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WEIGHTED NORM INEQUALITIES FOR THE CONJUGATE FUNCTION ON **a**-ADIC SOLENOIDS

Nakhlé Asmar and Phyllis Panman

ABSTRACT. In this paper we generalize a theorem of Hunt, Muckenhoupt, and Wheeden on weighted norm inequalities for the conjugate function. Our generalization to the cases of **a**-adic solenoids is formulated in terms of the ergodic A_p-condition.

1. Introduction

We consider an arbitrary noncyclic subgroup of $\mathbb{Q}$ and its compact dual group $\Sigma_{\mathbf{a}}$. There is an explicit construction for $\Sigma_{\mathbf{a}}$ which is called the **a**-adic solenoid. Since $\widehat{\Sigma_{\mathbf{a}}}$ is simply a subgroup of $\mathbb{Q}$, $\widehat{\Sigma_{\mathbf{a}}}$ inherits the order from $\mathbb{Q}$; that is, if we let $P = \widehat{\Sigma_{\mathbf{a}}} \cap (0, \infty)$ then P defines the order on $\widehat{\Sigma_{\mathbf{a}}}$. For $f \in \mathfrak{L}_2(\Sigma_{\mathbf{a}})$, we use the Fourier transform of f to define the conjugate function $\tilde{f}$ (with respect to the order P):

$$\tilde{f}^{\wedge}(\chi) = -i \operatorname{sgn}_P(\chi) \hat{f}(\chi) \quad (\chi \in \widehat{\Sigma_{\mathbf{a}}}) \tag{1.1}$$

where $\operatorname{sgn}_P(\chi) = -1$, 0, or 1 according to $\chi \in (-P)\setminus\{0\}$, $\chi = 0$, or $\chi \in P\setminus\{0\}$. The operator $f \mapsto \tilde{f}$ is clearly a norm-decreasing multiplier on $\mathfrak{L}_2(\Sigma_{\mathbf{a}})$. If $1 < p < \infty$, the operator $f \mapsto \tilde{f}$ extends from $\mathfrak{L}_2(\Sigma_{\mathbf{a}}) \cap \mathfrak{L}_p(\Sigma_{\mathbf{a}})$ to a bounded linear operator of $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ such that the identity (1.1) holds, and the inequality

$$\|\tilde{f}\|_p \leq M_p \|f\|_p$$

holds for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}})$, where M_p is independent of f (see [3], or [1, Theorem 7.2]). We ask for which measures, other than Haar measure, is the operator $f \mapsto \tilde{f}$ a bounded operator. More precisely, if $1 < p < \infty$, we seek to characterize those finite nonnegative Borel measures ν for which the operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu)$.

By way of background, we recall that Forelli [7] studied this problem in the case $G = \mathbb{T}$ (henceforth, $\mathbb{T}$ is parameterized by $[-\pi, \pi)$). He showed that if the operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\mathbb{T}, \nu) \cap \mathfrak{L}_1(\mathbb{T})$ into $\mathfrak{L}_p(\mathbb{T}, \nu)$, then ν must be absolutely continuous with respect to Lebesgue measure λ ($\nu \ll \lambda$), and hence there is a nonnegative function w in $\mathfrak{L}_1(\nu)$ where $d\nu = w \frac{dt}{2\pi}$. This result was later extended by Hunt, et al. [13], who showed that the operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\mathbb{T}, w) \cap \mathfrak{L}_1(\mathbb{T})$ into $\mathfrak{L}_p(\mathbb{T}, w)$ exactly when w satisfies a property called the A_p-condition. We state this result in the following definition and theorem:

Definition 1.1. (The A_p-condition on $\mathbb{T}$) Let $1 \leq p < \infty$. Let w be a nonnegative 2π-periodic measurable function. The function w satisfies the A_p-condition on $\mathbb{T}$ if

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there is a constant A_p independent of all intervals $I \subseteq \mathbb{R}$ such that

$$\sup_I \frac{1}{I}\int_I w(t)dt\Big(\frac{1}{I}\int_I w^{-1/(p-1)}(t)dt\Big)^{p-1} \leq A_p. \tag{1.2}$$

We say that $w \in A_p(\mathbb{T})$ if w satisfies (1.2), and we let $A_p(w)$ denote the least constant such that (1.2) holds. When $p = 1$, (1.2) is of the form $\sup_I \frac{1}{I}\int_I w(t)dt$ $\operatorname{ess\,sup}_{t\in I} \frac{1}{w(t)} \leq A_1$.

Theorem 1.1. *Let w be a nonnegative 2π-periodic measurable function. If $1 < p < \infty$, then $w \in A_p(\mathbb{T})$ if and only if for all $f \in \mathfrak{L}_p(\mathbb{T}, w)$,*

$$\Big(\int_{-\pi}^{\pi} |\tilde{f}(t)|^p w(t)dt\Big)^{1/p} \leq K_p\Big(\int_{-\pi}^{\pi} |f(t)|^p w(t)dt\Big)^{1/p} \tag{1.3}$$

where K_p is independent of f. If $1 \leq p < \infty$, $w \in A_p(\mathbb{T})$ if and only if for all $f \in \mathfrak{L}_p(\mathbb{T}, w)$,

$$\sup_{\tau>0} \tau^p \int_{-\pi}^{\pi} \mathbf{1}_{\{t\in[-\pi,\pi):|\tilde{f}(t)|>\tau\}}(t)w(t)dt \leq K_p^p \int_{-\pi}^{\pi} |f(t)|^p w(t)dt \tag{1.4}$$

where K_p is independent of f.

Remark 1.1. We note that from the proof of Theorem 1.1 ([13]), when (1.4) holds, $w \in A_p(\mathbb{T})$ with $A_p(w)$ less than or equal to $K_p^2(4\pi)^{2p}$. Also, by a modification of the proof in [13], it is enough to assume that (1.4) holds for all $f \in \mathfrak{L}_p(\mathbb{T}, w) \cap \mathfrak{L}_1(\mathbb{T})$.

Hewitt and Ritter in [8] and [9] make an extensive study of conjugate Fourier series on **a**-adic solenoids. In this paper, we study weighted norm inequalities on **a**-adic solenoids $\Sigma_{\mathbf{a}}$. Our main theorem (Theorem 4.4) gives a generalization of Theorem 1.1 in terms of the conjugate function on $\Sigma_{\mathbf{a}}$, obtaining a similar characterization as Hunt et al. [13] of those finite nonnegative Borel measures ν for which the operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu)$.

The plan of the paper is as follows. In Section 2, we give an explicit representation of $\Sigma_{\mathbf{a}}$ and define some other terms needed in our analysis. In Section 3, we show that if ν is a nonnegative Borel measure on $\Sigma_{\mathbf{a}}$, and the operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu)$, then ν is absolutely continuous with respect to Haar measure μ. This shows that we need only characterize those weights $w \in \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ that satisfy the property that the operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w)$. In Section 4, we state and prove a characterization of those weights that satisfy this last property (Theorem 4.4).

2. Preliminaries

2.1. The a-adic solenoid and its character group. Up to isomorphism, any non-cyclic subgroup of $\mathbb{Q}$ can be described as follows. Let $\mathbf{a} = (a_0, a_1, \dots)$ be a fixed infinite sequence of integers all greater than 1. Let

$$A_0 = 1,\ A_1 = a_0,\ A_2 = a_0a_1, \dots,\ A_n = a_0a_1\cdots a_{n-1}, \dots$$

Let $\mathbb{Q}_{\mathbf{a}}$ be the set of all rational numbers $\frac{l}{A_k}$, where $l \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. Clearly $\mathbb{Q}_{\mathbf{a}}$ is a non-cyclic additive subgroup of $\mathbb{Q}$, and as shown in [2], any non-cyclic subgroup of $\mathbb{Q}$ is of this form.

According to the Pontrjagin duality ([10, 24.8, p. 378]), the character group of $\mathbb{Q}_{\mathbf{a}}$ is a compact abelian group, which we denote by $\Sigma_{\mathbf{a}}$, and the character group of $\Sigma_{\mathbf{a}}$ is again $\mathbb{Q}_{\mathbf{a}}$. We let μ denote normalized Haar measure on $\Sigma_{\mathbf{a}}$. The group $\Sigma_{\mathbf{a}}$ can be realized as the set $[0,1) \times \Delta_{\mathbf{a}}$, which is described in detail in [10, Section

10]. The group $\Delta_{\mathbf{a}}$ consists of all infinite sequences $\mathbf{x} = (x_0, x_1, \ldots, x_k, \ldots)$ where each $x_k \in \{0, 1, \ldots, a_k - 1\}$. Addition in $\Delta_{\mathbf{a}}$ is defined coordinate-wise and carrying quotients (see [10, 10.2]). Also, the elements $\mathbf{u} = (1, 0, 0, \ldots)$ and $\mathbf{0} = (0, 0, 0, \ldots)$ are both in $\Delta_{\mathbf{a}}$, and addition on $[0, 1) \times \Delta_{\mathbf{a}}$ is defined by

$$(\xi, \mathbf{x}) + (\eta, \mathbf{y}) = (\xi + \eta - \lfloor \xi + \eta \rfloor, \mathbf{x} + \mathbf{y} - \lfloor \xi + \eta \rfloor \mathbf{u})$$

where $\lfloor \cdot \rfloor$ is the greatest integer function. The group $\Sigma_{\mathbf{a}}$ is a compact connected Abelian group admitting a continuous homomorphism $\varphi : \mathbb{R} \to \Sigma_{\mathbf{a}}$, where $\varphi(\mathbb{R})$ is a dense subgroup of $\Sigma_{\mathbf{a}}$ and

$$\varphi(s) = (s - \lfloor s \rfloor, \lfloor s \rfloor \mathbf{u}) \tag{2.1}$$

([10, Theorem 10.13], and [9, 3.2]). For $k = 1, 2, \ldots$, define the sets

$$\Lambda_k = \{(0, \mathbf{x}) \in \Sigma_{\mathbf{a}} : x_0 = x_1 = \cdots = x_{k-1} = 0\}.$$

The sets Λ_k are compact, closed subgroups of $\Sigma_{\mathbf{a}}$ ([10, Theorem 10.5, p. 110]), and we let μ_k denote normalized Haar measure on Λ_k. The measure μ_k is a singular Borel measure on $\Sigma_{\mathbf{a}}$, and the Fourier transform is equal to the indicator function of $(1/A_k)\mathbb{Z}$:

$$\widehat{\mu}_k = \mathbf{1}_{(1/A_k)\mathbb{Z}}$$

([9, 5.1ff, p. 825]). For all $k \in \mathbb{N}$, the quotient group $\Sigma_{\mathbf{a}} / \Lambda_k$ is topologically isomorphic to the circle group $\mathbb{T}$ (see [8, 3.1]). Indeed, the mapping

$$\pi_k(t, \mathbf{x}) = \chi_{\frac{1}{A_k}}(t, \mathbf{x}) \tag{2.2}$$

is a continuous homomorphism of $\Sigma_{\mathbf{a}}$ onto $\mathbb{T}$ with kernel Λ_k where

$$\chi_{\frac{1}{A_k}}((t, \mathbf{x})) = \exp\left(2\pi i \frac{1}{A_k}\left(t + \sum_{h=0}^{k-1} x_h A_h\right)\right)$$

is the character corresponding to the element $\frac{1}{A_k}$ of $\mathbb{Q}_{\mathbf{a}}$. Also, if $f \in \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ and f is constant on cosets of Λ_k, then $f = f * \mu_k$, and there is a function $f_k \in \mathfrak{L}_1(\mathbb{T})$ that satisfies $f = f * \mu_k = f_k \circ \pi_k$ and

$$\int_{\Sigma_{\mathbf{a}}} f d\mu = \int_{\Sigma_{\mathbf{a}}} f_k \circ \pi_k d\mu = \int_{\mathbb{T}} f_k dx \tag{2.3}$$

([11, 28.55] and [9, 5.1.3]).

Martingales on $\Sigma_{\mathbf{a}}$. If $f \in \mathfrak{L}_1(\Sigma_{\mathbf{a}})$, then the sequence $(f * \mu_k)_{k \geq 0}$ is a martingale relative to a sequence of σ-algebras $(\mathfrak{F}_k)_{k \geq 0}$ where $\mathfrak{F}_k$ consists of those Borel sets $F \subset \Sigma_{\mathbf{a}}$ such that $F + \Lambda_k = F$ (see [6, Theorem 5.4.1]). The functions $f * \mu_k$ also are known as the conditional expectations of f relative to $\mathfrak{F}_k$. It is a well-known theorem of Doob's that if $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}})$, then $f * \mu_k \to f$ in $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ as $k \to \infty$ (see [5], or [6, Theorem 5.2.6]).

The conjugate function on $\Sigma_{\mathbf{a}}$. It is easy to see that $\mathbb{Q}_{\mathbf{a}}$ admits exactly one order P under which 1 is in P; the order is the one inherited from the usual order on $\mathbb{R}$. We take this ordering on $\mathbb{Q}_{\mathbf{a}}$ where $P = \{\chi_\alpha \in \mathbb{Q}_{\mathbf{a}} : \alpha \geq 0\}$. For $f \in \mathfrak{L}_2(\Sigma_{\mathbf{a}})$, we use the Fourier transform and the order generated by P to define the conjugate function $\tilde{f}$:

$$\tilde{f}^\wedge(\chi_\alpha) = -i \operatorname{sgn}_P(\chi_\alpha) \hat{f}(\chi_\alpha) \ (\chi_\alpha \in \mathbb{Q}_{\mathbf{a}})$$

where $\operatorname{sgn}_P(\chi_\alpha) = -1$, 0, or 1, according to $\alpha < 0$, $\alpha = 0$, or $\alpha > 0$, respectively. As noted before, if $1 < p < \infty$, the operator $f \mapsto \tilde{f}$ extends from $\mathfrak{L}_2(\Sigma_{\mathbf{a}}) \cap \mathfrak{L}_p(\Sigma_{\mathbf{a}})$

to a bounded linear operator of $\mathfrak{L}_p(\Sigma_{\mathbf{a}})$ ([1, Theorem 7.2]). In addition, the conjugate function $\tilde{f}$ has an integral representation that exists μ-almost everywhere for all functions $f \in \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ ([1, 6.11(c) and Theorem 6.5]).

The ergodic A_p-condition on $\Sigma_{\mathbf{a}}$. Let $\varphi : \mathbb{R} \to \Sigma_{\mathbf{a}}$ be the continuous homomorphism defined in (2.1). If $1 \leq p < \infty$ and w is a nonnegative function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$, we say that w is in $A_p(\Sigma_{\mathbf{a}})$ if the following condition is satisfied: for almost every $x \in \Sigma_{\mathbf{a}}$,

$$\sup_I \frac{1}{|I|}\int_I w(x-\varphi(t))dt\Big(\frac{1}{|I|}\int_I w^{-1/(p-1)}(x-\varphi(t))dt\Big)^{p-1} \leq K_p \tag{2.4}$$

where K_p is a constant independent of x. We let $A_p(w)$ denote the least constant such that (2.4) holds.

3. The continuity of the conjugate function with respect to Borel measures

In this section, we show that if ν is a finite nonnegative Borel measure on $\Sigma_{\mathbf{a}}$, the continuity of the operator $f \mapsto \tilde{f}$ from $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu)$ implies that $\nu \ll \mu$ where μ is Haar measure on $\Sigma_{\mathbf{a}}$.

Theorem 3.1. *Let $1 < p < \infty$. Let ν be a finite nonnegative Borel measure on $\Sigma_{\mathbf{a}}$. Suppose that the inequality*

$$\|\tilde{f}\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},\nu)} \leq K_p\|f\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},\nu)} \tag{3.1}$$

is valid for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ where K_p is independent of f. Then $\nu \ll \mu$, and hence there is a nonnegative function w in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ such that $d\nu = w d\mu$.

Proof. Assuming that the linear operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu)$, we can continuously extend the operator to all of $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu)$. Let T denote the extended linear operator. Fix a real-valued function g in $\mathfrak{L}_q(\Sigma_{\mathbf{a}}, \nu) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then by Hölder's inequality, we have for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu)$,

$$\Big|\int_{\Sigma_{\mathbf{a}}} (Tf)g d\nu\Big| \leq \|Tf\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},\nu)}\|g\|_{\mathfrak{L}_q(\Sigma_{\mathbf{a}},\nu)} \leq K_p\|g\|_{\mathfrak{L}_q(\Sigma_{\mathbf{a}},\nu)}\|f\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},\nu)}.$$

Hence, if we define the linear functional $L_g : \mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu) \to \mathbb{C}$ by $L_g f = \int_{\Sigma_{\mathbf{a}}}(Tf)g d\nu$, then L_g is bounded. By the Riesz Representation Theorem ([15, p. 284]), there is a function $h \in \mathfrak{L}_q(\Sigma_{\mathbf{a}}, \nu)$ such that

$$L_g f = \int_{\Sigma_{\mathbf{a}}} (Tf)g d\nu = \int_{\Sigma_{\mathbf{a}}} hf d\nu \tag{3.2}$$

for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu)$.

We claim that h is real-valued ν-a.e. To see this, consider a continuous character $\chi_\alpha \in P \setminus \{0\}$ (so then $\alpha > 0$ and $\overline{\chi}_\alpha \in (-P)\setminus\{0\}$). By (3.2), we have

$$\begin{aligned}\int_{\Sigma_{\mathbf{a}}} (\operatorname{Re}\chi_\alpha)h d\nu &= \frac{1}{2}\int_{\Sigma_{\mathbf{a}}} (\chi_\alpha + \overline{\chi}_\alpha)h d\nu = \frac{1}{2}\int_{\Sigma_{\mathbf{a}}} (T\chi_\alpha + T\overline{\chi}_\alpha)g d\nu \\ &= \frac{1}{2}\int_{\Sigma_{\mathbf{a}}} (-i\chi_\alpha + i\overline{\chi}_\alpha)g d\nu \\ &= \int_{\Sigma_{\mathbf{a}}} (\operatorname{Im}\chi_\alpha)g d\nu.\end{aligned}$$

Since the last integral is real-valued, $\int_{\Sigma_{\mathbf{a}}} \operatorname{Re}\chi_\alpha \operatorname{Im} h d\nu = 0$. Similarly, $\int_{\Sigma_{\mathbf{a}}} \operatorname{Im}\chi_\alpha \operatorname{Im} h d\nu = 0$, so that $\int_{\Sigma_{\mathbf{a}}} \chi_\alpha \operatorname{Im} h d\nu = 0$. This is also true if χ_α is replaced by any trigonometric polynomial $\Sigma_{j=1}^{m}\chi_{\alpha_j}$ ($\chi_{\alpha_j} \in \mathbb{Q}_{\mathbf{a}}$). By the Stone-Weierstrass Theorem, the set of trigonometric polynomials on $\Sigma_{\mathbf{a}}$ is dense in the set of all continuous functions on $\Sigma_{\mathbf{a}}$; hence for all continuous functions f on $\Sigma_{\mathbf{a}}$, we have $\int_{\Sigma_{\mathbf{a}}} f \operatorname{Im} h d\nu = 0$. But then the signed measure $(\operatorname{Im} h d\nu) \equiv 0$, which means that $\operatorname{Im} h = 0$ ν-a.e. and h is real-valued ν-a.e.

We also claim that $(h + ig)d\nu$ is of analytic type in the sense that $(h + ig)d\nu$ has a Fourier transform vanishing for the negative characters in $\mathbb{Q}_{\mathbf{a}}$ (see [16, p. 197]). By (3.2), we have

$$-i\int_{\Sigma_{\mathbf{a}}} \chi_\alpha g d\nu = \int_{\Sigma_{\mathbf{a}}} h\chi_\alpha d\nu, \text{ for all } \chi_\alpha \in P\backslash\{0\}.$$

Thus,

$$\int_{\Sigma_{\mathbf{a}}} \chi_\alpha(h + ig)d\nu = 0, \text{ for all } \chi_\alpha \in P\backslash\{0\},$$

and equivalently

$$\int_{\Sigma_{\mathbf{a}}} \overline{\chi}_\alpha(h + ig)d\nu = 0, \text{ for all } \chi_\alpha \in (-P)\backslash\{0\}.$$

Thus, $(h + ig)d\nu$ is of analytic type.

By [12, Theorem 19.42, p. 326], we can write the Lebesgue decomposition of $d\nu$ as $d\nu = d\nu_s + d\nu_a$ where $d\nu_s$ is singular with respect to $d\mu$ ($d\nu_s \perp d\mu$), and $d\nu_a$ is absolutely continuous with respect to $d\mu$ ($d\nu_a \ll d\mu$). Then it is clear that the Lebesgue decomposition of $(h + ig)d\nu$ is

$$(h + ig)d\nu = (h + ig)d\nu_s + (h + ig)d\nu_a. \tag{3.3}$$

Since $(h + ig)d\nu$ is of analytic type,

$$\int_{\Sigma_{\mathbf{a}}} (h + ig)d\nu_s = 0$$

([16, Theorem 8.2.3, p. 200]). Since g and h are real-valued ν_s-a.e., $\int_{\Sigma_{\mathbf{a}}} g d\nu_s = 0$. This is true for every continuous real-valued $g \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$, so it is also true for the real and imaginary parts of every continuous complex-valued function g, and hence $\nu_s \equiv 0$. But then $\nu \ll \mu$, and by the Radon-Nikodym Theorem, ([12, Theorem 19.23, p. 315]), there is a nonnegative measurable function $w \in \mathfrak{L}_1^+(\Sigma_{\mathbf{a}})$ such that $\nu(A) = \int_A w d\mu$ for all Borel measurable subsets A of $\Sigma_{\mathbf{a}}$. □

Remark 3.1. We note that the proof of Theorem 3.1 does not depend on the structure of the **a**-adic solenoid $\Sigma_{\mathbf{a}}$. In fact, using the same argument, we can show that Theorem 3.1 holds for any compact connected abelian group G where the dual is ordered and the conjugate function $\tilde{f}$ is defined as in (1.1).

4. The A_p-condition on a-adic solenoids

We seek to characterize those finite nonnegative Borel measures ν for which the operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \nu)$. By Theorem 3.1, it suffices to characterize those weights $w \in \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ for which the operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w)$. In Theorem 4.4, we show that this property holds if and only if w satisfies the ergodic A_p-condition in (2.4).

We prove some propositions before proving Theorem 4.4. It is essential for our analysis to define the following classes of functions for $1 \le p < \infty$ and $w \in \mathfrak{L}_1(\Sigma_{\mathbf{a}})$:

$$\mathfrak{L}_p(\Sigma_{\mathbf{a}}, \mu) * \mu_k = \{f * \mu_k : f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}})\},$$
$$\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w * \mu_k) * \mu_k = \{f * \mu_k : f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w * \mu_k)\}.$$

From Hewitt and Ross [11, p. 95, Theorem 28.55], $\mathfrak{L}_p(\Sigma_{\mathbf{a}}) * \mu_k$ is isometrically isomorphic to $\mathfrak{L}_p(\Sigma_{\mathbf{a}}/\Lambda_k) \approx \mathfrak{L}_p(\mathbb{T})$. By a modification of the proof in [11], we also have $(\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w * \mu_k) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})) * \mu_k$ isometrically isomorphic to $\mathfrak{L}_p(\mathbb{T}, w_k) \cap \mathfrak{L}_1(\mathbb{T})$ where w_k is the function in $\mathfrak{L}_1(\mathbb{T})$ such that $w * \mu_k = w_k \circ \pi_k$ (see (2.3)).

We use the following notation. If ν is a nonnegative Borel measure on $\Sigma_{\mathbf{a}}$ and $1 \le p < \infty$, we define the Lorentz $\mathfrak{L}_{p,\infty}$ quasi-norm for a measurable function f as

$$\|f\|^*_{\mathfrak{L}_{p,\infty}(\nu)} = \sup_{\tau > 0} \tau \big(\nu(\{x \in \Sigma_{\mathbf{a}} : |f(x)| > \tau\})\big)^{1/p}.$$

(See [17, Ch.5, Sect.3]; note that $\|\cdot\|^*_{\mathfrak{L}_{p,\infty}(\nu)}$ actually defines a norm when $1 < p < \infty$.)

Proposition 4.1. *Let $1 \le p < \infty$. Let T denote the operator $f \mapsto \tilde{f}$, and let w be a nonnegative function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$. Then the following are equivalent:*

(i) *The inequality*

$$\|Tf\|^*_{\mathfrak{L}_{p,\infty}(\Sigma_{\mathbf{a}}, w)} \le K_p \|f\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w)}$$

is valid for every $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ where K_p is independent of f.

(ii) *For each $k = 1, 2, \dots$, the inequality*

$$\|T(f * \mu_k)\|^*_{\mathfrak{L}_{p,\infty}(\Sigma_{\mathbf{a}}, w*\mu_k)} \le K_p \|f * \mu_k\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w*\mu_k)}$$

is valid for every $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w * \mu_k) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ where K_p is independent of f and k.

Proof. (i)→(ii) Let f be a trigonometric polynomial on $\Sigma_{\mathbf{a}}$ and fix an integer $1 \le k < \infty$. Since f is bounded, $f * \mu_k$ is bounded and $f * \mu_k \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$. Then we have by Fubini's Theorem and the translation invariance of μ,

$$\begin{aligned}
&\sup_{\tau>0} \tau^p \int_{\Sigma_{\mathbf{a}}} \mathbf{1}_{\{x \in \Sigma_{\mathbf{a}} : |T(f*\mu_k)(x)| > \tau\}}(x) w * \mu_k(x) d\mu(x) \\
&= \sup_{\tau>0} \tau^p \int_{\Sigma_{\mathbf{a}}} \int_{\Sigma_{\mathbf{a}}} \mathbf{1}_{\{x \in \Sigma_{\mathbf{a}} : |T(f*\mu_k)(x)| > \tau\}}(x) w(x-y)\, d\mu_k(y) d\mu(x) \\
&= \sup_{\tau>0} \tau^p \int_{\Sigma_{\mathbf{a}}} \int_{\Sigma_{\mathbf{a}}} \mathbf{1}_{\{x \in \Sigma_{\mathbf{a}} : |T(f*\mu_k)(x)| > \tau\}}(x+y) w(x)\, d\mu(x) d\mu_k(y) \\
&= \sup_{\tau>0} \tau^p \int_{\Sigma_{\mathbf{a}}} \int_{\Sigma_{\mathbf{a}}} \mathbf{1}_{\{x \in \Sigma_{\mathbf{a}} : |T(f*\mu_k)(x+y)| > \tau\}}(x) w(x)\, d\mu(x) d\mu_k(y). \qquad (4.1)
\end{aligned}$$

Letting $(f * \mu_k)_y$ denote the function $x \mapsto (f * \mu_k)(x+y)$ and applying the hypothesis to $(f * \mu_k)_y$, we have from (4.1),

$$\begin{aligned}
&\sup_{\tau>0} \tau^p \int_{\Sigma_{\mathbf{a}}} \mathbf{1}_{\{x \in \Sigma_{\mathbf{a}} : |T(f*\mu_k)(x)| > \tau\}}(x) w * \mu_k(x) d\mu(x) \\
&\le K_p^p \int_{\Sigma_{\mathbf{a}}} \int_{\Sigma_{\mathbf{a}}} |(f * \mu_k)_y(x)|^p w(x)\, d\mu(x) d\mu_k(y) \\
&= K_p^p \int_{\Sigma_{\mathbf{a}}} |f * \mu_k(x)|^p w * \mu_k(x) d\mu(x).
\end{aligned}$$

It is easy to see that this is enough to show that (ii) holds.

(ii)→(i) Consider a trigonometric polynomial f on $\Sigma_{\mathbf{a}}$. Then it is clear that there is an integer $N \geq 1$ such that $f = f * \mu_k$ for all $k \geq N$. Fix $\tau > 0$. Since $\|w * \mu_k - w\|_{\mathfrak{L}_1(\Sigma_{\mathbf{a}})} \to 0$, and f is a bounded function, we have by the hypothesis

$$\begin{aligned}\tau^p \int_{\Sigma_{\mathbf{a}}} &\mathbf{1}_{\{x \in \Sigma_{\mathbf{a}}: |Tf(x)| > \tau\}}(x) w(x) d\mu(x) \\ &= \lim_{\substack{k \to \infty \\ k \geq N}} \tau^p \int_{\Sigma_{\mathbf{a}}} \mathbf{1}_{\{x \in \Sigma_{\mathbf{a}}: |T(f*\mu_k)(x)| > \tau\}}(x) w * \mu_k(x) d\mu(x) \\ &\leq K_p^p \lim_{\substack{k \to \infty \\ k \geq N}} \int_{\Sigma_{\mathbf{a}}} |f * \mu_k(x)|^p w * \mu_k(x) d\mu(x) \\ &= K_p^p \int_{\Sigma_{\mathbf{a}}} |f(x)|^p w(x) d\mu(x).\end{aligned}$$

It is easy to see that this is enough to show that (i) holds. □

Remark 4.1. We note that by slightly modifying the proof of Proposition (4.1), we can show that similar strong-type estimates hold.

Proposition 4.2. *Let $1 \leq p < \infty$. Suppose w is a nonnegative function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$ and w is constant on the cosets of Λ_k for some positive integer k. Let w_k denote the function in $\mathfrak{L}_1(\mathbb{T})$ such that $w = w * \mu_k = w_k \circ \pi_k$* (see (2.3)). *Then w is in $A_p(\Sigma_{\mathbf{a}})$ if and only if w_k is in $A_p(\mathbb{T})$. Moreover, in this case $A_p(w_k) \leq A_p(w)$.*

Proof. We show that the necessity part of the proposition holds. The sufficiency part follows by a similar argument. Assume that w is in $A_p(\Sigma_{\mathbf{a}})$ with bound $A_p(w)$. Let $I = (a, b)$ be an interval in $\mathbb{R}$. Let $(t, \mathbf{x})$ be an element in $\Sigma_{\mathbf{a}}$ such that (2.4) holds. As noted in (2.2) and the following, we have $\pi_k((t, \mathbf{x})) = \chi_{\frac{1}{A_k}}((t, \mathbf{x})) = \exp(2\pi i \frac{1}{A_k} t_0)$ where $t_0 = t + \sum_{h=0}^{k-1} x_h A_h$. We consider the expression

$$\frac{1}{|I|} \int_I w_k(\exp(is)) ds \left(\frac{1}{|I|} \int_I w_k^{-1/(p-1)}(\exp(is)) ds \right)^{p-1}. \tag{4.2}$$

Let $s = \frac{2\pi}{A_k}(t_0 - u)$, $ds = -\frac{2\pi}{A_k} du$, $a' = t_0 - \frac{A_k}{2\pi} a$, $b' = t_0 - \frac{A_k}{2\pi} b$, and $I' = (b', a')$. It is easily observed that $\pi_k(\varphi(u)) = \chi_{\frac{1}{A_k}}(\varphi(u)) = \exp(2\pi i \frac{1}{A_k} u)$ for all $u \in \mathbb{R}$ (see [9, 3.2.4 ff]). Since w is in $A_p(\Sigma_{\mathbf{a}})$ with bound $A_p(w)$ and $w = w * \mu_k = w_k \circ \pi_k$, we can use a change of variables to see that (4.2) is bounded by $A_p(w)$:

$$\begin{aligned}\frac{1}{|I|} \int_I &w_k(\exp(is)) ds \left(\frac{1}{|I|} \int_I w_k^{-1/(p-1)}(\exp(is)) ds \right)^{p-1} \\ &= \frac{1}{|I|} \left(\frac{-2\pi}{A_k} \right) \int_{a'}^{b'} w_k(\exp(2\pi i \frac{1}{A_k}(t_0 - u))) du \\ &\quad \times \left(\frac{1}{|I|} \frac{-2\pi}{A_k} \int_{a'}^{b'} w_k^{-1/(p-1)}(\exp(2\pi i \frac{1}{A_k}(t_0 - u))) du \right)^{p-1} \\ &= \frac{1}{|I'|} \int_{I'} w_k(\pi_k((t, \mathbf{x}) - \varphi(u))) du \\ &\quad \times \left(\frac{1}{|I'|} \int_{I'} w_k^{-1/(p-1)}(\pi_k((t, \mathbf{x}) - \varphi(u))) du \right)^{p-1} \\ &\leq A_p(w).\end{aligned}$$

This is true for any interval I, hence w_k is in $A_p(\mathbb{T})$ with bound less than or equal to $A_p(w)$. □

The next proposition shows that if $w \in A_p(\Sigma_{\mathbf{a}})$, then the operator $f \mapsto \tilde{f}$ is bounded from $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ into $\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w)$. The proof is similar to that of [14, Theorem 2.1 and Corollary 2.4], using the transference methods of Coifman and Weiss [4]. We include the proof for completeness.

Proposition 4.3. *Let T denote the operator $f \mapsto \tilde{f}$ and let w be a nonnegative function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$. If $1 < p < \infty$ and $w \in A_p(\Sigma_{\mathbf{a}})$, then the inequality*

$$\|Tf\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},w)} \leq A_p(w)\|f\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},w)}$$

is valid for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$, where $A_p(w)$ is independent of f. If $p = 1$ and $w \in A_1(\Sigma_{\mathbf{a}})$, then the inequality

$$\|Tf\|^*_{\mathfrak{L}_{1,\infty}(\Sigma_{\mathbf{a}},w)} \leq A_1(w)\|f\|_{\mathfrak{L}_1(\Sigma_{\mathbf{a}},w)}$$

is valid for all $f \in \mathfrak{L}_1(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$, where $A_1(w)$ is independent of f.

Proof. We show the proposition holds for the case $1 < p < \infty$. The case $p = 1$ follows by a similar argument. We assume that $w \in A_p(\Sigma_{\mathbf{a}})$ with bound $A_p(w)$ and show that the inequality

$$\|Tf\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},w)} \leq A_p(w)\|f\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},w)} \tag{4.3}$$

is valid for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$. Let $K_n = \{t : \frac{1}{n} \leq |t| \leq n\}$ and $k_n(t) = \frac{1}{\pi t}\mathbf{1}_{K_n}(t)$ and $H_n f(x) = \int_{\mathbb{R}} f(x-\varphi(t))k_n(t)dt$ where $\varphi : \mathbb{R} \to \Sigma_{\mathbf{a}}$ is the homomorphism defined in (2.1). To see that (4.3) holds, it is enough to show that for all $n \geq 1$, the inequality

$$\int_{\Sigma_{\mathbf{a}}} |H_n f(x)|^p w(x)d\mu(x) \leq A_p^p(w)(1 + \tfrac{1}{n}) \int_{\Sigma_{\mathbf{a}}} |f(x)|^p w(x)d\mu(x) \tag{4.4}$$

is valid for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$. (By [1, Theorem 6.5 and 6.11(c)], if $f \in \mathfrak{L}_1(\Sigma_{\mathbf{a}})$, then $|H_n f(x)| \to |Tf(x)|$ for μ-a.e. $x \in \Sigma_{\mathbf{a}}$. So assuming (4.4) holds, we can use Fatou's lemma to show that (4.3) is valid for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$.)

To see that (4.4) holds, fix $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w) \cap \mathfrak{L}_1(\Sigma_{\mathbf{a}})$ and let $n > 1$. Since $\mathbb{R}$ is amenable, we can choose a compact set K such that $\frac{|K-K_n|}{|K|} < 1 + \frac{1}{n}$ (see [4, 2.1, p. 8]). By the translation invariance of Haar measure μ and Fubini's theorem, we have

$$\begin{aligned}
&\int_{\Sigma_{\mathbf{a}}} |H_n f(x)|^p w(x)d\mu(x)\\
&= \frac{1}{|K|}\int_K \int_{\Sigma_{\mathbf{a}}} |H_n f(x-\varphi(t))|^p w(x-\varphi(t))\, d\mu(x)dt\\
&= \frac{1}{|K|}\int_{\Sigma_{\mathbf{a}}} \int_K \left|\int_{\mathbb{R}} f(x-\varphi(t-s))k_n(s)ds\right|^p w(x-\varphi(t))\, dtd\mu(x)\\
&= \frac{1}{|K|}\int_{\Sigma_{\mathbf{a}}} \int_K \left|\int_{\mathbb{R}} f(x-\varphi(t-s))\mathbf{1}_{K-K_n}(t-s)k_n(s)ds\right|^p w(x-\varphi(t))\, dtd\mu(x).
\end{aligned}$$

Let $g_x(t) = f(x-\varphi(t))\mathbf{1}_{K-K_n}(t)$ and $w_x(t) = w(x-\varphi(t))$. We have assumed that $w \in A_p(\Sigma_{\mathbf{a}})$, which means that for μ-a.e. $x \in \Sigma_{\mathbf{a}}$, $w_x(t)$ satisfies (2.4) with bound

$A_p(w)$. Then by the above equalities and [13, Theorem 9, p. 247], we have

$$\begin{aligned}\int_{\Sigma_{\mathbf{a}}} &|H_n f(x)|^p w(x) d\mu(x)\\ &= \frac{1}{|K|}\int_{\Sigma_{\mathbf{a}}}\int_K \left|\int_{\mathbb{R}} g_x(t-s)k_n(s)ds\right|^p w_x(t)\,dtd\mu(x)\\ &\le \frac{A_p^p(w)}{|K|}\int_{\Sigma_{\mathbf{a}}}\int_{\mathbb{R}} |g_x(t)|^p w_x(t)\,dtd\mu(x)\\ &= \frac{A_p^p(w)}{|K|}\int_{K-K_n}\int_{\Sigma_{\mathbf{a}}} |f(x-\varphi(t))|^p w(x-\varphi(t))\,d\mu(x)dt\\ &= A_p^p(w)\tfrac{|K-K_n|}{|K|}\int_{\Sigma_{\mathbf{a}}} |f(x)|^p w(x)d\mu(x).\end{aligned}$$

Since $\frac{|K-K_n|}{|K|} < 1+\frac{1}{n}$, we have shown that (4.4) holds, completing the proof of the proposition. □

Now we state and prove our main theorem.

Theorem 4.4. *Let T denote the operator $f \mapsto \tilde{f}$ and let w be a nonnegative function in $\mathfrak{L}_1(\Sigma_{\mathbf{a}})$. If $1 < p < \infty$, then $w \in A_p(\Sigma_{\mathbf{a}})$ if and only if the inequality*

$$\|Tf\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},w)} \le K_p \|f\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},w)} \tag{4.5}$$

is valid for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w)\cap\mathfrak{L}_1(\Sigma_{\mathbf{a}})$, where K_p is a constant independent of f. If $1 \le p < \infty$, then $w \in A_p(\Sigma_{\mathbf{a}})$ if and only if the inequality

$$\|Tf\|^*_{\mathfrak{L}_{p,\infty}(\Sigma_{\mathbf{a}},w)} \le K_p \|f\|_{\mathfrak{L}_p(\Sigma_{\mathbf{a}},w)} \tag{4.6}$$

is valid for all $f \in \mathfrak{L}_p(\Sigma_{\mathbf{a}}, w)\cap\mathfrak{L}_1(\Sigma_{\mathbf{a}})$, where K_p is a constant independent of f.

Proof. By Proposition (4.3), the necessity parts of the theorem hold. To prove the sufficiency parts of the theorem, let $1 \le p < \infty$ and assume that (4.6) holds. As noted before, $(\mathfrak{L}_p(\Sigma_{\mathbf{a}}, w*\mu_k)\cap\mathfrak{L}_1(\Sigma_{\mathbf{a}}))*\mu_k$ is isometrically isomorphic to $\mathfrak{L}_p(\mathbb{T}, w_k)\cap\mathfrak{L}_1(\mathbb{T})$ where w_k is a function in $\mathfrak{L}_1(\mathbb{T})$ such that $w*\mu_k = w_k\circ\pi_k$. So by Proposition (4.1), for $k = 1, 2, \ldots$, the inequality

$$\|Tf\|^*_{\mathfrak{L}_{p,\infty}(\mathbb{T},w_k)} \le K_p \|f\|_{\mathfrak{L}_p(\mathbb{T},w_k)}$$

is valid for all $f \in \mathfrak{L}_p(\mathbb{T}, w_k)\cap\mathfrak{L}_1(\mathbb{T})$ where Tf is the conjugate function of f defined on the circle. By Theorem 1.1, for $k = 1, 2, \ldots$, we have $w_k \in A_p(\mathbb{T})$ with bound less than or equal to $K_p^2(4\pi)^{2p}$. By Proposition (4.2), for $k = 1, 2, \ldots$, we have $w*\mu_k \in A_p(\Sigma_{\mathbf{a}})$ with bound less than or equal to $K_p^2(4\pi)^{2p}$. Fix an interval I. Since $\|w*\mu_k - w\|_{\mathfrak{L}_1(\Sigma_{\mathbf{a}})} \to 0$ as $k\to\infty$, by Fatou's lemma and Fubini's Theorem, there is a subsequence $(w*\mu_{k_l})_{l\ge 0}$ such that for μ-a.e. x in $\Sigma_{\mathbf{a}}$,

$$
\begin{aligned}
&\frac{1}{|I|}\int_I w(x-\varphi(s))ds\left(\frac{1}{|I|}\int_I w^{-1/(p-1)}(x-\varphi(s))ds\right)^{p-1}\\
&\quad\le \frac{1}{|I|}\liminf_l \int_I w*\mu_{k_l}(x-\varphi(s))ds\\
&\qquad\times\left(\frac{1}{|I|}\liminf_l \int_I (w*\mu_{k_l})^{-1/(p-1)}(x-\varphi(s))ds\right)^{p-1}\\
&\quad\le \frac{1}{|I|}\limsup_l \int_I w*\mu_{k_l}(x-\varphi(s))ds\\
&\qquad\times\limsup_l\left(\frac{1}{|I|}\int_I (w*\mu_{k_l})^{-1/(p-1)}(x-\varphi(s))ds\right)^{p-1}\\
&\quad= \frac{1}{|I|}\limsup_l \int_I w*\mu_{k_l}(x-\varphi(s))ds\\
&\qquad\times\left(\frac{1}{|I|}\int_I (w*\mu_{k_l})^{-1/(p-1)}(x-\varphi(s))ds\right)^{p-1}\\
&\quad\le K_p^2(4\pi)^{2p}.
\end{aligned}
$$

So, for each interval I, the above inequality holds for x in $\Sigma_{\mathbf{a}}$, except possibly on a set of measure 0 (depending on I). Thus, the inequality holds for μ-a.e. x in $\Sigma_{\mathbf{a}}$ and for all intervals with rational endpoints (countably many). Approximating an arbitrary interval I by an interval with rational endpoints, a straightforward argument shows that the above inequality still holds for μ-a.e. x in $\Sigma_{\mathbf{a}}$ and all intervals I, hence showing that (2.4) holds and $w \in A_p(\Sigma_{\mathbf{a}})$. □

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Department of Mathematics, University of Missouri, Columbia, Missouri 65211
E-mail: mathna@mizzou1.missouri.edu, saleem@csulb.edu