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A NOVEL UNIFORM EXPANSION FOR A SINGULARLY PERTURBED PARABOLIC PROBLEM WITH CORNER SINGULARITY

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ABSTRACT. A linear singularly perturbed parabolic equation defined in a quarter plane possesses internal layer behavior when there is an incompatible relationship between the initial data and the boundary data. A novel expansion for the solution is obtained in the form $\sum_{k=0}^{n-1} \varepsilon^k [u_{2k}(x,t;\varepsilon)+\varepsilon^{k/2} u_{2k+1}(x,t;\varepsilon)]+\varepsilon^n u^{(n+1)}(x,t;\varepsilon)$ by applying Temme's technique [1982] along with the device of splitting and combining the integral representation. The expansion $\sum_{k=0}^{\lfloor n/2 \rfloor} \varepsilon^k [u_{2k}(x,t;\varepsilon) + \varepsilon^{k/2} u_{2k+1}(x,t;\varepsilon)]$ is shown to be the exact solution when the initial function is a polynomial of degree n and the boundary function is a polynomial of degree $\lfloor n/2 \rfloor + 1$. Moreover, a comparison is made with some known results in the literature.

1. Introduction

Numerical solutions of boundary value problems of partial differential equations often use a variety of finite difference/element formulations of differential equations. On the other hand, it is also quite common to convert given problems into integral equations, from which one employs boundary element methods, see, for example, Chen and Zhou [6], to obtain numerical approximations.

For singularly perturbed problems of differential equations, methods of matched asymptotic expansions have been very popular in constructing composite expansions which are uniformly valid in the domain under consideration. Typically, a composite expansion consists of an outer expansion and an inner expansion; the outer expansion gives an excellent approximation to a given problem except for some narrow regions of a rapid variation, either near part of the boundary or along some internal curve, at each of which one is required to construct an inner expansion based on some stretched variable in the region of non-uniformity. It has been traditional to apply methods of matched asymptotic expansions or methods of multiple scales to obtain such theoretical approximations for singularly perturbed differential equations.

Sometimes, there is a necessity to adopt different approaches when methods of matched asymptotic expansions fail to give satisfactory results for constructing a uniform approximate solution. To gain the asymptotic behavior of a singularly perturbed problem, one often constructs an integral representation for the solution in terms of Green's function, from which one analyzes the asymptotic behavior by using asymptotic approximations of integrals; see, for example, Grasman [8] and Temme [14] on investigating parabolic boundary layer and corner layer structures, respectively, arising from some singularly perturbed problems for the elliptic differential equation in

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the form

$$-\varepsilon \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\} + \frac{\partial u}{\partial y} = 0,$$

in the domain $x > 0$, $y > 0$.

The goal of this study is to unveil internal layer structures of some linear singularly perturbed parabolic problems with corner singularity by finding the asymptotic behavior of an integral form of its solution. An integral representation for the solution of a linear singularly perturbed parabolic differential equation involves the spatial variable x, the temporal variable t, and the small parameter ε satisfying $0 < \varepsilon \ll 1$. Thus, it is important to derive an asymptotic expansion which is uniformly valid in x, t as $\varepsilon \downarrow 0$. One obtains the outer expansion of the singularly perturbed problem if the classical Laplace's method is employed. Our method of obtaining a uniform expansion has the following major steps:

(i) obtain an integral representation for a given singularly perturbed problem;
(ii) transform each integral appearing in the solution representation obtained in (i) into a standard form;
(iii) construct a formal uniform expansion for each canonical integral;
(iv) construct an error bound for the expansion; and
(v) investigate the asymptotic properties of the expansion and compare with methods of matched asymptotic expansions.

There is a large class of parabolic problems which have solutions in integral form. For example, the equation

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + p \frac{\partial u}{\partial x} = F(x,t)$$

with $p(x,t) = x\, q(t)$ can be converted into the heat equation via a transformation in both independent variables. But, for the sake of clarity of illustrating our techniques, in this paper, we consider only the case where p is a positive constant. Moreover, without loss of generality, we choose $F(x,t) = 0$ since the function $F(x,t)$ gives its contribution only to the first outer function $u_0(x,t)$ defined by

$$\frac{\partial u_0}{\partial t} + p \frac{\partial u_0}{\partial x} = F(x,t),$$

and to other higher-order outer functions but not in the structure of the inner expansion. One standard form of the transformed integrals related to the initial data of the given singularly perturbed problem is

$$I(x,t) = \int_a^b f(s;x,t) \, \exp\left(-\frac{s^2}{\varepsilon}\right) \, ds$$

where a may be finite or $-\infty$ and b finite or ∞. A uniform asymptotic expansion of the integral of this type was studied by Temme [15] using a technique of subtraction and integration by parts. Specifically, one writes $I(x,t)$ as

$$I(x,t) = \int_a^b f(0;x,t) \, \exp\left(-\frac{s^2}{\varepsilon}\right) \, ds + \int_a^b [f(s;x,t) - f(0;x,t)] \, \exp\left(-\frac{s^2}{\varepsilon}\right) \, ds.$$

Then integrating the second integral by parts gives the resultant integral of the same form as $I(x,t)$. Repeating this process of subtraction and integration by parts gives an asymptotic expansion of the integral $I(x,t)$ in ε. The technique of repeated integrations by parts is a simple and often effective way of deriving the asymptotic expansion of an integral containing a parameter. For instance, in determining the asymptotic behavior of the complementary error function $\operatorname{erfc}(x)$ as x approaches infinity, one may employ integration by parts to derive

$$\operatorname{erfc}(x) \sim \frac{\exp(-x^2)}{\sqrt{\pi}\, x} \sum_{k=0}^{\infty} (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{(2x^2)^k}, \qquad x \uparrow \infty.$$

For $x \in (0, \infty)$, the error term does not exceed the first neglected term in the series in absolute value and has the same sign; see, for example, Olver [11, p. 67]. The other standard form of integrals associated with the boundary data of the given singularly perturbed problem requires additional manipulations of splitting and combining after employing the process of subtraction and integration by parts. The device of splitting a difficult integral into two to derive its asymptotic behavior in this paper is similar to the one in Olver [12].

Now, we define the complementary error function erfc by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-s^2)\, ds,$$

with the following properties

$$\begin{aligned} &\operatorname{erfc}(0) = 1, \qquad \operatorname{erfc}(-x) = 2 - \operatorname{erfc}(x), \\ &\operatorname{erfc}(x) \sim \frac{1}{\sqrt{\pi x}} \exp(-x^2), \qquad \text{as } x \uparrow \infty. \end{aligned} \tag{1.1}$$

Next, the first iterated integral of the complementary error function ierfc is defined by

$$\operatorname{ierfc}(x) = \int_x^{\infty} \operatorname{erfc}(s)\, ds = \frac{1}{\sqrt{\pi}} \exp(-x^2) - x \operatorname{erfc}(x), \tag{1.2}$$

which has the properties

$$\operatorname{ierfc}(0) = \frac{1}{\sqrt{\pi}}, \qquad \operatorname{ierfc}'(0) = -1, \qquad \operatorname{ierfc}(-x) = 2x + \operatorname{ierfc}(x), \tag{1.3}$$

$$\operatorname{ierfc}(x) \sim \frac{1}{2\sqrt{\pi}\, x^2} \exp(-x^2) \quad \text{as } x \uparrow \infty, \qquad \operatorname{ierfc}(x) \sim -2x \quad \text{as } x \downarrow -\infty.$$

In general, the n^{th} iterated integral of the complementary error function $\mathrm{i}^n \operatorname{erfc}$ is defined by

$$\mathrm{i}^n \operatorname{erfc}(x) = \int_x^{\infty} \mathrm{i}^{n-1} \operatorname{erfc}(s)\, ds = \frac{1}{2n}\, \mathrm{i}^{n-2} \operatorname{erfc}(x) - \frac{x}{n}\, \mathrm{i}^{n-1} \operatorname{erfc}(x), \tag{1.4}$$

for $n = 2, 3, 4, \ldots$. Moreover, we have

$$\mathrm{i}^2 \operatorname{erfc}(x) \sim \frac{1}{4\sqrt{\pi}\, x^3} \exp(-x^2) \quad \text{as } x \uparrow \infty, \quad \mathrm{i}^2 \operatorname{erfc}(-x) = x^2 + \frac{1}{2} - \mathrm{i}^2 \operatorname{erfc}(x), \tag{1.5}$$

$$\mathrm{i}^3 \operatorname{erfc}(x) \sim \frac{1}{8\sqrt{\pi}\, x^4} \exp(-x^2) \quad \text{as } x \uparrow \infty, \quad \mathrm{i}^3 \operatorname{erfc}(-x) = \frac{x^3}{3} + \frac{x}{2} + \mathrm{i}^3 \operatorname{erfc}(x). \tag{1.6}$$

For more information on these functions, see Abramowitz and Stegun [1].

Linear singularly perturbed parabolic problems with corner singularity have been studied by Bobisud [4], Howes [9], and Joseph [10]. Bobisud investigated a linear singularly perturbed parabolic problem defined in the square domain $0 < x < 1$, $0 < t < 1$ when $x = 0$ and $t = 0$ are the inflow boundaries, so that there is a boundary layer at $x = 1$ and an internal layer along the characteristic curve of the reduced problem, emanating at the origin. The asymptotic expansion constructed with the boundary layer term was shown to be of the order $\sqrt{\varepsilon}$ under the assumption of continuity between the initial data and the boundary data at the origin. To improve the order of validity, one is required to construct an internal layer term. Howes obtained some exponential upper bound for the internal layer function of a linear singularly perturbed problem under the same assumption for the initial and boundary functions. As far as we know, an explicit construction of internal layer functions is not available in the literature.

The organization of this paper is as follows. Section 2 provides an integral representation of the solution for a singularly perturbed parabolic problem defined in a quarter plane. An expansion is constructed in Section 3 from the obtained integrals. Some properties of this expansion are given in Section 4. A comparison is made with some related results of Howes and Joseph in Section 5.

2. Integral representation of solution

Assume that the function $f(x)$ and $g(t)$ are smooth in their respective domains $x > 0$, $t > 0$ and satisfy the growth conditions

$$|f(x)| \le C_1 \, \exp\big(C_2 x^{1+\alpha}\big) \qquad x > 0, \tag{2.1}$$

$$|g(t)| \le C_1 \, \exp\big(C_2 t^{1+\alpha}\big) \qquad t > 0, \tag{2.2}$$

where C_1 and C_2 are positive constants and $0 \le \alpha < 1$. From Cannon [5, p. 50], we have the following theorem.

Theorem 2.1. *The heat equation*

$$\frac{\partial u}{\partial t} = \varepsilon \, \frac{\partial^2 u}{\partial x^2} \qquad x > 0, \quad t > 0,$$

with a parameter ε *satisfying* $0 < \varepsilon \ll 1$, *subject to the initial condition*

$$u(x, 0) = f(x) \qquad x > 0,$$

and the Dirichlet boundary condition

$$u(0, t) = g(t) \qquad t > 0,$$

has the solution of the form

$$u(x,t) = \int_0^\infty G(x,\eta,t) f(\eta)\, d\eta + \varepsilon \int_0^t \frac{\partial G}{\partial \eta}(x, 0, t-\tau) g(\tau)\, d\tau \tag{2.3}$$

where $G(x,\eta,t)$ *is the Green's function of the heat operator* $\partial/\partial t - \varepsilon\, \partial^2/\partial x^2$ *over the quarter plane defined by*

$$G(x,\eta,t) = K(x-\eta,t) - K(x+\eta,t), \tag{2.4}$$

with the fundamental solution $K(x,t)$ *of the heat operator* $\partial/\partial t - \varepsilon\, \partial^2/\partial x^2$ *given by*

$$K(x,t) = \frac{1}{2\sqrt{\pi t \varepsilon}} \, \exp\left(-\frac{x^2}{4t\varepsilon}\right). \tag{2.5}$$

Now this result can be extended to a convection-diffusion problem.

Theorem 2.2. *The convection-diffusion equation*

$$\frac{\partial u}{\partial t} + p \, \frac{\partial u}{\partial x} = \varepsilon \, \frac{\partial^2 u}{\partial x^2} \qquad x > 0, \quad t > 0, \tag{2.6}$$

with a constant $p > 0$ *and a parameter* ε *satisfying* $0 < \varepsilon \ll 1$, *subject to the initial condition*

$$u(x, 0) = f(x) \qquad x > 0, \tag{2.7}$$

and the Dirichlet boundary condition

$$u(0,t) = g(t) \qquad t > 0, \tag{2.8}$$

has the solution of the form

$$u(x,t) = \exp\left(\frac{px}{2\varepsilon} - \frac{p^2 t}{4\varepsilon}\right) \left\{\int_0^\infty G(x,\eta,t) f(\eta) \exp\left(-\frac{p\eta}{2\varepsilon}\right) d\eta \right. \\ \left. + \varepsilon \int_0^t \frac{\partial G}{\partial \eta}(x,0,t-\tau) g(\tau) \exp\left(\frac{p^2\tau}{4\varepsilon}\right) d\tau \right\} \tag{2.9}$$

where $G(x,\eta,t)$ is the Green's function of the heat operator $\partial/\partial t - \varepsilon\, \partial^2/\partial x^2$ over the quarter plane defined by (2.4).

Proof. Making the substitution

$$u(x,t) = v(x,t) \exp\left(\frac{px}{2\varepsilon} - \frac{p^2 t}{4\varepsilon}\right)$$

converts the initial boundary value problem (2.6), (2.7), (2.8) to the problem

$$\frac{\partial v}{\partial t} = \varepsilon \frac{\partial^2 v}{\partial x^2} \qquad x > 0, \quad t > 0,$$

$$v(x,0) = f(x) \exp\left(-\frac{px}{2\varepsilon}\right), \qquad v(0,t) = g(t) \exp\left(\frac{p^2 t}{4\varepsilon}\right).$$

The desired result follows from (2.3). □

To study the asymptotic behavior of $u(x,t)$ given by (2.9) for small values of ε, we reduce it to the following form.

Theorem 2.3. *The solution $u(x,t)$ of the initial boundary value problem* (2.6), (2.7), (2.8) *can be expressed as*

$$u(x,t) = I_1(x,t) - I_2(x,t) \exp\left(\frac{px}{\varepsilon}\right) + I_3(x,t) + I_4(x,t) \exp\left(\frac{px}{\varepsilon}\right) \tag{2.10}$$

where

$$I_1(x,t) = \frac{1}{\sqrt{\pi\varepsilon}} \int_{-\infty}^{\frac{x-pt}{2\sqrt{t}}} f(x - pt - 2\sqrt{t}\sigma) \exp\left(-\frac{\sigma^2}{\varepsilon}\right) d\sigma, \tag{2.11}$$

$$I_2(x,t) = \frac{1}{\sqrt{\pi\varepsilon}} \int_{\frac{x+pt}{2\sqrt{t}}}^{\infty} f(-x - pt + 2\sqrt{t}\sigma) \exp\left(-\frac{\sigma^2}{\varepsilon}\right) d\sigma, \tag{2.12}$$

$$I_3(x,t) = \int_0^t g(t-s) \frac{x+ps}{4\sqrt{\pi s^3 \varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds, \tag{2.13}$$

$$I_4(x,t) = \int_0^t g(t-s) \frac{x-ps}{4\sqrt{\pi s^3 \varepsilon}} \exp\left[-\frac{(x+ps)^2}{4s\varepsilon}\right] ds. \tag{2.14}$$

Proof. First of all, from (2.9), the contribution of the initial data $f(x)$ is

$$\begin{aligned} u_i(x,t) &= \exp\left(\frac{px}{2\varepsilon} - \frac{p^2 t}{4\varepsilon}\right) \int_0^\infty G(x,\eta,t) f(\eta) \exp\left(-\frac{p\eta}{2\varepsilon}\right) d\eta \\ &= \exp\left(\frac{px}{2\varepsilon} - \frac{p^2 t}{4\varepsilon}\right) \int_0^\infty [K(x-\eta,t) - K(x+\eta,t)] f(\eta) \exp\left(-\frac{p\eta}{2\varepsilon}\right) d\eta \\ &= \exp\left(-\frac{p^2 t}{4\varepsilon}\right) \int_{-\infty}^x K(s,t) f(x-s) \exp\left(\frac{ps}{2\varepsilon}\right) ds \\ &\quad - \exp\left(\frac{px}{\varepsilon} - \frac{p^2 t}{4\varepsilon}\right) \int_x^\infty K(s,t) f(s-x) \exp\left(-\frac{ps}{2\varepsilon}\right) ds \\ &= I_1(x,t) - I_2(x,t) \exp\left(\frac{px}{\varepsilon}\right), \end{aligned}$$

with

$$\begin{aligned} I_1(x,t) &= \int_{-\infty}^x \frac{f(x-s)}{2\sqrt{\pi t \varepsilon}} \exp\left[-\frac{(s-pt)^2}{4t\varepsilon}\right] ds, \\ I_2(x,t) &= \int_x^\infty \frac{f(s-x)}{2\sqrt{\pi t \varepsilon}} \exp\left[-\frac{(s+pt)^2}{4t\varepsilon}\right] ds. \end{aligned}$$

A change of variable gives the desired form for these two integrals.

Next, we reduce the contribution of the boundary data $g(t)$ to a simpler form:

$$\begin{aligned} u_b(x,t) &= \varepsilon \exp\left(\frac{px}{2\varepsilon} - \frac{p^2 t}{4\varepsilon}\right) \int_0^t \frac{\partial G}{\partial \eta}(x,0,t-\tau) g(\tau) \exp\left(\frac{p^2 \tau}{4\varepsilon}\right) d\tau \\ &= \varepsilon \exp\left(\frac{px}{2\varepsilon}\right) \int_0^t \frac{\partial G}{\partial \eta}(x,0,s) g(t-s) \exp\left(-\frac{p^2 s}{4\varepsilon}\right) ds \\ &= -2\varepsilon \exp\left(\frac{px}{2\varepsilon}\right) \int_0^t \frac{\partial K}{\partial x}(x,s) g(t-s) \exp\left(-\frac{p^2 s}{4\varepsilon}\right) ds \\ &= \int_0^t \frac{x g(t-s)}{2\sqrt{\pi s^3 \varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds. \end{aligned}$$

To make $u_b(x,t)$ more tractable, we now split it into two integrals:

$$\begin{aligned} u_b(x,t) &= \int_0^t \frac{(x+ps) g(t-s)}{4\sqrt{\pi s^3 \varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds \\ &\quad + \int_0^t \frac{(x-ps) g(t-s)}{4\sqrt{\pi s^3 \varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds \\ &= I_3(x,t) + \exp\left(\frac{px}{\varepsilon}\right) I_4(x,t), \end{aligned}$$

with $I_3(x,t)$, $I_4(x,t)$ given by (2.13), (2.14), respectively. This completes the proof. □

3. Construction of an expansion

To investigate the asymptotic behavior for the solution $u(x,t)$ of the singularly perturbed problem (2.6), (2.7), (2.8) for small values of ε, one analyzes the integrals $I_1(x,t)$, $I_2(x,t)$, $I_3(x,t)$, $I_4(x,t)$ for small values of ε. Laplace's method is known to give *only* the outer expansion of $u(x,t)$, which is not uniformly valid in a neighborhood of the curve $x = pt$ with $t \geq 0$. In the terminology of the asymptotics of integrals, we have a classic problem. For example, along the curve $x = pt$ of the non-uniformity for the singularly perturbed problem (2.6), (2.7), (2.8), the end point $(x - pt)/(2\sqrt{t})$ of the integration in σ in the integral $I_1(x,t)$ coalesces with the saddle point at $\sigma=0$, while the end point t of the integration in s in the integral $I_3(x,t)$ coalesces with the saddle point at $s=p/x$. Several contributions in the literature deal with this aspect, for instance, Bleistein [2], Bleistein and Hadelsman [3], Erdélyi [7], Olver [11], and Wong [16]. Our method of obtaining a uniform expansion in each integral $I_k(x,t)$, $k=1,2,3,4$, is motivated by Temme [15] with additional manipulations of splitting and combining for $I_k(x,t)$, $k=3,4$. We illustrate our technique by obtaining an expansion with an arbitrary number of terms. Now, we obtain some preliminary integral results.

Lemma 3.1.

$$\frac{1}{\sqrt{\pi\varepsilon}}\int_{-\infty}^{x}\exp\left(-\frac{\sigma^2}{\varepsilon}\right)\,d\sigma=1-\frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{\varepsilon}}\right), \tag{3.1}$$

$$\frac{1}{\sqrt{\pi\varepsilon}}\int_{x}^{\infty}\exp\left(-\frac{\sigma^2}{\varepsilon}\right)\,d\sigma=\frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{\varepsilon}}\right), \tag{3.2}$$

$$\int_0^t\frac{x+ps}{4\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]\,ds=\frac{1}{2}\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right), \tag{3.3}$$

$$\begin{aligned}\int_0^t\frac{x^2-p^2s^2}{2\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]\,ds&=2\sqrt{\frac{t\varepsilon}{\pi}}\exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right]\\&-\frac{\varepsilon}{p}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}.\end{aligned} \tag{3.4}$$

Proof. The first two equations (3.1), (3.2) are obtained easily. The third equation (3.3) follows from the substitution

$$\sigma=\frac{x-ps}{2\sqrt{s\varepsilon}},\qquad\frac{d\sigma}{ds}=-\frac{x+ps}{4\sqrt{s^3\varepsilon}}.$$

For (3.4), noting

$$\frac{d}{ds}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]=\frac{x^2-p^2s^2}{4s^2\varepsilon}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right], \tag{3.5}$$

we integrate by parts to get

$$\begin{aligned}&\int_0^t\frac{x^2-p^2s^2}{2\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]\,ds\\&=2\sqrt{\frac{t\varepsilon}{\pi}}\exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right]-2\varepsilon\int_0^t\frac{s}{2\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]\,ds.\end{aligned}$$

To evaluate the new integral, we split it into two integrals as follows:

$$\begin{aligned}\int_0^t\frac{s}{2\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]\,ds&=\frac{1}{p}\int_0^t\frac{x+ps}{4\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]\,ds\\&-\frac{1}{p}\exp\left(\frac{px}{\varepsilon}\right)\int_0^t\frac{x-ps}{4\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x+ps)^2}{4s\varepsilon}\right]\,ds.\end{aligned}$$

Then, we have (3.4) by virtue of (3.3). □

To make our techniques clearer, some integral results are listed in the following theorems under the assumption that the function $\Psi(s;x,t)$ satisfies the growth condition for (2.1) or (2.2) when s is large. The next theorem can be employed as often as necessary to obtain a higher-order expansion for the integrals associated with the initial data of the given problem.

Theorem 3.1. *The integral $I_a(x,t)$ defined by*

$$I_a(x,t) = \frac{1}{\sqrt{\pi\varepsilon}} \int_{-\infty}^{\frac{x-pt}{2\sqrt{t}}} \Psi(\sigma;x,t) \ \exp\left(-\frac{\sigma^2}{\varepsilon}\right) \ d\sigma$$

can be expanded as

$$I_a(x,t) = \sum_{k=0}^{2} \varepsilon^{k/2} \ I_a^{(k/2)}(x,t), \tag{3.6}$$

where the terms are given by

$$\begin{aligned} I_a^{(0)}(x,t) &= \Psi(0;x,t) - \frac{1}{2}\Psi\left(\frac{x-pt}{2\sqrt{t}};x,t\right) \ \text{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right), \\ I_a^{(1/2)}(x,t) &= \frac{\sqrt{t}}{x-pt}\left[\Psi(0;x,t) - \Psi\left(\frac{x-pt}{2\sqrt{t}};x,t\right)\right] \text{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right), \\ I_a^{(1)}(x,t) &= \frac{1}{2\sqrt{\pi\varepsilon}} \int_{-\infty}^{\frac{x-pt}{2\sqrt{t}}} \Psi^{(1)}(\sigma;x,t) \ \exp\left(-\frac{\sigma^2}{\varepsilon}\right) \ d\sigma, \end{aligned} \tag{3.7}$$

with

$$\Psi^{(1)}(\sigma;x,t) = \frac{d}{d\sigma}\left[\frac{1}{\sigma}\{\Psi(\sigma;x,t) - \Psi(0;x,t)\}\right].$$

The integral $I_b(x,t)$ defined by

$$I_b(x,t) = \frac{1}{\sqrt{\pi\varepsilon}} \int_{\frac{x+pt}{2\sqrt{t}}}^{\infty} \Psi(\sigma;x,t) \ \exp\left(-\frac{\sigma^2}{\varepsilon}\right) \ d\sigma$$

can be expanded as

$$I_b(x,t) = \sum_{k=0}^{2} \varepsilon^{k/2} \ I_b^{(k/2)}(x,t) \tag{3.8}$$

where the terms are given by

$$\begin{aligned} I_b^{(0)}(x,t) &= \frac{1}{2}\Psi\left(\frac{x+pt}{2\sqrt{t}};x,t\right) \ \text{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right), \\ I_b^{(1/2)}(x,t) &= \frac{\sqrt{t}}{x+pt}\left[\Psi\left(\frac{x+pt}{2\sqrt{t}};x,t\right) - \Psi(0;x,t)\right] \text{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right), \\ I_b^{(1)}(x,t) &= \frac{1}{2\sqrt{\pi\varepsilon}} \int_{\frac{x+pt}{2\sqrt{t}}}^{\infty} \Psi^{(1)}(\sigma;x,t) \ \exp(-\frac{\sigma^2}{\varepsilon}) \ d\sigma. \end{aligned} \tag{3.9}$$

Moreover, if $\Psi(\sigma;x,t)$ is a polynomial of degree n in σ, then $\Psi^{(1)}(\sigma;x,t)$ is a polynomial of degree $n-2$ in σ.

Proof. Rewrite the integral $I_a(x,t)$ as

$$\begin{aligned} I_a &= \frac{1}{\sqrt{\pi\varepsilon}} \int_{-\infty}^{\frac{x-pt}{2\sqrt{t}}} \Psi(0;x,t) \ \exp\left(-\frac{\sigma^2}{\varepsilon}\right) \ d\sigma \\ &+ \frac{1}{\sqrt{\pi\varepsilon}} \int_{-\infty}^{\frac{x-pt}{2\sqrt{t}}} [\Psi(\sigma;x,t) - \Psi(0;x,t)] \ \exp\left(-\frac{\sigma^2}{\varepsilon}\right) \ d\sigma, \end{aligned}$$

due to the fact that the main contribution of the integral $I_a(x,t)$ takes place at $\sigma = 0$. Using the integral formula (3.1) in the first integral and performing integration by parts in the second integral give

$$
\begin{aligned}
I_a &= \Psi(0;x,t) - \frac{1}{2}\Psi(0;x,t)\ \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad - \frac{1}{2}\sqrt{\frac{\varepsilon}{\pi}} \int_{-\infty}^{\frac{x-pt}{2\sqrt{t}}} \frac{\Psi(\sigma;x,t) - \Psi(0;x,t)}{\sigma}\ d\exp\left(-\frac{\sigma^2}{\varepsilon}\right) \\
&= \Psi(0;x,t) - \frac{1}{2}\Psi(0;x,t)\ \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad + \sqrt{\frac{t\varepsilon}{\pi}} \frac{\Psi(0;x,t) - \Psi\left(\frac{x-pt}{2\sqrt{t}};x,t\right)}{x-pt} \exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right] + \varepsilon\, I_a^{(1)}(x,t),
\end{aligned}
$$

with $I_a^{(1)}(x,t)$ defined by (3.7). The desired result (3.6) follows by using (1.2).

Applying the above procedure of subtraction and integration by parts to the integral $I_b(x,t)$ along with the use of (3.2) yields

$$
\begin{aligned}
I_b &= \frac{1}{2}\Psi(0;x,t)\ \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad + \sqrt{\frac{t\varepsilon}{\pi}} \frac{\Psi\left(\frac{x+pt}{2\sqrt{t}};x,t\right) - \Psi(0;x,t)}{x+pt} \exp\left[-\frac{(x+pt)^2}{4t\varepsilon}\right] + \frac{\varepsilon}{2}\, I_b^{(1)}(x,t),
\end{aligned}
$$

with $I_b^{(1)}(x,t)$ given by (3.9). Using (1.2), we then have (3.8). □

The integrals associated with the boundary data are difficult to expand. One needs to use the next theorem to obtain the first expansion.

Theorem 3.2. *The integral $I_c(x,t)$ defined by*

$$
I_c(x,t) = \int_0^t \Psi(s;x,t) \frac{x+ps}{4\sqrt{\pi s^3 \varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds
$$

can be expressed as

$$
I_c(x,t) = \sum_{k=0}^{2} \varepsilon^{k/2}\, I_c^{(k/2)}(x,t) \tag{3.10}
$$

where the terms are defined by

$$
\begin{aligned}
I_c^{(0)}(x,t) &= \frac{1}{2}\Psi(t;x,t)\ \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right), \\
I_c^{(1/2)}(x,t) &= \frac{\sqrt{t}}{x-pt}\left[\Psi(t;x,t) - \Psi\left(\frac{x}{p};x,t\right)\right] \operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right), \\
I_c^{(1)}(x,t) &= \frac{1}{\sqrt{\pi\varepsilon}} \int_0^t \frac{d}{ds}\left\{\sqrt{s}\, \frac{\Psi(x/p;x,t) - \Psi(s;x,t)}{x-ps}\right\} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds.
\end{aligned} \tag{3.11}
$$

Proof. The main contribution of the integral $I_c(x,t)$ comes from a neighborhood of the point $s = x/p$, so we then express $I_c(x,t)$ as

$$
\begin{aligned}
I_c &= \int_0^t \Psi\left(\frac{x}{p};x,t\right) \frac{x+ps}{4\sqrt{\pi s^3\varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds \\
&\quad + \int_0^t \left[\Psi(s;x,t) - \Psi\left(\frac{x}{p};x,t\right)\right] \frac{x+ps}{4\sqrt{\pi s^3\varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds.
\end{aligned}
$$

Noting (3.5), we use the integral formula (3.3) in the first integral and perform integration by parts in the second integral to give

$$\begin{aligned}
I_c &= \frac{1}{2}\Psi\left(\frac{x}{p};x,t\right)\ \mathrm{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad + \sqrt{\frac{\varepsilon}{\pi}}\int_0^t \sqrt{s}\frac{\Psi(s;x,t)-\Psi(x/p;x,t)}{x-ps}\ d\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] \\
&= \frac{1}{2}\Psi\left(\frac{x}{p};x,t\right)\ \mathrm{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad + \sqrt{\frac{t\varepsilon}{\pi}}\frac{\Psi(t;x,t)-\Psi(x/p;x,t)}{x-pt}\ \exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right] + \varepsilon\, I_c^{(1)}(x,t).
\end{aligned}$$

with $I_c^{(1)}(x,t)$ defined by (3.11). Then (3.10) follows by using (1.2). □

From this theorem, we then have the following result.

Corollary 3.1. *The integrals associated with the boundary data can be expanded as*

$$I_3(x,t) + I_4(x,t)\ \exp\left(\frac{px}{\varepsilon}\right) = \sum_{k=0}^{2} \varepsilon^{k/2}\, I_{34}^{(k/2)}(x,t) \tag{3.12}$$

where the terms are given by

$$\begin{aligned}
I_{34}^{(0)}(x,t) &= \frac{g(0)}{2}\left\{\mathrm{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\ \mathrm{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}, \\
I_{34}^{(1/2)}(x,t) &= \frac{\sqrt{t}}{x-pt}\left\{g(0) - g\left(t-\frac{x}{p}\right)\right\}\ \mathrm{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad + \frac{\sqrt{t}}{x+pt}\left\{g(0) - g\left(t+\frac{x}{p}\right)\right\}\ \exp\left(\frac{px}{\varepsilon}\right)\ \mathrm{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right), \\
I_{34}^{(1)}(x,t) &= \frac{1}{\sqrt{\pi\varepsilon}}\int_0^t \frac{d}{ds}\{\sqrt{s}\ g_{34}(s;x,t)\}\ \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]\ ds,
\end{aligned}$$

with

$$g_{34}(s;x,t) = \frac{g(t-x/p)-g(t-s)}{x-ps} + \frac{g(t+x/p)-g(t-s)}{x+ps}.$$

Furthermore, if g is a polynomial of degree n, then $g_{34}(s;x,t)$ is a polynomial of degree $n-2$ in s.

Lemma 3.2. *The integral $M(x,t)$ defined by*

$$M(x,t) = \int_0^t \Psi(s;x,t)\ \frac{x^2-p^2s^2}{2\sqrt{\pi s^3\varepsilon}}\ \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]\ ds$$

can be expressed as

$$\begin{aligned}
M(x,t) &= -\frac{\varepsilon}{p}\ \Psi\left(\frac{x}{p};x,t\right)\left\{\mathrm{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\ \mathrm{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
&\quad + 2\sqrt{\frac{t\varepsilon}{\pi}}\ \Psi(t;x,t)\ \exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right] \\
&\quad - 2\sqrt{\frac{\varepsilon}{\pi}}\int_0^t \frac{d}{ds}\left\{\sqrt{s}\left[\Psi(s;x,t) - \Psi\left(\frac{x}{p};x,t\right)\right]\right\}\ \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]\ ds.
\end{aligned} \tag{3.13}$$

Proof. We split $M(x,t)$ into two integrals

$$M = \int_0^t \Psi\left(\frac{x}{p};x,t\right) \frac{x^2-p^2s^2}{2\sqrt{\pi s^3\varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds + \int_0^t \left[\Psi(s;x,t)-\Psi\left(\frac{x}{p};x,t\right)\right] \frac{x^2-p^2s^2}{2\sqrt{\pi s^3\varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds.$$

Integrating the second integral by parts with (3.5) gives

$$\begin{aligned}
&\int_0^t \left[\Psi(s;x,t)-\Psi\left(\frac{x}{p};x,t\right)\right] \frac{x^2-p^2s^2}{2\sqrt{\pi s^3\varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds \\
&= 2\sqrt{\frac{\varepsilon}{\pi}} \int_0^t \sqrt{s}\left[\Psi(s;x,t)-\Psi\left(\frac{x}{p};x,t\right)\right] d\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] \\
&= 2\sqrt{\frac{t\varepsilon}{\pi}} \left[\Psi(t;x,t)-\Psi\left(\frac{x}{p};x,t\right)\right] \exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right] \\
&\quad - 2\sqrt{\frac{\varepsilon}{\pi}} \int_0^t \frac{d}{ds}\left\{\sqrt{s}\left[\Psi(s;x,t)-\Psi\left(\frac{x}{p};x,t\right)\right]\right\} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds.
\end{aligned}$$

With the use of (3.4), we obtain the result (3.13). □

Theorem 3.3. *The integral $N(x,t)$ defined by*

$$N(x,t) = \frac{1}{\sqrt{\pi\varepsilon}} \int_0^t \frac{d}{ds}\left\{\sqrt{s}\,\Psi(s;x,t)\right\} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds \tag{3.14}$$

can be expressed as

$$N(x,t) = \sum_{k=0}^{2} \varepsilon^{k/2} N^{(k/2)}(x,t) \tag{3.15}$$

where the terms are given by

$$\begin{aligned}
N^{(0)}(x,t) &= \frac{1}{2}\Psi_c(t;x,t)\,\mathrm{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \frac{1}{2}\Psi_d(t;x,t)\exp\left(\frac{px}{\varepsilon}\right)\mathrm{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right), \\
N^{(1/2)}(x,t) &= \frac{\sqrt{t}}{x-pt}\left[\Psi_c(t;x,t)-\Psi_c\left(\frac{x}{p};x,t\right)\right] \mathrm{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad + \frac{\sqrt{t}}{x+pt}\left[\Psi_d(t;x,t)-\Psi_d\left(-\frac{x}{p};x,t\right)\right] \exp\left(\frac{px}{\varepsilon}\right)\mathrm{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad - \frac{4}{p^2}\sqrt{\frac{t}{\pi}}\,\Psi_1(t;x,t)\exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right], \\
N^{(1)}(x,t) &= \frac{2}{p^3}\Psi_1\left(\frac{x}{p};x,t\right)\left\{\mathrm{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\mathrm{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
&\quad + N^{(10)}(x,t),
\end{aligned}$$

with

$$\Psi_1(s;x,t) = \frac{d}{ds}\Psi(s;x,t), \tag{3.16}$$

$$\Psi_c(s;x,t) = \frac{1}{p}\Psi(s;x,t) + \frac{2x}{p^2}\Psi_1(s;x,t), \tag{3.17}$$

$$\Psi_d(s;x,t) = \frac{-1}{p}\Psi(s;x,t) + \frac{2x}{p^2}\Psi_1(s;x,t), \tag{3.18}$$

$$N^{(10)}(x,t) = \frac{1}{\sqrt{\pi\varepsilon}}\int_0^t \frac{d}{ds}\left\{\sqrt{s}\,\Psi^{(1)}(s;x,t)\right\}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]ds, \tag{3.19}$$

and

$$\Psi^{(1)}(s;x,t) = \frac{\Psi_c(x/p;x,t)-\Psi_c(s;x,t)}{x-ps} + \frac{\Psi_d(-x/p;x,t)-\Psi_d(s;x,t)}{x+ps} + \frac{4}{p^2}\left[\Psi_1(s;x,t) - \Psi_1\left(\frac{x}{p};x,t\right)\right].$$

Moreover, if $\Psi(s;x,t)$ *is a polynomial of degree* n *in* s*, then* $\Psi^{(1)}(s;x,t)$ *is a polynomial of degree* $n-1$ *in* s*.*

Proof. Differentiation gives

$$N(x,t) = N_a(x,t) + N_b(x,t)$$

where

$$N_a = \frac{1}{2\sqrt{\pi\varepsilon}}\int_0^t \frac{1}{\sqrt{s}}\,\Psi(s;x,t)\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]ds,$$

$$N_b = \frac{1}{\sqrt{\pi\varepsilon}}\int_0^t \sqrt{s}\,\Psi_1(s;x,t)\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]ds,$$

with $\Psi_1(s;x,t)$ defined by (3.16). The device of splitting yields

$$N_a(x,t) = \frac{1}{p}\int_0^t \Psi(s;x,t)\,\frac{x+ps}{4\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]ds + \frac{-1}{p}\exp\left(\frac{px}{\varepsilon}\right)\int_0^t \Psi(s;x,t)\,\frac{x-ps}{4\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x+ps)^2}{4s\varepsilon}\right]ds,$$

and

$$N_b(x,t) = \frac{-2}{p^2}\int_0^t \Psi_1(s;x,t)\,\frac{x^2-p^2s^2}{2\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]ds + \frac{2x}{p^2}\int_0^t \Psi_1(s;x,t)\,\frac{x+ps}{4\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right]ds + \frac{2x}{p^2}\exp\left(\frac{px}{\varepsilon}\right)\int_0^t \Psi_1(s;x,t)\,\frac{x-ps}{4\sqrt{\pi s^3\varepsilon}}\exp\left[-\frac{(x+ps)^2}{4s\varepsilon}\right]ds.$$

It then follows that

$$N(x,t) = N_1(x,t) + N_c(x,t) + N_d(x,t)\exp\left(\frac{px}{\varepsilon}\right)$$

where we have

$$N_1 = \frac{-2}{p^2} \int_0^t \Psi_1(s;x,t) \, \frac{x^2 - p^2 s^2}{2\sqrt{\pi s^3 \varepsilon}} \, \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] \, ds,$$

$$N_c = \int_0^t \Psi_c(s;x,t) \, \frac{x+ps}{4\sqrt{\pi s^3 \varepsilon}} \, \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] \, ds,$$

$$N_d = \int_0^t \Psi_d(s;x,t) \, \frac{x-ps}{4\sqrt{\pi s^3 \varepsilon}} \, \exp\left[-\frac{(x+ps)^2}{4s\varepsilon}\right] \, ds,$$

with $\Psi_c(s;x,t)$, $\Psi_d(s;x,t)$ defined by (3.17), (3.18), respectively. The desired result (3.15) follows with the use of (3.10) and (3.13). □

Based on Theorem 3.1, Corollary 3.1, and Theorem 3.3, we obtain the following important result.

Theorem 3.4. *For each positive integer n, the solution $u(x,t)$ of the initial boundary value problem* (2.6), (2.7), (2.8) *can be expressed as*

$$u(x,t) = \sum_{k=0}^{n-1} \varepsilon^k \left[u_{2k}(x,t;\varepsilon) + \sqrt{\varepsilon}\, u_{2k+1}(x,t;\varepsilon)\right] + \varepsilon^n \, u^{(n)}(x,t;\varepsilon) \tag{3.20}$$

where the terms in the expansion are given by

$$\begin{aligned} u_{2k}(x,t;\varepsilon) = {} & \psi_{2k}^a(x,t) + \psi_{2k}^b(x,t) \, \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \\ & + \psi_{2k}^c(x,t) \, \exp\left(\frac{px}{\varepsilon}\right) \, \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right), \end{aligned} \tag{3.21}$$

$$\begin{aligned} u_{2k+1}(x,t;\varepsilon) = \sqrt{t}\Big\{ & \psi_{2k+1}^a(x,t) \, \operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \\ & + \psi_{2k+1}^b(x,t) \, \exp\left(\frac{px}{\varepsilon}\right) \, \operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right) \\ & + \psi_{2k+1}^c(x,t) \, \exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right] \Big\}, \end{aligned} \tag{3.22}$$

with

$$\psi_{2k}^a(x,t) = \frac{1}{2^k} \lim_{s\downarrow 0} \phi_1^{(k)}(s;x,t),$$

$$\psi_{2k}^b(x,t) = \begin{cases} \dfrac{-1}{2^{k+1}} \, \phi_1^{(k)}\left(\dfrac{x-pt}{2\sqrt{t}};x,t\right) + \dfrac{1}{2}\phi_3^{(k)}(t;x,t), & k=0,1, \\ \dfrac{-1}{2^{k+1}} \, \phi_1^{(k)}\left(\dfrac{x-pt}{2\sqrt{t}};x,t\right) + \dfrac{1}{2}\phi_3^{(k)}(t;x,t) & \\ \qquad + \dfrac{2}{p^3}\Phi_{34}^{(k-1)}\left(\dfrac{x}{p};x,t\right), & k \geq 2, \end{cases}$$

$$\psi_{2k}^c(x,t) = \begin{cases} \dfrac{-1}{2^{k+1}} \, \phi_2^{(k)}\left(\dfrac{x+pt}{2\sqrt{t}};x,t\right) + \dfrac{1}{2}\phi_4^{(k)}(t;x,t), & k=0,1, \\ \dfrac{-1}{2^{k+1}} \, \phi_2^{(k)}\left(\dfrac{x+pt}{2\sqrt{t}};x,t\right) + \dfrac{1}{2}\phi_4^{(k)}(t;x,t) & \\ \qquad - \dfrac{2}{p^3}\Phi_{34}^{(k-1)}\left(\dfrac{x}{p};x,t\right), & k \geq 2, \end{cases}$$

$$\psi^a_{2k+1}(x,t) = \frac{\phi_1^{(k)}(0;x,t) - \phi_1^{(k)}\left(\frac{x-pt}{2\sqrt{t}};x,t\right)}{2^k(x-pt)} + \frac{\phi_3^{(k)}(t;x,t) - \phi_3^{(k)}(x/p;x,t)}{x-pt},$$

$$\psi^b_{2k+1}(x,t) = \frac{\phi_2^{(k)}(0;x,t) - \phi_2^{(k)}\left(\frac{x+pt}{2\sqrt{t}};x,t\right)}{2^k(x+pt)} + \frac{\phi_4^{(k)}(t;x,t) - \phi_4^{(k)}(-x/p;x,t)}{x+pt},$$

$$\psi^c_{2k+1}(x,t) = \frac{-4}{p^2\sqrt{\pi}}\,\Phi_{34}^{(k)}(t;x,t),$$

and

$$\phi_1^{(k)}(s;x,t) = \begin{cases} f(x-pt-2\sqrt{t}s), & k=0,\\ \dfrac{d}{ds}\left[\dfrac{1}{s}\left\{\phi_1^{(k-1)}(s;x,t) - \phi_1^{(k-1)}(0;x,t)\right\}\right], & k\ge 1,\end{cases} \tag{3.23}$$

$$\phi_2^{(k)}(s;x,t) = \begin{cases} f(-x-pt+2\sqrt{t}s), & k=0,\\ \dfrac{d}{ds}\left[\dfrac{1}{s}\left\{\phi_2^{(k-1)}(s;x,t) - \phi_2^{(k-1)}(0;x,t)\right\}\right], & k\ge 1,\end{cases} \tag{3.24}$$

$$\phi_3^{(k)}(s;x,t) = \begin{cases} g(t-s), & k=0,\\ \dfrac{1}{p}\,\phi_{34}^{(k)}(s;x,t) + \dfrac{2x}{p^2}\,\Phi_{34}^{(k)}(s;x,t), & k\ge 1,\end{cases} \tag{3.25}$$

$$\phi_4^{(k)}(s;x,t) = \begin{cases} g(t-s), & k=0,\\ \dfrac{-1}{p}\,\phi_{34}^{(k)}(s;x,t) + \dfrac{2x}{p^2}\,\Phi_{34}^{(k)}(s;x,t), & k\ge 1,\end{cases} \tag{3.26}$$

$$\Phi_{34}^{(k)}(s;x,t) = \frac{d}{ds}\phi_{34}^{(k)}(s;x,t), \tag{3.27}$$

$$\phi_{34}^{(k)}(s;x,t) = \begin{cases} 0, & k=0,\\ \dfrac{\phi_3^{(k-1)}(x/p;x,t) - \phi_3^{(k-1)}(s;x,t)}{x-ps} & \\ \quad + \dfrac{\phi_4^{(k-1)}(-x/p;x,t) - \phi_4^{(k-1)}(s;x,t)}{x+ps} & \\ \quad + \dfrac{4}{p^2}\left[\Phi_{34}^{(k-1)}(s;x,t) - \Phi_{34}^{(k-1)}(x/p;x,t)\right], & k\ge 1.\end{cases} \tag{3.28}$$

The remainder $u^{(n)}(x,t;\varepsilon)$ *is given by*

$$u^{(n)}(x,t;\varepsilon) = \frac{2}{p^3}\Phi_{34}^{(n-1)}\left(\frac{x}{p};x,t\right)\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$+ \frac{1}{2^n}\left\{I_1^{(n)}(x,t) - I_2^{(n)}(x,t)\exp\left(\frac{px}{\varepsilon}\right)\right\} + J_{34}^{(n)}(x,t) \tag{3.29}$$

where $I_1^{(n)}(x,t)$, $I_2^{(n)}(x,t)$, *and* $J_{34}^{(n)}(x,t)$ *are defined by*

$$I_1^{(n)}(x,t) = \frac{1}{\sqrt{\pi\varepsilon}} \int_{-\infty}^{\frac{x-pt}{2\sqrt{t}}} \phi_1^{(n)}(\sigma;x,t)\, \exp\left(-\frac{\sigma^2}{\varepsilon}\right) d\sigma, \tag{3.30}$$

$$I_2^{(n)}(x,t) = \frac{1}{\sqrt{\pi\varepsilon}} \int_{\frac{x+pt}{2\sqrt{t}}}^{\infty} \phi_2^{(n)}(\sigma;x,t)\, \exp\left(-\frac{\sigma^2}{\varepsilon}\right) d\sigma, \tag{3.31}$$

$$J_{34}^{(n)}(x,t) = \frac{1}{\sqrt{\pi\varepsilon}} \int_0^t \frac{d}{ds}\left\{\sqrt{s}\,\phi_{34}^{(n)}(s;x,t)\right\} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds, \tag{3.32}$$

respectively.

Proof. To obtain an expansion of $u(x,t)$ defined by (2.10), one is required to carry out an expansion for each of integrals $I_1(x,t)$, $I_2(x,t)$, $I_3(x,t)$, and $I_4(x,t)$. There are three steps.

Step One: From (3.6) and (3.8), we obtain

$$\begin{aligned}
&I_1(x,t) - I_2(x,t)\, \exp\left(\frac{px}{\varepsilon}\right) \\
&= f(x-pt) - \frac{f(0)}{2}\left[\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right) \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right] \\
&\quad + \sqrt{t\varepsilon}\left[\frac{f(x-pt)-f(0)}{x-pt}\ \operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)\right. \\
&\quad \left. + \frac{f(-x-pt)-f(0)}{x+pt}\ \exp\left(\frac{px}{\varepsilon}\right) \operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right] \\
&\quad + \frac{\varepsilon}{2}\left[I_1^{(1)}(x,t) - I_2^{(1)}(x,t)\, \exp\left(\frac{px}{\varepsilon}\right)\right]
\end{aligned} \tag{3.33}$$

where $I_1^{(1)}(x,t)$ and $I_2^{(1)}(x,t)$ are given by (3.30) and (3.31), respectively, with $n=1$.
Next, putting (3.33), (3.12) into (2.10), we have

$$u(x,t) = u_0(x,t;\varepsilon) + \sqrt{\varepsilon}\, u_1(x,t;\varepsilon) + \varepsilon\, u^{(1)}(x,t;\varepsilon)$$

where $u_0(x,t,\varepsilon)$ and $u_1(x,t,\varepsilon)$ are defined by (3.21) and (3.22), respectively, with $k=0$. The remainder $u^{(1)}(x,t;\varepsilon)$ is defined by

$$u^{(1)}(x,t;\varepsilon) = \frac{1}{2}\left\{I_1^{(1)}(x,t) - I_2^{(1)}(x,t)\, \exp\left(\frac{px}{\varepsilon}\right)\right\} + J_{34}^{(1)}(x,t)$$

where $I_1^{(1)}(x,t)$, $I_2^{(1)}(x,t)$, and $J_{34}^{(1)}(x,t)$ are defined by (3.30), (3.31), and (3.32), respectively, with $n=1$.

Step Two: To obtain a higher-order expansion for $u(x,t)$ in ε, the remainder term $u^{(1)}(x,t;\varepsilon)$ is expanded in ε similarly. First, since the function $I_1^{(1)}(x,t)$ defined by (3.30) is of the same form as $I_1(x,t)$, and $I_2^{(1)}(x,t)$ defined by (3.31) is of the same form as $I_2(x,t)$, they can be expanded in ε analogously. Thus, we obtain from (3.6) and (3.8)

$$\begin{aligned}
&\frac{1}{2}\left\{I_1^{(1)}(x,t) - I_2^{(1)}(x,t)\, \exp\left(\frac{px}{\varepsilon}\right)\right\} \\
&= \frac{1}{2}\phi_1^{(1)}(0;x,t) - \frac{1}{4}\phi_1^{(1)}\left(\frac{x-pt}{2\sqrt{t}};x,t\right) \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)
\end{aligned}$$

$$
\begin{aligned}
& - \frac{1}{4}\phi_2^{(1)}\left(\frac{x+pt}{2\sqrt{t}};x,t\right) \exp\left(\frac{px}{\varepsilon}\right) \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right) \\
& + \sqrt{t\varepsilon}\left\{ \frac{\phi_1^{(1)}(0;x,t) - \phi_1^{(1)}\left(\frac{x-pt}{2\sqrt{t}};x,t\right)}{2(x-pt)} \operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \right. \\
& \left. + \frac{\phi_2^{(1)}(0;x,t) - \phi_2^{(1)}\left(\frac{x+pt}{2\sqrt{t}};x,t\right)}{2(x+pt)} \exp\left(\frac{px}{\varepsilon}\right) \operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right) \right\} \\
& + \frac{\varepsilon}{4}\left\{ I_1^{(2)}(x,t) - I_2^{(2)}(x,t) \exp\left(\frac{px}{\varepsilon}\right)\right\}
\end{aligned} \tag{3.34}
$$

where we have

$$
\phi_1^{(1)}(0;x,t) = \lim_{\sigma\downarrow 0}\phi_1^{(1)}(\sigma;x,t), \qquad \phi_2^{(1)}(0;x,t) = \lim_{\sigma\downarrow 0}\phi_2^{(1)}(\sigma;x,t),
$$

and $I_1^{(2)}(x,t)$, $I_2^{(2)}(x,t)$ are defined by (3.30), (3.31), respectively, with $n=2$.

Moreover, the function $J_{34}^{(1)}(x,t)$ defined by (3.32) is of the same form as $N(x,t)$ defined by (3.14). It then follows from (3.15) that we have

$$
\begin{aligned}
J_{34}^{(1)}(x,t) = {} & \frac{1}{2}\phi_3^{(1)}(t;x,t) \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \frac{1}{2}\phi_4^{(1)}(t;x,t) \exp\left(\frac{px}{\varepsilon}\right) \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right) \\
& + \sqrt{t\varepsilon}\left\{ \frac{\phi_3^{(1)}(t;x,t) - \phi_3^{(1)}(x/p;x,t)}{x-pt} \operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \right. \\
& - \frac{4}{p^2\sqrt{\pi}}\, \Phi_{34}^{(1)}(t;x,t) \exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right] \\
& \left. + \frac{\phi_4^{(1)}(t;x,t) - \phi_4^{(1)}(-x/p;x,t)}{x+pt} \exp\left(\frac{px}{\varepsilon}\right) \operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& + \frac{2\varepsilon}{p^3}\, \Phi_{34}^{(1)}\left(\frac{x}{p};x,t\right)\left\{ \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right) \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} + \varepsilon\, J_{34}^{(2)}(x,t)
\end{aligned} \tag{3.35}
$$

where $\phi_3^{(1)}(s;x,t)$, $\phi_4^{(1)}(s;x,t)$, $\Phi_{34}^{(1)}(s;x,t)$ are defined by (3.25), (3.26), (3.27) with $k=1$ and $J_{34}^{(2)}(x,t)$ by (3.32) with $n=2$, respectively.

It follows from (3.34), (3.35) that

$$
u^{(1)}(x,t;\varepsilon) = u_2(x,t;\varepsilon) + \sqrt{\varepsilon}\, u_3(x,t;\varepsilon) + \varepsilon\, u^{(2)}(x,t;\varepsilon)
$$

where $u_2(x,t;\varepsilon)$ and $u_3(x,t;\varepsilon)$ are defined by (3.21), (3.22), respectively, with $k=1$. The remainder $u^{(2)}(x,t;\varepsilon)$ is given by (3.29) with $n=2$.

Step Three: We continue as in Step Two to expand $u^{(2)}(x,t;\varepsilon)$ in ε. We first use (3.6) and (3.8) to obtain

$$
\begin{aligned}
& \frac{1}{4}\left\{ I_1^{(2)}(x,t) - I_2^{(2)}(x,t) \exp\left(\frac{px}{\varepsilon}\right)\right\} \\
& \quad = \frac{1}{4}\phi_1^{(2)}(0;x,t) - \frac{1}{8}\phi_1^{(2)}\left(\frac{x-pt}{2\sqrt{t}};x,t\right) \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8}\phi_2^{(2)}\left(\frac{x+pt}{2\sqrt{t}};x,t\right)\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\\
&+\sqrt{t\varepsilon}\left\{\frac{\phi_1^{(2)}(0;x,t)-\phi_1^{(2)}\left(\frac{x-pt}{2\sqrt{t}};x,t\right)}{4(x-pt)}\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)\right.\\
&\left.+\frac{\phi_2^{(2)}(0;x,t)-\phi_2^{(2)}\left(\frac{x+pt}{2\sqrt{t}};x,t\right)}{4(x+pt)}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&+\frac{\varepsilon}{8}\left\{I_1^{(3)}(x,t)-I_2^{(3)}(x,t)\exp\left(\frac{px}{\varepsilon}\right)\right\}
\end{aligned}
\tag{3.36}
$$

where we have

$$
\phi_1^{(2)}(0;x,t)=\lim_{\sigma\downarrow 0}\phi_1^{(2)}(\sigma;x,t),\qquad \phi_2^{(2)}(0;x,t)=\lim_{\sigma\downarrow 0}\phi_2^{(2)}(\sigma;x,t),
$$

and $I_1^{(3)}(x,t)$, $I_2^{(3)}(x,t)$ are defined by (3.30), (3.31), respectively, with $n=3$.

Next, from (3.15), we obtain

$$
\begin{aligned}
J_{34}^{(2)}(x,t)=&\frac{1}{2}\phi_3^{(2)}(t;x,t)\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\frac{1}{2}\phi_4^{(2)}(t;x,t)\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\\
&+\sqrt{t\varepsilon}\left\{\frac{\phi_3^{(2)}(t;x,t)-\phi_3^{(2)}(x/p;x,t)}{x-pt}\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)\right.\\
&-\frac{4}{p^2\sqrt{\pi}}\Phi_{34}^{(2)}(t;x,t)\exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right]\\
&\left.+\frac{\phi_4^{(2)}(t;x,t)-\phi_4^{(2)}(-x/p;x,t)}{x+pt}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&+\frac{2\varepsilon}{p^3}\Phi_{34}^{(2)}\left(\frac{x}{p};x,t\right)\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&+\varepsilon\, J_{34}^{(3)}(x,t)
\end{aligned}
\tag{3.37}
$$

where $\phi_3^{(2)}(s;x,t)$, $\phi_4^{(2)}(s;x,t)$, $\Phi_{34}^{(2)}(s;x,t)$ are given by (3.25), (3.26), (3.27) with $k=2$, and $J_{34}^{(3)}(x,t)$ by (3.32) with $n=3$, respectively.

It follows from (3.29) with $n=2$, (3.36), and (3.37) that

$$
u^{(2)}(x,t;\varepsilon)=u_4(x,t;\varepsilon)+\sqrt{\varepsilon}\,u_5(x,t;\varepsilon)+\varepsilon\,u^{(3)}(x,t;\varepsilon)
$$

where $u_4(x,t;\varepsilon)$ and $u_5(x,t;\varepsilon)$ are defined by (3.21) and (3.22), respectively, with $k=2$. The remainder $u^{(3)}(x,t;\varepsilon)$ is given by (3.29) with $n=3$. This third step can be continued as many times as necessary. □

4. Properties of the expansion

Note that the expansion (3.20) is valid not only in a neighborhood of $x=pt$ but also in the whole domain $x\geq 0$, $t\geq 0$ for all values of ε. More precisely, it is identical to the exact solution, and thus it can be differentiated as many times as one wishes. Moreover, it gives a uniform approximation to the given singularly perturbed problem for small values of ε since the function $u^{(n)}(x,t;\varepsilon)$ in the expansion is bounded uniformly in ε.

Theorem 4.1. *The solution $u(x,t)$ of the initial boundary value problem* (2.6), (2.7), (2.8) *has an expansion*

$$u(x,t) = \sum_{k=0}^{n-1} \varepsilon^k \left[u_{2k}(x,t;\varepsilon) + \sqrt{\varepsilon}\, u_{2k+1}(x,t;\varepsilon)\right] + \mathcal{O}(\varepsilon^n) \qquad \text{as } \varepsilon \downarrow 0,$$

which is uniformly valid in $x \geq 0$, $0 \leq t \leq T$ for some $T > 0$.

Proof. The function $u^{(n)}(x,t;\varepsilon)$ is bounded uniformly in ε due to the following facts:

(i) The function

$$\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right) \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)$$

is bounded uniformly in ε for all $x \geq 0$, $t \geq 0$.

(ii) By using the boundedness of $\phi_1^{(n)}$ and $\phi_2^{(n)}$, we have

$$\begin{aligned}
&\left|I_1^{(n)}(x,t) - I_2^{(n)}(x,t) \exp\left(\frac{px}{\varepsilon}\right)\right| \\
&\quad \leq \frac{1}{2} \max\left\{\left|\phi_1^{(n)}\right|, \left|\phi_2^{(n)}\right|\right\} \left\{\operatorname{erfc}\left(\frac{pt-x}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right) \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},
\end{aligned}$$

which is bounded uniformly in ε for all $x \geq 0$, $t \geq 0$.

(iii) The function $J_{34}^{(n)}(x,t)$ is bounded uniformly in ε for all $x \geq 0$, $0 \leq t \leq T$ since

$$\int_0^t \frac{s}{2\sqrt{\pi s^3 \varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds = \frac{1}{2p}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right) \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},$$

$$\begin{aligned}
\int_0^t \frac{s^2}{2\sqrt{\pi s^3 \varepsilon}} \exp\left[-\frac{(x-ps)^2}{4s\varepsilon}\right] ds &= \frac{px+2\varepsilon}{2p^3} \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad + \frac{px-2\varepsilon}{2p^3} \exp\left(\frac{px}{\varepsilon}\right) \operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right) \\
&\quad - \frac{2}{p^2}\sqrt{\frac{t\varepsilon}{\pi}} \exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right].
\end{aligned}$$

□

Theorem 4.2. *The solution $u(x,t)$ of the initial boundary value problem* (2.6), (2.7), (2.8) *is of the form*

$$u(x,t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \varepsilon^k \left[u_{2k}(x,t;\varepsilon) + \sqrt{\varepsilon}\, u_{2k+1}(x,t;\varepsilon)\right]$$

when $f(x)$ is a polynomial of degree n, and $g(t)$ is a polynomial of degree $\lfloor n/2 \rfloor + 1$. The symbol $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the number x.

Proof. The degree of $\phi_1^{(n)}(\sigma;x,t)$ in σ decreases by two after every step of the expansion. So does that of $\phi_2^{(n)}(\sigma;x,t)$ in σ. On the other hand, the degree of $\phi_{34}^{(1)}(\sigma;x,t)$ in σ drops by two from that of $g(t)$ in t. Moreover, the degree of $\phi_{34}^{(n)}(\sigma;x,t)$ in σ drops by one after each step of the expansion. □

As an example, consider the initial function

$$f(x) = a_r x^3 + b_r x^2 + c_r x + d_r, \tag{4.1}$$

and the boundary function

$$g(t) = a_\ell t^3 + b_\ell t^2 + c_\ell t + d_\ell. \tag{4.2}$$

We then obtain the following theorem.

Theorem 4.3. *The solution $u(x,t)$ of the initial boundary value problem* (2.6), (2.7), (2.8) *with* (4.1), (4.2) *can be expressed as*

$$u(x,t) = \sum_{k=0}^{1} \varepsilon^k \, v_k^r(x,t) + \sum_{k=0}^{5} \varepsilon^{k/2} \, w_k^r(x,t;\varepsilon) \tag{4.3}$$

where the functions are given by

$$v_0^r(x,t) = a_r(x-pt)^3 + b_r(x-pt)^2 + c_r(x-pt) + d_r, \tag{4.4}$$

$$v_1^r(x,t) = 2t[3a_r(x-pt) + b_r], \tag{4.5}$$

$$w_0^r(x,t;\varepsilon) = \frac{d_\ell - d_r}{2}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}, \tag{4.6}$$

$$w_1^r(x,t;\varepsilon) = \left(c_r + \frac{c_\ell}{p}\right)\sqrt{t}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}, \tag{4.7}$$

$$w_2^r(x,t;\varepsilon) = 4\left(\frac{b_\ell}{p^2} - b_r\right)t\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}, \tag{4.8}$$

$$\begin{aligned} w_3^r(x,t;\varepsilon) = {} & 24\left(\frac{a_\ell}{p^3} + a_r\right)t^{3/2}\left\{\mathrm{i}^3\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^3\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\ & - \frac{2}{p^3}b_\ell\sqrt{t}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}, \end{aligned} \tag{4.9}$$

$$w_4^r(x,t;\varepsilon) = -\frac{24}{p^4}a_\ell t\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}, \tag{4.10}$$

$$w_5^r(x,t;\varepsilon) = \frac{12}{p^5}a_\ell\sqrt{t}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}. \tag{4.11}$$

Proof. From $\phi_1^{(2)}(s;x,t) = 0$, $\phi_2^{(2)}(s;x,t) = 0$, and $\phi_{34}^{(3)}(s;x,t) = 0$, we have $u^{(3)}(x,t;\varepsilon) = 0$ and

$$u(x,t) = \sum_{k=0}^{2} \varepsilon^k \left[u_{2k}(x,t;\varepsilon) + \sqrt{\varepsilon}\, u_{2k+1}(x,t;\varepsilon)\right]$$

where the functions $u_{2k}(x,t;\varepsilon)$, $u_{2k+1}(x,t;\varepsilon)$ defined by (3.21), (3.22), respectively, become

$$\begin{aligned} u_0 = {} & a_r(x-pt)^3 + b_r(x-pt)^2 + c_r(x-pt) + d_r \\ & + \frac{d_\ell - d_r}{2}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}, \end{aligned}$$

$$\begin{aligned} u_1 = {} & \sqrt{t}\left\{a_r(x-pt)^2 + b_r(x-pt) + c_r + \frac{1}{p}\left[a_\ell\left(t-\frac{x}{p}\right)^2 + b_\ell\left(t-\frac{x}{p}\right) + c_\ell\right]\right\} \\ & \times \operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \sqrt{t}\left\{-a_r(x+pt)^2 + b_r(x+pt) - c_r\right. \\ & \left. - \frac{1}{p}\left[a_\ell\left(t+\frac{x}{p}\right)^2 + b_\ell\left(t+\frac{x}{p}\right) + c_\ell\right]\right\}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right), \end{aligned}$$

$$u_2 = 2t\left[3a_r(x-pt)+b_r\right] + \left\{-t\left[a_r(x-pt)+b_r\right] - \frac{x}{p^4}\left[2a_\ell(x-pt)-b_\ell p\right]\right\}$$
$$\times \operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \left\{t\left[a_r(x+pt)-b_r\right] - \frac{x}{p^4}\left[2a_\ell(x+pt)+b_\ell p\right]\right\}$$
$$\times \exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right),$$
$$u_3 = \sqrt{t}\left\{\left[4a_r t + \frac{2a_\ell x}{p^4}\right]\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \left[-4a_r t + \frac{2a_\ell x}{p^4}\right]\right.$$
$$\left.\times \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} + \frac{8a_\ell x\sqrt{t}}{p^4\sqrt{\pi}}\exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right],$$
$$u_4 = -\frac{6a_\ell x}{p^5}\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \frac{6a_\ell x}{p^5}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right),$$
$$u_5 = 0.$$

Then, we obtain

$$\sum_{k=0}^{5}\varepsilon^{k/2}\,u_k(x,t;\varepsilon) = \sum_{k=0}^{1}\varepsilon^k\,v_k^r(x,t) + \sum_{k=0}^{1}\varepsilon^{k/2}\,w_k^r(x,t;\varepsilon)$$
$$+\,\varepsilon\left\{b_r\,B_r(x,t;\varepsilon) + b_\ell\,B_\ell(x,t;\varepsilon)\right\}$$
$$+\,\varepsilon^{3/2}\left\{a_r\,A_r(x,t;\varepsilon) + a_\ell\,A_\ell(x,t;\varepsilon)\right\}$$

where $v_0^r(x,t)$, $v_1^r(x,t)$, $w_0^r(x,t;\varepsilon)$, and $w_1^r(x,t;\varepsilon)$ are defined by (4.4), (4.5), (4.6), and (4.7), respectively, with

$$B_r = \sqrt{t}\left\{\frac{x-pt}{\sqrt{\varepsilon}}\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \frac{x+pt}{\sqrt{\varepsilon}}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$-\,t\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},$$
$$B_\ell = \frac{1}{p^2}\sqrt{t}\left\{\frac{pt-x}{\sqrt{\varepsilon}}\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \frac{x+pt}{\sqrt{\varepsilon}}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$+\,\frac{x}{p^3}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},$$
$$A_r = \sqrt{t}\left\{\frac{(x-pt)^2}{\varepsilon}\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \frac{(x+pt)^2}{\varepsilon}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$+\,t\left\{\frac{pt-x}{\sqrt{\varepsilon}}\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \frac{x+pt}{\varepsilon}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$+\,4t^{3/2}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},$$
$$A_\ell = \frac{\sqrt{t}}{p^3}\left\{\frac{(pt-x)^2}{\varepsilon}\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \frac{(x+pt)^2}{\varepsilon}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$+\,\frac{2x}{p^4}\left\{\frac{pt-x}{\sqrt{\varepsilon}}\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \frac{x+pt}{\sqrt{\varepsilon}}\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$+\,\frac{2x\sqrt{t}}{p^4}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$

$$+\frac{8x\sqrt{t}}{p^4\sqrt{\pi}}\exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right]$$
$$-\frac{6x\sqrt{\varepsilon}}{p^5}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}.$$

By using the identities

$$\frac{pt-x}{4\sqrt{t\varepsilon}}\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)=\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\frac{1}{4}\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right), \tag{4.12}$$
$$\frac{x+pt}{4\sqrt{t\varepsilon}}\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)=-\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)+\frac{1}{4}\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right), \tag{4.13}$$

we obtain

$$B_r=-4t\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},$$
$$B_\ell=\frac{4t}{p^2}\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$-\frac{1}{p^3}\left\{(pt-x)\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+(x+pt)\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},$$
$$A_r=4t\left\{\frac{pt-x}{\sqrt{\varepsilon}}\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\frac{x+pt}{\sqrt{\varepsilon}}\exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$+4t^{3/2}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},$$
$$A_\ell=\frac{4t}{p^3}\left\{\frac{pt-x}{\sqrt{\varepsilon}}\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\frac{x+pt}{\sqrt{\varepsilon}}\exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$+\left\{\left(\frac{2x}{p^4}-\frac{t}{p^3}\right)\frac{pt-x}{\sqrt{\varepsilon}}\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\left(\frac{2x}{p^4}+\frac{t}{p^3}\right)\frac{x+pt}{\sqrt{\varepsilon}}\exp\left(\frac{px}{\varepsilon}\right)\right.$$
$$\left.\times\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}+\frac{2x\sqrt{t}}{p^4}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$+\frac{8x\sqrt{t}}{p^4\sqrt{\pi}}\exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right]-\frac{6x\sqrt{\varepsilon}}{p^5}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}.$$

Moreover, with the use of the identities

$$\frac{pt-x}{2\sqrt{t\varepsilon}}\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)=\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\frac{1}{\sqrt{\pi}}\exp\left[-\frac{(x-pt)^2}{4t\varepsilon}\right], \tag{4.14}$$
$$\frac{x+pt}{2\sqrt{t\varepsilon}}\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)=-\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)+\frac{1}{\sqrt{\pi}}\exp\left[-\frac{(x+pt)^2}{4t\varepsilon}\right], \tag{4.15}$$

we have

$$B_\ell=\frac{4t}{p^2}\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}$$
$$-\frac{2}{p^3}\sqrt{t\varepsilon}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},$$

$$
\begin{aligned}
A_\ell = \frac{4t}{p^3}&\left\{\frac{pt-x}{\sqrt{\varepsilon}}\,\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\frac{x+pt}{\sqrt{\varepsilon}}\,\exp\left(\frac{px}{\varepsilon}\right)\,\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&+\frac{2t^{3/2}}{p^3}\left\{-\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\exp\left(\frac{px}{\varepsilon}\right)\,\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&+\frac{6x\sqrt{t}}{p^4}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\exp\left(\frac{px}{\varepsilon}\right)\,\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&-\frac{6x\sqrt{\varepsilon}}{p^5}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\,\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}.
\end{aligned}
$$

Finally, with the identities

$$
\begin{aligned}
\mathrm{i}^3\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)&=\frac{1}{6}\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\frac{x-pt}{6\sqrt{t\varepsilon}}\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right),\\
\mathrm{i}^3\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)&=\frac{1}{6}\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)-\frac{x+pt}{6\sqrt{t\varepsilon}}\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right),
\end{aligned}
$$

we get

$$
\begin{aligned}
A_r &= 24t^{3/2}\left\{\mathrm{i}^3\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\,\mathrm{i}^3\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},\\
A_\ell &= \frac{24t^{3/2}}{p^3}\left\{\mathrm{i}^3\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\,\mathrm{i}^3\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&\quad+\frac{6\sqrt{t}}{p^4}\left\{(x-pt)\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+(x+pt)\exp\left(\frac{px}{\varepsilon}\right)\,\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&\quad-\frac{6x\sqrt{\varepsilon}}{p^5}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\,\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}.
\end{aligned}
$$

By virtue of (4.12), (4.13), we have

$$
\begin{aligned}
A_\ell &= \frac{24t^{3/2}}{p^3}\left\{\mathrm{i}^3\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\,\mathrm{i}^3\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&\quad-\frac{24t\sqrt{\varepsilon}}{p^4}\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\exp\left(\frac{px}{\varepsilon}\right)\,\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&\quad+\frac{6\sqrt{\varepsilon}}{p^5}\left\{(pt-x)\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+(x+pt)\exp\left(\frac{px}{\varepsilon}\right)\,\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}.
\end{aligned}
$$

One more substitution with (4.14), (4.15) gives

$$
\begin{aligned}
A_\ell = \frac{24t^{3/2}}{p^3}&\left\{\mathrm{i}^3\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)-\exp\left(\frac{px}{\varepsilon}\right)\,\mathrm{i}^3\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&-\frac{24t\sqrt{\varepsilon}}{p^4}\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\exp\left(\frac{px}{\varepsilon}\right)\,\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}\\
&+\frac{12\varepsilon\sqrt{t}}{p^5}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right)+\exp\left(\frac{px}{\varepsilon}\right)\,\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}.
\end{aligned}
$$

It follows that the constructed expansion is given by (4.3). □

Note that the function (4.3) is also checked by Maple V release 3 to be the exact solution of the singularly perturbed problem (2.6), (2.7), (2.8) with (4.1), (4.2). Thus the contribution of the initial data to the exact solution is

$$\begin{aligned}
u_i = {} & a_r(x-pt)^3 + b_r(x-pt)^2 + c_r(x-pt) + d_r + 2t\varepsilon\left[3a_r(x-pt)+b_r\right] \\
& - \frac{d_r}{2}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& + c_r\sqrt{t\varepsilon}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& - 4b_r t\varepsilon\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& + 24a_r(t\varepsilon)^{3/2}\left\{\mathrm{i}^3\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^3\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\},
\end{aligned}$$

while the contribution of the boundary data to the exact solution is

$$\begin{aligned}
u_b = {} & \frac{d_\ell}{2}\left\{\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& + \frac{c_\ell}{p}\sqrt{t\varepsilon}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& + 4\frac{b_\ell}{p^2}t\varepsilon\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& - \frac{2}{p^3}b_\ell\varepsilon\sqrt{t\varepsilon}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& + 24\frac{a_\ell}{p^3}(t\varepsilon)^{3/2}\left\{\mathrm{i}^3\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^3\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& - \frac{24}{p^4}a_\ell t\varepsilon^2\left\{\mathrm{i}^2\operatorname{erfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) + \exp\left(\frac{px}{\varepsilon}\right)\mathrm{i}^2\operatorname{erfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\} \\
& + \frac{12}{p^5}a_\ell\varepsilon^2\sqrt{t\varepsilon}\left\{\operatorname{ierfc}\left(\frac{x-pt}{2\sqrt{t\varepsilon}}\right) - \exp\left(\frac{px}{\varepsilon}\right)\operatorname{ierfc}\left(\frac{x+pt}{2\sqrt{t\varepsilon}}\right)\right\}.
\end{aligned}$$

We then conclude that the internal layer structure of the initial boundary value problem is more complicated than that of the initial value problem. Contrary to the initial value problem studied in Shih [13], we are not aware of any method of matched asymptotic expansions which is able to construct the internal layer function $w_0^r(x,t;\varepsilon)$ or $w_1^r(x,t;\varepsilon)$ *precisely*. For example, using the variables (ξ, t) with the stretched variable $\xi = (x-pt)/\sqrt{\varepsilon}$ to replace the given independent variables (x,t) in the internal layer region, one finds by using (1.1) that

$$w_0^r(x,t;\varepsilon) = \frac{d_\ell - d_r}{2}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{t}}\right) + \mathcal{O}(\sqrt{\varepsilon})$$

in a neighborhood of $x = pt$ for all $t > 0$.

On the other hand, the constructed solution gives rise to a close relationship between internal layer functions and corner singularity conditions for the reduced problem. For example, the first layer function $w_0^r(x,t;\varepsilon)$ is nonzero if $f(0) \neq g(0)$, while the second layer function $w_1^r(x,t;\varepsilon)$ is nonzero if $g'(0) + pf'(0) \neq 0$.

5. Historical survey

Two papers related to this work are Howes [9] and Joseph [10]. In studying the development of boundary layers for the linear singularly perturbed parabolic system

$$\frac{\partial u}{\partial t} + A(x,t)\,\frac{\partial u}{\partial x} = \varepsilon\,\frac{\partial^2 u}{\partial x^2},$$

Joseph first considered the scalar equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \varepsilon\,\frac{\partial^2 u}{\partial x^2} \tag{5.1}$$

in the first quadrant $x > 0$, $t > 0$ with a parameter ε satisfying $0 < \varepsilon \ll 1$. Equation (5.1) is subject to the auxiliary conditions

$$u(x,0) = 0, \qquad u(0,t) = u_b(t), \tag{5.2}$$

with a smooth function $u_b(t)$ satisfying $u_b(0) = 0$. Then, the outer function $u^0(x,t)$ is defined by (5.1) with $\varepsilon = 0$ and (5.2). It is claimed that the relation

$$u(x,t) = u^0(x,t) + \mathcal{O}(\varepsilon)$$

is *uniformly* valid in $x \geq 0$, $0 \leq t \leq T$ for some $T > 0$. It is clear that this result is inconsistent with ours of adding an internal layer function of order $\mathcal{O}(\sqrt{\varepsilon})$ to the outer function $u^0(x,t)$. For instance, from (4.3) and (1.3), the exact solution of the problem (5.1), (5.2) with $u_b(t) = t$ is

$$u(x,t) = u^0(x,t) + \sqrt{\varepsilon}\, w_1(x,t;\varepsilon),$$

with

$$u^0(x,t) = \begin{cases} t - x, & x < t, \\ 0, & x > t, \end{cases}$$

$$w_1(x,t;\varepsilon) = \begin{cases} \sqrt{t}\left\{ \operatorname{ierfc}\left(\dfrac{t-x}{2\sqrt{t\varepsilon}}\right) - \exp\left(\dfrac{x}{\varepsilon}\right)\, \operatorname{ierfc}\left(\dfrac{x+t}{2\sqrt{t\varepsilon}}\right)\right\}, & x < t, \\ \sqrt{t}\left\{ \operatorname{ierfc}\left(\dfrac{x-t}{2\sqrt{t\varepsilon}}\right) - \exp\left(\dfrac{x}{\varepsilon}\right)\, \operatorname{ierfc}\left(\dfrac{x+t}{2\sqrt{t\varepsilon}}\right)\right\}, & x > t, \end{cases}$$

and $w_1(x,t;\varepsilon) = \mathcal{O}(1)$ in $x \geq 0$, $0 \leq t \leq T$ as $\varepsilon \downarrow 0$.

Howes studied the internal layer behavior for the parabolic equation (5.1) defined in $0 < x < 1, 0 < t < T$ subject to the initial condition

$$u(x,0) = \varphi(x) \tag{5.3}$$

for $0 \leq x \leq 1$ and the boundary conditions

$$u(0,t) = A(t), \qquad u(1,t) = B(t), \tag{5.4}$$

for $0 \leq t \leq T$ with smooth functions $\varphi(x)$, $A(t)$, $B(t)$ satisfying $A(0) = \varphi(0)$. Under the assumption $\varphi'(0) + A'(0) \neq 0$, Howes found that the internal layer function of this problem is of the order

$$\sqrt{\varepsilon}\,|\varphi'(0) + A'(0)|\, \exp\left(-\frac{|x-t|}{\sqrt{\varepsilon}}\right).$$

As a comparison, it is found in the present work that the quarter-plane problem (5.1), (5.3), (5.4) satisfying $A(0) = \varphi(0)$ has the dominant internal layer function

$$\sqrt{t\varepsilon}\, \left\{ c_1(x,t)\, \operatorname{ierfc}\left(\frac{x-t}{2\sqrt{t\varepsilon}}\right) + c_2(x,t)\, \exp\left(\frac{x}{\varepsilon}\right)\, \operatorname{ierfc}\left(\frac{x+t}{2\sqrt{t\varepsilon}}\right)\right\}$$

with

$$c_1(x,t) = \frac{\varphi(x-t) - \varphi(0) + A(0) - A(t-x)}{x-t},$$
$$c_2(x,t) = \frac{\varphi(-x-t) - \varphi(0) + A(0) - A(t+x)}{x+t}.$$

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