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A DERIVATION OF GARDNER'S EQUATION

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Dedicated to Martin D. Kruskal on the occasion of his 70th birthday and to Clifford S. Gardner who made it look easy.

ABSTRACT. During the early studies of the Korteweg–de Vries (KdV) equation, C. S. Gardner introduced the nonlinear evolution equation

$$w_t - 6(w + \varepsilon^2 w^2)w_x + w_{xxx} = 0$$

in his proof of the existence of an infinite number of conservation laws. Gardner's equation can be obtained via a Galilean transformation of the modified Korteweg–de Vries equation and its solution can be related directly to the solution of the Korteweg–de Vries equation. Here Gardner's equation is shown to arise naturally as an approximation to the class of quasilinear partial differential equations

$$F^2(y_t;\delta)y_{tt} = G^2(y_x;\delta)y_{xx} + \delta^2 H(y_t, y_x;\delta)y_{xxxx}, \quad \delta^2 \ll 1$$

where F, G, and H are $O(1)$ and have polynomial approximations in δ. The derivation uses a modification of the Riemann invariants for the associated second-order equation, i.e., with $\delta = 0$, and requires setting one of the modified Riemann invariants equal to zero at the initial time. A physical example in this class of equations is the continuous approximation of the anharmonic lattice equations. Subsequently, Gardner's equation has been derived in studies of surface and internal waves in fluid mechanics when quadratic and cubic nonlinearities are comparable.

1. Introduction

In the summer of 1953, at the Los Alamos Scientific Laboratory, Fermi, Pasta, and Ulam (FPU) began a series of numerical experiments on nonlinear problems using the newly built electronic computer called MANIAC. This work culminated in their report [2], cf. [15]. Initially, the problem under study was to determine the "rate of approach to the equipartition of energy among the various degrees of freedom" of a nonlinear continuous string. If we let $y(x,t)$ denote the longitudinal displacement of the position on the string which in equilibrium is situated at x, then the equation of motion is given by

$$c^{-2}y_{tt} = [1 + \delta n(y_x)]y_{xx} \tag{1.1}$$

where subscripts denote partial differentiations with respect to t and x, c is the linear wave speed, δ is a measure of the relative magnitude of the nonlinearity, and n specifies the nonlinearity. Equation (1.1) was discretized by FPU [2] for the case $n = y_x$, and

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they solved the discrete equations numerically over many periods of the linearized oscillations. While following the energy distribution in the various linear modes of vibration, they observed near-recurrence behavior, i.e., the initial energy originally contained in the lowest mode of vibration flowed into other modes of vibration, but then almost completely returned to the first mode.

On the other hand, Zabusky [16] solved the equation exactly using the hodograph transformation. The important qualitative feature revealed was that the solution could become multivalued and, therefore, was unacceptable as a solution to the physical problem. This is clear since nonlinear hyperbolic partial differential equations can have characteristics of the same family crossing each other.

The contradiction between the numerical and exact results was resolved by Kruskal and Zabusky [6, 7, 17] who showed that the discretized version of (1.1) used by FPU [2] actually approximated a fourth-order quasilinear partial differential equation given by

$$c^{-2}y_{tt} = (1+\delta n(y_x))y_{xx} + \frac{h^2}{12}y_{xxxx} \tag{1.2}$$

where h is the spatial discretization distance.

Introduce the Riemann invariants for (1.1) (see Section 2 for the general case)

$$r^{\pm} = \frac{1}{2}\left\{\pm c^{-1}y_t + \int_0^{y_x}(1+\delta n(s))^{1/2}ds\right\}. \tag{1.3}$$

For $n = y_x$, (1.3) can be inverted approximately for y_t and y_x in terms of $r^+ - r^-$ and $r^+ + r^-$, respectively. In this case, (1.2) is approximated to $O(\delta^2)$ and $O(\delta h^2)$ by

$$c^{-1}r_t^{\pm} \mp \left[1 + \frac{1}{2}\delta(r^+ + r^-)\right]r_x^{\pm} \mp \frac{h^2}{24}(r^+ + r^-)_{xxx} = 0. \tag{1.4}$$

For the periodic initial-value problem, by choosing special initial data, $r^+(x,0) = 0$, and assuming $r^-(x,0)$ has zero mean, one can show that $r^+(x,t) = O(\delta^2)$ (Zabusky [17]). Then translating to a uniformly moving frame of reference with speed c and changing variables to $u \equiv -6r^-$, $\tau \equiv \sqrt{3}\delta^{3/2}ct/h$, and $\xi \equiv 2\sqrt{3\delta}x$ yields the Korteweg–de Vries (KdV) equation

$$u_\tau - 6uu_\xi + u_{\xi\xi\xi} = 0. \tag{1.5}$$

Subsequently, various researchers have discovered many important properties of this equation and its solutions, e.g., solitons and an explanation of the recurrence phenomenon (see Miura [11] for a survey of results and references).

One of the first properties discovered by Miura et al. [12] was the existence of an infinite number of conservation laws. The simple elegant proof presented in [12] is due to Gardner and uses his equation

$$w_\tau - 6(w + \varepsilon^2 w^2)w_\xi + w_{\xi\xi\xi} = 0. \tag{1.6}$$

Gardner's equation can be obtained by a Galilean transformation applied to the modified Korteweg–de Vries equation [12].

Since then, Gardner's equation has been derived in various studies to describe surface and internal waves in fluid mechanics ([3, 4, 5, 9, 10]). These derivations are based on the argument that in these applications, the quadratic and cubic nonlinearities are of comparable "significance." Chow [1] and Marchant and Smyth [8] derived an "extended Korteweg–de Vries" equation which includes higher-order nonlinearities as well as a fifth-order spatial derivative term. Under a special scaling and ignoring the higher-order dispersion terms [8], this is reduced to Gardner's equation.

In this paper, Gardner's equation is shown to represent an approximation to the class of quasilinear partial differential equations

$$F^2(y_t;\delta)y_{tt} = G^2(y_x;\delta)y_{xx} + \delta^2 H(y_t, y_x;\delta)y_{xxxx}, \qquad \delta^2 \ll 1, \tag{1.7}$$

using modifications of the Riemann invariants for the associated second-order equation, i.e., with $\delta = 0$. Properties of Gardner's equation and its solutions follow from properties of the modified Korteweg–de Vries equation using a Galilean transformation [12].

2. Approximation using modified Riemann invariants

The reduced second-order form of (1.7), i.e., with $\delta = 0$, has the Riemann invariants, $r^{\pm}$, given by

$$r^{\pm} \equiv \frac{1}{2}\left[\pm \int_0^{y_t} F(p;\delta)dp + \int_0^{y_x} G(q;\delta)dq\right], \tag{2.1}$$

so that

$$r^+ + r^- = \int_0^{y_x} G(q;\delta)dq, \qquad r^+ - r^- = \int_0^{y_t} F(q;\delta)dq, \tag{2.2}$$

which satisfy

$$r_t^{\pm} \mp \frac{G(y_x;\delta)}{F(y_t;\delta)} r_x^{\pm} = 0 \tag{2.3}$$

(see Kruskal and Zabusky [7]).

For the more general equation (1.7) treated here, the procedure of Kruskal and Zabusky [10] is followed. Thus (1.7) becomes

$$r_t^{\pm} \mp \frac{G(y_x;\delta)}{F(y_t;\delta)} r_x^{\pm} \mp \frac{\delta^2 H(y_t, y_x;\delta)}{2F(y_t;\delta)} y_{xxxx} = 0. \tag{2.4}$$

The coefficients F, G, and H are assumed to depend on δy_t and δy_x and to have asymptotic approximations (obtained by simple Taylor series) given by

$$F(p;\delta) = F_0 + \delta F_1 p + \delta^2 F_2 p^2 + O(\delta^3), \tag{2.5}$$

$$G(q;\delta) = G_0 + \delta G_1 q + \delta^2 G_2 q^2 + O(\delta^3), \tag{2.6}$$

$$H(p,q;\delta) = H_0 + O(\delta), \tag{2.7}$$

where F_i, G_i, and H_i, $i = 0, 1, 2$, are constants and $F_0 \neq 0$, $G_0 \neq 0$, $H_0 \neq 0$. The definitions of $r^{\pm}$ and (2.5)–(2.7) yield the approximations

$$r^+ + r^- = G_0 y_x + \frac{\delta G_1}{2} y_x^2 + O(\delta^2), \tag{2.8}$$

$$r^+ - r^- = F_0 y_t + \frac{\delta F_1}{2} y_t^2 + O(\delta^2). \tag{2.9}$$

Solving recursively for y_t and y_x yields

$$y_x = \frac{r^+ + r^-}{G_0} - \frac{\delta G_1}{2G_0^3}(r^+ + r^-)^2 + O(\delta^2), \tag{2.10}$$

$$y_t = \frac{r^+ - r^-}{F_0} - \frac{\delta F_1}{2F_0^3}(r^+ - r^-)^2 + O(\delta^2). \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.4) and expanding for small δ using (2.5)–(2.7) results in

$$\begin{aligned} r_t^\pm \mp \frac{G_0}{F_0}\Bigg\{1 + \delta\left[\frac{G_1}{G_0^2}(r^+ + r^-) - \frac{F_1}{F_0^2}(r^+ - r^-)\right] \\ + \delta^2\Bigg[\frac{2G_0G_2 - G_1^2}{2G_0^4}(r^+ + r^-)^2 + \frac{3F_1^2 - 2F_0F_2}{2F_0^4}(r^+ - r^-)^2 \\ - \frac{F_1G_1}{F_0^2G_0^2}(r^+ + r^-)(r^+ - r^-)\Bigg]\Bigg\} r_x^\pm \mp \frac{\delta^2 H_0}{2F_0G_0}(r^+ + r^-)_{xxx} + O(\delta^3) = 0. \end{aligned} \tag{2.12}$$

Here all terms through $O(\delta^2)$ have been kept.

The difficulty in simplifying (2.12) further is the mixed occurrences of r^+ and r^-. To overcome this difficulty, an idea due to Tang and Tappert [14] is used. Introduce new dependent variables as modifications of the Riemann invariants (2.1), namely

$$u^\pm \equiv r^\pm + \frac{\delta^2 H_0}{4G_0^2}(r^+ + r^-)_{xx}. \tag{2.13}$$

Thus

$$u^+ + u^- = r^+ + r^- + O(\delta^2), \qquad u^+ - u^- = r^+ - r^-. \tag{2.14}$$

Recursive solution of (2.13) for $r^\pm$ yields

$$r^\pm = u^\pm - \frac{\delta^2 H_0}{4G_0^2}(u^+ + u^-)_{xx} + O(\delta^4). \tag{2.15}$$

Also, the leading-order terms from (2.12) and (2.14)–(2.15) yield

$$(u^+ + u^-)_t = \frac{G_0}{F_0}(u^+ - u^-)_x + O(\delta). \tag{2.16}$$

Thus (2.12) is approximated through $O(\delta^2)$ by

$$\begin{aligned} u_t^\pm \mp \frac{G_0}{F_0}\left\{1 + \delta(A_1u^+ + B_1u^-) + \delta^2\left[A_2(u^+)^2 + B_2(u^-)^2 + C_2u^+u^-\right]\right\}u_x^\pm \\ \mp \frac{\delta^2 H_0}{2F_0G_0}u_{xxx}^\pm = 0 \end{aligned} \tag{2.17}$$

where

$$A_1 \equiv \frac{G_1}{G_0^2} - \frac{F_1}{F_0^2}, \qquad B_1 \equiv \frac{G_1}{G_0^2} + \frac{F_1}{F_0^2}, \tag{2.18}$$

$$A_2 \equiv \frac{2G_0G_2 - G_1^2}{2G_0^4} - \frac{2F_0F_2 - 3F_1^2}{2F_0^4} - \frac{F_1G_1}{F_0^2G_0^2}, \tag{2.19}$$

$$B_2 \equiv \frac{2G_0G_2 - G_1^2}{2G_0^4} - \frac{2F_0F_2 - 3F_1^2}{2F_0^4} + \frac{F_1G_1}{F_0^2G_0^2}, \tag{2.20}$$

$$C_2 \equiv \frac{2G_0G_2 - G_1^2}{G_0^4} + \frac{2F_0F_2 - 3F_1^2}{F_0^4}. \tag{2.21}$$

These two evolution equations for $u^{\pm}$ are first-order in time and third-order in x and form a coupled system of nonlinear dispersive equations. With the specified scalings, the solutions $u^{\pm}$ should be accurate up to $O(\delta^3)$.

3. Reduction to Gardner's equation

Finally, reduction of (2.17) to Gardner's equation follows from:

Theorem. *If $u^{\pm}(x,t)$ are classical solutions of (2.17), $u^{\pm}$ and u_x^- are bounded, and $u^+(x,0) \equiv 0$, $-\infty < x < +\infty$, with bounded energy at time t, defined by*

$$E(t) \equiv \int_{-\infty}^{\infty} [u^+(x,t)]^2 dx < \infty, \tag{3.1}$$

then $u^+(x,t) \equiv 0$ and $u^-(x,t)$ satisfies Gardner's equation

$$u_t^- + \frac{G_0}{F_0}\left[1 + \delta B_1 u^- + \delta^2 B_2 (u^-)^2\right] u_x^- + \frac{\delta^2 H_0}{2F_0G_0} u_{xxx}^- = 0. \tag{3.2}$$

Proof. A standard energy argument is used; namely, multiply (2.17) for u^+ by u^+ and integrate over $-\infty < x < \infty$ with $u^{\pm}$ and their derivatives going to zero as $|x| \to \infty$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} [u^+(x,t)]^2 dx = \frac{1}{2}\frac{dE(t)}{dt} \tag{3.3}$$

$$= -\frac{G_0}{F_0}\int_{-\infty}^{\infty} \left[\frac{\delta}{2}B_1 + \delta^2\left(B_2 u^- + \frac{1}{3}C_2 u^+\right)\right](u^+)^2 u_x^- \, dx.$$

Then since for any t, there is some constant $M > 0$ such that

$$\left|\frac{G_0}{F_0}\left[\frac{\delta}{2}B_1 + \delta^2\left(B_2 u^- + \frac{1}{3}C_2 u^+\right)\right] u_x^-\right| \le \frac{1}{2}M, \qquad -\infty < x < \infty, \tag{3.4}$$

there is the inequality

$$\frac{dE}{dt} \le ME \tag{3.5}$$

with solution

$$E(t) \le E(0)e^{Mt}. \tag{3.6}$$

Now $u^+(x,0) = 0$ implies $E(0) = 0$, so that $E(t) = 0$ for all $t > 0$. From continuity of u^+ in x, this implies $u^+(x,t) = 0$ for all $-\infty < x < \infty$, $0 < t$. Thus the equation for u^- becomes (3.2).

To transform (3.2) into a standard form of Gardner's equation, introduce the new variables

$$\xi \equiv \frac{\varepsilon G_0 B_1}{\delta\sqrt{-3H_0B_2}}(x - \frac{G_0}{F_0}t), \tag{3.7}$$

$$\tau \equiv -\frac{\varepsilon^3 G_0^2 B_1^3}{6\delta F_0 B_2\sqrt{-3H_0B_2}}t, \tag{3.8}$$

$$w \equiv \frac{\delta B_2}{\varepsilon^2 B_1}u^-, \tag{3.9}$$

then (3.2) becomes

$$w_\tau - 6(w + \varepsilon^2 w^2)w_\xi + w_{\xi\xi\xi} = 0. \tag{3.10}$$

Gardner's equation is related to the Korteweg–de Vries equation (1.5) by the identification

$$u = w + \varepsilon w_\xi + \varepsilon^2 w^2, \tag{3.11}$$

so that

$$\begin{aligned} 0 &= u_\tau - 6uu_\xi + u_{\xi\xi\xi} \\ &= (1 + \varepsilon\frac{\partial}{\partial\xi} + 2\varepsilon^2 w)[w_\tau - 6(w + \varepsilon^2 w^2)w_\xi + w_{\xi\xi\xi}]. \end{aligned} \tag{3.12}$$

□

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