* (1.1) *to a single ordinary differential equation for* $w(z)$ *if and only if the overdetermined system of partial differential equations defined by* (1.1) *and* (4.7) *is compatible.*

Thus, there is a one-to-one correspondence between direct reductions of the partial differential equation (1.1) and compatible first-order quasi-linear differential constraints. Solutions of (4.7) are the functions which are invariant under the one-parameter group generated by the vector field

$$\mathbf{w} = \xi(x,t)\partial_x + \tau(x,t)\partial_t + \phi(x,t,u)\partial_u. \tag{4.8}$$

Hence $\mathbf{w}$ generates a group of "*fibre-preserving transformations*" since ξ and τ are independent of u.

In the direct method, one requires that the ansatz (4.6) reduces the partial differential equation (1.1) to a single ordinary differential equation. In the nonclassical method, one requires that the differential constraint (4.7), which requires the solutions to be invariant under the group generated by $\mathbf{w}$, be compatible with the original partial differential equation (1.1) in the sense that the overdetermined system of partial differential equations defined by (1.1) and (4.7) has no integrability conditions. The general nonclassical method, which allows arbitrary point transformation symmetry groups so that ξ and τ in (4.8) also can depend upon u, is similarly equivalent to the more general (though considerably harder to deal with) ansatz

$$u(x,t) = U(x,t,w(z)), \qquad z = z(x,t,u). \tag{4.9}$$

Applying the direct method with the ansatz (4.6) does **not** always find all reductions that are obtained using the classical methods, as shown in the following example.

Example 4.4.

Consider the equation

$$u_x u_{xx} - (\alpha u u_x - \beta u_t)(1 - t u_x)^3 = 0 \tag{4.10}$$

where α and β are arbitrary constants. Applying the classical method to this equation yields the infinitesimals

$$\begin{gathered} \xi = \frac{1}{3}(\kappa_1 + \kappa_3)(x + 2tu) + \kappa_4 u - \kappa_2 t + \kappa_5, \\ \tau = \kappa_3 t + \kappa_4, \qquad \phi = \kappa_1 u + \kappa_2 \beta/\alpha \end{gathered} \tag{4.11}$$

where $(\alpha+\beta)(\kappa_1+\kappa_3)=0$ and $\kappa_1, \kappa_2, \ldots, \kappa_5$ are arbitrary constants. In the special case when $\kappa_1=\kappa_2=\kappa_3=\kappa_5=0$ and $\kappa_4=1$, the invariant surface condition becomes

$$uu_x + u_t = 0,$$

which has solution

$$u(x,t) = w(z), \qquad z = x - ut. \tag{4.12}$$

Substituting this back into (4.10) yields

$$w'[w'' - (\alpha+\beta)w] = 0.$$

For convenience, we set $\alpha=\beta=\frac{1}{2}$, and thus we obtain the implicit solution

$$u(x,t) = A\exp\{x - u(x,t)t\} + B\exp\{-x + u(x,t)t\} \tag{4.13}$$

where A and B are arbitrary constants, which defines $u(x,t)$ by a transcendental function. Applying the direct method with the ansatz (4.6) will not obtain such a reduction.

It is not clear how the direct method, developed by Clarkson and Kruskal [22] for finding symmetry reductions of partial differential equations, may be applied to equations which contain arbitrary functions such as the nonlinear heat equation

$$u_t = u_{xx} + f(u) \tag{4.14}$$

where $f(u)$ is an arbitrary sufficiently differentiable function and subscripts denote partial derivatives. This equation arises in several important physical applications including microwave heating (where $f(u)$ is the rate of absorption of microwave energy [66, 72]), in the theory of chemical reactions (where $f(u)$ is the temperature dependent reaction rate [5, 6, 33]), and in mathematical biology (where $f(u)$ represents the reaction kinetics in a diffusion process [54]. Clarkson and Mansfield [23] used the nonclassical method in conjunction with the method of differential Gröbner bases [52] to find the conditions on $f(u)$ in (4.14) under which symmetries other than the trivial spatial and temporal translational symmetries exist and then solved the determining equations for the infinitesimals. A complete catalogue of symmetry reductions is given in [23] for the nonlinear heat equation (4.14); in particular, a classification of exact solutions of (4.14) for $f(u)=(u-a)(u-b)(u-c)$ expressed in terms of the roots a, b and c of the cubic is given.

The use of differential Gröbner bases has made the analysis of overdetermined systems of partial differential equations, such as those arising as the determining equations for classical and nonclassical symmetries, more tractable. Whilst the `diffgrob2` [48], [49] package needs to be used interactively at present, nevertheless it has proved effective in solving such overdetermined systems [23–28, 50, 51].

It appears to be the case that for some partial differential equations, one of the direct or nonclassical methods is simpler to apply than the others. One difference between the two methods is that the direct method yields the symmetry reduction in one step whereas in the nonclassical method, one first solves for the infinitesimals and then, given the infinitesimals, one solves the invariant surface condition, which is a two-step procedure.

To conclude, we make some remarks comparing the classical Lie, direct, and nonclassical methods.

- Classical Lie Method. The positive aspects of this method are that the determining equations are linear and the associated vector fields have a Lie algebraic structure, which has many useful applications. Also there exist several symbolic manipulation programs which generate the determining equations; further some of these programs also solve the determining equations. However, as we have seen, the method does not find all reductions for all partial differential equations.
- Clarkson-Kruskal Direct Method. This method is more general than the classical Lie method, except for implicit reductions, has no associated group framework, and one can choose the dimension of the reduced equation. Furthermore, the direct method is a one-step procedure. However, the determining equations are nonlinear, the associated vector fields have no Lie algebraic structure, and there are only limited symbolic manipulation programs available.
- Bluman-Cole Nonclassical Method. This method is even more general than the other two and can be viewed as a modification of the classical theory. As for the direct method, the determining equations are nonlinear, the associated vector fields have no Lie algebraic structure, and there are only limited symbolic manipulation programs. In contrast to the direct method, it is a two-step procedure.

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References

1. M. J. Ablowitz and R. Haberman, *Resonantly coupled nonlinear evolution equations*, J. Math. Phys. **18** (1975), 2301–2305.
2. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *The inverse scattering transform — Fourier analysis for nonlinear problems*, Stud. Appl. Math. **53** (1974), 249–315.
3. M. J. Ablowitz, A. Ramani, and H. Segur, *Nonlinear evolution equations and ordinary differential equations of Painlevé type*, Phys. Rev. Lett. **23** (1978), 333–338.
4. ———, *A connection between nonlinear evolution equations and ordinary differential equations of P-type. I*, J. Math. Phys. **21** (1980), 715–721.
5. W. F. Ames, *Nonlinear Partial Differential Equations in Engineering. I*, Academic Press, New York, 1967.
6. R. Aris, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, Vols. I and II, Oxford University Press, Oxford, 1975.
7. D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion and nerve propagation.* In: Partial Differential Equations and Related Topics (Ed. J. A. Goldstein), Lect. Notes Math. **446**, Springer-Verlag, Berlin, 1975, pp. 5–49.
8. ———, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. Math. **30** (1978), 33–76.
9. D. Arrigo, P. Broadbridge, and J. M. Hill, *Nonclassical symmetry solutions and the methods of Bluman-Cole and Clarkson-Kruskal*, J. Math. Phys. **34** (1993), 4692–4703.
10. T. B. Benjamin, J. L. Bona, and J. Mahoney, *Model equations for long waves in nonlinear dispersive systems*, Phil. Trans. R. Soc. Lond. Ser. A **272** (1972), 47–78.
11. G. W. Bluman and J. D. Cole, *The general similarity of the heat equation*, J. Math. Mech. **18** (1969) 1025–1042.
12. G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Appl. Math. Sci. **81**, Springer-Verlag, Berlin, 1989.

13. G. W. Bluman and V. Shtelen, *Developments in similarity methods related to the pioneering work of Julian Cole.* In: Mathematics is for Solving Problems (Eds. L. P. Cook, V. Roytburd, and M. Turin), SIAM, Philadelphia, 1996, pp. 105–117.
14. J. Boussinesq, *Théorie de l'intumescence appelée onde solitaire ou de translation se propagente dans un canal rectangulaire*, Comptes Rendus **72** (1871), 755–759.
15. ______, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblemant parielles de la surface au fond*, J. Pure Appl. **7** (1872), 55–108.
16. G. I. Burde, *The construction of special explicit solutions of the boundary-layer equations. Steady flows*, Q. J. Mech. Appl. Math. **47** (1994), 247–260.
17. ______, *The construction of special explicit solutions of the boundary-layer equations. Unsteady flows*, Q. J. Mech. Appl. Math. **48** (1995), 611–633.
18. ______, *New similarity solutions of the steady-state boundary-layer equations*, J. Phys. A: Math. Gen. **29** (1996), 1665–1983.
19. P. A. Clarkson, *New exact solutions for the Boussinesq equation*, Europ. J. Appl. Math. **1** (1990), 279–300.
20. ______, *Nonclassical symmetry reductions of the Boussinesq equation*, Chaos, Solitons and Fractals **5** (1995), 2261–2301.
21. P. A. Clarkson and S. Hood, *Nonclassical symmetry reductions and exact solutions of the Zabalotskaya-Khokhlov equation*, Europ. J. Appl. Math. **3** (1992), 381–414.
22. P. A. Clarkson and M. D. Kruskal, *New similarity solutions of the Boussinesq equation*, J. Math. Phys. **30** (1989), 2201–2213.
23. P. A. Clarkson and E. L. Mansfield, *Symmetry reductions and exact solutions of a class of nonlinear heat equations*, Physica D **70** (1994), 250–288.
24. ______, *Algorithms for the nonclassical method of symmetry reductions*, SIAM J. Appl. Math. **54** (1994), 1693–1719.
25. ______, *On a shallow water wave equation*, Nonlinearity **7** (1994), 975–1000.
26. ______, *Symmetry reductions and exact solutions of shallow water wave equations*, Acta Appl. Math. **39** (1995), 245–276.
27. P. A. Clarkson, E. L. Mansfield, and A. E. Milne, *Symmetries and exact solutions of a* $2+1$*-dimensional Sine-Gordon system*, Phil. Trans. R. Soc. Lond. A **354** (1996), 1807–1835.
28. P. A. Clarkson, E. L. Mansfield, and T. J. Priestley, *Symmetries of a class of nonlinear third order partial differential equations*, Math. Comp. Model. **25** (1997), 195–212.
29. P. A. Clarkson and P. Winternitz, *Nonclassical symmetry reductions for the Kadomtsev-Petviashvili equation*, Physica **49D** (1991), 257–272.
30. R. Conte and M. Musette, *Algorithmic method for deriving Lax pairs from the invariant Painlevé analysis of nonlinear partial differential equations*, J. Math. Phys. **32**, (1991), 1450–1457.
31. A. Espinosa and J. Fujioka, *Hydrodynamic foundation and Painlevé analysis of the Hirota-Satsuma-type equations*, J. Phys. Soc. Japan **63** (1994), 1289–1294.
32. R. Fitzhugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophysical J. **1** (1961), 445–466.
33. D. A. Frank-Kamenetskii, *Diffusion and Heat Exchange in Chemical Kinetics*, Princeton University Press, Princeton, 1955.
34. W. I. Fushchych, *Conditional symmetry of the equations of mathematical physics* Ukrain. Math. J. **43** (1991), 1456–1470.
35. ______, *Ansatz '95*, J. Nonlinear Mathematical Physics **2** (1995), 216–235.
36. W. I. Fushchych, W. M. Shtelen, and N. I. Serov, *Symmetry Analysis and Exact Solutions of the Equations of Mathematical Physics*, Kluwer, Dordrecht, 1993.
37. V. A. Galaktionov, *On new exact blow-up solutions for nonlinear heat conduction equations with source and applications*, Diff. and Int. Eqns. **3** (1990), 863–874.
38. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Method for solving the KdV equation*, Phys. Rev. Lett. **19** (1967), 1095–1097.
39. A. G. Hansen, *Similarity Analyses of Boundary Value Problems in Engineering*, Prentice-Hall, Englewood Cliffs, 1964.

40. W. Hereman, *Symbolic software for Lie symmetry analysis.* In: Lie Group Analysis of Differential equations. III. New Trends in Theoretical Developments an Computational Methods (Ed. N. H. Ibragimov), CRC Press, Boca Raton, 1996, Chapter XII, pp. 367–413.
41. J. Hietarinta, *Hirota's bilinear method and partial integrability.* In: Partially Integrable Evolution Equations in Physics (Eds. R. Conte and N. Boccara), NATO ASI Series C: Mathematical and Physical Sciences **310** Kluwer, Dordrecht, 1990, pp. 459–478.
42. R. Hirota, *Direct methods in soliton theory.* In: Solitons (Eds. R. K. Bullough and P. J. Caudrey), Topics in Current Physics, **17**, Springer-Verlag, Berlin, 1980, 157–176.
43. R. Hirota and J. Satsuma, *N-soliton solutions of model equations for shallow water waves*, J. Phys. Soc. Japan **40** (1976), 611–612.
44. S. Hood, *New exact solutions of Burgers' equation — an extension to the direct method of Clarkson and Kruskal*, J. Math. Phys. **36** (1995), 1971–1990.
45. E. L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.
46. D. Levi and P. Winternitz, *Nonclassical symmetry reduction: example of the Boussinesq equation*, J. Phys. A: Math. Gen. **22** (1989), 2915–2924.
47. S.-Y. Lou, *A note on the new similarity reductions of the Boussinesq equation*, Phys. Lett. **151A** (1990), 133–135.
48. E. L. Mansfield, *diffgrob2: A symbolic algebra package for analysing systems of PDE using Maple*, `ftp ftp.ukc.ac.uk`, login: anonymous, password: your email address, directory: `pub/Liz/Maple`, files:`diffgrob2_src.tar.Z, diffgrob2_man.tex.Z`, 1993.
49. ———, *diffgrob2: A symbolic algebra package for analysing systems of PDE using MAPLE*, preprint, Department of Mathematics, University of Exeter, U.K., 1993.
50. ———, *The differential algebra package diffgrob2*, Maple Technical Newsletter **3** (1996), 33-37.
51. E. L. Mansfield and P. A. Clarkson, *Applications of the differential algebra package diffgrob2 to classical symmetries of PDEs*, J. Symb. Comp. **23** (1997), 517–533.
52. E. L. Mansfield and E. Fackerell, *Differential Gröbner bases*, preprint **92/108**, Macquarie University, Sydney, Australia, 1992.
53. J. B. McLeod and P. J. Olver, *The connection between partial differential equations soluble by inverse scattering and ordinary differential equations of Painlevé type*, SIAM J. Math. Anal. **14** (1983), 488–506.
54. J. D. Murray, *Mathematical Biology*, Springer-Verlag, New York, 1989.
55. J. S. Nagumo, S. Arimoto, and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proc. IRE **50** (1962), 2061–2070.
56. T. Nishitani and M. Tajiri, *On similarity solutions of the Boussinesq equation*, Phys. Lett. **89A** (1982), 379–380.
57. M. C. Nucci and P. A. Clarkson, *The nonclassical method is more general than the direct method for symmetry reductions: an example of the Fitzhugh-Nagumo equation*, Phys. Lett. **164A** (1992), 49–56.
58. P. J. Olver, *Applications of Lie Groups to Differential Equations*, Second edition, Graduate Texts Math. **107**, Springer-Verlag, New York, 1993.
59. ———, *Direct reduction and differential constraints*, Proc. R. Soc. Lond. A **444** (1994), 509–523.
60. P. J. Olver and P. Rosenau, *The construction of special solutions to partial differential equations*, Phys. Lett. **114A** (1986), 107–112.
61. ———, *Group-invariant solutions of differential equations*, SIAM J. Appl. Math. **47** (1987), 263–275.
62. P. J. Olver and E. M. Vorob'ev, *Nonclassical and conditional symmetries.* In: Lie Group Analysis of Differential equations. III. New Trends in Theoretical Developments an Computational Methods (Ed. N. H. Ibragimov), CRC Press, Boca Raton, 1996, Chapter X, pp. 291–328.
63. J. Ondich, *A differential constraints approach to partial invariance*, Europ. J. Appl. Math. **6** (1995), 631–637.
64. L. V. Ovsiannikov, *Group Analysis of Differential Equations*, [Trans. W. F. Ames], Academic, New York, 1982.
65. H. Peregrine, *Calculations of the development of an undular bore*, J. Fluid Mech. **25** (1966), 321–330.
66. A. H. Pincombe and N. F. Smyth, *Initial boundary value-problems for the Korteweg – de Vries equation*, Proc. Roy. Soc. Lond. A **433** (1991), 479–498.

67. T. J. Priestley, *Methods of Symmetry Reduction and their Application*, Ph. D. thesis, Institute of Mathematics and Statistics, University of Kent at Canterbury, U.K., 1995.
68. E. Pucci, *Similarity reductions of partial differential equations*, J. Phys. A: Math. Gen. **25** (1992), 2631–2640.
69. G. R. W. Quispel, F. W. Nijhoff, and H. W. Capel, *Linearization of the Boussinesq equation and the modified Boussinesq equation*, Phys. Lett. **91A** (1982), 143–145.
70. P. Rosenau and J. L. Schwarzmeier, *On similarity solutions of Boussinesq type equations*, Phys. Lett. **115A** (1986), 75–77.
71. A. C. Scott, *The application of Bäcklund transforms to physical problems.* In: Bäcklund Transformations (Ed. R. M. Miura), Lect. Notes Math. **515**, Springer, Berlin, 1975, pp. 80–105.
72. N. F. Smyth, *The effect of conductivity on hotspots*, J. Aust. Math Soc., Ser. B **33** (1992), 403–413.
73. M. Toda, *Studies of a nonlinear lattice*, Phys. Rep. **8** (1975), 1–125.
74. F. Ursell, *The long-wave paradox in the theory of gravity waves*, Proc. Camb. Phil. Soc. **49** (1953), 685–694.
75. E. M. Vorob'ev, *Symmetries of compatibility conditions for systems of differential equations*, Acta Appl. Math. **24** (1991), 1–24.
76. J. Weiss, *The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative*, J. Math. Phys. **24** (1983), 1405–1413.
77. J. Weiss, M. Tabor, and G. Carnevale, *The Painlevé property for partial differential equations*, J. Math. Phys. **24** (1983), 522–526.
78. G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.
79. E. E. Whittaker and G. M. Watson, *Modern Analysis*, 4th Edition, Cambridge University Press, Cambridge, 1927.
80. N. N. Yanenko, *Theory of consistency and methods of integrating systems of nonlinear partial differential equations.* In: Proceedings of the Fourth All-Union Mathematics Congress, Leningrad, 1964, pp. 247–259. (in Russian)
81. V. E. Zakharov, *On stochastization of one-dimensional chains of nonlinear oscillations*, Sov. Phys. JETP **38** (1974), 108–110.
82. N. J. Zabusky, *A synergetic approach to problems of nonlinear dispersive wave propagation and interaction.* In: Nonlinear Partial Differential Equations (Ed. W. F. Ames), Academic, New York, 1967, pp. 233–258.

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