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A SINGULARLY PERTURBED SEMILINEAR SYSTEM

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ABSTRACT. A constructive existence proof is given for solutions of boundary-layer type for the singularly perturbed semilinear system $\varepsilon^2 d^2x/dt^2 = H(t, x, \varepsilon)$ subject to either Dirichlet or general Robin boundary conditions. The required assumptions involve only natural conditions that are induced by the O'Malley construction.

1. Introduction

We consider the following second-order system

$$\varepsilon^2 \frac{d^2x}{dt^2} = H(t, x, \varepsilon) \tag{1.1}$$

for solutions $x = x(t, \varepsilon)$ on the interval $0 < t < 1$ for small values of ε ($\varepsilon \to 0^+$) subject either to the Dirichlet boundary conditions

$$x(0, \varepsilon) = \alpha(\varepsilon), \qquad x(1, \varepsilon) = \beta(\varepsilon) \tag{1.2}$$

where x, α, β, and H are n-dimensional real vector-valued functions, or to the general Robin boundary conditions

$$B\big(x(0, \varepsilon),\ x(1, \varepsilon),\ x'(0, \varepsilon),\ x'(1, \varepsilon),\ \varepsilon\big) = 0 \tag{1.3}$$

where B is a given $2n$-dimensional vector-valued function of $x(0, \varepsilon)$, $x(1, \varepsilon)$, $x'(0, \varepsilon)$, $x'(1, \varepsilon)$, and ε.

The scalar case ($n = 1$) of the problem (1.1)–(1.2) has been considered by many authors including Brish [2], Vasil'eva and Tupčiev [23], Boglaev [1], Vasil'eva and Butuzov [22], Fife [4, 5], Yarmish [24], Smith [20, 21], O'Malley [19], Howes [8–10], van Harten [6, 7], Chang and Howes [3], and others. The vector case has been considered in Kelley [13, 14], Howes and O'Malley [11], and O'Donnell [16]. O'Donnell [16] assumes that H has a special structure which permits the system to be decoupled, and then the scalar theory can be applied to each component of the system. Kelley [13, 14] and Chang and Howes [3] assume stability conditions which imply, in particular, that all the eigenvalues of $H_x(t, X_0(t), 0)$ have positive real parts for $0 \leq t \leq 1$ for a suitable outer solution $X_0(t)$.

The scalar case ($n = 1$) of the problem (1.1)–(1.3) is considered in O'Malley [17, 18] and van Harten [6]. The vector problem is considered in Chang and Howes [3]; however, their assumptions impose certain restrictive conditions on the structure of H and B, and spatial coupling of the boundary conditions is excluded.

For both the Dirichlet and Robin problems, we use the O'Malley construction to obtain an approximate solution; then we linearize the original problem about the proposed approximate solution and apply the Banach-Picard fixed point theorem to prove the existence of a locally unique exact solution along with error estimates between the

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exact solution and the approximate solution. We employ natural conditions induced by the O'Malley construction. The matrix $H_x(t, X_0(t), 0)$ is assumed to be nonsingular and its eigenvalues are excluded from lying on the negative real axis for a suitable outer solution $X_0(t)$. This requirement is necessary so as to exclude strictly oscillatory solutions to the associated linearized system and thus allow the construction of an approximate solution that exhibits boundary-layer behavior (see equations (4.1), (5.9), (5.13), and (6.5)).

Sections 2 and 3 contain discussions of our assumptions for the Dirichlet and Robin problems, respectively. Section 4 contains a proof of Lemma 1 which is needed to construct certain fundamental solutions satisfying appropriate exponential dichotomies. The approximate solutions for the Dirichlet and Robin problems are constructed in Sections 5 and 6, respectively. Section 7 contains the statements and proofs of existence and local uniqueness. Examples are provided in Section 8.

2. Assumptions for the Dirichlet problem

Assumption D1. There exists a continuous solution $X_0(t)$ to the reduced equation

$$H(t, X_0(t), 0) = 0 \tag{2.1}$$

such that the $n \times n$ matrix $H_x(t, X_0(t), 0)$ is nonsingular and its eigenvalues do not lie on the negative real axis.

Assumption D2. There exist decaying solutions $\hat{X}_0$ and $\tilde{X}_0$ to the boundary-layer differential equations

$$\begin{aligned} \frac{d^2 \hat{X}_0}{d\tau^2} &= H(0, X_0(0) + \hat{X}_0(\tau), 0), \\ \hat{X}_0(0) &= \alpha(0) - X_0(0), \end{aligned} \tag{2.2}$$

$$\begin{aligned} \frac{d^2 \tilde{X}_0}{d\sigma^2} &= H(1, X_0(1) + \tilde{X}_0(\sigma), 0), \\ \tilde{X}_0(0) &= \beta(0) - X_0(1). \end{aligned} \tag{2.3}$$

For our next assumption, we consider the following two linear systems

$$\frac{d}{d\tau}\hat{\xi} = \begin{pmatrix} 0 & I_n \\ H_x(0, X_0(0) + \hat{X}_0(\tau), 0) & 0 \end{pmatrix} \hat{\xi}, \tag{2.4}$$

$$\frac{d}{d\sigma}\tilde{\xi} = -\begin{pmatrix} 0 & I_n \\ H_x(1, X_0(1) + \tilde{X}_0(\sigma), 0) & 0 \end{pmatrix} \tilde{\xi}. \tag{2.5}$$

It follows from Assumption D1, Lemma 1 (see Section 4), and Lemma 6.1 of Jeffries and Smith [12] that there exist fundamental solutions $\hat{\xi}(\tau)$ and $\tilde{\xi}(\sigma)$ satisfying the exponential dichotomies

$$\begin{aligned} \left|\hat{\xi}(\tau) P \hat{\xi}^{-1}(u)\right| &\le K e^{-\nu(\tau - u)} \qquad u \le \tau, \\ \left|\hat{\xi}(\tau)(I - P)\hat{\xi}^{-1}(u)\right| &\le K e^{-\nu(u - \tau)} \qquad \tau \le u, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \left|\tilde{\xi}(\sigma)(I - P)\tilde{\xi}^{-1}(u)\right| &\le K e^{-\nu(\sigma - u)} \qquad u \le \sigma, \\ \left|\tilde{\xi}(\sigma) P \tilde{\xi}^{-1}(u)\right| &\le K e^{-\nu(u - \sigma)} \qquad \sigma \le u \end{aligned} \tag{2.7}$$

where K and ν are positive constants and $P := \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$. Given two such fundamental solutions, we make the following assumption.

Assumption D3. The columns of $(I_n\ 0)\hat{\xi}(0)P\hat{\xi}^{-1}(0)$ span R^n, and the columns of $(I_n\ 0)\tilde{\xi}(0)(I-P)\tilde{\xi}^{-1}(0)$ span R^n.

Note that this assumption is independent of the particular choice made for the fundamental solutions $\hat{\eta}$ and $\tilde{\eta}$, as long as they satisfy the corresponding exponential dichotomies (2.6) and (2.7).

Assumption D4. There exist positive constants ε_1 and δ_1 such that for $0 < \varepsilon \leq \varepsilon_1$, $H(t,x,\varepsilon)$ is of class C^{N+1} with respect to (t,x) on $\mathcal{N}$

$$\mathcal{N} := \left\{(t,x) : 0 \leq t \leq 1,\ \left|x - (X_0(t) + \hat{X}_0(\tfrac{t}{\varepsilon}) + \tilde{X}_0(\tfrac{1-t}{\varepsilon}))\right| \leq \delta_1\right\}, \tag{2.8}$$

and its derivatives are uniformly bounded. Furthermore, we assume that $H(t,x,\varepsilon)$, $\alpha(\varepsilon)$, and $\beta(\varepsilon)$ possess expansions in ε of the form

$$\begin{pmatrix} H(t,x,\varepsilon) \\ \alpha(\varepsilon) \\ \beta(\varepsilon) \end{pmatrix} = \sum_{k=0}^{N} \begin{pmatrix} H_k(t,x) \\ \alpha_k \\ \beta_k \end{pmatrix} \varepsilon^k + \begin{pmatrix} H_{N+1}(t,x,\varepsilon) \\ \alpha_{N+1}(\varepsilon) \\ \beta_{N+1}(\varepsilon) \end{pmatrix} \varepsilon^{N+1} \tag{2.9}$$

where the coefficient functions $H_k(t,x)$ are of class C^{N-k+1}, and $H_{N+1}(t,x,\varepsilon)$, $\alpha_{N+1}(\varepsilon)$, $\beta_{N+1}(\varepsilon)$ are uniformly bounded.

3. Assumptions for the Robin problem

Letting q, r, s, z represent the boundary values $x(0,\varepsilon)$, $x(1,\varepsilon)$, $x'(0,\varepsilon)$, $x'(1,\varepsilon)$, respectively, we may regard B as a function of q, r, s, z, and ε. As an example, consider the following set of boundary conditions ($n = 2$)

$$\begin{aligned} x_1(0,\varepsilon) - x_1'(0,\varepsilon) &= 2, \\ x_2(1,\varepsilon) + x_2'(0,\varepsilon) &= 3, \\ x_1(1,\varepsilon) + x_1'(1,\varepsilon) &= 1, \\ x_2(0,\varepsilon) - x_2'(1,\varepsilon) &= -1. \end{aligned} \tag{3.1}$$

In this case, $B = B(q,r,s,z,\varepsilon)$ would have the form

$$B(q,r,s,z,\varepsilon) = \begin{bmatrix} q_1 - s_1 - 2 \\ r_2 + s_2 - 3 \\ r_1 + z_1 - 1 \\ q_2 - z_2 + 1 \end{bmatrix}. \tag{3.2}$$

Note that spatial coupling of the boundary values is allowed. We now are ready to state our assumptions.

Assumption R1. Same as Assumption D1.

Assumption R2. There exist n-dimensional vectors α_0 and β_0 such that

$$B(X_0(0), X_0(1), \alpha_0, \beta_0, 0) = 0, \tag{3.3}$$

and the following $2n \times 2n$ matrix is nonsingular

$$\left[B_s(X_0(0), X_0(1), \alpha_0, \beta_0, 0)\ \middle|\ B_z(X_0(0), X_0(1), \alpha_0, \beta_0, 0)\right]. \tag{3.4}$$

Assumption R3. There exist positive constants ε_2 and δ_2 such that for $0 < \varepsilon \leq \varepsilon_2$, the given data function $H(t,x,\varepsilon)$ is of class C^{N+1}, $N \geq 2$, with respect to (t,x) on $\mathcal{N}_1$

$$\mathcal{N}_1 := \{(t,x) : 0 \leq t \leq 1,\ |x - X_0(t)| \leq \delta_2\}, \tag{3.5}$$

B is of class C^{N+1} with respect to (q,r,s,z) on $\mathcal{N}_2$

$$\begin{aligned}\mathcal{N}_2 := \big\{(q,r,s,z) :\ &|q - X_0(0)| \leq \delta_2,\ |r - X_0(1)| \leq \delta_2,\\ &|s - \alpha_0| \leq \delta_2,\ |z - \beta_0| \leq \delta_2\big\},\end{aligned} \tag{3.6}$$

and the derivatives of H and B are uniformly bounded. Furthermore, we assume that H and B possess an asymptotic expansion in ε of the form

$$\begin{pmatrix} H(t,x,\varepsilon) \\ B(q,r,s,z,\varepsilon) \end{pmatrix} = \sum_{k=0}^{N} \begin{pmatrix} H_k(t,x) \\ B_k(q,r,s,z) \end{pmatrix} \varepsilon^k + \begin{pmatrix} H_{N+1}(t,x,\varepsilon) \\ B_{N+1}(q,r,s,z,\varepsilon) \end{pmatrix} \varepsilon^{N+1} \tag{3.7}$$

where the coefficient functions H_k and B_k are of class C^{N+1-k}, and H_{N+1} and B_{N+1} are uniformly bounded.

4. Lemma 1

Lemma 1. *There exist fundamental solutions $\hat{\eta}$ and $\tilde{\eta}$ to the following linear systems*

$$\frac{d}{d\tau}\hat{\eta} = \begin{pmatrix} 0 & I_n \\ H_{0,x}(0, X_0(0)) & 0 \end{pmatrix} \hat{\eta}, \tag{4.1}$$

$$\frac{d}{d\sigma}\tilde{\eta} = \begin{pmatrix} 0 & -I_n \\ -H_{0,x}(1, X_0(1)) & 0 \end{pmatrix} \tilde{\eta},$$

satisfying the exponential dichotomies

$$\begin{aligned} |\hat{\eta}(\tau)P\hat{\eta}^{-1}(u)| &\leq Ke^{-\nu(\tau-u)} \qquad \tau \geq u,\\ |\hat{\eta}(\tau)(I-P)\hat{\eta}^{-1}(u)| &\leq Ke^{-\nu(u-\tau)} \qquad u \geq \tau, \end{aligned} \tag{4.2}$$

$$\begin{aligned} |\tilde{\eta}(\sigma)(I-P)\tilde{\eta}^{-1}(u)| &\leq Ke^{-\nu(\sigma-u)} \qquad \sigma \geq u,\\ |\tilde{\eta}(\sigma)P\tilde{\eta}^{-1}(u)| &\leq Ke^{-\nu(u-\sigma)} \qquad u \geq \sigma \end{aligned} \tag{4.3}$$

where K and ν are positive constants and $P = \left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Furthermore, the columns of each of the following two matrices span R^n

$$\begin{pmatrix} 0 & I_n \end{pmatrix} P_1, \qquad \begin{pmatrix} 0 & I_n \end{pmatrix} P_2 \tag{4.4}$$

where $P_1 := \hat{\eta}(0)P\hat{\eta}^{-1}(0)$ and $P_2 := \tilde{\eta}(0)(I-P)\tilde{\eta}^{-1}(0)$.

Proof. We first consider $\hat{\eta}$. Since $H_{0,x}(0, X_0(0))$ is nonsingular and its eigenvalues do not lie on the negative real axis, they may be expressed as $\lambda_1^2, \ldots, \lambda_n^2$ where $Re(\lambda_i) > 0$ for $i = 1, \ldots, n$. Let S transform $H_{0,x}(0, X_0(0))$ into its Jordan canonical form, i.e., $S^{-1}H_{0,x}(0, X_0(0))S = J =$ diagonal $\{J_1, J_2, \ldots, J_r\}$ where J_i is of size $m_i \times m_i$ and has the form

$$J_i = \begin{pmatrix} \lambda_{q_i}^2 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_{q_i}^2 \end{pmatrix} \tag{4.5}$$

where $q_i = m_1 + \cdots + m_{i-1} + 1$. It follows that if S^j is the j^{th} column of S then

$$HS^1 = \lambda^2 S^1,$$
$$HS^j = \lambda^2 S^j + S^{j-1} \quad \text{for } j = 2, \ldots, m_1 \tag{4.6}$$

where, for notational purposes, we have set $H = H_{0,x}(0, X_0(0))$ and $\lambda^2 = \lambda^2_{q_1}$. We will now construct an eigenvector V^1 with eigenvalue $-\lambda$ and $m_1 - 1$ general eigenvectors such that

$$\begin{pmatrix} 0 & I_n \\ H & 0 \end{pmatrix} V^1 = -\lambda V^1,$$
$$\begin{pmatrix} 0 & I_n \\ H & 0 \end{pmatrix} V^j = -\lambda V^j + V^{j-1} \quad \text{for } j = 2, \ldots, m_1. \tag{4.7}$$

Defining V^1 and V^2 by

$$V^1 := \begin{pmatrix} S^1 \\ (-\lambda)S^1 \end{pmatrix} \text{ and } V^2 := \begin{pmatrix} 2(-\lambda)S^2 \\ 2(-\lambda)^2 S^2 + S^1 \end{pmatrix}, \tag{4.8}$$

it can be verified easily that they satisfy (4.8). To construct the remaining general eigenvectors, we assume that we have constructed eigenvectors V^l for $l = 2, \ldots, k$ such that

$$V^l = \begin{pmatrix} \sum_{i=1}^{l} C_{l,i}(-\lambda)^{i-1} V^i \\ \sum_{i=1}^{l} D_{l,i}(-\lambda)^i V^i \end{pmatrix} \tag{4.9}$$

and $C_{l,l} = D_{l,l} > 0$. We now will show that we can construct V^{k+1} with the same properties. Let V^{k+1} have the following form where the coefficients $C_{k+1,i}$ are to be determined

$$V^{k+1} = \begin{pmatrix} \sum_{i=1}^{k+1} C_{k+1,i}(-\lambda)^{i-1} V^i \\ \sum_{i=1}^{k+1} C_{k+1,i}(-\lambda)^i V^i + \sum_{i=1}^{k} C_{k,i}(-\lambda)^{i-1} V^i \end{pmatrix}. \tag{4.10}$$

It follows that V^{k+1} will satisfy

$$\begin{pmatrix} 0 & I_n \\ H & 0 \end{pmatrix} V^{k+1} = -\lambda V^{k+1} + V^k \tag{4.11}$$

provided

$$\sum_{i=2}^{k+1} C_{k+1,i}(-\lambda)^{i-1} V^{i-1} = \sum_{i=1}^{k} [C_{k,i} + D_{k,i}](-\lambda)^i V^i. \tag{4.12}$$

Re-indexing the sum on the left, we have

$$\sum_{i=1}^{k} C_{k+1,i+1}(-\lambda)^i V^i = \sum_{i=1}^{k} [C_{k,i} + D_{k,i}](-\lambda)^i V^i. \tag{4.13}$$

The above equation can be solved, and we find that $C_{k+1,k+1} = C_{k,k} + D_{k,k}$. By induction, we have $C_{k+1,k+1} = 2C_{k,k} > 0$. These vectors generate m_1 independent exponentially decaying solutions $U_1 Y_1(\tau)$ where $U_1 := \left[V^1, \ldots, V^{m_1}\right]$ and

$$Y_1(\tau) = e^{-\lambda_{q_1}\tau} \begin{pmatrix} 1 & \tau & & \frac{\tau^{m_1-1}}{(m_1-1)!} \\ & \ddots & \ddots & \\ & & \ddots & \tau \\ & & & 1 \end{pmatrix}. \tag{4.14}$$

Note that span $\left\{\left(0\ I_n\right)\left[V^1, \ldots, V^{m_1}\right]\right\}$ = span $\left\{S^1, \ldots, S^{m_1}\right\}$. In a strictly analogous fashion, we can construct an eigenvector $\bar{V}^1$ and $m_1 - 1$ general eigenvectors $\bar{V}^j$ corresponding to the eigenvalue λ_{q_1}. These generate m_1 exponentially increasing solutions $\bar{U}_1 \bar{Y}_1(\tau)$ where $\bar{U}_1 := \left[\bar{V}^1, \ldots, \bar{V}^{m_1}\right]$ and

$$\bar{Y}_1(\tau) = e^{\lambda_{q_1}\tau} \begin{pmatrix} 1 & \tau & & \frac{\tau^{m1-1}}{(m_1-1)!} \\ & \ddots & \ddots & \\ & & \ddots & \tau \\ & & & 1 \end{pmatrix} \tag{4.15}$$

such that span $\left\{\left(0\ I_n\right)\left[\bar{V}^1, \ldots, \bar{V}^{m_1}\right]\right\}$ = span $\left\{S^1, \ldots, S^{m_1}\right\}$. This can be done for each block J_i to produce m_i exponentially decreasing solutions $U_i Y_i(\tau)$ where $U_i := \left[V^{q_i}, \ldots, V^{q_{i+1}-1}\right]$ and

$$Y_i(\tau) = e^{-\lambda_{q_i}\tau} \begin{pmatrix} 1 & \tau & & \frac{\tau^{m_i-1}}{(m_i-1)!} \\ & \ddots & \ddots & \\ & & \ddots & \tau \\ & & & 1 \end{pmatrix} \tag{4.16}$$

such that span $\left\{\left(0\ I_n\right)\left[V^{q_i}, \ldots, V^{q_{i+1}-1}\right]\right\}$ = span $\left\{\left[S^{q_i}, \ldots, S^{q_{i+1}-1}\right]\right\}$, and m_i exponentially increasing solutions $\bar{U}_i \bar{Y}_i(\tau)$ where $\bar{U}_i := \left[\bar{V}^{q_i}, \ldots, \bar{V}^{q_{i+1}-1}\right]$ and

$$\bar{Y}_i(\tau) = e^{\lambda_{q_i}\tau} \begin{pmatrix} 1 & \tau & & \frac{\tau^{m_i-1}}{(m_i-1)!} \\ & \ddots & \ddots & \\ & & \ddots & \tau \\ & & & 1 \end{pmatrix} \tag{4.17}$$

such that span$\left\{\left(0\ I_n\right)\left[\bar{V}^{q_i}, \ldots, \bar{V}^{q_{i+1}-1}\right]\right\}$ = span $\left\{\left[S^{q_i}, \ldots, S^{q_{i+1}-1}\right]\right\}$. Setting

$$\hat{\eta}(\tau) = \left[U_1, \ldots, U_r \,|\bar{U}_1, \ldots, \bar{U}_r\right] \text{diagonal}\left\{Y_1(\tau), \ldots, Y_r(\tau)\,|\bar{Y}_1(\tau), \ldots, \bar{Y}_r(\tau)\right\}, \tag{4.18}$$

it follows that the fundamental solution $\hat{\eta}$ satisfies an exponential dichotomy and, since

$$\text{span}\left\{\left(0\ I_n\right)\hat{\eta}(0)P\hat{\eta}^{-1}(0)\right\} = \text{span}\left\{\left(0\ I_n\right)\left[U_1, \ldots, U_r\right]\right\} = \text{span}\left\{S\right\}, \tag{4.19}$$

the columns of $\begin{pmatrix}0 & I_n\end{pmatrix}\hat{\eta}(0)P\hat{\eta}^{-1}(0)$ span R^n.

For $\tilde{\eta}$, we construct a fundamental solution Q to the linear system

$$\frac{d}{d\tau}Q = \begin{pmatrix} 0 & I_n \\ H_{0,x}(1, X_0(1)) & 0 \end{pmatrix} Q, \tag{4.20}$$

just as we did for $\hat{\eta}$, such that

$$\begin{aligned} |Q(\sigma)PQ^{-1}(u)| &\le Ke^{-\nu(\sigma-u)} & \sigma \ge u, \\ |Q(\sigma)(I-P)Q^{-1}(u)| &\le Ke^{-\nu(u-\sigma)} & u \ge \sigma, \end{aligned} \tag{4.21}$$

and the columns of $\begin{pmatrix}0 & I_n\end{pmatrix}Q(0)(I-P)Q^{-1}(0)$ span R^n. It follows that $\tilde{\eta}(\sigma) := Q(-\sigma)$ satisfies (4.2) and (4.3). □

5. The approximate solution for the Dirichlet problem

In this section, we construct an approximate solution to the problem (1.1)–(1.2) using the O'Malley construction. We write the approximate solution $X^N(t,\varepsilon)$ as the sum of an outer solution and boundary-layer correction functions

$$X^N(t,\varepsilon) = X(t,\varepsilon) + \hat{X}(\tau,\varepsilon) + \tilde{X}(\sigma,\varepsilon) \tag{5.1}$$

where $\tau := \frac{t}{\varepsilon}$, $\sigma := \frac{1-t}{\varepsilon}$, and X, $\hat{X}$, and $\tilde{X}$ possess expansions in ε of the form

$$\begin{pmatrix} X(t,\varepsilon) \\ \hat{X}(\tau,\varepsilon) \\ \tilde{X}(\sigma,\varepsilon) \end{pmatrix} = \sum_{k=0}^{N} \begin{pmatrix} X_k(t) \\ \hat{X}_k(\tau) \\ \tilde{X}_k(\sigma) \end{pmatrix} \varepsilon^k. \tag{5.2}$$

The outer solution coefficient functions X_k are determined by requiring that the outer solution satisfy the differential equation up to $O(\varepsilon^N)$, i.e.,

$$\varepsilon^2 \frac{d^2X}{dt^2} = H(t,X,\varepsilon) + \bar{\rho}(t,\varepsilon) \qquad \text{for} \quad 0 < t < 1 \tag{5.3}$$

where $\bar{\rho}(t,\varepsilon)$ is a continuous function of t and is uniformly of $O(\varepsilon^{N+1})$. Inserting the expansion for $X(t,\varepsilon)$ and expanding about $\varepsilon = 0$, we find that the higher-order terms X_k for $k = 1, \dots, N$ must satisfy linear (algebraic) equations of the form

$$H_{0,x}(t, X_0(t))X_k = P_{k-1}(t) \tag{5.4}$$

where $P_{k-1}(t)$ is a suitable function that is known in terms of the preceding coefficient functions. Since $H_{0,x}(t, X_0(t))$ is nonsingular (see Assumptions D1 and D4) the linear system (5.4) is uniquely solvable.

The boundary-layer correction functions $\hat{X}$ and $\tilde{X}$ are determined by requiring that $X^N(t,\varepsilon)$ satisfy the full problem to $O(\varepsilon^N)$, i.e.,

$$\begin{gathered} \varepsilon^2 \frac{d^2}{dt^2} X^N(t,\varepsilon) = H(t, X^N(t,\varepsilon), \varepsilon) + \rho(t,\varepsilon), \\ \alpha(\varepsilon) - X^N(0,\varepsilon) = \phi_1(\varepsilon), \qquad \beta(\varepsilon) - X^N(1,\varepsilon) = \phi_2(\varepsilon) \end{gathered} \tag{5.5}$$

where $\rho(t,\varepsilon)$ is a continuous function of t and is uniformly of $O(\varepsilon^{N+1})$, and $\phi_1(\varepsilon)$ and $\phi_2(\varepsilon)$ are of $O(\varepsilon^{N+1})$. Because each of the boundary-layer functions is negligible where the other is not, we may consider them separately. Hence, we require that $X + \hat{X}$ satisfy the differential equation and the left boundary conditions to $O(\varepsilon^N)$

$$\begin{aligned}\varepsilon^2 \frac{d^2}{dt^2}(X + \hat{X}) &= H(t, X + \hat{X}, \varepsilon) + \hat{\rho}(t, \varepsilon),\\ \alpha(\varepsilon) - (X(0,\varepsilon) + \hat{X}(0,\varepsilon)) &= \hat{\phi}_1(\varepsilon)\end{aligned} \tag{5.6}$$

where $\hat{\rho}(t,\varepsilon)$ is a continuous function of t and is uniformly of $O(\varepsilon^{N+1})$, and $\hat{\phi}_1(\varepsilon)$ is of $O(\varepsilon^{N+1})$. In a like manner, we require that $X + \tilde{X}$ satisfy the differential equation and the right boundary conditions to $O(\varepsilon^N)$

$$\begin{aligned}\varepsilon^2 \frac{d^2}{dt^2}(X + \tilde{X}) &= H(t, X + \tilde{X}, \varepsilon) + \tilde{\rho}(t, \varepsilon),\\ \beta(\varepsilon) - (X(1,\varepsilon) + \tilde{X}(0,\varepsilon)) &= \tilde{\phi}_2(\varepsilon)\end{aligned} \tag{5.7}$$

where $\tilde{\rho}(t,\varepsilon)$ is a continuous function of t and is uniformly of $O(\varepsilon^{N+1})$, and $\tilde{\phi}_2(\varepsilon)$ is of $O(\varepsilon^{N+1})$. We first consider the left boundary-layer coefficient functions. Using the results for the outer solution, writing in terms of τ, and expanding about $\varepsilon = 0$, we find that the leading left boundary-layer function $\hat{X}_0$ must satisfy

$$\begin{aligned}\frac{d^2}{d\tau^2}\hat{X}_0 &= H_0(0, X_0(0) + \hat{X}_0(\tau)),\\ \hat{X}_0(0) &= \alpha(0) - X_0(0),\end{aligned} \tag{5.8}$$

and the higher-order boundary-layer correction functions must satisfy

$$\begin{aligned}\frac{d^2}{d\tau^2}\hat{X}_k &= H_{0,x}(0, X_0(0) + \hat{X}_0(\tau))\hat{X}_k + \hat{P}_{k-1},\\ \hat{X}_k(0) &= \alpha_k - X_k(0)\end{aligned} \tag{5.9}$$

for suitable functions $\hat{P}_{k-1}$ that are known successively in terms of the preceding coefficient functions $\hat{X}_j$ for $j \leq k-1$. Furthermore, if the preceding coefficient functions are exponentially decaying, then $\hat{P}_{k-1}$ also is exponentially decaying. An exponentially decaying solution to (5.8) is given by Assumption D2. Using the fundamental solution $\hat{\xi}(\tau)$ and imposing the matching condition $\hat{X}_k(\tau) \to 0$ as $\tau \to \infty$, we can solve for $\hat{X}_k$ to find

$$\begin{aligned}\begin{pmatrix}\hat{X}_k\\ \frac{d}{d\tau}\hat{X}_k\end{pmatrix} &= \hat{\xi}(\tau)P\hat{\xi}^{-1}(0)\hat{\gamma}_k + \int_0^\tau \hat{\xi}(\tau)P\hat{\xi}^{-1}(u)\begin{pmatrix}0\\ \hat{P}_{k-1}(u)\end{pmatrix}du\\ &\quad - \int_\tau^\infty \hat{\xi}(\tau)(I-P)\hat{\xi}^{-1}(u)\begin{pmatrix}0\\ \hat{P}_{k-1}(u)\end{pmatrix}du\end{aligned} \tag{5.10}$$

where $\hat{\gamma}_k$ is an arbitrary real $2n$-dimensional vector. Imposing the initial condition $\hat{X}_k(0) = \alpha_k - X_k(0)$, we require that

$$\begin{aligned}\begin{pmatrix}I_n & 0\end{pmatrix}\hat{\xi}(0)P\hat{\xi}^{-1}(0)\hat{\gamma}_k &= \alpha_k - X_k(0)\\ &\quad + \begin{pmatrix}I_n & 0\end{pmatrix}\int_0^\infty \hat{\xi}(0)(I-P)\hat{\xi}^{-1}(u)\begin{pmatrix}0\\ \hat{P}_{k-1}(u)\end{pmatrix}du.\end{aligned} \tag{5.11}$$

It follows from Assumption D3 that there exists a solution $\hat{\gamma}_k$. Furthermore, since $\hat{\xi}(\tau)$ satisfies an exponential dichotomy and $\hat{P}_{k-1}$ is exponentially decaying, $\hat{X}_k$ and $\frac{d}{d\tau}\hat{X}_k$ are exponentially decaying.

We now consider the right boundary-layer correction function. Using the results for the outer solution, writing in terms of σ, and expanding about $\varepsilon = 0$, we find that the leading right boundary-layer function $\tilde{X}_0$ must satisfy

$$\begin{aligned}\frac{d^2\tilde{X}_0}{d\sigma^2} &= H(1, X_0(1) + \tilde{X}_0(\sigma), 0),\\ \tilde{X}_0(0) &= \beta(0) - X_0(1),\end{aligned} \tag{5.12}$$

and the higher-order boundary-layer correction functions must satisfy

$$\begin{aligned}\frac{d^2}{d\sigma^2}\tilde{X}_k &= H_{0,x}(1, X_0(1) + \tilde{X}_0(\sigma))\tilde{X}_k + \tilde{P}_{k-1},\\ \tilde{X}_k(0) &= \beta_k - X_k(1)\end{aligned} \tag{5.13}$$

for suitable functions $\tilde{P}_{k-1}$ that are known successively in terms of the preceding coefficient functions $\tilde{X}_j$ for $j \leq k-1$. Furthermore, if the preceding coefficient functions are exponentially decaying, then $\tilde{P}_{k-1}$ also is exponentially decaying. An exponentially decaying solution to (5.12) is given by Assumption D2. Defining $\tilde{Y}_k := -\frac{d}{d\sigma}\tilde{X}_k$, we convert the second-order system (5.13) to the following first-order system

$$\frac{d}{d\sigma}\begin{pmatrix}\tilde{X}_k\\ \tilde{Y}_k\end{pmatrix} = \begin{pmatrix}0 & -I_n\\ -H_x(1, X_0(1) + \tilde{X}_0(\sigma), 0) & 0\end{pmatrix}\begin{pmatrix}\tilde{X}_k\\ \tilde{Y}_k\end{pmatrix} + \begin{pmatrix}0\\ \tilde{P}_{k-1}\end{pmatrix}. \tag{5.14}$$

Using the fundamental solution $\tilde{\xi}$ and imposing the matching condition $\tilde{X}_k(\sigma) \to 0$ as $\sigma \to \infty$, we can solve the linear system (5.16) to find

$$\begin{aligned}\begin{pmatrix}\tilde{X}_k\\ \tilde{Y}_k\end{pmatrix} &= \tilde{\xi}(\sigma)(I-P)\tilde{\xi}^{-1}(0)\tilde{\gamma}_k + \int_0^\sigma \tilde{\xi}(\sigma)(I-P)\tilde{\xi}^{-1}(u)\begin{pmatrix}0\\ \tilde{P}_{k-1}(u)\end{pmatrix}du\\ &\quad - \int_\sigma^\infty \tilde{\xi}(\sigma)P\tilde{\xi}^{-1}(u)\begin{pmatrix}0\\ \tilde{P}_{k-1}(u)\end{pmatrix}du\end{aligned} \tag{5.15}$$

where $\tilde{\gamma}_k$ is an arbitrary real $2n$-dimensional vector. Imposing the initial condition $\tilde{X}_k(0) = \beta_k - X_k(1)$, we require that

$$\begin{pmatrix}I_n & 0\end{pmatrix}(I-P_2)\tilde{\gamma}_k = \beta_k - X_k(1) + \begin{pmatrix}I_n & 0\end{pmatrix}\int_0^\infty \tilde{\xi}(0)P\tilde{\xi}^{-1}(u)\begin{pmatrix}0\\ \hat{P}_{k-1}(u)\end{pmatrix}du. \tag{5.16}$$

It follows from Assumption D3 that there exists a solution $\tilde{\gamma}_k$. Furthermore, since $\tilde{\xi}(\sigma)$ satisfies an exponential dichotomy and $\tilde{P}_{k-1}$ is exponentially decaying, $\tilde{X}_k$ and $\frac{d}{d\tau}\tilde{X}_k$ are exponentially decaying. Finally, defining

$$\begin{aligned}\rho(t,\varepsilon) &:= \varepsilon^2\frac{d^2}{dt^2}X^N(t,\varepsilon) - H(t, X^N(t,\varepsilon), \varepsilon),\\ \phi_1(\varepsilon) &:= \alpha(\varepsilon) - X^N(0,\varepsilon),\\ \phi_2(\varepsilon) &:= \beta(\varepsilon) - X^N(1,\varepsilon),\end{aligned} \tag{5.17}$$

it follows that ρ is a continuous function and is uniformly of $O(\varepsilon^{N+1})$, and $\phi_1(\varepsilon)$ and $\phi_2(\varepsilon)$ are of $O(\varepsilon^{N+1})$.

6. The approximate solution for the Robin problem

In this section, we construct an approximate solution to the problem (1.1)–(1.3) using the O'Malley construction. We write the approximate solution $X^N(t,\varepsilon)$ as the sum of an outer solution and boundary-layer correction functions

$$X^N(t,\varepsilon) = X(t,\varepsilon) + \varepsilon\hat{X}(\tau,\varepsilon) + \varepsilon\tilde{X}(\sigma,\varepsilon) \tag{6.1}$$

where $X(t,\varepsilon)$, $\hat{X}(\tau,\varepsilon)$, $\tilde{X}(\sigma,\varepsilon)$ possess expansions in ε of the form

$$\begin{pmatrix} X(t,\varepsilon) \\ \hat{X}(\tau,\varepsilon) \\ \tilde{X}(\sigma,\varepsilon) \end{pmatrix} = \sum_{k=0}^{N} \begin{pmatrix} X_k(t) \\ \hat{X}_k(\tau) \\ \tilde{X}_k(\sigma) \end{pmatrix} \varepsilon^k. \tag{6.2}$$

The outer solution coefficient functions, $X_k(t)$, are determined as in the Dirichlet problem (see (5.3)–(5.5)). The boundary-layer correction functions are determined by requiring that the approximate solution $X^N(t,\varepsilon)$ satisfy the full problem (1.1)–(1.3) up to $O(\varepsilon^N)$, that is,

$$\begin{aligned} \varepsilon^2 \frac{d^2}{dt^2} X^N(t,\varepsilon) &= H(t, X^N(t,\varepsilon),\varepsilon) + \rho(t,\varepsilon), \\ B(X^N(0,\varepsilon), X^N(1,\varepsilon), \frac{d}{dt}X^N(0,\varepsilon), &\frac{d}{dt}X^N(1,\varepsilon)) = \phi(\varepsilon) \end{aligned} \tag{6.3}$$

where $\rho(t,\varepsilon)$ is a continuous function of t and $\rho(t,\varepsilon)$ and $\phi(\varepsilon)$ are of $O(\varepsilon^{N+1})$. As before, we may consider the left and right boundary-layer correction functions separately. $\hat{X}(\tau,\varepsilon)$ must satisfy

$$\varepsilon^2 \frac{d^2}{dt^2}(X + \varepsilon\hat{X}) = H(t, X + \varepsilon\hat{X}) + \hat{\rho}(t,\varepsilon) \tag{6.4}$$

where $\hat{\rho}(t,\varepsilon)$ is a continuous function of t and is of $O(\varepsilon^{N+1})$. The left boundary-layer correction functions, $\hat{X}_k(\tau)$, must satisfy

$$\frac{d^2}{d\tau^2}\hat{X}_k = H_{0,x}(0, X_0(0))\hat{X}_k + \hat{P}_{k-1}(\tau) \tag{6.5}$$

for suitable functions $\hat{P}_{k-1}(\tau)$ that are known successively in terms of the preceding boundary-layer correction functions. Furthermore, if the preceding boundary-layer correction functions are exponentially decaying then so is $\hat{P}_{k-1}(\tau)$. Using the fundamental solution $\hat{\eta}(\tau)$ and imposing the matching condition $\hat{X}_k(\tau) \to 0$ as $\tau \to \infty$, we may solve for $\hat{X}_k(\tau)$ to find

$$\begin{aligned} \begin{pmatrix} \hat{X}_k \\ \frac{d}{d\tau}\hat{X}_k \end{pmatrix} &= \hat{\eta}(\tau)P\hat{\eta}^{-1}(0)\hat{\gamma}_k + \int_0^{\tau} \hat{\eta}(\tau)P\hat{\eta}^{-1}(u) \begin{pmatrix} 0 \\ \hat{P}_{k-1}(u) \end{pmatrix} du \\ &\quad - \int_{\tau}^{\infty} \hat{\eta}(\tau)(I-P)\hat{\eta}^{-1}(u) \begin{pmatrix} 0 \\ \hat{P}_{k-1}(u) \end{pmatrix} du \end{aligned} \tag{6.6}$$

where $\hat{\gamma}_k$ is an arbitrary $2n$-dimensional vector. Imposing the initial condition $\frac{d}{d\tau}\hat{X}_k(0) = \alpha_k$ where α_k is an arbitrary n-dimensional vector, we have

$$\begin{pmatrix} 0 & I_n \end{pmatrix} P_1\hat{\gamma}_k = \alpha_k + \begin{pmatrix} 0 & I_n \end{pmatrix} \int_0^{\infty} \hat{\eta}(0)(I-P)\hat{\eta}^{-1}(u) \begin{pmatrix} 0 \\ \hat{P}_{k-1}(u) \end{pmatrix} du. \tag{6.7}$$

It follows from Lemma 1 that there exists a solution $\hat{\gamma}_k$. Furthermore, since $\hat{\eta}(\tau)$ satisfies an exponential dichotomy and $\hat{P}_{k-1}$ is exponentially decaying, $\hat{X}_k$ and $\frac{d}{d\tau}\hat{X}_k$ are exponentially decaying. A similar construction may be done for the right boundary-layer correction function $\tilde{X}(\sigma,\varepsilon)$ such that $\tilde{X}_k(\sigma)$ and $\frac{d}{d\sigma}\tilde{X}_k(\sigma)$ are exponentially decaying and $-\frac{d}{d\sigma}\tilde{X}_k(0)=\beta_k$ where β_k is an arbitrary n-dimensional vector.

We now must choose α_k and β_k, $k=1,\dots,N$, such that

$$B(X^N(0,\varepsilon),X^N(1,\varepsilon),\frac{d}{dt}X^N(0,\varepsilon),\frac{d}{dt}X^N(1,\varepsilon),\varepsilon)=\phi(\varepsilon) \tag{6.8}$$

where $\phi(\varepsilon)$ is of $O(\varepsilon^{N+1})$. By Assumption R3, there exist constants α_0 and β_0 such that

$$B_0(X_0(0),X_0(1),\alpha_0,\beta_0)=0. \tag{6.9}$$

Expanding about $\varepsilon=0$, we find that

$$\begin{aligned}B_{0,q}(\zeta)X_k(0)+B_{0,r}(\zeta)X_k(1)+B_{0,s}(\zeta)(\frac{d}{dt}X_k(0)+\alpha_k)\\ +B_{0,z}(\zeta)(\frac{d}{dt}X_k(1)+\beta_k)=\gamma_{k-1}\end{aligned} \tag{6.10}$$

where $\zeta=(X_0(0),X_0(1),\alpha,\beta)$, and γ_{k-1} is known in terms of the previous values $\alpha_0,\beta_0,\dots,\alpha_{k-1},\beta_{k-1}$. It follows from Assumption R3 that there exists a unique solution α_k,β_k to the linear equation (6.10).

7. Existence and local uniqueness

Theorem 1. *Given Assumptions* D1–D4, *there exist constants* ε_D *and* D_N *such that the Dirichlet problem* (1.1)–(1.2) *has an exact solution* $x(t,\varepsilon)$ *satisfying the estimates*

$$\begin{aligned}\left|x(t,\varepsilon)-X^N(t,\varepsilon)\right|&\le D_N\varepsilon^{N+1},\\ \left|\frac{d}{dt}x(t,\varepsilon)-\frac{d}{dt}X^N(t,\varepsilon)\right|&\le D_N\varepsilon^N\end{aligned} \tag{7.1}$$

uniformly on the region $0\le t\le 1$, $0<\varepsilon\le\varepsilon_D$ *where* $X^N(t,\varepsilon)$ *is the approximate solution constructed in Section* 3. *Moreover,* $x(t,\varepsilon)$ *is unique subject to the estimates of* (7.1).

Theorem 2. *Given Assumptions* R1–R3, *there exist constants* ε_R *and* C_N *such that the Robin problem* (1.1)–(1.3) *has an exact solution* $x(t,\varepsilon)$ *satisfying the estimates*

$$\begin{aligned}\left|x(t,\varepsilon)-X^N(t,\varepsilon)\right|&\le C_N\varepsilon^{N+1},\\ \left|\frac{d}{dt}x(t,\varepsilon)-\frac{d}{dt}X^N(t,\varepsilon)\right|&\le C_N\varepsilon^N\end{aligned} \tag{7.2}$$

uniformly on the region $0\le t\le 1$, $0<\varepsilon\le\varepsilon_R$, *where* $X^N(t,\varepsilon)$ *is the approximate solution constructed in Section* 4. *Moreover,* $x(t,\varepsilon)$ *is unique subject to the estimates of* (7.2).

We give the proof for Theorem 2. Theorem 1 is proved in a like manner.

Proof. Defining $\bar{x}(t,\varepsilon):=x(t,\varepsilon)-X^N(t,\varepsilon)$ and $\bar{y}:=\varepsilon\frac{d\bar{x}}{dt}$, we find that $\bar{x}$ and $\bar{y}$ must satisfy the first-order system

$$\frac{d}{dt}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\frac{1}{\varepsilon}\begin{pmatrix}0 & I_n\\ H_x(t,X^N(t,\varepsilon),\varepsilon) & 0\end{pmatrix}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}+\frac{1}{\varepsilon}\begin{pmatrix}0\\ E(t,\bar{x},\varepsilon)+\rho(t,\varepsilon)\end{pmatrix} \tag{7.3}$$

subject to the boundary conditions

$$L(\varepsilon)\begin{pmatrix}\bar{x}(0,\varepsilon)\\ \bar{y}(0,\varepsilon)\end{pmatrix}+R(\varepsilon)\begin{pmatrix}\bar{x}(1,\varepsilon)\\ \bar{y}(1,\varepsilon)\end{pmatrix}=-\varepsilon[\phi(\varepsilon)+F(\bar{x},\bar{y},\varepsilon)] \tag{7.4}$$

where

$$E(t,\bar{x},\varepsilon):=\int_0^1(1-s)\frac{d^2}{ds^2}H(t,X^N(t,\varepsilon)+s\bar{x},\varepsilon)ds, \tag{7.5}$$

$$F(\bar{x},\bar{y},\varepsilon):=\int_0^1(1-s)\frac{d^2}{ds^2}B(\zeta^N(\varepsilon)+s\zeta(\bar{x},\bar{y},\varepsilon))ds, \tag{7.6}$$

$$\begin{aligned}L(\varepsilon)&:=\left[\varepsilon B_q(\zeta^N(\varepsilon),\varepsilon)|B_s(\zeta^N(\varepsilon),\varepsilon)\right],\\ R(\varepsilon)&:=\left[\varepsilon B_r(\zeta^N(\varepsilon),\varepsilon)|B_z(\zeta^N(\varepsilon),\varepsilon)\right],\end{aligned} \tag{7.7}$$

$$\begin{aligned}\zeta^N(\varepsilon)&:=(X^N(0,\varepsilon),X^N(1,\varepsilon),\frac{d}{dt}X^N(0,\varepsilon),\frac{d}{dt}X^N(0,\varepsilon)),\\ \zeta(\bar{x},\bar{y},\varepsilon)&:=(\bar{x}(0,\varepsilon),\bar{x}(1,\varepsilon),\frac{1}{\varepsilon}\bar{y}(0,\varepsilon),\frac{1}{\varepsilon}\bar{y}(1,\varepsilon)).\end{aligned} \tag{7.8}$$

It follows from Assumption R3 that there exist positive constants $|E|$ and $|F|$ such that for all sufficiently small $\bar{x}$ and $\bar{y}$

$$\begin{aligned}|E(t,\bar{x},\varepsilon)|&\le|E|\,|\bar{x}|^2,\\ |E(t,\bar{x}_1,\varepsilon)-E(t,\bar{x}_2,\varepsilon)|&\le\max\{|\bar{x}_1|,|\bar{x}_2|\}\,|E|\,|\bar{x}_1-\bar{x}_2|,\end{aligned} \tag{7.9}$$

$$\begin{aligned}|F(\bar{x},\bar{y},\varepsilon)|&\le|F|\Big\{|\bar{x}(0,\varepsilon)|+|\bar{x}(1,\varepsilon)|+\frac{1}{\varepsilon}|\bar{y}(0,\varepsilon)|+\frac{1}{\varepsilon}|\bar{y}(1,\varepsilon)|\Big\}^2,\\ |F(\bar{x}_1,\bar{y}_1,\varepsilon)-F(\bar{x}_2,\bar{y}_2,\varepsilon)|&\le M|F|\Big\{|\bar{x}_1(0,\varepsilon)-\bar{x}_2(0,\varepsilon)|+|\bar{x}_1(1,\varepsilon)-\bar{x}_2(1,\varepsilon)|\\ &\quad+\frac{1}{\varepsilon}|\bar{y}_1(0,\varepsilon)-\bar{y}_2(0,\varepsilon)|+\frac{1}{\varepsilon}|\bar{y}_1(1,\varepsilon)-\bar{y}_2(1,\varepsilon)|\Big\}\end{aligned} \tag{7.10}$$

where $M:=\max_{i=1,2}\left\{|\bar{x}_i(0,\varepsilon)|,|\bar{x}_i(1,\varepsilon)|,\frac{1}{\varepsilon}|\bar{y}_i(0,\varepsilon)|,\frac{1}{\varepsilon}|\bar{y}_i(1,\varepsilon)|\right\}$. From Lemma 6.2 of Jeffries and Smith [12], there exists a fundamental solution $Z(t,\varepsilon)$ to the linear system

$$\frac{d}{dt}Z=\frac{1}{\varepsilon}\begin{pmatrix}0 & I_n\\ H_x(t,X^N(t,\varepsilon),\varepsilon) & 0\end{pmatrix}Z \tag{7.11}$$

satisfying the exponential dichotomy

$$\begin{aligned}\left|Z(t,\varepsilon)PZ^{-1}(s,\varepsilon)\right|&\le K_1e^{-\nu_1\frac{(t-s)}{\varepsilon}}\quad\text{for}\quad 0\le s\le t\le 1,\\ \left|Z(t,\varepsilon)(I-P)Z^{-1}(s,\varepsilon)\right|&\le K_1e^{-\nu_1\frac{(s-t)}{\varepsilon}}\quad\text{for}\quad 0\le t\le s\le 1\end{aligned} \tag{7.12}$$

where K_1 and v_1 are positive constants. Applying Lemma 6.3 of Jeffries and Smith [12], we may conclude that there exist fundamental solutions $\hat{\eta}_1(\tau,\varepsilon)$ and $\tilde{\eta}_1(\sigma,\varepsilon)$ to the following linear systems

$$\frac{d}{d\tau}\hat{\eta}_1=\begin{pmatrix}0 & I_n\\ H_x(0,X_0(0)+\hat{X}_0(\tau),0) & 0\end{pmatrix}\hat{\eta}_1\quad\text{for}\quad 0\le\tau\le\frac{1}{\nu_2}\ln\frac{1}{\varepsilon}, \tag{7.13}$$

$$\frac{d}{d\sigma}\tilde{\eta}_1=-\begin{pmatrix}0 & I_n\\ H_x(1,X_0(1)+\tilde{X}_0(\sigma),0) & 0\end{pmatrix}\tilde{\eta}_1\quad\text{for}\quad 0\le\sigma\le\frac{1}{\nu_2}\ln\frac{1}{\varepsilon}, \tag{7.14}$$

satisfying the exponential dichotomies

$$\begin{aligned}\left|\hat{\eta}_1(\tau,\varepsilon)P\hat{\eta}_1^{-1}(u,\varepsilon)\right| &\le K_2 e^{-\nu_2(\tau-u)} && \text{for} \quad u \le \tau,\\ \left|\hat{\eta}_1(\tau,\varepsilon)(I-P)\hat{\eta}_1^{-1}(u,\varepsilon)\right| &\le K_2 e^{-\nu_2(u-\tau)} && \text{for} \quad \tau \le u,\end{aligned} \tag{7.15}$$

$$\begin{aligned}\left|\tilde{\eta}_1(\sigma,\varepsilon)(I-P)\tilde{\eta}_1(u,\varepsilon)\right| &\le K_2 e^{-\nu_2(\sigma-u)} && \text{for} \quad u \le \sigma,\\ \left|\tilde{\eta}_1(\sigma,\varepsilon)P\tilde{\eta}_1(u,\varepsilon)\right| &\le K_2 e^{-\nu_2(u-\sigma)} && \text{for} \quad \sigma \le u,\end{aligned} \tag{7.16}$$

and the estimates

$$\begin{aligned}\hat{\eta}_1(0,\varepsilon)P\hat{\eta}_1^{-1}(0,\varepsilon) &= Z(0,\varepsilon)PZ^{-1}(0,\varepsilon) + O(\varepsilon \ln \frac{1}{\varepsilon}),\\ \tilde{\eta}_1(0,\varepsilon)(I-P)\tilde{\eta}_1(0,\varepsilon) &= Z(1,\varepsilon)(I-P)Z^{-1}(1,\varepsilon) + O(\varepsilon \ln \frac{1}{\varepsilon})\end{aligned} \tag{7.17}$$

where ν_2 and K_2 are positive constants. From Lemma 6.4 of Jeffries and Smith [12], we may conclude that there exist bounded nonsingular matrices $\hat{S}(\varepsilon)$ and $\tilde{S}(\varepsilon)$, with bounded inverses, such that

$$\begin{aligned}\hat{\eta}_1(0,\varepsilon)P\hat{\eta}_1^{-1}(0,\varepsilon) &= P_1\hat{S}(\varepsilon) + O(\varepsilon),\\ \tilde{\eta}_1(0,\varepsilon)(I-P)\tilde{\eta}_1(0,\varepsilon) &= (I-P_2)\tilde{S}(\varepsilon) + O(\varepsilon).\end{aligned} \tag{7.18}$$

Using the fundamental solution $Z(t,\varepsilon)$, we may write (7.3) as an integral equation

$$\begin{aligned}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} &= Z(t,\varepsilon)PZ^{-1}(0,\varepsilon)C_L + Z(t,\varepsilon)(I-P)Z^{-1}(1,\varepsilon)C_R\\ &\quad + \frac{1}{\varepsilon}\int_0^t Z(t,\varepsilon)PZ^{-1}(s,\varepsilon)\begin{pmatrix}0\\ E(s,\bar{x}(s,\varepsilon),\varepsilon)+\rho(s,\varepsilon)\end{pmatrix} ds\\ &\quad - \frac{1}{\varepsilon}\int_t^1 Z(t,\varepsilon)(I-P)Z^{-1}(s,\varepsilon)\begin{pmatrix}0\\ E(s,\bar{x}(s,\varepsilon),\varepsilon)+\rho(s,\varepsilon)\end{pmatrix} ds\end{aligned} \tag{7.19}$$

where C_L and C_R are to be determined. Imposing the boundary conditions, we find that C_L and C_R must satisfy the linear system

$$M_L(\varepsilon)C_L + M_R(\varepsilon)C_R = b(\bar{x},\bar{y},\varepsilon) \tag{7.20}$$

where

$$\begin{aligned}M_L(\varepsilon) &:= L(\varepsilon)Z(0,\varepsilon)PZ^{-1}(0,\varepsilon) + R(\varepsilon)Z(1,\varepsilon)PZ^{-1}(0,\varepsilon),\\ M_R(\varepsilon) &:= L(\varepsilon)Z(0,\varepsilon)(I-P)Z^{-1}(1,\varepsilon) + R(\varepsilon)Z(1,\varepsilon)(I-P)Z^{-1}(1,\varepsilon),\end{aligned} \tag{7.21}$$

$$\begin{aligned}b(\bar{x},\bar{y},\varepsilon) &:= -\varepsilon[\phi(\varepsilon) + F(\bar{x},\bar{y},\varepsilon)]\\ &\quad + \frac{1}{\varepsilon}L(\varepsilon)\int_0^1 Z(0,\varepsilon)(I-P)Z^{-1}(s,\varepsilon)\begin{pmatrix}0\\ E(s,\bar{x}(s,\varepsilon),\varepsilon)+\rho(s,\varepsilon)\end{pmatrix} ds\\ &\quad - \frac{1}{\varepsilon}R(\varepsilon)\int_0^1 Z(1,\varepsilon)PZ^{-1}(s,\varepsilon)\begin{pmatrix}0\\ E(s,\bar{x}(s,\varepsilon),\varepsilon)+\rho(s,\varepsilon)\end{pmatrix} ds.\end{aligned} \tag{7.22}$$

From the estimates of (7.17)–(7.18), we have

$$\begin{aligned} M_L(\varepsilon)\hat{S}^{-1}(\varepsilon)\hat{\eta}(0) &= \left\{ [0|B_{0,s}(\zeta)] \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} + O(\varepsilon \ln \frac{1}{\varepsilon}) \right\}, \\ M_R(\varepsilon)\tilde{S}^{-1}(\varepsilon)\tilde{\eta}(0) &= \left\{ [0|B_{0,z}(\zeta)] \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} + O(\varepsilon \ln \frac{1}{\varepsilon}) \right\} \end{aligned} \tag{7.23}$$

where

$$A := \begin{pmatrix} 0 & I_n \end{pmatrix} \hat{\eta}(0) \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & I_n \end{pmatrix} \tilde{\eta}(0) \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \tag{7.24}$$

Assumption R2 and Lemma 1 imply that the linear system

$$\left[M_L(\varepsilon)\hat{S}^{-1}(\varepsilon)\hat{\eta}(0) + M_R(\varepsilon)\tilde{S}^{-1}(\varepsilon)\tilde{\eta}(0) \right] \begin{pmatrix} u \\ v \end{pmatrix} = b(\bar{x}, \bar{y}, \varepsilon) \tag{7.25}$$

is invertible. The boundary conditions are satisfied by letting

$$C_L = \hat{S}^{-1}(\varepsilon)\hat{\eta}(0) \begin{pmatrix} u \\ v \end{pmatrix}, \qquad C_R = \tilde{S}^{-1}(\varepsilon)\tilde{\eta}(0) \begin{pmatrix} u \\ v \end{pmatrix}. \tag{7.26}$$

We now may apply the Banach-Picard fixed-point theorem to conclude that there exists a unique fixed point to the integral equation (7.19) satisfying the estimates of (7.2). For the details of such a proof, see Jeffries and Smith [12, pp.26–30]. □

8. Examples

We now consider the following second-order system

$$\begin{aligned} \varepsilon^2 x_1'' &= 2x_2 + t(1-t)x_1^2, \\ \varepsilon^2 x_2'' &= -8x_1 - 16t \end{aligned} \tag{8.1}$$

on the interval $0 \le t \le 1$ subject to the Dirichlet boundary conditions

$$\begin{aligned} x_1(0) &= 1, & x_1(1) &= 3, \\ x_2(0) &= 2, & x_2(1) &= 0. \end{aligned} \tag{8.2}$$

The reduced system

$$\begin{aligned} 0 &= 2X_{2,0} + t(1-t)X_{1,0}^2, \\ 0 &= -8X_{1,0} - 16t \end{aligned} \tag{8.3}$$

has solution $X_{1,0}(t) = -2t, X_{2,0}(t) = -2t^3(1-t)$. $H_{0,x}(t, X_0(t))$ is given by

$$\begin{bmatrix} -4t^2(1-t) & 2 \\ -8 & 0 \end{bmatrix} \tag{8.4}$$

and has eigenvalues $-2t^2(1-t)\pm 4i\sqrt{1-.25t^4(1-t)^2}$. The boundary-layer differential equations are given by

$$\begin{aligned}
\frac{d^2\hat{X}_{1,0}}{d\tau^2} &= 2\hat{X}_{2,0}, & \hat{X}_{1,0}(0) &= 1,\\
\frac{d^2\hat{X}_{2,0}}{d\tau^2} &= -8\hat{X}_{1,0}, & \hat{X}_{2,0}(0) &= 2,\\
\frac{d^2\tilde{X}_{1,0}}{d\sigma^2} &= 2\tilde{X}_{2,0}, & \tilde{X}_{1,0}(0) &= 5,\\
\frac{d^2\tilde{X}_{2,0}}{d\sigma^2} &= -8\tilde{X}_{1,0}, & \tilde{X}_{2,0}(0) &= 0,
\end{aligned} \tag{8.5}$$

with solutions

$$\begin{aligned}
\begin{pmatrix}\hat{X}_{1,0}(\tau)\\ \hat{X}_{2,0}(\tau)\end{pmatrix} &= \begin{pmatrix}\cos(\sqrt{2}\tau)-2\sin(\sqrt{2}\tau)\\ 2\cos(\sqrt{2}\tau)+\sin(\sqrt{2}\tau)\end{pmatrix} e^{-\sqrt{2}\tau},\\
\begin{pmatrix}\tilde{X}_{1,0}(\sigma)\\ \tilde{X}_{2,0}(\sigma)\end{pmatrix} &= \begin{pmatrix}5\cos(\sqrt{2}\sigma)\\ 10\sin(\sqrt{2}\sigma)\end{pmatrix} e^{-\sqrt{2}\sigma}.
\end{aligned} \tag{8.6}$$

The fundamental solution $\hat{\xi}(\tau) = A\hat{Z}(\tau)$ where A equals

$$\begin{pmatrix}
1 & 0 & 1 & 0\\
0 & 2 & 0 & -2\\
-\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2}\\
2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2} & -2\sqrt{2}
\end{pmatrix} \tag{8.7}$$

and $\hat{Z}(\tau)$ equals

$$\begin{pmatrix}
e^{-\sqrt{2}\tau}\begin{bmatrix}\cos(\sqrt{2}\tau) & -\sin(\sqrt{2}\tau)\\ \sin(\sqrt{2}\tau) & \cos(\sqrt{2}\tau)\end{bmatrix} & 0\\
0 & e^{\sqrt{2}\tau}\begin{bmatrix}\cos(\sqrt{2}\tau) & \sin(\sqrt{2}\tau)\\ \sin(\sqrt{2}\tau) & -\cos(\sqrt{2}\tau)\end{bmatrix}
\end{pmatrix}, \tag{8.8}$$

satisfies the appropriate exponential dichotomy, as does $\tilde{\xi}(\sigma) := \hat{\xi}(-\sigma)$. We therefore may conclude that the problem (8.1)–(8.2) has an exact solution $x_1(t,\varepsilon), x_2(t,\varepsilon)$ satisfying the estimates

$$\begin{aligned}
\begin{pmatrix}x_1(t,\varepsilon)\\ x_2(t,\varepsilon)\end{pmatrix} &= \begin{pmatrix}-2t\\ -2t^3(t-1)\end{pmatrix} + \begin{pmatrix}\cos(\frac{\sqrt{2}t}{\varepsilon})-2\sin(\frac{\sqrt{2}t}{\varepsilon})\\ 2\cos(\frac{\sqrt{2}t}{\varepsilon})+\sin(\frac{\sqrt{2}t}{\varepsilon})\end{pmatrix} e^{-\frac{\sqrt{2}t}{\varepsilon}}\\
&\quad + \begin{pmatrix}5\cos(\frac{\sqrt{2}(1-t)}{\varepsilon})\\ 5\sin(\frac{\sqrt{2}(1-t)}{\varepsilon})\end{pmatrix} e^{-\frac{\sqrt{2}(1-t)}{\varepsilon}} + O(\varepsilon).
\end{aligned} \tag{8.9}$$

Next, we consider the following second-order system

$$\begin{aligned}
\varepsilon x_1'' &= -2x_1 + x_2,\\
\varepsilon x_2'' &= (1-x_1)x_2
\end{aligned} \tag{8.10}$$

subject to the Robin boundary conditions

$$\begin{aligned} x_1(0,\varepsilon) - x_1'(0,\varepsilon) &= 2, \\ x_2(1,\varepsilon) + x_2'(0,\varepsilon) &= 3, \\ x_1(1,\varepsilon) + x_1'(1,\varepsilon) &= 1, \\ x_2(0,\varepsilon) - x_2'(1,\varepsilon) &= -1. \end{aligned} \tag{8.11}$$

The reduced system

$$\begin{aligned} 0 &= -2x_1 + x_2, \\ 0 &= (1 - x_1)x_2, \end{aligned} \tag{8.12}$$

has two sets of solutions

$$\begin{aligned} X_{1,0}(t) &= 1, & \bar{X}_{1,0}(t) &= 0, \\ X_{2,0}(t) &= 2, & \bar{X}_{2,0}(t) &= 0. \end{aligned} \tag{8.13}$$

However, only the first solution satisfies Assumption R1. The boundary-layer differential equations are given by

$$\begin{aligned} \frac{d^2\hat{X}_{1,0}}{d\tau^2} &= -2\hat{X}_{1,0} + \hat{X}_{2,0}, & \frac{d\hat{X}_{1,0}}{d\tau}(0) &= -1, \\ \frac{d^2\hat{X}_{2,0}}{d\tau^2} &= -2\hat{X}_{1,0}, & \frac{d\hat{X}_{2,0}}{d\tau}(0) &= -3, \end{aligned} \tag{8.14}$$

and

$$\begin{aligned} \frac{d^2\tilde{X}_{1,0}}{d\sigma^2} &= 2\tilde{X}_{1,0} - \tilde{X}_{2,0}, & \frac{d\tilde{X}_{1,0}}{d\sigma}(0) &= -2, \\ \frac{d^2\tilde{X}_{2,0}}{d\sigma^2} &= 2\tilde{X}_{1,0}, & \frac{d\tilde{X}_{2,0}}{d\sigma}(0) &= 5. \end{aligned} \tag{8.15}$$

They have solutions

$$\begin{pmatrix} \hat{X}_{1,0}(\tau) \\ \hat{X}_{2,0}(\tau) \end{pmatrix} = \frac{e^{-\mu_1\tau}}{\sqrt{2}} \begin{pmatrix} (\mu_1+\mu_2)\cos(\mu_2\tau) + (\mu_1-\mu_2)\sin(\mu_2\tau) \\ -2\mu_2\cos(\mu_2\tau) - 2\mu_1\sin(\mu_2\tau) \end{pmatrix} \tag{8.16}$$

and

$$\begin{pmatrix} \tilde{X}_{1,0}(\sigma) \\ \tilde{X}_{2,0}(\sigma) \end{pmatrix} = -\frac{3e^{-\mu_1\sigma}}{\sqrt{2}} \begin{pmatrix} \mu_2\cos(\mu_2\sigma) + \mu_1\sin(\mu_2\sigma) \\ (\mu_1-\mu_2)\cos(\mu_2\sigma) - (\mu_1+\mu_2)\sin(\mu_2\sigma) \end{pmatrix} \tag{8.17}$$

where $\mu_1 = \sqrt{\frac{\sqrt{2}-1}{2}}$ and $\mu_2 = \sqrt{\frac{\sqrt{2}+1}{2}}$. We may conclude that the problem (8.9)–(8.10) has an exact solution $x_1(t,\varepsilon), x_2(t,\varepsilon)$ satisfying the estimates

$$\begin{pmatrix} x_1(t,\varepsilon) \\ x_2(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \varepsilon \begin{pmatrix} \hat{X}_{1,0}(\frac{t}{\varepsilon}) \\ \hat{X}_{2,0}(\frac{t}{\varepsilon}) \end{pmatrix} + \varepsilon \begin{pmatrix} \tilde{X}_{1,0}(\frac{1-t}{\varepsilon}) \\ \tilde{X}_{2,0}(\frac{1-t}{\varepsilon}) \end{pmatrix} + O(\varepsilon^2). \tag{8.18}$$

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