Methods and Applications of Analysis
2 (1) 1995, pp. 112–119

ISSN 1073–2772

Modified Sturm-Liouville expansions

A. McD. Mercer and Peter R. Mercer

1. Introduction

Let q be a real-valued continuous function defined on the compact interval $[a,b]$. Then the Sturm-Liouville expansion of an "arbitrary" function in terms of the solutions of the differential equation

$$\frac{d^2y}{dx^2} + [\lambda - q(x)]y = 0 \tag{1}$$

(given certain appropriate boundary conditions) is classical. There is an extensive literature devoted to these series and their generalizations.

Variants of these expansions are concerned with the different types of convergence involved and with the several ways the conditions on the function q and on the interval $[a,b]$ can be altered. A prime source for the analysis of many of these variants is [1].

Other expansions, which instead of being orthogonal are "bi-orthogonal", can be derived if we start from a related form of (1) which has a given nontrivial function specified on the right-hand side. Such a family of expansions was investigated in [2]. While such bi-orthogonal expansions possess theoretical interest, the loss of convenience for applications, which goes with the loss of pure orthogonality, is considerable.

It is the purpose of this paper to present a new and somewhat extensive family of orthogonal expansions associated with an inhomogeneous form of (1). Here we shall be concerned exclusively with the simplest conditions on the function q and the interval $[a,b]$, and the type of convergence to be considered is pointwise convergence. We hope to deal with some other variations in a future paper.

2. Preliminaries and statement of results

Let q be a real-valued continuous function on $[a,b]$ and let $\phi(x,\lambda)$ and $\chi(x,\lambda)$ satisfy (1) and the end conditions

$$\begin{aligned} \phi(a,\lambda) &= \sin\alpha, \quad \phi'(a,\lambda) = -\cos\alpha, \\ \chi(b,\lambda) &= \sin\beta, \quad \chi'(b,\lambda) = -\cos\beta. \end{aligned} \tag{2}$$

Note. To avoid repetitious analysis, we shall assume throughout that $\sin\alpha \neq 0 \neq \sin\beta$.

The Wronskian $\omega(\lambda) = \phi(x,\lambda)\chi'(x,\lambda) - \phi'(x,\lambda)\chi(x,\lambda)$ of ϕ and χ is independent of x. It is an entire function whose zeros are all real and simple and can be enumerated as $\lambda_1 < \lambda_2 < \cdots \to \infty$. When λ is one of the λ_n, then $\phi(x,\lambda_n)$ and $\chi(x,\lambda_n)$ are linearly dependent, say

$$\chi(x,\lambda_n) = k_n\phi(x,\lambda_n). \tag{3}$$

Received January 24, 1994, revised December 5, 1994.
Research of the second author supported in part by N.S.E.R.C. Canada.

The functions $\{\phi(x,\lambda_n)\}_1^\infty$ satisfy the relations

$$\int_a^b \phi(x,\lambda_n)\phi(x,\lambda_m)dx = \delta_{n,m}\frac{\omega'(\lambda_n)}{k_n}. \tag{4}$$

This orthogonality condition leads to the classical result:

Theorem A. ([1, Theorem 1.9]). *If $f \in L(a,b)$ then the Sturm-Liouville series associated with f, namely*

$$\sum_{n=1}^\infty \frac{k_n}{\omega'(\lambda_n)}\phi(x,\lambda_n)\int_a^b \phi(y,\lambda_n)f(y)dy,$$

behaves as regards convergence (pointwise) in the same way as an ordinary Fourier series.

Now let us introduce another function $\mathbf{w}(x)$. This function will be non-trivial, real-valued, and absolutely continuous on $[a,b]$. Furthermore, for reasons which will appear later, we shall assume that $\mathbf{w}(x)$ is in "general position" with respect to the functions $\phi(x,\lambda_n)$ — i.e., it is not orthogonal to any of them. (This condition could be relaxed at the expense of some complication of detail.)

With $\mathbf{w}(x)$ as above, let $\theta(x,\lambda)$ be the solution of the problem

$$\begin{aligned} y'' + [\lambda - q(x)]y &= \omega(\lambda)\mathbf{w}(x),\\ y(a)\cos\alpha + y'(a)\sin\alpha &= 0,\\ y(b)\cos\beta + y'(b)\sin\beta &= 0, \end{aligned} \tag{5}$$

and set

$$m(\lambda) = \int_a^b \mathbf{w}(x)\theta(x,\lambda)dx. \tag{6}$$

Like $\omega(\lambda)$, $m(\lambda)$ is an entire function, and as we shall see (Lemma 1 below), all of its zeros are real and simple and can be enumerated as $\mu_1 < \mu_2 < \cdots \to \infty$. In Lemma 2 below, we shall see that the functions $\theta(x,\mu_n)$ satisfy the relations

$$\int_a^b \theta(x,\mu_n)\theta(x,\mu_m)dx = -\delta_{n,m}\omega(\mu_n)m'(\mu_n). \tag{7}$$

Our purpose in this paper is to establish the following.

Theorem 1. *If $f \in L(a,b)$, then the series*

$$\theta_o(x)\int_a^b f(y)\theta_o(y)dy + \sum_{n=1}^\infty \frac{-1}{\omega(\mu_n)m'(\mu_n)}\theta(x,\mu_n)\int_a^b f(y)\theta(y,\mu_n)dy$$

behaves as regards convergence (pointwise) in the same way as an ordinary Fourier series. Here $\theta_0(x)$ is the normalized form of $\mathbf{w}(x)$, namely

$$\theta_o(x) = \frac{\mathbf{w}(x)}{\|\mathbf{w}\|}, \quad \textit{where} \quad \|\mathbf{w}\|^2 = \int_a^b [\mathbf{w}(x)]^2 dx.$$

Note. The leading term in this series, which has no counterpart in the series of Theorem A, appears because the subspace generated by the set of functions $\{\theta(x,\mu_n)\}_1^\infty$ is orthogonal to $\mathbf{w}(x)$ by virtue of (6).

In the above, we have followed the notation of [1] and we will continue to do so whenever possible. Several of the results to be found there will be of use in our analysis and so we quote these. We write $\lambda = s^2$ and $s = \sigma + it$, where σ and t are real.

Lemma A. *$\phi(x,\lambda)$ satisfies the integral equation*

$$\phi(x,\lambda) = \sin\alpha\cos s(x-a) - \cos\alpha\frac{\sin s(x-a)}{s} + \frac{1}{s}\int_a^x \sin s(x-y)q(y)\phi(y,\lambda)dy.$$

Lemma B. *As $|s|\to\infty$, we have*

$$\phi(x,\lambda) = \sin\alpha\cos s(x-a) + O\left(\frac{e^{|t|(x-a)}}{|s|}\right)$$

uniformly in $[a,b]$.

Lemmas A and B have their obvious analogues involving the function χ.

We shall also use various complex forms of the Riemann-Lebesgue Lemma, typified by the following.

Lemma C. *Let $f\in L(a,b)$. We have*

$$\int_a^b f(x)\sin s(x-a)dx = o\left(e^{|t|(b-a)}\right) \quad \text{as } |s|\to\infty.$$

We close this section with a remark concerning the assumption that $\mathbf{w}(x)$ is in general position with respect to $\{\phi(x,\lambda_n)\}_1^\infty$.

Observing (2), the method of variation of parameters when applied to (5) gives

$$\theta(x,\lambda) = \chi(x,\lambda)\int_a^x \phi(y,\lambda)\mathbf{w}(y)dy + \phi(x,\lambda)\int_x^b \chi(y,\lambda)\mathbf{w}(y)dy. \tag{8}$$

Now if λ is put equal to λ_n in (5), the right side vanishes making the problem homogeneous, and so there arises the possibility that the function $\theta(x,\lambda_n)$ is merely the trivial solution. However, put $\lambda=\lambda_n$ in (8) and use (3) to see that

$$\theta(x,\lambda_n) = k_n\phi(x,\lambda_n)\int_a^b \mathbf{w}(x)\phi(x,\lambda_n)dx. \tag{9}$$

Hence because of our assumption on $\mathbf{w}(x)$, none of these integrals is zero and so none of the functions $\theta(x,\lambda_n)$ is trivial.

3. Preparatory lemmas

Recall that $\theta(x,\lambda)$ and $\theta(x,\mu)$ satisfy, respectively,

$$\begin{aligned}\theta''(x,\lambda) + [\lambda - q(x)]\theta(x,\lambda) &= \omega(\lambda)\mathbf{w}(x),\\ \theta''(x,\mu) + [\mu - q(x)]\theta(x,\mu) &= \omega(\mu)\mathbf{w}(x),\end{aligned} \tag{10}$$

and that $m(\lambda) = \int_a^b \mathbf{w}(x)\theta(x,\lambda)dx$. We shall need the following lemmas.

Lemma 1. *The zeros of the entire function $m(\lambda)$ are all real, simple, and satisfy $\lambda_1 < \mu_1 < \lambda_2 < \mu_2 \cdots \to \infty$.*

Lemma 2. *The orthogonality relation (7) holds.*

Lemma 3. *As $|s| \to \infty$, we have*

$$\theta(x,\lambda) = \sin\alpha \sin\beta \frac{\sin s(b-a)}{s} \mathbf{w}(x) + o\left(\frac{e^{|t|(b-a)}}{|s|}\right).$$

Lemma 4. *As $|s| \to \infty$, we have*

$$m(\lambda) = \|\mathbf{w}\|^2 \sin\alpha \sin\beta \frac{\sin s(b-a)}{s} + o\left(\frac{e^{|t|(b-a)}}{|s|}\right).$$

A final lemma is needed concerning the zeros of $m(\lambda)$. In [1], and in what follows, a certain closed contour Γ_N in the complex λ-plane is used. With $\lambda = s^2$, Γ_N is the image of the contour $\gamma_N = ABCD$ in the upper half s-plane whose vertices are $A = (C_N, 0)$, $B = (C_N, C_N)$, $C = (-C_N, C_N)$, $D = (-C_N, 0)$, where $C_N = (N + \frac{1}{2})\pi/(b-a)$.

Lemma 5. *For N sufficiently large, the contour Γ_N encloses precisely the zeros $\lambda_1, \ldots, \lambda_{N+1}$ and $\mu_1, \ldots, \mu_N$.*

4. Proofs of the lemmas

Proof of Lemma 1. Denote the normalized form of $\phi(x,\lambda_n)$ by $\tilde{\phi}(x,\lambda_n)$ That is, $\int_a^b \tilde{\phi}(x,\lambda_n)\tilde{\phi}(x,\lambda_m)dx = \delta_{n,m}$. Also, set $c_n = \int_a^b \mathbf{w}(x)\tilde{\phi}(x,\lambda_n)dx$ $(n = 1, 2, \ldots)$ and $A_n(\lambda) = \int_a^b \theta(x,\lambda)\tilde{\phi}(x,\lambda_n)dx$. If we multiply the upper member of (10) by $\tilde{\phi}(x,\lambda_n)$, integrate over $[a,b]$, and use (1) and (2), we obtain

$$A_n(\lambda) = \frac{\omega(\lambda)c_n}{\lambda - \lambda_n} \quad (\lambda \neq \lambda_n).$$

These are the coefficients of the normalized Sturm-Liouville expansion of $\theta(x,\lambda)$, and since for each λ this function is both continuous and of bounded variation, we actually have

$$\theta(x,\lambda) = \sum_{n=1}^{\infty} \frac{\omega(\lambda)c_n}{\lambda - \lambda_n}\tilde{\phi}(x,\lambda_n) \quad (a < x < b,\ \lambda \neq \lambda_n).$$

Multiplying this throughout by $\mathbf{w}(x)$ and integrating over $[a,b]$, we obtain

$$m(\lambda) = \omega(\lambda)\sum_{n=1}^{\infty} \frac{c_n^2}{\lambda - \lambda_n}. \tag{11}$$

Now since $\mathbf{w} \in L^2(a,b)$, we have $\sum c_n^2 < +\infty$, and so the series in (11) is uniformly convergent throughout the complex λ-plane punctured by disks of radius $\delta > 0$ centered at the points λ_n. Now $m(\lambda)$ will be represented by (11) throughout the whole complex λ-plane provided we define the right side of (11) at the λ_n by continuity.

The λ_n are all real and by assumption none of the c_n vanishes. Thus there is precisely one real zero of $m(\lambda)$ lying between each of the λ_n. It can also be seen from (11) that $m(\lambda)$ and $\omega(\lambda)$ cannot vanish simultaneously since none of the $\omega'(\lambda_n)$ is zero.

Finally, if we write $c_n^2/(\lambda-\lambda_n)=(c_n^2/|\lambda-\lambda_n|^2)(\bar{\lambda}-\lambda_n)$ in (11) and then assume the existence of a non-real zero μ_o of $m(\mu)$, we arrive at a contradiction. The proof of Lemma 1 is complete.

Proof of Lemma 2. If we multiply the upper member of (10) by $\theta(x,\lambda)$ and subtract the lower member multiplied by $\theta(x,\mu)$, then integrate the result over $[a,b]$, we obtain

$$[\theta'(x,\lambda)\theta(x,\mu)-\theta'(x,\mu)\theta(x,\lambda)]_a^b+(\lambda-\mu)\int_a^b\theta(x,\lambda)\theta(x,\mu)dx=\omega(\lambda)m(\mu)-\omega(\mu)m(\lambda).$$

Using the boundary conditions in (5), this reduces to

$$\int_a^b\theta(x,\lambda)\theta(x,\mu)dx=\frac{\omega(\lambda)m(\mu)-\omega(\mu)m(\lambda)}{\lambda-\mu}. \tag{12}$$

Observing Lemma 1, we set $\mu=\mu_n$ in (12) and let $\lambda\to\mu_n$ to obtain (7). Thus Lemma 2 is proved.

Remark. In a similar fashion, from (12) we may deduce that

$$\int_a^b\theta(x,\lambda_n)\theta(x,\lambda_m)dx=\delta_{n,m}\omega'(\lambda_n)m(\lambda_n), \tag{13}$$

which is just another form of (4). Indeed, multiplying (9) by $\mathbf{w}(x)$ and integrating over $[a,b]$, we obtain

$$m(\lambda_n)=k_n\left(\int_a^b\mathbf{w}(x)\phi(x,\lambda_n)dx\right)^2. \tag{14}$$

Now by substituting (9) into (13), and using (14), we obtain (4).

Proof of Lemma 3. If we insert the result of Lemma B into the right side of Lemma A, we obtain the following estimate for $\phi(x,\lambda)$: As $|s|\to\infty$, we have

$$\begin{aligned}\phi(x,\lambda)&=\sin\alpha\cos s(x-a)-\frac{1}{s}\cos\alpha\sin s(x-a)\\&\quad+\frac{1}{2s}\sin\alpha\sin s(x-a)\int_a^x q(y)dy+o\left(\frac{e^{|t|(x-a)}}{|s|}\right),\end{aligned} \tag{15}$$

uniformly in $[a,b]$. Using the analogues of Lemmas A and B, we see in the same way that as $|s|\to\infty$, we have

$$\begin{aligned}\chi(x,\lambda)&=\sin\beta\cos s(b-x)+\frac{1}{s}\cos\beta\sin s(b-x)\\&\quad+\frac{1}{2s}\sin\beta\sin s(b-x)\int_x^b q(y)dy+o\left(\frac{e^{|t|(b-x)}}{|s|}\right)\end{aligned} \tag{16}$$

uniformly in $[a,b]$.

From (15), we get

$$\begin{aligned}\int_a^x\phi(u,\lambda)\mathbf{w}(u)du&=\sin\alpha\int_a^x\mathbf{w}(u)\cos s(u-a)du-\frac{1}{s}\cos\alpha\int_a^x\mathbf{w}(u)\sin s(u-a)du\\&\quad+\frac{1}{2s}\sin\alpha\int_a^x\mathbf{w}(u)\sin s(u-a)\int_a^u q(y)dydu+\int_a^x\mathbf{w}(u)o\left(\frac{e^{|t|(u-a)}}{|s|}\right)du.\end{aligned}$$

Now when we combine this last estimate with (16) in order to estimate the former member of (8), namely $\chi(x,\lambda)\int_a^x \phi(y,\lambda)\mathbf{w}(y)dy$, we obtain sixteen terms. Fortunately each of these terms is simple to analyze. During this analysis, use is made of integration by parts and of results of the type appearing in Lemma C. In the end we get: As $|s|\to\infty$, we have

$$\chi(x,\lambda)\int_a^x \phi(y,\lambda)\mathbf{w}(y)dy \tag{17}$$
$$= \frac{1}{s}\sin\alpha\sin\beta\sin s(x-a)\cos s(b-x)\mathbf{w}(x) + o\left(\frac{|t|(b-a)}{|s|}\right).$$

The corresponding result for the right-most member of (8) is: As $|s|\to\infty$, we have

$$\phi(x,\lambda)\int_x^b \chi(y,\lambda)\mathbf{w}(y)dy \tag{18}$$
$$= \frac{1}{s}\sin\alpha\sin\beta\cos s(x-a)\sin s(b-x)\mathbf{w}(x) + o\left(\frac{|t|(b-a)}{|s|}\right).$$

Finally, Lemma 3 is proved upon addition of (17) and (18).

Proof of Lemma 4. This follows from Lemma 3 upon multiplying by $\mathbf{w}(x)$ and integrating over $[a,b]$.

Proof of Lemma 5. The former statement was proved in [1] by applying Rouché's Theorem to

$$\omega(\lambda) = \sqrt{\lambda}\sin[\sqrt{\lambda}(b-a)]\sin\alpha\sin\beta + O(e^{|t|(b-a)}) \tag{19}$$
$$(= \text{ the } f(\lambda)+g(\lambda) \text{ of Rouché's Theorem}),$$

taken round the closed contour Γ_N with N large. The latter statement follows similarly by using the result of Lemma 4. (The function f in this case has one fewer zero in Γ_N than does the f in (19) since $\lambda = 0$ is no longer a zero.)

5. The proof of Theorem 1

Let $f \in L(a,b)$. By (9) and (13), the series in Theorem A can be written as

$$\sum_{n=1}^{\infty} \frac{1}{\omega'(\lambda_n)m(\lambda_n)}\int_a^b f(v)\theta(v,\lambda_n)dv\theta(x,\lambda_n),$$

and so, in view of Theorem A, to prove Theorem 1 it suffices to show that as $N\to\infty$ we have

$$\sum_{n=1}^{N} \frac{1}{\omega(\mu_n)m'(\mu_n)}\int_a^b f(v)\theta(v,\mu_n)dv\theta(x,\mu_n)$$
$$+\sum_{n=1}^{N+1} \frac{1}{\omega'(\lambda_n)m(\lambda_n)}\int_a^b f(v)\theta(v,\lambda_n)dv\theta(x,\lambda_n) \to \theta_o(x)\int_a^b f(v)\theta_o(v)dv.$$

By Lemma 5, the left-hand side here is the sum of the residues of the function

$$\frac{1}{\omega(\lambda)m(\lambda)}\int_a^b f(v)\theta(v,\lambda)dv\theta(x,\lambda)$$

inside the closed contour Γ_N when N is large; so it suffices to show that as $N \to \infty$ we have

$$\frac{1}{2\pi i}\int_{\Gamma_N}\frac{1}{\omega(\lambda)m(\lambda)}\int_a^b f(v)\theta(v,\lambda)dv\theta(x,\lambda)d\lambda \to \theta_o(x)\int_a^b f(v)\theta_o(v)dv.$$

We put $\lambda = s^2$, which transforms this integral to

$$\frac{1}{2\pi i}\int_{\gamma_N}\frac{1}{\omega(\lambda)m(\lambda)}\int_a^b f(v)\theta(v,\lambda)dv\theta(x,\lambda)2sds,$$

and we examine the size of each of its terms as $N \to \infty$ and, correspondingly, as $|s| \to \infty$ on the contour.

From [1], we have

$$\frac{1}{\omega(\lambda)} = \frac{1}{s\sin\alpha\sin\beta\sin s(b-a)}\left[1+O\left(\frac{1}{|s|}\right)\right], \tag{20}$$

and similarly, from Lemma 4, we have

$$\frac{1}{m(\lambda)} = \frac{1}{\|\mathbf{w}\|^2\sin\alpha\sin\beta\sin s(b-a)}[1+o(1)]. \tag{21}$$

Each of these results depends on the observation that $|\sin s(b-a)| > A\,e^{|t|(b-a)}$ on γ_N, where A is some absolute constant. This observation also allows us to deduce from Lemma 3 that

$$\theta(x,\lambda) = \sin\alpha\sin\beta\frac{\sin s(b-a)}{s}\mathbf{w}(x)[1+o(1)], \tag{22}$$

and

$$\begin{aligned}\int_a^b f(v)\theta(v,\lambda)dv &= \sin\alpha\sin\beta\frac{\sin s(b-a)}{s}\int_a^b \mathbf{w}(v)f(v)dv + o\left(\frac{e^{|t|(b-a)}}{|s|}\right)\\ &= \sin\alpha\sin\beta\frac{\sin s(b-a)}{s}\int_a^b \mathbf{w}(v)f(v)dv[1+o(1)],\end{aligned} \tag{23}$$

again from Lemma 3. The results (20)–(23) together give

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\gamma_N}\frac{1}{\omega(\lambda)m(\lambda)}\int_a^b f(v)\theta(v,\lambda)dv\theta(x,\lambda)2sds\\ &\quad= \frac{\mathbf{w}(x)}{\|\mathbf{w}\|^2}\int_a^b f(v)\mathbf{w}(v)dv\frac{1}{2\pi i}\int_{\gamma_N}\frac{1}{s}[1+o(1)]2ds\\ &\quad= \frac{\mathbf{w}(x)}{\|\mathbf{w}\|^2}\int_a^b f(v)\mathbf{w}(v)dv\frac{1}{2\pi i}\int_{\Gamma_N}\frac{1}{\lambda}[1+o(1)]d\lambda.\end{aligned}$$

As $N \to \infty$, the integral here tends to unity and the term multiplying it is $\theta_o(x)\int_a^b f(v)\theta_o(v)dv$, so the proof of Theorem 1 is complete.

References

1. E. C. Titchmarsh, *Eigenfunction Expansions–Part 1*, 2nd ed., Oxford University Press, 1962.
2. A. McD. Mercer, *A class of bi-orthogonal expansions arising from the Sturm-Liouville equation*, Quart. J. Math. (Oxford) **18** (1967), 207–217.

Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario N1G 2W1, Canada

Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599–3250 U.S.A.

Current address: Department of Mathematics, Purdue University, West Lafayette, IN. 47907–1395 U.S.A.