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Kramer-type sampling theorems associated with Fredholm integral operators

Mahmoud H. Annaby and M. A. El-Sayed

Abstract. In a recent paper of Zayed [18], a new Kramer-type sampling theorem was established. In Kramer's theorem, it is assumed that the kernel associated with the theorem arises from a self-adjoint boundary-value problem whose eigenvalues are all real and simple and the eigenfunctions are all generated by one single function. In Zayed's theorem, the class of the integral transforms has been extended to a larger one. For instance, the kernel associated with the theorem may arise from a non-self-adjoint boundary-value problem with repeated eigenvalues. The technique used in the theorem is based on the use of Green's functions for boundary-value problems to reconstruct the interpolating functions. In this paper we make use of Zayed's idea to obtain a sampling theorem associated with a Fredholm integral operator of the second kind.

1. Introduction

Kramer's sampling theorem [9], which generalizes the Whittaker-Shannon-Kotel'nikov sampling theorem (WSKST), has been studied extensively, see [2,3]. It imposes many questions. One of these questions is from what situation do the kernel and the sampling points associated with the theorem come? This question is answered partially by Kramer in [9]. He assumed that the kernel and the sampling points arise naturally when one solves certain self-adjoint boundary-value problems whose eigenvalues are all real and simple and all eigenfunctions are generated by one single function. More precisely, consider the boundary-value problem

$$\ell(y) = \lambda y, \tag{1.1}$$

$$U_j(y) = 0, \quad j = 1, 2, \ldots, n, \tag{1.2}$$

where

$$\ell(y) = \sum_{i=0}^{n} p_i(x) y^{(n-i)}(x), \quad -\infty < a \le x \le b < \infty, \tag{1.3}$$

is any self-adjoint differential expression [5,11], and

$$U_j(y) = \sum_{i=1}^{n} \alpha_{ij} y^{(i-1)}(a) + \beta_{ij} y^{(i-1)}(b) \tag{1.4}$$

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are n linearly independent forms of $y^{(i-1)}(a)$, $y^{(i-1)}(b)$, $i = 1, 2, \ldots, n$, such that conditions (1.2) are self-adjoint, cf. [11]. Here $p_i(x)$, $i = 0, 1, \ldots, n$, are assumed to be complex-valued continuous functions with $n - i$ continuous derivatives, $p_0(x) \neq 0$ for $x \in [a, b]$. Assume that $\phi(x, \lambda)$ is a solution of (1.1) and $\{\lambda_k\}$, $\{\phi(x, \lambda_k)\}$ are the sequences of eigenvalues and their corresponding eigenfunctions. Let $f(x) \in L^2(a, b)$ and

$$F(\lambda) = \int_a^b f(x)\phi(x, \lambda)\, dx. \tag{1.5}$$

Then $F(\lambda)$ will admit the sampling series

$$F(\lambda) = \sum_{k=0}^{\infty} F(\lambda_k) \frac{\int_a^b \phi(x, \lambda)\overline{\phi(x, \lambda_k)}\, dx}{\int_a^b |\phi(x, \lambda_k)|^2 dx}. \tag{1.6}$$

Other questions which are motivated by Kramer's theorem involve the relationship between Kramer's theorem and WSKST on one hand, and on the other hand, another generalization of WSKST, which is due to Paley and Wiener [12]. Campbell [3] showed that Kramer's theorem gives nothing more than WSKST if problem (1.1)–(1.2) is of the first or second order. Recent papers [1,15-18] dealt with the second part of the question and showed that Kramer's theorem gives nothing more than the Paley-Wiener (Lagrange interpolation) one.

To the best of our knowledge, in the literature dealing with Kramer's theorem and its motivations, the kernel and the sampling points were always assumed to come from ordinary differential operators. It is worth mentioning that the idea of using ordinary differential operators in sampling theory goes back to Weiss [14]. It is known that any boundary-value problem of the type (1.1)–(1.2) can be transformed into an integral equation, but no sampling theorem associated with integral equations has been developed so far. This paper is devoted to this task. In more detail, let $G(x, \xi)$ be the Green's function associated with the differential equation $\ell(y) = 0$ and the boundary conditions (1.2). Then it is known [5,11,13] that the boundary-value problem (1.1)–(1.2), which is not necessarily self-adjoint, is equivalent to the Fredholm integral equation of the second kind

$$y(x) = \lambda \int_a^b G(x, \xi) y(\xi)\, d\xi. \tag{1.7}$$

Also it is known that the Green's function $G(x, \xi, \lambda)$ associated with problem (1.1)–(1.2) is the resolvent kernel associated with equation (1.7) when problem (1.1)–(1.2) is self-adjoint [4,5]. In the next sections, we are going to derive Kramer-type sampling theorems associated with the integral equation

$$y(x) = \lambda \int_a^b K(x, \xi) y(\xi)\, d\xi, \tag{1.8}$$

where $K(x, \xi)$ is any L^2-kernel. The resolvent kernel $R(x, \xi, \lambda)$ plays, as expected, the role played by the Green's function in [7,18] (see Theorem 4.1). In Theorem 5.1, the kernel of the integral transform, associated with the sampling theorem, is expressed in terms of a solution of the inhomogeneous integral equation. In both cases we confine ourselves to the case where K is symmetric and the poles of the resolvent kernel are simple.

The main results of the paper are formulated in Theorems 4.1 and 5.1 below. Section 7 is devoted to some illustrative examples concerning the sampling theorems associated with the Fredholm integral equation.

2. The use of Green's functions in sampling theory

In this section, we mention briefly Zayed's idea of how Green's functions can be used to derive sampling theorems.

Consider the boundary-value problem

$$\ell(y) = \lambda y, \tag{2.1}$$

$$U_j(y) = 0, \quad j = 1, 2, \ldots, n, \tag{2.2}$$

where $\ell(\cdot)$, $U_j(\cdot)$, $j = 1, 2, \ldots, n$, are given in (1.3), (1.4). Both $\ell(\cdot)$ and $U_j(\cdot)$ are not necessarily self-adjoint. Let $G(x, \xi, \lambda)$ be the Green's function associated with (2.1)–(2.2). The construction of this function is given in [5,11]. Let $\{\lambda_k\}$ be the sequence of the eigenvalues of the problem (2.1)–(2.2). Assume that they have the asymptotic behaviour

$$\lambda_k = O\left\{\left(\frac{k\pi}{b-a}\right)^n\right\}, \quad |k| \to \infty, \tag{2.3}$$

where n is the order of the differential equation (2.1). Now define the entire function

$$P(\lambda) = \begin{cases} \lambda \displaystyle\prod_{k=1}^{\infty}(1 - \frac{\lambda}{\lambda_k}), & \text{if } \lambda_0 = 0 \text{ is an eigenvalue,} \\ \displaystyle\prod_{k=0}^{\infty}(1 - \frac{\lambda}{\lambda_k}), & \text{if } 0 \text{ is not an eigenvalue.} \end{cases} \tag{2.4}$$

Obviously, $P(\lambda)$ is well defined for $n > 1$. If $n = 1$ then $P(\lambda)$ will take the form

$$P(\lambda) = \begin{cases} \lambda \displaystyle\prod_{k=1}^{\infty}(1 - \frac{\lambda}{\lambda_k}) \exp(\frac{\lambda}{\lambda_k}), & \text{if } \lambda_0 = 0 \text{ is an eigenvalue,} \\ \displaystyle\prod_{k=0}^{\infty}(1 - \frac{\lambda}{\lambda_k}) \exp(\frac{\lambda}{\lambda_k}), & \text{if } 0 \text{ is not an eigenvalue.} \end{cases} \tag{2.5}$$

If the poles of $G(x, \xi, \lambda)$, which are precisely the eigenvalues of the problem, are assumed to be simple, then the function

$$\Phi(x, \lambda) = P(\lambda) G(x, \xi_0, \lambda) \tag{2.6}$$

is an entire function of λ for $x \in [a, b]$. Here ξ_0 is chosen in $[a, b]$ as indicated in [18].

Theorem 2.1 ([18]). *If $f \in L^2(a, b)$ and*

$$F(\lambda) = \int_a^b \overline{f(x)} \Phi(x, \lambda)\, dx, \tag{2.7}$$

then $F(\lambda)$ is an entire function of λ of order not exceeding $1/n$ that admits the sampling representation

$$F(\lambda) = \sum_{k=0}^{\infty} F(\lambda_k) \frac{P(\lambda)}{(\lambda - \lambda_k) P'(\lambda_k)}. \tag{2.8}$$

Moreover, if problem (2.1)–(2.2) *is self-adjoint or if f satisfies the boundary conditions adjoint to* (2.2), *then series* (2.8) *is uniformly convergent on any compact subset of the complex λ-plane.*

3. Preliminaries

Consider the integral equation

$$y(x) = \lambda \int_a^b K(x,\xi) y(\xi)\, d\xi, \tag{3.1}$$

where $K(x,\xi)$ is an $L^2(S)$-function and $\lambda \in \mathbb{C}$. Here S is the square $[a,b] \times [a,b]$. Moreover K is assumed to be symmetric, i.e.,

$$K(x,\xi) = \overline{K(\xi,x)}, \quad a \le x, \xi \le b, \tag{3.2}$$

If (3.1) has a non-trivial solution $\phi(x)$ for some λ, we say that λ is an eigenvalue of equation (3.1), or of the kernel $K(x,\xi)$, and $\phi(x)$ is an eigenfunction of equation (3.1), or of the kernel $K(x,\xi)$, corresponding to the eigenvalue λ. In the case that an eigenvalue λ has more than one eigenfunction we define the multiplicity of λ to be the number of the linearly independent eigenfunctions corresponding to λ. It is well-known [4,6] that the multiplicity of any eigenvalue of equation (3.1) is finite.

Equation (3.1) may have no solution except the trivial one, i.e., it has no eigenvalue. The Volterra kernel ($K(x,\xi) = 0$, for $x > \xi$) is an example of this case [6,13]. In some other cases, equation (3.1) may have only a finite number of eigenvalues. For instance degenerate (finite rank) kernels, i.e., kernels of the form

$$K(x,\xi) = \sum_{i=1}^{r} a_i(x) b_i(\xi),$$

can have only a finite number of eigenvalues.

The case of interest is when equation (3.1) has infinitely many eigenvalues. In this case [4,6], the set of eigenvalues is countable and has no finite limit point. The only possible limit points are $\pm\infty$. For this reason the eigenvalues of equation (3.1) can be ordered according to their absolute values in a sequence

$$0 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_k| \le \cdots, \tag{3.3}$$

where $|\lambda_k| \to \infty$ when $k \to \infty$. We can repeat an eigenvalue in the above sequence as many times as its multiplicity. Then we can also consider the sequence of corresponding eigenfunctions to be

$$\phi_1(x), \phi_2(x), \ldots, \phi_k(x), \ldots. \tag{3.4}$$

Since the symmetric kernel $K(x,\xi)$ generates a self-adjoint operator in $L^2(a,b)$, the eigenvalues of equation (3.1) are all real [6,13], and the eigenfunctions corresponding to different eigenvalues are orthogonal. So we may assume that the set $\{\phi_k(x)\}$ is orthogonal since we can apply the Gram-Schmidt orthogonalization scheme if necessary.

If the equation

$$\int_a^b K(x,\xi) y(\xi)\, d\xi = 0 \tag{3.5}$$

has a nontrivial solution $\phi(x)$, then we say that ∞ is an eigenvalue of the equation (3.1) with an eigenfunction $\phi(x)$. Degenerate kernels defined above [4] are examples of such kernels. If ∞ is not an eigenvalue of equation (3.1), then the kernel $K(x,\xi)$ is called a closed kernel. The Green's function in equation (1.7) is an example of a closed kernel. This fact directly follows from the completeness of the eigenfunctions of the Green's function $G(x,\xi)$ [5], and that a kernel K is closed if and only if $\{\phi_k(x)\}$ is complete [4]. From now on we consider closed kernels only. Thus we can assume that the set $\{\phi_k(x)\}$ is a complete orthonormal set in $L^2(a,b)$.

Let $R(x,\xi,\lambda)$ be the resolvent kernel associated with equation (3.1) (see [4] for definition). Then we have the following

Lemma 3.1 ([4]). *The resolvent kernel $R(x,\xi,\lambda)$ is an L^2-function for any $\lambda \neq \lambda_k$, that admits the expansion*

$$R(x,\xi,\lambda) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\overline{\phi_k(\xi)}}{\lambda_k - \lambda}, \quad \lambda \neq \lambda_k. \tag{3.6}$$

The convergence is in the L^2-norm.

Since each eigenvalue may have more than one eigenfunction, relation (3.6) may be written in the form

$$R(x,\xi,\lambda) = \sum_{k=1}^{\infty}\sum_{\nu=1}^{\nu_k} \frac{\phi_{k,\nu}(x)\overline{\phi_{k,\nu}(\xi)}}{\lambda_k - \lambda}, \quad \lambda \neq \lambda_k, \tag{3.7}$$

where ν_k is the multiplicity of λ_k. Since $\{\phi_k\}$ are L^2-functions, each of them exists almost everywhere. Let $\xi_0 \in [a,b]$ such that $\phi_k(\xi_0)$ is finite for all k. Define the function $\phi(x,\lambda)$ to be

$$\phi(x,\lambda) = R(x,\xi_0,\lambda) = \sum_{k=1}^{\infty} \frac{\overline{\phi_k(\xi_0)}}{\lambda_k - \lambda}\phi_k(x) = \sum_{k=1}^{\infty}\sum_{\nu=1}^{\nu_k} \frac{\phi_{k,\nu}(x)\overline{\phi_{k,\nu}(\xi_0)}}{\lambda_k - \lambda}, \quad \lambda \neq \lambda_k. \tag{3.8}$$

Since $\{\phi_k(x)\}$ is complete, (3.8) can be viewed as the Fourier expansion of $\phi(x,\lambda)$ with the Fourier coefficients $\frac{\overline{\phi_k(\xi_0)}}{\lambda_k-\lambda}, \lambda \neq \lambda_k$. Also, $\phi(x,\lambda)$ is a meromorphic function with simple poles λ_k. The residue at each pole λ_k is

$$r_k = \sum_{\nu=1}^{\nu_k} \phi_{k,\nu}(x)\overline{\phi_{k,\nu}(\xi_0)}. \tag{3.9}$$

The sequence of the eigenvalues $\{\lambda_k\}$ given in (3.3) satisfies (see [4, pp. 48, 88])

$$\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\right)^2 < \infty. \tag{3.10}$$

Define the entire function $\omega(\lambda)$ to be

$$\omega(\lambda) = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_k}\right), \tag{3.11}$$

when $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$, and

$$\omega(\lambda) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right) \exp(\frac{\lambda}{\lambda_k}), \tag{3.12}$$

otherwise. Due to the self-adjointness of the problem in hand, the poles of the resolvent kernel are all simple. Then in the canonical products (3.11)–(3.12), each λ_k will appear once. This does not require that the λ_k be simple eigenvalues, but rather simple zeros of $\omega(\lambda)$.

The function

$$\Phi(x, \lambda) = \omega(\lambda)\phi(x, \lambda) \tag{3.13}$$

is an entire function of λ for each fixed x.

4. The sampling theorem

In this section we derive a sampling theorem associated with the Fredholm integral equation (3.1). Here the kernel of the transformation is $\Phi(x, \lambda)$ defined in (3.13) above.

Theorem 4.1. *Let $f \in L^2(a, b)$ and*

$$F(\lambda) = \int_a^b \overline{f(x)}\Phi(x, \lambda)\, dx. \tag{4.1}$$

Then $F(\lambda)$ is an entire function of λ of order not exceeding 2 that admits the sampling representation

$$F(\lambda) = \sum_{k=1}^{\infty} F(\lambda_k) \frac{\omega(\lambda)}{(\lambda - \lambda_k)\omega'(\lambda_k)}. \tag{4.2}$$

Moreover the sampling series (4.2) *is uniformly convergent on any compact subset of the complex λ-plane.*

Proof. By the Cauchy-Schwarz inequality, we have

$$|F(\lambda)|^2 \le \left(\int_a^b |f(x)|^2\, dx\right) \left(\int_a^b |\Phi(x, \lambda)|^2\, dx\right) < \infty. \tag{4.3}$$

Thus $F(\lambda)$ is well-defined. Also, that $F(\lambda)$ is an entire function of λ with order ≤ 2 follows directly from (4.3) and the fact that $\Phi(x, \lambda)$ satisfies these properties [4, p. 50].

Since both f and Φ are L^2-functions and $\{\phi_k(x)\}$ is a complete orthonormal set in $L^2(a, b)$, then

$$f(x) = \sum_{k=1}^{\infty} \langle f, \phi_k\rangle \phi_k(x), \tag{4.4}$$

and

$$\Phi(x, \lambda) = \sum_{k=1}^{\infty} \langle \Phi, \phi_k\rangle \phi_k(x), \tag{4.5}$$

are the Fourier series of f and Φ, respectively. Here $\langle f, \phi_k\rangle$ and $\langle \Phi, \phi_k\rangle$ are the Fourier coefficients. Using Parseval's identity, we get

$$F(\lambda) = \sum_{k=1}^{\infty} \overline{\langle f, \phi_k\rangle} \langle \Phi, \phi_k\rangle. \tag{4.6}$$

Equation (4.6) can be rewritten in the form

$$F(\lambda) = \sum_{k=1}^{\infty} \sum_{\nu=1}^{\nu_k} \overline{\langle f, \phi_{k,\nu}\rangle} \langle \Phi, \phi_{k,\nu}\rangle. \tag{4.7}$$

From (3.8) and (3.13), we have

$$\langle \Phi, \phi_{k,\nu}\rangle = \frac{\omega(\lambda)}{\lambda_k - \lambda} \overline{\phi_{k,\nu}(\xi_0)}. \tag{4.8}$$

From (3.13) and (4.1), we have

$$F(\lambda) = \omega(\lambda) \int_a^b \overline{f(x)} \phi(x, \lambda)\, dx.$$

Therefore, using (3.8), we get

$$\begin{aligned} F(\lambda_k) &= \lim_{\lambda\to\lambda_k} \frac{\omega(\lambda)}{\lambda - \lambda_k} \int_a^b (\lambda - \lambda_k) \overline{f(x)} \phi(x, \lambda) dx \\ &= -\omega'(\lambda_k) \sum_{\nu=1}^{\nu_k} \overline{\phi_{k,\nu}(\xi_0)} \int_a^b \overline{f(x)} \phi_{k,\nu}(x) dx \\ &= -\omega'(\lambda_k) \sum_{\nu=1}^{\nu_k} \overline{\phi_{k,\nu}(\xi_0)}\; \overline{\langle f, \phi_{k,\nu}\rangle}. \end{aligned} \tag{4.9}$$

Substituting (4.8), (4.9) in (4.7), one gets (4.2).

Finally, to prove the uniform convergence of the series (4.2), we use (4.6) and the Cauchy-Schwarz inequality to obtain

$$|F(\lambda) - \sum_{k=1}^{N-1} \overline{\langle f, \phi_k\rangle} \langle \Phi, \phi_k\rangle| \le \left(\sum_{k=N}^{\infty} |\langle f, \phi_k\rangle|^2 \right)^{1/2} \left(\sum_{k=N}^{\infty} |\langle \Phi, \phi_k\rangle)|^2 \right)^{1/2}$$

But, in view of Bessel's inequality [4], the series

$$\sum_{k=1}^{\infty} |\langle f, \phi_k\rangle|^2$$

converges. Again from Bessel's inequality, we have

$$\sum_{k=N}^{\infty} |\langle \Phi, \phi_k\rangle)|^2 \le \|\Phi(x, \lambda)\|^2.$$

From equation (2), p. 50 in [4], $\|\Phi(x,\lambda)\|^2$ is uniformly bounded on compact subsets of the complex λ-plane. Thus, if M is a compact subset of the complex plane, then there exists a positive constant $C(M)$ such that

$$\sum_{k=N}^{\infty} |\langle \Phi, \phi_k \rangle|^2 \leq C(M), \quad \lambda \in M.$$

Hence, for $\lambda \in M$

$$\left| F(\lambda) - \sum_{k=1}^{N-1} \overline{\langle f, \phi_k \rangle} \langle \Phi, \phi_k \rangle \right| \leq \sqrt{C(M)} \left(\sum_{k=N}^{\infty} |\langle f, \phi_k \rangle|^2 \right)^{1/2} \to 0 \text{ as } N \to \infty.$$

Remark 1. Note that Theorem 4.1 gives a family of sampling theorems due to the fact that the kernel $\Phi(x,\lambda)$ depends on the choice of $\xi_0 \in [a,b]$. In some cases (see, for instance, example 1 below) it may happen that for some $\xi_0 \in [a,b]$

$$\phi_k(\xi_0) = 0, \quad \text{for all } k. \tag{4.10}$$

On the other hand, as example 2 shows, there is no $\xi_0 \in [a,b]$ such that (4.10) takes place. Therefore the values of ξ_0 that make (4.10) true depend on the nature of the system of eigenfunctions. So, it might be difficult to classify ξ_0's which fulfill (4.10), i.e., those ξ_0's that are inadequate for constructing a (nontrivial) sampling theorem. However, since $\{\phi_k\}$ is a complete orthonormal system, there are always $\xi_0 \in [a,b]$ that could be used to achieve our aim.

Remark 2. As in [18], the kernel $\Phi(x,\lambda)$ is a Kramer-type kernel [9] only when all eigenvalues of the problem are simple. In this case we have

(1) $\Phi(x,\lambda) \in L^2(a,b), \quad \lambda \in \mathbb{C}$,

(2) $\{\Phi(x,\lambda_k)\} = \{\omega'(\lambda_k)\overline{\phi_k(\xi_0)}\phi_k(x)\}$ which is a complete orthogonal set in $L^2(a,b)$.

Remark 3. Although in Theorem 4.1 the upper estimate of the order of the sampled function is 2, the order of the entire function $F(\lambda)$ may be very small (cf. example 2). This fact directly follows from the dependence of the order on the convergence exponent of the sequence $\{\lambda_k\}$, i.e., the infimum of the positive numbers γ such that $\sum |\lambda_k|^{-\gamma} < \infty$.

Remark 4. The previous results can be extended to the non-self-adjoint operators, i.e., when $K(x,\xi)$ is not symmetric. In this case, to maintain a sampling theorem, we assume the poles of the resolvent kernel to be simple. To prove this, we follow the same line established in Theorem 4.1 above and use the following facts.

(1) $R(x,\xi,\lambda)$ is a meromorphic function of λ having simple poles at each eigenvalue λ_k. The corresponding residue is

$$r_k = \sum_{\nu=1}^{\nu_k} \phi_{k,\nu}(x)\overline{\psi_{k,\nu}(\xi)},$$

where ν_k is the multiplicity of the eigenvalue λ_k, and $\{\psi_{k,\nu}(x)\}$ are the eigenfunctions of the equation adjoint to (3.1) corresponding to the eigenvalue λ_k [4,8].

(2) The sets of the eigenfunctions of (3.1) and its adjoint are complete [4, Chapter 14].

(3) The set of the eigenvalues $\{\lambda_k\}$ satisfies $\sum |\lambda_k|^{-2} < \infty$.

Remark 5. The condition for uniform convergence of the sampling series in [18] is superfluous here.

5. Another sampling theorem

In this section, we derive another sampling theorem connected with (3.1). The kernel of the integral transform, associated with this sampling theorem, is expressed in terms of a solution of the inhomogeneous integral equation instead of the resolvent kernel. The advantage of this approach is twofold. First, it resembles the ideas in one of the main directions to construct sampling theorems for differential equations. Second, it eliminates the need for defining ξ_0 mentioned above.

Note that the kernel of the integral transform associated with the sampling theorem must be defined at each $\lambda \in \mathbb{C}$. But, since (3.1) has only a discrete set of eigenvalues, we use the inhomogenuous equation instead.

Consider the homogeneous Fredhom integral equation

$$y(x) = \lambda \int_a^b K(x,\xi)y(\xi)\,d\xi \tag{5.1}$$

and its corresponding inhomogeneous equation

$$y(x) = \lambda \int_a^b K(x,\xi)y(\xi)\,d\xi + h(x), \quad h \in L^2(a,b). \tag{5.2}$$

Let $\{\lambda_k\}$ be the sequence of eigenvalues of equation (5.1) and $\{\phi_k(x)\}$ the set of the corresponding eigenfunctions. We assume, without loss of generality, that $\{\lambda_k\}$ are all simple. Equation (5.2), cf. [4], has a unique solution, $\psi(x,\lambda)$, $\lambda \neq \lambda_k$, which can be written in the form

$$\psi(x,\lambda) = h(x) + \lambda \int_a^b R(x,\xi,\lambda)h(\xi)\,d\xi, \tag{5.3}$$

where $R(x,\xi,\lambda)$ is the resolvent kernel defined above. Also, [4, p. 195], $\psi(x,\lambda)$ may be expanded in the form

$$\psi(x,\lambda) = h(x) + \lambda \sum_{k=1}^{\infty} \frac{\widehat{h}(k)}{\lambda_k - \lambda}\phi_k(x), \tag{5.4}$$

where $\widehat{h}(k)$ are the Fourier coefficients of h. Define

$$\Psi(x,\lambda) = \omega(\lambda)\psi(x,\lambda), \tag{5.5}$$

where

$$\omega(\lambda) = \begin{cases} \prod_{k=1}^{\infty}\left(1 - \dfrac{\lambda}{\lambda_k}\right), & \text{if } \sum_{k=1}^{\infty} \dfrac{1}{\lambda_k} < \infty, \\ \prod_{k=1}^{\infty}\left(1 - \dfrac{\lambda}{\lambda_k}\right)\exp(\lambda/\lambda_k), & \text{otherwise.} \end{cases} \tag{5.6}$$

Set

$$\psi_1(x,\lambda) = \sum_{k=1}^{\infty} \frac{\widehat{h}(k)}{\lambda_k - \lambda} \phi_k(x), \quad \lambda \neq \lambda_k. \tag{5.7}$$

Then we have the following sampling theorem

Theorem 5.1. *Let $f(x) \in L^2(a,b)$ and*

$$F(\lambda) = \int_a^b \overline{f(x)} \Psi(x,\lambda)\, dx. \tag{5.8}$$

Then $F(\lambda)$ is an entire function of λ with order ≤ 2 that admits the sampling expansion

$$F(\lambda) = \sum_{k=1}^{\infty} F(\lambda_k) \frac{\omega(\lambda)}{(\lambda - \lambda_k)\omega'(\lambda_k)}. \tag{5.9}$$

The series (5.9) is uniformly convergent on compact subsets of the complex λ-plane.

Proof. Using the Cauchy-Schwarz inequality, equation (5.3), and that both $\omega(\lambda)$ and $\omega(\lambda)R(x,\xi,\lambda)$ are entire functions of order ≤ 2 [4, p. 50], we conclude that the function $F(\lambda)$ exists, entire of order not exceeding 2.

From (5.4) we get

$$F(\lambda) - \omega(\lambda) \int_a^b \overline{f(x)} h(x)\, dx = \lambda \int_a^b \overline{f(x)} \psi_1(x,\lambda). \tag{5.10}$$

Parseval's identity and expansion (5.7) lead to

$$F(\lambda) - \omega(\lambda) \int_a^b \overline{f(x)} h(x)\, dx = \lambda \sum_{k=1}^{\infty} \overline{\widehat{f}(k)}\, \widehat{h}(k) \frac{\omega(\lambda)}{\lambda_k - \lambda}. \tag{5.11}$$

But we have

$$\begin{aligned} F(\lambda_k) &= \lim_{\lambda \to \lambda_k} \int_a^b \overline{f(x)} \Psi(x,\lambda)\, dx \\ &= \lim_{\lambda \to \lambda_k} \frac{\omega(\lambda)}{\lambda - \lambda_k} \int_a^b \lambda(\lambda - \lambda_k) \overline{f(x)} \psi_1(x,\lambda)\, dx \\ &= -\omega'(\lambda_k)\lambda_k \overline{\widehat{f}(k)}\, \overline{h(k)}. \end{aligned} \tag{5.12}$$

Hence, combining (5.11), (5.12), we get

$$F(\lambda) = \omega(\lambda) \int_a^b \overline{f(x)} h(x)\, dx + \sum_{k=1}^{\infty} F(\lambda_k) \frac{\lambda \omega(\lambda)}{\lambda_k (\lambda - \lambda_k)\omega'(\lambda_k)}. \tag{5.13}$$

Applying Parseval's identity, we obtain

$$\int_a^b \overline{f(x)} h(x)\, dx = \sum_{k=1}^{\infty} \widehat{h}(k) \overline{\widehat{f}(k)}.$$

Using (5.12), one gets

$$\sum_{k=1}^{\infty} \widehat{h}(k) \overline{\widehat{f}(k)} = \sum_{k=1}^{\infty} \frac{-F(\lambda_k)}{\lambda_k \omega'(\lambda_k)}.$$

Thus

$$F(\lambda) = \sum_{k=1}^{\infty} -F(\lambda_k)\frac{\omega(\lambda)}{\lambda_k \omega'(\lambda_k)} + \sum_{k=1}^{\infty} F(\lambda_k)\frac{\lambda\omega(\lambda)}{\lambda_k(\lambda-\lambda_k)\omega'(\lambda_k)}.$$

A simple manipulation yields

$$F(\lambda) = \sum_{k=1}^{\infty} F(\lambda_k)\frac{\omega(\lambda)}{(\lambda-\lambda_k)\omega'(\lambda_k)}.$$

To prove the uniform convergence of series (5.9), it sufficies to show that the series

$$\sum_{k=1}^{\infty} F(\lambda_k)\frac{\lambda\omega(\lambda)}{\lambda_k(\lambda-\lambda_k)\omega'(\lambda_k)} \tag{5.14}$$

converges uniformly on every compact subset M of the complex λ-plane.

Indeed, from (5.12), we have

$$\sum_{k=1}^{\infty} F(\lambda_k)\frac{\lambda\omega(\lambda)}{\lambda_k(\lambda-\lambda_k)\omega'(\lambda_k)} = \sum_{k=1}^{\infty} \widehat{h}(k)\overline{\widehat{f}(k)}\frac{\lambda\omega(\lambda)}{\lambda_k-\lambda}.$$

But there exists a positive constant $C(M)$ such that, cf. [18],

$$\left|\frac{\lambda\omega(\lambda)}{\lambda_k-\lambda}\right| \le C(M), \quad \lambda \in M, \ k = 1, 2, \ldots .$$

Thus

$$\left|\widehat{h}(k)\overline{\widehat{f}(k)}\frac{\lambda\omega(\lambda)}{\lambda_k-\lambda}\right| \le C(M)|\widehat{h}(k)\widehat{f}(k)|.$$

Since h, f are L^2-functions, then

$$\sum_{k=1}^{\infty} |\widehat{h}(k)\widehat{f}(k)| \le \left(\sum_{k=1}^{\infty} |\widehat{h}(k)|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |\widehat{f}(k)|^2\right)^{1/2} < \infty.$$

The uniform convergence of (5.14) now follows from the Weierstrass test for uniform convergence.

Remark 6. The idea of using the inhomogenuous equations to obtain sampling theorems was suggested by Professor Gilbert G. Walter during the Conference of Mathematical Analysis and Signal Processing held in Cairo University, Jan. 3–9, 1994. He suggested that the kernel $\Psi(x, \lambda)$ of the integral transform associated with the sampling theorem should be a solution of the equation

$$y(x) = \lambda \int_a^b K(x, \xi) y(\xi)\, d\xi + \omega(\lambda),$$

i.e., $\psi(x, \lambda)$ is a solution of (5.2) when $h(x) = 1$.

Remark 7. Note also that the sampling theorem, 5.1, depends on h. Therefore, we have again a family of sampling theorems.

6. Some extensions

The results of Sections 3, 4, and 5 above can be extended to the equations

$$y(x) = \int_0^\infty K(x,\xi) y(\xi)\, d\xi \tag{6.1}$$

and

$$y(x) = \int_{-\infty}^\infty K(x,\xi) y(\xi)\, d\xi, \tag{6.2}$$

where $K(x,\xi)$ is a symmetric L^2-function on the corresponding domain. In fact, we can consider a transformation of x and ξ that carries the infinite interval to a finite one [13, pp.151–152]. However, it might happen that the new kernel has some singularities, but, in spite of this, it is an L^2-kernel.

7. Examples

Here we give some examples illustrating how to obtain sampling expansions associated with Fredholm integral equation. The first two examples are applications of Theorem 4.1. The next two examples are devoted to the case discussed in §6. The last example illustrates how to obtain sampling expansions associated with inhomogenuous equations.

Example 1. Let $K(x,\xi)$ be the kernel

$$K(x,\xi) = \min_{0\le x,\xi\le 1}(x,\xi) - x\xi. \tag{7.1}$$

The eigenvalues of this kernel [10] are $\lambda_k = k^2\pi^2$, $k = 1,2,\dots$ and the corresponding eigenfunctions are $\phi_k(x) = \sqrt{2}\sin k\pi x$. If ξ_0 is chosen in $]0,1[$, then the kernel is

$$\Phi(x,\lambda) = \omega(\lambda) R(x,\xi_0,\lambda),$$

where

$$\omega(\lambda) = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{k^2\pi^2}\right) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}$$

and

$$R(x,\xi,\lambda) = \sum_{k=1}^{\infty} 2\frac{\sin k\pi x \sin k\pi\xi}{k^2\pi^2}, \quad \lambda \ne k^2\pi^2.$$

The sampling series takes the form

$$F(\lambda) = \sum_{k=1}^{\infty}(-1)^k F(k^2\pi^2)\frac{2k^2\pi^2\sin\sqrt{\lambda}}{\sqrt{\lambda}(\lambda - k^2\pi^2)}.$$

Remark 8. In the above example, the resolvent kernel $R(x,\xi,\lambda)$ [4,5] is the Green's function associated with the boundary-value problem

$$-y'' = \lambda y, \quad y(0) = y(1) = 0,$$

which takes the form

$$R(x,\xi,\lambda) = G(x,\xi,\lambda) = \begin{cases} \dfrac{\sin\sqrt{\lambda}x \ \sin\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}\ \sin\sqrt{\lambda}}, & 0 \le x \le \xi, \\ \dfrac{\sin\sqrt{\lambda}\xi \ \sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}\ \sin\sqrt{\lambda}}, & 0 \le \xi \le x. \end{cases}$$

Remark 9. In the above example, there are only two points that lead to the trivial case, viz. $\xi_0 = 0, 1$.

Example 2. Let $K(x,\xi) = K(x-\xi)$, $-\pi \le x,\ \xi \le \pi$, where $K(x)$ is an even function which is periodically extended to the entire x-axis so that

$$K(x-\xi) = K(\xi - x). \tag{7.2}$$

In this case [10]

$$\phi_0(x) = (2\pi)^{-1/2}, \quad \phi_k^{(1)}(x) = \frac{1}{\sqrt{\pi}}\cos kx, \quad \phi_k^{(2)}(x) = \frac{1}{\sqrt{\pi}}\sin kx$$

are the eigenfunctions of the integral equation corresponding to the eigenvalues

$$\lambda_0 = (\pi a_0)^{-1}, \quad \lambda_k = (\pi a_k)^{-1}, \quad k = 1, 2, \dots ,$$

where

$$a_0 = \int_{-\pi}^{\pi} K(x)\, dx, \quad a_k = \int_{-\pi}^{\pi} K(x)\cos kx\, dx.$$

Hence

$$R(x,\xi,\lambda) = \frac{1}{2\pi(\lambda_0 - \lambda)} + \sum_{k=1}^{\infty} \frac{\cos k(x-\xi)}{\pi(\lambda_k - \lambda)}.$$

Choose $\xi_0 \in [-\pi, \pi]$. Then the sampling series has the form

$$F(\lambda) = \sum_{k=0}^{\infty} F(\frac{1}{\pi a_k}) \frac{\omega(\lambda)}{(\lambda - \lambda_k)\omega'(\lambda_k)},$$

where

$$\omega(\lambda) = \prod_{k=0}^{\infty} (1 - \pi\lambda a_k)\,.$$

Remark 10. In example 2 we can choose $K(x)$ sufficiently smooth so that the convergence exponent of the sequence $\{\lambda_k\}$ is very small. For instance:

(1) If $K(x) = x^2, \quad -\pi \le x \le \pi$, then [10]

$$\lambda_0 = 3(2\pi^3)^{-1}, \quad \lambda_k = \frac{(-1)^k}{(4\pi)}k^2, \quad k = 1, 2, \dots .$$

(2) If $K(x) = 3(5 - 4\cos x)^{-1}, \quad -\pi \le x \le \pi$, then [10]

$$\lambda_k = 2^k, \quad k = 1, 2, \dots .$$

Note, however, that there is no ξ in $[-\pi, \pi]$ that leads to the trivial case.

Example 3. Take the kernel

$$K(x, \xi) = e^{(x+\xi)/2} \int_{\max(x,\xi)}^{\infty} \frac{e^{-\tau}}{\tau} \, d\tau, \quad 0 \le x, \xi < \infty. \tag{7.3}$$

The eigenvalues of the kernel (7.3) [4,10] are $\lambda_k = k + 1$, $k = 0, 1, \dots$, and the corresponding eigenfunctions are

$$\phi_k(x) = \frac{e^{-x/2}}{k!} L_k(x),,$$

where $L_k(x)$ are the Laguerre polynomials. This set of eigenfunctions is a complete orthonormal set in $L^2(0, \infty)$ [4,10]. The resolvent kernel $R(x, \xi, \lambda)$ will take the form

$$R(x, \xi, \lambda) = \sum_{k=0}^{\infty} \frac{e^{-(x+\xi)/2} L_k(x) L_k(\xi)}{(k!)^2 \left((k+1) - \lambda\right)}.$$

If ξ_0 and $\Phi(x, \lambda)$ are taken as described above, then the sampling representation will take the form

$$F(\lambda) = \sum_{k=0}^{\infty} F(k+1) \frac{\omega(\lambda)}{(\lambda - k - 1)\omega'(k+1)},$$

where

$$\omega(\lambda) = \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{k+1}\right) \exp\left[\lambda/(k+1)\right].$$

Example 4. Let $K(x, \xi)$ be the kernel

$$K(x, \xi) = e^{(x^2+\xi^2)/2} \int_{\infty}^{\min(x,\xi)} e^{-\tau^2} \, d\tau \int_{\max(x,\xi)}^{\infty} e^{-\tau^2} \, d\tau, \quad -\infty < x, \xi < \infty. \tag{7.4}$$

The eigenvalues of this kernel [4,10] are $\lambda_k = 2k + 2$, $k = 0, 1, \dots$, and the corresponding eigenfunctions are

$$\phi_k(x) = \left(2^k \sqrt{\pi}(k!)\right)^{-1/2} e^{-k^2/2} H_k(x),$$

where $H_k(x)$ are the Hermite polynomials. This set of eigenfunctions is a complete orthonormal set in $L^2(-\infty, \infty)$ [4,10]. The resolvent kernel $R(x, \xi, \lambda)$ will take the form

$$R(x, \xi, \lambda) = \sum_{k=0}^{\infty} \frac{e^{-(x^2+\xi^2)/2} H_k(x) H_k(\xi)}{2^k (k!) \sqrt{\pi} \left(2(k+1) - \lambda\right)}.$$

Let ξ_0 and $\Phi(x,\lambda)$ be taken as described above, then the sampling representation will take the form

$$F(\lambda)=\sum_{k=0}^{\infty}F(2k+2)\frac{\omega(\lambda)}{(\lambda-2k-2)\omega'(2k+2)},$$

where

$$\omega(\lambda)=\prod_{k=0}^{\infty}\left(1-\frac{1}{2k+2}\right)\exp[\lambda/(2k+2)].$$

Example 5. Consider the integral equation

$$y(x)=\lambda\int_0^1 K(x,\xi)y(\xi)\,d\xi+x, \tag{7.5}$$

where

$$K(x,\xi)=\begin{cases}-e^{-\xi}\sinh x, & 0\le x\le\xi,\\ -e^{-x}\sinh\xi, & \xi\le x\le 1.\end{cases} \tag{7.6}$$

The eigenvalues of the kernel (7.6) [10] are $\lambda_k=-1-\mu_k^2$ and the corresponding eigenfunctions are $\phi_k(x)=\sin\mu_k x$, where μ_k are the zeros of the equation $\tan\mu=\mu$, $\mu>0$. Hence, in view of Theorem 5.1,

$$\psi(x,\lambda)=x+\lambda\sum_{k=1}^{\infty}\frac{\widehat{h}(k)}{\lambda_k-\lambda}\sin\mu_k x,\quad \lambda\neq\lambda_k,$$

and

$$\Psi(x,\lambda)=\omega(\lambda)\psi(x,\lambda),\quad \lambda\in\mathbb{C},$$

where

$$\omega(\lambda)=\prod_{k=1}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right),$$

and

$$\widehat{h}(k)=\int_0^1 x\sin\mu_k x\,dx.$$

Now if $f\in L^2(0,1)$ and

$$F(\lambda)=\int_0^1\overline{f(x)}\Psi(x,\lambda)\,dx,$$

then

$$F(\lambda)=\sum_{k=1}^{\infty}F(\lambda_k)\frac{\omega(\lambda)}{(\lambda-\lambda_k)\omega'(\lambda_k)}.$$

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Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
E-mail, Annaby: mnaby@frcu.eun.eg, El-Sayed: masayed@frcu.eun.eg