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DYNAMICS OF AN ALMOST PERIODIC LOGISTIC INTEGRODIFFERENTIAL EQUATION

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ABSTRACT. Sufficient conditions are derived for the existence of a globally attractive positive almost-periodic solution of the logistic integrodifferential equation

$$\frac{dN(t)}{dt} = N(t)\left[a(t) - b(t)\int_0^\infty K_\alpha(s)N(t-s)ds\right], \qquad t \geq 0,\ \alpha \in (0,\infty),$$

in which $a(t), b(t)$ are continuous positive almost-periodic functions defined on $(-\infty,\infty)$ and $K_\alpha : [0,\infty) \to [0,\infty)$ is piecewise continuous and integrable on $[0,\infty)$, where α is a positive-valued parameter. We obtain sufficient conditions for all positive solutions to have "level-crossings" about the unique almost-periodic solution. Existence of a positive solution with no "level-crossings" about the almost-periodic solution is also discussed.

1. Introduction

It is well-known that the environments of most natural populations change with time and that such changes induce variation in the growth characteristics of populations. For instance, favourable weather conditions stimulate an increase in the body size and reproduction while unfavourable environments can lead to a decline in the birth rate and an increase in mortality. Temporal variations of an environment of a population are usually incorporated in model systems by the introduction of time-dependent parameters in governing equations. Such governing equations are nonautonomous, and studies of nonautonomous equations have not attained a satisfactory level of maturity comparable to that of autonomous equations. The reader is referred to the recent monograph of Gopalsamy [8] for an extensive discussion of multispecies dynamics in temporally uniform environments governed by autonomous differential equations with discrete and continuously distributed delays.

The purpose of this article is to derive a set of algebraic sufficient conditions for the existence of a globally attractive positive almost-periodic solution of the logistic integrodifferential equation

$$\frac{dN(t)}{dt} = N(t)\left[a(t) - b(t)\int_0^\infty K_\alpha(s)N(t-s)ds\right], \qquad t \geq 0,\ \alpha \in (0,\infty), \quad (1.1)$$

in which $a(t), b(t)$ are continuous positive almost-periodic functions defined on $[0,\infty)$ and $K_\alpha : [0,\infty) \to [0,\infty)$ is piecewise continuous and integrable on $[0,\infty)$ for each $\alpha \in [0,\infty)$. For a general discussion of almost-periodic differential equations with

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and without delays, we refer to the works of Fink [7], Yoshizawa [26, 27], and Corduneanu [5]. It will be found that the contents of this article provide a generalization of the recent results in Zhang and Gopalsamy [29, 30] and Gopalsamy et.al.[15] where equations of the form

$$\frac{dx(t)}{dt} = r(t)x(t)\left[1 - \frac{x(t-n\tau)}{K(t)}\right], \tag{1.2}$$

$$\frac{dv(t)}{dt} = r(t)v(t)\left[1 - \frac{v(t-\tau(t))}{K}\right], \tag{1.3}$$

$$\frac{dN(t)}{dt} = N(t)\left[a(t) - b(t)\int_0^\infty K(s)N(t-s)ds\right], \tag{1.4}$$

in which the coefficients are periodic positive functions with a common period, have been investigated. We like to add that although (1.1) and (1.4) are alike, a, b of (1.1) are almost-periodic functions while a, b of (1.4) are periodic with a common period. We note that it is possible for the various components of biological and physical environments (reproduction rates, resource regeneration, etc.) of a population system can be periodic with rationally independent periods, and therefore it is not unreasonable to consider the various parameters of model systems to be changing "almost-periodically" rather than periodically with a common period. There exists an extensive literature on differential equations with periodic coefficients and their applications; for example, we refer to the articles by Arino et al. [1], Burton [4], Cushing [6], Gopalsamy [9–13], Halanay [17], Hamaya [19], Nisbet and Gurney [22], Qichang [23], and Zhang and Gopalsamy [28]. Almost-periodic integrodifferential systems modelling population dynamics have been discussed by Seifert [24, 25], Murakami [21], and Hamaya and Yoshizawa [20]. The analyses and results of these authors depend crucially on an assumption that there exists a negative stabilising feedback mechanism in the dynamics acting without delay. The methods of analysis of these authors are not directly applicable to equations of the form (1.1). Our interest in the study of (1.1) has evolved from a desire to generalise equations like (1.2)–(1.4) to those with almost-periodic coefficients. We discuss furthermore the delay- (or memory-) induced level-crossing or absence of level-crossing of solutions of (1.1) about a certain nonstationary equilibrium-like solution of (1.1).

2. A priori bounds

In this section, we obtain *a priori* upper and lower bounds of positive solutions of the nonautonomous logistic integrodifferential equation

$$\frac{dN(t)}{dt} = N(t)\left[a(t) - b(t)\int_0^\infty K_\alpha(s)N(t-s)ds\right], \quad t \geq 0, \quad \alpha \in (0,\infty), \tag{2.1}$$

having an initial condition of the type

$$N(s) = \phi(s) \geq 0 \quad \text{for } s \leq 0 \text{ and } \phi(0) > 0, \tag{2.2}$$

where ϕ is bounded and continuous on $(-\infty, 0]$. We assume that a, b are nonnegative, continuous on $\mathbb{R}$, and satisfy

$$0 < a_o \leq a(t) \leq a^o, \quad 0 < b_o \leq b(t) \leq b^o, \quad \text{for } t \in \mathbb{R}, \tag{2.3}$$

$$\int_0^\infty K_\alpha(s)ds = 1, \quad \sigma_\alpha \equiv \int_0^\infty sK_\alpha(s)ds < \infty. \tag{2.4}$$

In particular, we will be interested in delay kernels of the form

$$K_\alpha(s) = \frac{\alpha^{n+1}}{n!} s^n e^{-\alpha s}; \quad s \in (0, \infty),\ n = 0, 1, 2, \cdots,\ \alpha > 0.$$

Theorem 2.1. *Let a, b, and K_α satisfy the above assumptions. Let $N(t)$ be any solution of (2.1) and (2.2). Then $N(t) > 0$ for all $t \geq 0$, and furthermore*

$$\limsup_{t\to\infty} N(t) \leq M_\alpha \equiv \frac{a^o}{b_o \int_0^\infty K_\alpha(s) e^{-a^o s} ds} \quad \text{for } \alpha > 0, \tag{2.5}$$

where a^o and b_o are defined by (2.3).

Proof. The positivity of the solution $N(t)$ of (2.1) and (2.2) for $t > 0$ is immediate from the form of (2.1) and the assumptions on the initial values. By the positivity of $N(t)$ and (2.1), we have

$$\frac{dN(t)}{dt} \leq N(t)\left[a^o - b_o \int_0^\infty K_\alpha(s) N(t-s) ds\right] \tag{2.6}$$

$$\leq a^o N(t) \quad \text{for } t > 0. \tag{2.7}$$

For $t - s \geq 0$, it follows from (2.7) that

$$\frac{dN(t)}{N(t)} \leq a^o dt. \tag{2.8}$$

Integrating (2.8) on $[t-s,\ t]$, we derive

$$N(t) \leq N(t-s) e^{a^o s},$$

that is

$$N(t-s) \geq N(t) e^{-a^o s} \quad \text{for } t \geq s \geq 0. \tag{2.9}$$

From (2.6) and (2.9),

$$\begin{aligned} \frac{dN(t)}{dt} &\leq N(t)\left[a^o - b_o \int_0^t K_\alpha(s) N(t-s) ds\right] \\ &\leq N(t)\left[a^o - b_o\left(\int_0^t K_\alpha(s) e^{-a^o s} ds\right) N(t)\right] \\ &= N(t)\left[a^o - b_o f(t) N(t)\right], \end{aligned} \tag{2.10}$$

where $f(t) = \int_0^t K_\alpha(s) e^{-a^o s} ds > 0$ and satisfies $f(t) \to f^* = \int_0^\infty K_\alpha(s) e^{-a^o s} ds < \infty$ as $t \to \infty$. By comparison, $N(t) \leq y(t)$ for $t > 0$, where $y(t)$ satisfies

$$\frac{dy(t)}{dt} = y(t)\left[a^o - b_o f(t) y(t)\right], \quad y(0) = N(0). \tag{2.11}$$

Solving (2.11), we have

$$\frac{1}{y(t)} = \frac{1}{y(0)} e^{-a^o t} + b_o \frac{\int_0^t f(s) e^{a^o s} ds}{e^{a^o t}}.$$

Since $f(t) \to f^* > 0$ as $t \to \infty$, one can derive that $\lim_{t\to\infty} \int_0^t f(s)e^{a^o s}ds/e^{a^o t} = f^*/a^o$, and hence $1/y(t) \to b_o f^*/a^o$ as $t \to \infty$. Thus

$$\limsup_{t\to\infty} N(t) \le \limsup_{t\to\infty} y(t) = \frac{a^o}{b_o \int_0^\infty K_\alpha(s)e^{-a^o s}ds},$$

and the proof is complete.

Theorem 2.2. *Suppose that α and K_α are such that for some $\epsilon_o > 0$,*

$$\int_0^\infty K_\alpha(s)e^{-(a_o - b^o M_\alpha - \epsilon_o)s}ds < \infty,$$

in which M_α is defined by (2.5). If $N(t)$ denotes any solution of (2.1) and (2.2), then $N(t)$ satisfies

$$\liminf_{t\to\infty} N(t) \ge m_\alpha \equiv \frac{a_o}{b^o \int_0^\infty K_\alpha(s)e^{-(a_o - b^o M_\alpha)s}ds}. \tag{2.12}$$

Proof. Let $N(t)$ be any positive solution of (2.1) and (2.2). It follows from Theorem 2.1 that for $\delta > 0$ small enough (say $\delta < \epsilon_o/(2b^o)$), there exists a $t_1 > 0$ such that

$$N(t) \le M_\alpha + \delta \qquad \text{for } t \ge t_1. \tag{2.13}$$

By the positivity of $N(t)$ and (2.1), we have for $t \ge t_1$,

$$\begin{aligned}\frac{dN(t)}{dt} &\ge N(t)\left[a_o - b^o \int_0^\infty K_\alpha(s)N(t-s)ds\right]\\ &= N(t)\left[a_o - b^o \int_0^{t-t_1} K_\alpha(s)N(t-s)ds - b^o \int_{t-t_1}^\infty K_\alpha(s)N(t-s)ds\right]\end{aligned} \tag{2.14}$$

which together with (2.13) implies

$$\begin{aligned}\frac{dN(t)}{dt} \ge N(t)\Big[&\left(a_o - b^o \int_{t-t_1}^\infty K_\alpha(s)N(t-s)ds\right)\\ &- b^o \int_0^{t-t_1} K_\alpha(s)(M_\alpha + \delta)ds\Big].\end{aligned} \tag{2.15}$$

Let $c_\alpha(t)$ be defined by

$$c_\alpha(t) = a_o - b^o(M_\alpha + \delta)\int_0^{t-t_1} K_\alpha(s)ds - b^o \int_{t-t_1}^\infty K_\alpha(s)N(t-s)ds, \quad t > t_1.$$

Then, by the boundedness of $N(t)$ and (2.4),

$$\lim_{t\to\infty} c_\alpha(t) = a_o - b^o(M_\alpha + \delta), \tag{2.16}$$

and also from (2.15),

$$\frac{dN(t)}{dt} \ge c_\alpha(t)N(t) \quad \text{for } t \ge t_1,$$

which leads to

$$N(t-s) \le N(t)e^{-\int_{t-s}^t c_\alpha(r)dr} \quad \text{for } t-s > t_1. \tag{2.17}$$

Hence, for $t \geq t_1$, we have from (2.14) and (2.17) that

$$\begin{aligned}\frac{dN(t)}{dt} &\geq N(t)\left[a_o - b^o \int_0^\infty K_\alpha(s)N(t-s)ds\right] \\ &\geq N(t)\left[a_o - b^o \int_{t-t_1}^\infty K_\alpha(s)N(t-s)ds\right. \\ &\quad \left. - b^o\left(\int_0^{t-t_1} K_\alpha(s)e^{-\int_{t-s}^t c_\alpha(r)dr}ds\right)N(t)\right]. \end{aligned} \tag{2.18}$$

We note from

$$\lim_{t\to\infty} b^o \int_{t-t_1}^\infty K_\alpha(s)N(t-s)ds = 0,$$

that for arbitrary $0 < \epsilon_1 < \epsilon_o/2$, there exists a $T_1 \geq t_1$ large enough such that

$$b^o \int_{t-t_1}^\infty K_\alpha(s)N(t-s)ds < \epsilon_1, \quad \text{for } t > T_1. \tag{2.19}$$

Also from (2.16) we have

$$[a_o - b^o(M_\alpha + \delta)] - \epsilon_1 < c_\alpha(t) < [a_o - b^o(M_\alpha + \delta)] + \epsilon_1, \quad \text{for } t \geq T_1. \tag{2.20}$$

By (2.20),

$$b^o \int_0^{t-t_1} K_\alpha(s)e^{-\int_{t-s}^t c_\alpha(r)dr}ds < b^o \int_0^\infty K_\alpha(s)e^{-(a_o-b^o(M_\alpha+\delta))s}e^{\epsilon_1 s}ds \quad \text{for } t \geq T_1. \tag{2.21}$$

From (2.18)–(2.21), we derive that, for $t \geq T_1$,

$$\frac{dN(t)}{dt} \geq N(t)\left[\left(a_o - \epsilon_1\right) - \left(b^o \int_0^\infty K_\alpha(s)e^{-(a_o-b^o(M_\alpha+\delta))s}e^{\epsilon_1 s}ds\right)N(t)\right]. \tag{2.22}$$

We can obtain from the hypothesis and (2.22) that there exists a T_2 satisfying $T_2 > T_1$ such that

$$N(t) \geq \frac{a_o - \epsilon_1}{b^o \int_0^\infty K_\alpha(s)e^{-(a_o-b^o(M_\alpha+\delta))s}e^{\epsilon_1 s}ds} \equiv E(\epsilon_1, \delta) \quad \text{for } t \geq T_2. \tag{2.23}$$

On the other hand, we have from

$$E(\epsilon_1, \delta) \to m_\alpha \quad \text{as } \epsilon_1, \delta \to 0^+,$$

where m_α is defined by (2.12), and by the continuity of $E(\epsilon_1, \delta)$ with respect to ϵ_1 and δ, that for arbitrary $\epsilon > 0$, there exists $0 < \delta_1 < \epsilon_o$ such that

$$E(\epsilon_1, \delta) > m_\alpha - \epsilon \quad \text{for } \epsilon_1 < \delta_1 \text{ and } \delta < \delta_1. \tag{2.24}$$

For this δ_1, it then follows from the above analysis that there exists a $T \geq T_2$ such that

$$N(t) \geq m_\alpha - \epsilon \quad \text{for } t \geq T,$$

from which (2.12) follows.

Remark. In the case of the single constant discrete delay logistic equation

$$\frac{dN(t)}{dt} = N(t)[a(t) - b(t)N(t-\alpha)], \tag{2.25}$$

one can obtain by other methods (see [15]) that

$$M_\alpha = \frac{a^o}{b_o} e^{a^o\alpha}, \quad m_\alpha = \frac{a_o}{b^o} e^{(a_o - b^o M_\alpha)\alpha} = \frac{a_o}{b^o} \exp\left[a_o\alpha\left(1 - \frac{a^o b^o}{a_o b_o} e^{a^o\alpha}\right)\right]. \tag{2.26}$$

It is intuitive that if a, b are positive constants, then the "oscillation width" induced by the discrete delay α in (2.25) is measured by $M_\alpha - m_\alpha$. Figure 1 illustrates the effects of the discrete delay on the oscillation width.

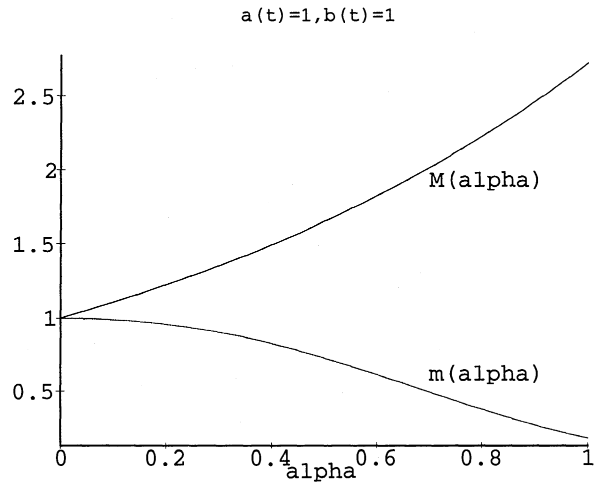

FIGURE 1.

3. Existence of a globally attracting almost-periodic solution

In this section, we derive sufficient conditions for the existence of a globally attracting positive almost-periodic solution of (2.1) when a, b are strictly positive continuous almost-periodic functions of $t \in R$. Our first result is concerned with the stability characteristics of positive solutions of (2.1) and is almost the same as Theorem 2.3 in Gopalsamy et al. [15]. We have included it here with a brief indication of proof for the sake of completeness.

Theorem 3.1. *Assume that a, b are strictly positive, bounded, and continuous on $\mathbb{R}$. Furthermore, suppose that $\int_0^\infty s^2 K_\alpha(s)ds < \infty$ and*

$$(b^o)^2 \sigma_\alpha \left(\frac{a^o}{b_o \int_0^\infty K_\alpha(s) e^{-a^o s} ds}\right) < b_o, \tag{3.1}$$

where

$$\sigma_\alpha = \int_0^\infty s K_\alpha(s) ds < \infty, \quad b_o = \inf_{t\in\mathbb{R}} b(t), \quad b^o = \sup_{t\in\mathbb{R}} b(t).$$

If $N_1(t)$ and $N_2(t)$ denote any two positive solutions of (2.1), *then*

$$\lim_{t\to\infty} [N_1(t) - N_2(t)] = 0. \tag{3.2}$$

Proof. We let

$$z_1(t) = \ln[N_1(t)], \quad z_2(t) = \ln[N_2(t)]$$

and derive that z_1, z_2 satisfy

$$\begin{aligned}\frac{d}{dt}&\left[z_1(t) - z_2(t) - \int_0^\infty K_\alpha(s)\left\{\int_{t-s}^t b(v+s)\left[e^{z_1(v)} - e^{z_2(v)}\right]dv\right\}ds\right] \\ &= -\left[e^{z_1(t)} - e^{z_2(t)}\right]\int_0^\infty K_\alpha(s)b(t+s)ds.\end{aligned}$$

We choose a $\delta > 0$ small enough satisfying $(b^o)^2\sigma_\alpha(M_\alpha + \delta) < b_o$, which is possible due to the assumption in (3.1). The details of the proof are based on the observations that N_1 and N_2 are eventually bounded in the sense that for the above δ, there exists $T_o > 0$ such that

$$N_1(t) \le M_\alpha + \delta, \quad N_2(t) \le M_\alpha + \delta \quad \text{for } t \ge T_o,$$

where

$$M_\alpha = \frac{a^o}{b_o \int_0^\infty K_\alpha(s)e^{-a^o s}ds},$$

and the functional $V(t) = V(z_1, z_2)(t)$ defined by

$$\begin{aligned}V(t) = &\left[z_1(t) - z_2(t) - \int_0^\infty K_\alpha(s)\left\{\int_{t-s}^t b(v+s)\left[e^{z_1(v)} - e^{z_2(v)}\right]dv\right\}ds\right]^2 \\ &+ (b^o)^2\int_0^\infty K_\alpha(s)\left\{\int_{t-s}^t\left(\int_u^t\left[e^{z_1(v)} - e^{z_2(v)}\right]^2 dv\right)du\right\}ds\end{aligned}$$

is a Lyapunov-type functional for the dynamics of $z_1(t) - z_2(t)$. The remaining details of the proof are similar to those of Theorem 2.3 in Gopalsamy et al. [15] and hence are omitted.

We remark that the sufficient condition in (3.1) is better than that obtained in Theorem 2.3 of Gopalsamy et al. [15]. The next result provides sufficient conditions for the existence of almost-periodic solutions of (2.1). With respect to the theory of almost-periodic functions, this paper is self-contained since the only fact from the theory which is used below is that of Bochner's criterion for almost-periodicity (see, for instance, Besicovitch [3]). Such a criterion says that *a function $g(t)$, continuous on $(-\infty, \infty)$ is almost-periodic if and only if, for every sequence of numbers $\{\tau_k\}_{k=1}^\infty$, there exists a subsequence $\{\tau_{m_k}\}_{k=1}^\infty$ such that the sequence of translates $\{g(t + \tau_{m_k})\}_{k=1}^\infty$ converges uniformly on $(-\infty, \infty)$.*

Theorem 3.2. *Let a, b be strictly positive almost-periodic functions defined on $\mathbb{R}$. Let $K_\alpha : [0, \infty) \to [0, \infty)$ be piecewise continuous nonnegative and satisfy the hypothesis of Theorem 2.2. Suppose the coefficients a and b satisfy*

$$M_\alpha^2 b^o \sigma_\alpha < m_\alpha, \tag{3.3}$$

where

$$M_\alpha = \frac{a^o}{b_o \int_0^\infty K_\alpha(s) e^{-a^o s} ds}, \qquad m_\alpha = \frac{a_o}{b^o \int_0^\infty K_\alpha(s) e^{-(a_o - b^o M_\alpha)s} ds}.$$

Then (2.1) *has a unique positive globally attracting almost-periodic solution* $N(t)$ *in the sense that if* $P(t)$ *is any other positive solution of* (2.1), *then*

$$\lim_{t \to \infty} [P(t) - N(t)] = 0.$$

Proof. It follows from (3.3) that there exist two positive numbers $\epsilon_1 > 0$, $\epsilon_2 > 0$ such that

$$(M_\alpha + \epsilon_1)^2 b^o \sigma_\alpha < m_\alpha - \epsilon_2. \tag{3.4}$$

From Theorems 2.1 and 2.2, it follows that all solutions of (2.1) remain eventually bounded above by $M_\alpha + \epsilon_1$ and below by $m_\alpha - \epsilon_2$. By using the almost-periodicity of $a(\cdot)$ and $b(\cdot)$, it can be shown that there exists a solution, say $N^*(t)$, of (2.1) such that

$$m_\alpha - \epsilon_2 \le N^*(t) \le M_\alpha + \epsilon_1 \quad \text{for } t \in R. \tag{3.5}$$

A detailed proof of this is contained in Lemma 2 of Murakami [21]. Let $\{t_m\}$ be an arbitrary sequence such that

$$t_m > t_n \quad \text{for} \quad m > n \quad \text{and} \quad \lim_{n \to \infty} t_n = +\infty.$$

We define $x^{(n)}$ and $x^{(m)}$ by

$$x^{(n)}(t) = N^*(t + t_n), \qquad x^{(m)}(t) = N^*(t + t_m). \tag{3.6}$$

We note that $x^{(n)}(t)$ and $x^{(m)}(t)$ satisfy

$$\frac{d}{dt} x^{(n)}(t) = x^{(n)}(t) \left[a(t + t_n) - b(t + t_n) \int_0^\infty K_\alpha(s) x^{(n)}(t - s) ds \right], \tag{3.7}$$

$$\frac{d}{dt} x^{(m)}(t) = x^{(m)}(t) \left[a(t + t_m) - b(t + t_m) \int_0^\infty K_\alpha(s) x^{(m)}(t - s) ds \right]. \tag{3.8}$$

We now let

$$u^{(m)}(t) = \ln \left[x^{(m)}(t) \right], \qquad u^{(n)} = \ln \left[x^{(n)}(t) \right], \tag{3.9}$$

and derive from (3.7)–(3.9) that

$$\begin{aligned}
\frac{d}{dt} \left[u^{(m)}(t) - u^{(n)}(t) \right] &= \frac{d}{dt} \left[\ln \left(x^{(m)}(t) \right) - \ln \left(x^{(n)}(t) \right) \right] \\
&= \left[a(t + t_m) - a(t + t_n) \right] - \left[b(t + t_m) - b(t + t_n) \right] \int_0^\infty K_\alpha(s) x^{(m)}(t - s) ds \\
&\quad - b(t + t_n) \left(\int_0^\infty K_\alpha(s) \left[x^{(m)}(t - s) - x^{(n)}(t - s) \right] ds \right) \\
&= -b(t + t_n) \left(\int_0^\infty K_\alpha(s) \left[x^{(m)}(t - s) - x^{(n)}(t - s) \right] ds \right) - g_1(t, t_m, t_n),
\end{aligned} \tag{3.10}$$

where

$$g_1(t,t_m,t_n) = \Big[a(t+t_m) - a(t+t_n)\Big] \\ - \Big[b(t+t_m) - b(t+t_n)\Big] \int_0^\infty K_\alpha(s) x^{(m)}(t-s)\,ds. \tag{3.11}$$

Let

$$w(t,t_m,t_n) = u^{(m)}(t) - u^{(n)}(t). \tag{3.12}$$

Then, by the mean value theorem,

$$\begin{aligned} x^{(m)}(t-s) - x^{(n)}(t-s) &= e^{u^{(m)}(t-s)} - e^{u^{(n)}(t-s)} \\ &= e^{\theta(t-s,t_m,t_n)}\Big[u^{(m)}(t-s) - u^{(n)}(t-s)\Big] \\ &= e^{\theta(t-s,t_m,t_n)} w(t-s,t_m,t_n), \end{aligned} \tag{3.13}$$

where $\theta(t-s,t_m,t_n)$ lies between $u^{(m)}(t-s)$ and $u^{(n)}(t-s)$. We have from (3.5) that

$$0 < m_\alpha - \epsilon_2 \le e^{\theta(t-s,t_m,t_n)} \le M_\alpha + \epsilon_1 \quad \text{for } t,s \in R. \tag{3.14}$$

It follows from (3.10), (3.12), and (3.13) that

$$\begin{aligned} &\frac{d}{dt} w(t,t_m,t_n) \\ &= -\,b(t+t_n) \int_0^\infty K_\alpha(s) e^{\theta(t-s,t_m,t_n)} w(t-s,t_m,t_n)\,ds + g_1(t,t_m,t_n) \\ &= -\,b(t+t_n) \Big(\int_0^\infty K_\alpha(s) e^{\theta(t-s,t_m,t_n)}\,ds \Big) w(t,t_m,t_n) + g_1(t,t_m,t_n) \\ &\qquad + b(t+t_n) \int_0^\infty K_\alpha(s) e^{\theta(t-s,t_m,t_n)} \Big[w(t,t_m,t_n) - w(t-s,t_m,t_n) \Big] ds \\ &= -\,b(t+t_n) \Big(\int_0^\infty K_\alpha(s) e^{\theta(t-s,t_m,t_n)}\,ds \Big) w(t,t_m,t_n) + g_1(t,t_m,t_n) \\ &\qquad + b(t+t_n) \int_0^\infty K_\alpha(s) e^{\theta(t-s,t_m,t_n)} s w'(t-\beta s,t_m,t_n)\,ds, \end{aligned} \tag{3.15}$$

with $0 < \beta = \beta(t,s) < 1$. Then

$$\begin{aligned}
\frac{d}{dt}&w(t,t_m,t_n)\\
&= - b(t+t_n)\left(\int_0^\infty K_\alpha(s)e^{\theta(t-s,t_m,t_n)}ds\right)w(t,t_m,t_n) + g_1(t,t_m,t_n)\\
&\quad + b(t+t_n)\int_0^\infty sK_\alpha(s)e^{\theta(t-s,t_m,t_n)}\Big[g_1(t-\beta s,t_m,t_n)\\
&\qquad - b(t-\beta s+t_n)\int_0^\infty K_\alpha(r)e^{\theta(t-\beta s-r,t_m,t_n)}w(t-\beta s-r,t_m,t_n)dr\Big]ds\\
&= - b(t+t_n)\left(\int_0^\infty K_\alpha(s)e^{\theta(t-s,t_m,t_n)}ds\right)w(t,t_m,t_n)\\
&\quad - b(t+t_n)\int_0^\infty sK_\alpha(s)e^{\theta(t-s,t_m,t_n)}\\
&\qquad \times\left[b(t-\beta s+t_n)\int_0^\infty K_\alpha(r)e^{\theta(t-\beta s-r,t_m,t_n)}w(t-\beta s-r,t_m,t_n)dr\right]ds\\
&\quad + g(t,t_m,t_n),
\end{aligned}\tag{3.16}$$

where

$$g(t,t_m,t_n) = g_1(t,t_m,t_n) + b(t+t_n)\int_0^\infty sK_\alpha(s)e^{\theta(t-s,t_m,t_n)}g_1(t-\beta s,t_m,t_n)ds. \tag{3.17}$$

We derive from (3.16) and (3.14) that

$$\begin{aligned}
\frac{D^+}{Dt}|w(t,t_m,t_n)| \le& - b(t+t_n)(m_\alpha-\epsilon_2)|w(t,t_m,t_n)|\\
&+ b(t+t_n)b^o(M_\alpha+\epsilon_1)^2\left(\int_0^\infty sK_\alpha(s)ds\right)|\tilde{w}(t,t_m,t_n)|\\
&+ |g(t,t_m,t_n)|,
\end{aligned}\tag{3.18}$$

where

$$|\tilde{w}(t,t_m,t_n)| = \sup_{s\le t}|w(s,t_m,t_n)|.$$

Since $\sigma_\alpha = \int_0^\infty sK_\alpha(s)ds < +\infty$, we have from (3.17) and (3.11) that

$$\begin{aligned}
|g(t,t_m,t_n)| &\le |g_1(t,t_m,t_n)| + b^o(M_\alpha+\epsilon_1)\sigma_\alpha|\tilde{g}_1(t,t_m,t_n)|\\
&\le [1+b^o\sigma_\alpha(M_\alpha+\epsilon_1)]\sup_{s\le t}\Big[\big|a(s+t_m)-a(s+t_n)\big|\\
&\quad + (M_\alpha+\epsilon_1)\big|b(s+t_m)-b(s+t_n)\big|\Big] \equiv G(t,t_m,t_n).
\end{aligned}\tag{3.19}$$

Hence we obtain from (3.18) and (3.19) that

$$\begin{aligned}
\frac{D^+}{Dt}|w(t,t_m,t_n)| \le& - b(t+t_n)\Big[(m_\alpha-\epsilon_2)|w(t,t_m,t_n)|\\
&- (M_\alpha+\epsilon_1)^2b^o\sigma_\alpha|\tilde{w}(t,t_m,t_n)|\Big] + G(t,t_m,t_n).
\end{aligned}\tag{3.20}$$

We can rewrite (3.20) in the form

$$\frac{D^+}{Dt}\left[|w(t,t_m,t_n)|e^{(m_\alpha-\epsilon_2)\int_{t_o}^t b(r+t_n)dr}\right]$$
$$\le e^{(m_\alpha-\epsilon_2)\int_{t_o}^t b(r+t_n)dr}$$
$$\times\left[b^o\sigma_\alpha(M_\alpha+\epsilon_1)^2 b(t+t_n)|\tilde{w}(t,t_m,t_n)|+G(t,t_m,t_n)\right], \tag{3.21}$$

where $t_o>0$ is some constant. Integrating (3.21) on $[t_o,t]$ and using the nondecreasinging nature of G in t, we have

$$\begin{aligned}|w(t,t_m,t_n)| &\le |w(t_o,t_m,t_n)|e^{-(m_\alpha-\epsilon_2)\int_{t_o}^t b(r+t_n)dr}\\
&+\int_{t_o}^t e^{-(m_\alpha-\epsilon_2)\int_s^t b(r+t_n)dr}\left[b^o\sigma_\alpha(M_\alpha+\epsilon_1)^2 b(s+t_n)|\tilde{w}(s,t_m,t_n)|\right]ds\\
&\quad+\int_{t_o}^t e^{-(m_\alpha-\epsilon_2)\int_s^t b(r+t_n)dr}G(s,t_m,t_n)ds\\
&\le |w(t_o,t_m,t_n)|+G(t,t_n,t_m)\int_{t_o}^t e^{-(m_\alpha-\epsilon_2)\int_s^t b_o dr}ds\\
&\quad+\left(\int_{t_o}^t e^{-(m_\alpha-\epsilon_2)\int_s^t b(r+t_m)dr}b^o\sigma_\alpha(M_\alpha+\epsilon_1)^2 b(s+t_n)ds\right)|\tilde{w}(t,t_m,t_n)|\\
&\le |w(t_o,t_m,t_n)|+\frac{1}{b_o(m_\alpha-\epsilon_2)}G(t,t_m,t_n)\\
&\quad+\frac{b^o\sigma_\alpha(M_\alpha+\epsilon_1)^2}{(m_\alpha-\epsilon_2)}|\tilde{w}(t,t_m,t_n)|,\end{aligned} \tag{3.22}$$

which leads to

$$\begin{aligned}|\tilde{w}(t,t_m,t_n)| &\le |w(t_o,t_m,t_n)|+\frac{1}{b_o(m_\alpha-\epsilon_2)}G(t,t_m,t_n)\\
&\quad+\frac{b^o\sigma_\alpha(M_\alpha+\epsilon_1)^2}{(m_\alpha-\epsilon_2)}|\tilde{w}(t,t_m,t_n)| \quad \text{for } t>t_o.\end{aligned} \tag{3.23}$$

From (3.4) and (3.23) we derive that

$$\begin{aligned}|w(t,t_m,t_n)| &\le |\tilde{w}(t,t_m,t_n)|\\
&\le \frac{m_\alpha-\epsilon_2}{(m_\alpha-\epsilon_2)-b^o\sigma_\alpha(M_\alpha+\epsilon_1)^2}\\
&\quad\times\left[|w(t_o,t_m,t_n)|+\frac{1}{b_o(m_\alpha-\epsilon_2)}G(t,t_m,t_n)\right].\end{aligned} \tag{3.24}$$

Thus

$$\begin{aligned}\big|\ln[x(t+t_m)]-\ln[x(t+t_n)]\big| &\le \frac{m_\alpha-\epsilon_2}{(m_\alpha-\epsilon_2)-b^o\sigma_\alpha(M_\alpha+\epsilon_1)^2}\\
&\quad\times\left[|w(t_o,t_m,t_n)|+\frac{1}{b_o(m_\alpha-\epsilon_2)}G(t,t_m,t_n)\right].\end{aligned} \tag{3.25}$$

The sequence $\{x(t+t_m)\}$ is uniformly bounded and equicontinuous, and hence on each compact subinterval of $[t_o, \infty)$ there exists a subsequence which converges uniformly in t belonging to the compact subinterval. If necessary restricting our further analysis to such a subsequence $\{\tilde{t}_n\}$, we can conclude that $|w(t_o, \tilde{t}_m, \tilde{t}_n)| = |x(t_o+\tilde{t}_m) - x(t_o+\tilde{t}_n)|$ converges to zero as $\tilde{t}_m, \tilde{t}_n \to \infty$. By the almost-periodicity of $a(t)$ and $b(t)$, it will follow that $G(t, \tilde{t}_m, \tilde{t}_n)$ converges to zero as $\tilde{t}_m, \tilde{t}_n \to \infty$. Thus we can conclude that a subsequence of $\{x(t+t_m)\}$ is uniformly convergent on $[t_o, \infty)$. It follows that $x(t)$ and hence $N^*(t)$ is asymptotically almost-periodic. One can proceed in the standard manner (see Hale [18], Yoshizawa [27]) to show that the almost-periodic part of $N^*(t)$ is a solution of (2.1). Thus the existence of an almost-periodic solution of (2.1) follows. To prove the global attractivity of this almost-periodic solution, it is sufficient to show that (3.3) implies (3.1). For instance, we define α_1 and α_2 as

$$\alpha_1 = \frac{(b^o)^2}{b_o}\sigma_\alpha M_\alpha, \qquad \alpha_2 = b^o \sigma_\alpha \frac{M_\alpha^2}{m_\alpha}.$$

We have immediately from (3.3) that $\alpha_2 < 1$. However,

$$\begin{aligned}
\frac{\alpha_1}{\alpha_2} &= \frac{(b^o)^2}{b_o}\sigma_\alpha M_\alpha \frac{m_\alpha}{b^o \sigma_\alpha M_\alpha^2} = \frac{b^o m_\alpha}{b_o M_\alpha} \\
&= \frac{b^o}{b_o}\frac{a_o}{b^o}\frac{1}{\int_0^\infty K_\alpha(s) e^{-(a_o - b^o M_\alpha)s} ds}\frac{b_o}{a^o}\int_0^\infty K_\alpha(s) e^{-a^o s} ds \\
&= \frac{a_o}{a^o}\frac{\int_0^\infty K_\alpha(s) e^{-a^o s} ds}{\int_0^\infty K_\alpha(s) e^{-(a_o - b^o M_\alpha)s} ds} < 1
\end{aligned}$$

which implies $\alpha_1 < \alpha_2 < 1$ and hence (3.3) implies (3.1). The uniqueness and the global attractivity of the almost-periodic solution follow from Theorem 3.1. The proof is complete.

Note: On p. 43 of [15], it is stated that the arguments of [15] are applicable to the almost-periodic case. We have obtained in the above theorem, sufficient conditions which are better than those obtainable with the techniques of [15]. It can be found from [15] that the delay distributed over $[0, \infty)$ has been reduced to that over one period by means of a hypothesis on the delay kernel. Such a reduction in the case of an almost-periodic system is not possible. While techniques of [15] are applicable for delays distributed over finite intervals, we need a new way of handling delays distributed over unbounded intervals. It is in this sense that Theorem 3.1 is significantly different from the results of [15].

4. Solutions with and without level-crossings

We have shown that under certain sufficient conditions, the equation (2.1) has a unique positive almost-periodic solution. Once we know the existence and attractivity of the almost-periodic solution, it is of interest to know the nature of attraction of the almost-periodic solution, that is, whether convergence of solutions of (2.1) to the almost-periodic solution is monotone or oscillatory. The following definition is derived from Gopalsamy et al. [15].

Definition 4.1. Let $y(t)$ be any solution of (2.1). We say that positive solutions of (2.1) have *level-crossings* about $y(t)$ if for any solution $N(t)$ of (2.1), there exists at least one $t^* \in (-\infty, \infty)$ such that

$$N(t^*) - y(t^*) = 0.$$

If no such solutions $N(t)$ exist, then the system (2.1) is said to have solutions with *no level-crossings* about $y(t)$.

In this section, sufficient conditions for all solutions of (2.1) and (2.2) to have one or more level-crossings about the almost-periodic solution are obtained first. Then we derive sufficient conditions for the existence of a positive solution without any level-crossings about the almost-periodic solution. In the special case of (2.1) with constant coefficients, these conditions coincide with the sufficient and necessary conditions for the equation

$$\frac{dN(t)}{dt} = N(t)\left[a - b\int_0^\infty K(s)N(t-s)ds\right]$$

to have level-crossings about its positive equilibrium a/b (for details, see Gopalsamy and Lalli [16]).

Theorem 4.2. *Suppose the hypotheses of Theorem* 3.2 *hold and let* $y(t)$ *denote the unique almost-periodic solution of* (2.1). *Furthermore assume that* $K_\alpha : [0,\infty) \to [0,\infty)$ *is piecewise continuous and the transcendental equation*

$$F(\lambda) \equiv \lambda + b_o m_\alpha \int_0^\infty K_\alpha(s)e^{-\lambda s}ds = 0 \tag{4.1}$$

has no real roots. If $N(t)$ *denotes any positive solution of* (2.1) *distinct from* y, *then* N *has level-crossings about* y *in the sense that*

$$N(t^*) - y(t^*) = 0 \qquad \textit{for some } t^* \in (-\infty, \infty).$$

Proof. Since any root of $F(\lambda) = 0$ has to be negative, we can consider (4.1) with λ replaced by $-\lambda$ so that $F(\lambda) = 0$ leads to

$$b_o m_\alpha \int_0^\infty K_\alpha(s)e^{\lambda s}ds = \lambda \qquad \text{for } \lambda \in (0,\infty).$$

If $b_o m_\alpha \int_0^\infty K_\alpha(s)e^{\lambda_o s}ds - \lambda_o < 0$ for some λ_o, then we can see from

$$\begin{aligned} b_o m_\alpha \int_o^\infty K_\alpha(s)e^{\lambda s}ds - \lambda &\geq b_o m_\alpha \int_o^\infty K_\alpha(s)\frac{\lambda^2 s^2}{2}ds - \lambda \\ &= \lambda\left[\frac{b_o m_\alpha}{2}\int_0^\infty K_\alpha(s)\lambda s^2 ds - 1\right] \\ &\to \infty \qquad \text{as } \lambda \to \infty \end{aligned}$$

that there exists a $\tilde{\lambda} > \lambda_o$ such that

$$b_o m_\alpha \int_o^\infty K_\alpha(s)e^{\tilde{\lambda} s}ds = \tilde{\lambda}.$$

Thus if $F(\lambda) = 0$ has no real roots, then

$$b_o m_\alpha \int_0^\infty K_\alpha(s)e^{\lambda s}ds > \lambda \qquad \text{for } \lambda \in (0,\infty),$$

that is,

$$F(\lambda) \equiv b_o m_\alpha \int_0^\infty K_\alpha(s) \frac{e^{\lambda s}}{\lambda} ds > 1 \qquad \text{for } \lambda \in (0, \infty).$$

Let P and Q denote subsets of $\mathbb{R}_+$ defined by

$$P = \{\lambda \in \mathbb{R}_+ | F(\lambda) = \infty\}, \qquad Q = \{\lambda \in \mathbb{R}_+ | F(\lambda) < \infty\}.$$

For $\lambda \in P$, it is true that there exist positive numbers ϵ, μ with $0 < \epsilon < m_\alpha$ and $\mu > 0$ such that

$$\frac{b_o(m_\alpha - \epsilon)}{1 + \mu} \int_0^\infty K_\alpha(s) \frac{e^{\lambda s}}{\lambda} ds \ > \ 1. \tag{4.2}$$

Suppose $\lambda \in Q$, and let $\{\epsilon_n\}$ and $\{\mu_n\}$ be two sequences of positive numbers such that $\epsilon_n \to 0$ and $\mu_n \to 0$ as $n \to \infty$, and $\epsilon_n < m_\alpha$. An application of Fatou's Lemma leads to

$$\liminf_{n \to \infty} \frac{b_o(m_\alpha - \epsilon_n)}{1 + \mu_n} \int_0^\infty K_\alpha(s) \frac{e^{\lambda s}}{\lambda} ds \geq b_o m_\alpha \int_0^\infty K_\alpha(s) \frac{e^{\lambda s}}{\lambda} ds > 1.$$

Hence there exist ϵ and μ with $0 < \epsilon < m_\alpha$ and $0 < \mu$ such that (4.2) holds. Now we suppose that there exists a solution of (2.1) and (2.2), say $x(t)$, without level-crossings about the almost-periodic solution $y(t)$ and show that this leads to a contradiction. Without loss of generality, we assume that

$$x(t) > y(t) \qquad \text{for } t \in (-\infty, +\infty). \tag{4.3}$$

Define $u(t)$, $v(t)$, and $w(t)$ as

$$u(t) = \ln[x(t)], \qquad v(t) = \ln[y(t)], \qquad w(t) = u(t) - v(t).$$

Then $w(t)$ is governed by

$$\begin{aligned} \frac{dw(t)}{dt} &= \frac{d}{dt}[u(t) - v(t)] \\ &= -b(t) \int_0^\infty K_\alpha(s) \left[e^{u(t-s)} - e^{v(t-s)}\right] ds \\ &= -b(t) \int_0^\infty K_\alpha(s) e^{v(t-s)} \left[e^{u(t-s)-v(t-s)} - 1\right] ds \qquad (4.4) \\ &\leq -b_o \int_0^\infty K_\alpha(s) y(t-s) w(t-s) ds, \qquad (4.5) \end{aligned}$$

on using (4.3). Since $y(t)$ is the almost-periodic solution of (2.1), by Theorems 2.1 and 2.2 we know that there exist $\delta > 0$ and $t_1 > 0$ such that

$$m_\alpha - \epsilon \leq y(t) \leq M_\alpha + \delta \qquad \text{for } t \geq t_1. \tag{4.6}$$

Using (2.4), we can choose $\beta > \eta > 0$ such that $\int_\eta^\beta K_\alpha(s) ds > 0$. From (4.5) and (4.6),

$$\frac{dw(t)}{dt} + b_o(m_\alpha - \epsilon) \int_\eta^\beta K_\alpha(s) w(t-s) ds \leq 0 \qquad \text{for } t \geq \beta + t_1. \tag{4.7}$$

We have from (4.3) that $w(t) > 0$ for $t \in (-\infty, \infty)$. Then it is easy to see from (4.7) that there exists a $t_2 \geq t_1$ such that $w(t)$ is nonincreasing for $t \geq t_2$, and consequently

$$\frac{dw(t)}{dt} + Bw(t-\eta) \leq 0 \qquad \text{for } t \geq t_2, \tag{4.8}$$

in which $B = b_o(m_\alpha - \epsilon)\int_\eta^\beta K_\alpha(s)ds > 0$. Integrating both sides of (4.8) on $[t-\eta/2, t]$ $(t \geq t_3 \geq t_2 + \eta)$,

$$w(t) - w(t-\frac{\eta}{2}) + B\int_{t-\eta/2}^{t} w(s-\eta)ds \leq 0 \qquad \text{for } t \geq t_3. \tag{4.9}$$

As a consequence of (4.9),

$$B\int_{t-\eta/2}^{t} w(s-\eta)ds \leq w(t-\frac{\eta}{2}) \qquad \text{for } t \geq t_3. \tag{4.10}$$

By the eventual nonincreasing nature of w, we derive from (4.10) that

$$B\frac{\eta}{2}w(t-\eta) \leq w(t-\frac{\eta}{2}),$$

or equivalently

$$w(t-\eta) \leq \Big(\frac{2}{B\eta}\Big)w(t-\frac{\eta}{2}). \tag{4.11}$$

Similarly, we can derive that

$$w(t-\frac{\eta}{2}) \leq \Big(\frac{2}{B\eta}\Big)w(t) \qquad \text{for } t \geq t_3, \tag{4.12}$$

which together with (4.11) leads to

$$w(t-\eta) \leq \Big(\frac{2}{B\eta}\Big)^2 w(t) \qquad \text{for all large } t. \tag{4.13}$$

Now we define a set

$$\Lambda = \Big\{\lambda \geq 0 : \frac{dw(t)}{dt} + \lambda w(t) \leq 0, \quad \text{for large } t\Big\}. \tag{4.14}$$

Clearly $\lambda = 0 \in \Lambda$ and Λ is a subinterval of $[0, \infty)$. We shall first show that Λ is bounded above. In fact, integrating both sides of (4.4) on $[t-\eta, t]$ $(t \geq t_3)$, we derive that

$$w(t) - w(t-\eta) + \int_{t-\eta}^{t} b(r)\int_0^\infty K_\alpha(s)\Big[e^{u(r-s)} - e^{v(r-s)}\Big]dsdr = 0,$$

and this implies

$$\int_{t-\eta}^{t} b(r)\int_0^\infty K_\alpha(s)\Big[e^{u(r-s)} - e^{v(r-s)}\Big]dsdr \leq w(t-\eta),$$

which can be rewritten as

$$b(t)\int_{t-\eta}^{t} \frac{b(r)}{b(t)}\int_0^\infty K_\alpha(s)\Big[e^{u(r-s)} - e^{v(r-s)}\Big]dsdr \leq w(t-\eta). \tag{4.15}$$

It is found from (4.3) and (4.4) that $e^{u(t)} - e^{v(t)} = x(t) - y(t)$ is nonincreasing, and therefore we have from (4.15) that

$$b(t)\eta\frac{b_o}{b^o}\int_0^\infty K_\alpha(s)\Big[e^{u(t-s)} - e^{v(t-s)}\Big]ds < w(t-\eta),$$

or equivalently (using (4.13))

$$b(t)\int_0^\infty K_\alpha(s)\Big[e^{u(t-s)} - e^{v(t-s)}\Big]ds < \frac{b^o}{b_o}\frac{1}{\eta}w(t-\eta) \le \frac{b^o}{b_o}\frac{1}{\eta}\Big(\frac{2}{B\eta}\Big)^2 w(t), \quad (4.16)$$

eventually. Now, from (4.4) and (4.16),

$$\begin{aligned} 0 &= \frac{dw(t)}{dt} + b(t)\int_0^\infty K_\alpha(s)\Big[e^{u(t-s)} - e^{v(t-s)}\Big]ds \\ &< \frac{dw(t)}{dt} + \frac{b^o}{b_o}\frac{1}{\eta}\Big(\frac{2}{B\eta}\Big)^2 w(t) \qquad \text{for some } t, \end{aligned} \quad (4.17)$$

which implies that

$$\frac{b^o}{b_o}\frac{1}{\eta}\Big(\frac{2}{B\eta}\Big)^2$$

is an upper bound of the set Λ. Thus Λ is bounded. As a consequence, for the μ chosen in (4.2) there must exist a $\lambda \in \Lambda$ such that $\lambda(1+\mu) \notin \Lambda$, that is,

$$0 < \frac{dw(t)}{dt} + \lambda(1+\mu)w(t) \qquad \text{for large } t. \quad (4.18)$$

Since $\lambda \in \Lambda$, there exists a $t_4 \ge t_3$ such that

$$\frac{dw(t)}{dt} + \lambda w(t) \le 0 \qquad \text{for } t \ge t_4$$

and hence, for $t > t_4$, $t - s > t_4$,

$$\begin{aligned} w(t-s) &= \left[\frac{w(t-s)}{w(t)}\right]w(t) \\ &= w(t)\exp\left(-\ln\frac{w(t)}{w(t-s)}\right) \\ &= w(t)\exp\left(-\int_{t-s}^t \frac{w'(s)}{w(s)}ds\right) \\ &\ge e^{\lambda s}w(t). \end{aligned} \quad (4.19)$$

From (4.18) and (4.19) we derive that for $t \geq t_4$,

$$
\begin{aligned}
0 &< \frac{dw(t)}{dt} + \lambda(1+\mu)w(t) \\
&= -b(t)\int_0^\infty K_\alpha(s)\left[e^{u(t-s)} - e^{v(t-s)}\right]ds + \lambda(1+\mu)w(t) \\
&\leq -b_o\int_0^{t-t_4} K_\alpha(s)y(t-s)w(t-s)ds + \lambda(1+\mu)w(t) \\
&\leq -b_o(m_\alpha - \epsilon)\int_0^{t-t_4} K_\alpha(s)e^{\lambda s}w(t)ds + \lambda(1+\mu)w(t) \\
&= \left[-b_o(m_\alpha - \epsilon)\int_0^{t-t_4} K_\alpha(s)e^{\lambda s}ds + \lambda(1+\mu)\right]w(t),
\end{aligned} \tag{4.20}
$$

which leads to

$$
b_o(m_\alpha - \epsilon)\int_0^{t-t_4} K_\alpha(s)e^{\lambda s}ds < \lambda(1+\mu). \tag{4.21}
$$

We let $t \to \infty$ in (4.21) and obtain

$$
b_o m_\alpha \int_0^\infty K_\alpha(s)e^{\lambda s}ds \leq \lambda(1+\mu)
$$

which contradicts (4.2). Therefore the result follows.

We shall now proceed to obtain a result on the existence of a solution of (2.1) with no level-crossing about the almost-periodic solution. Our result is established by an application of the Schauder-Tychonoff fixed point theorem.

Theorem 4.3. *Suppose that equation* (2.1) *with* (2.2) *has an almost-periodic solution, say* $y(t)$. *If there exists a* $\beta > 0$ *such that*

$$
b^o M_\alpha \int_0^\infty K_\alpha(s)e^{\beta s}ds < \beta, \tag{4.22}
$$

then there exists a positive solution $n^*(t)$ *of* (2.1) *on* $(-\infty, \infty)$ *such that*

$$
|n^*(t) - y(t)| > 0 \qquad \text{for } t \in (-\infty, \infty).
$$

Proof. We first note that if $N(t)$ is any solution of (2.1) and (2.2) and if $y(t)$ is the positive almost-periodic solution of (2.1), then we can define n, Y, x as

$$
n(t) = \ln[N(t)], \qquad Y(t) = \ln[y(t)], \qquad x(t) = n(t) - Y(t).
$$

Note that $x(t)$ is governed by

$$
\frac{dx(t)}{dt} = -b(t)\int_0^\infty K_\alpha(s)y(t-s)\left[e^{x(t-s)} - 1\right]ds. \tag{4.23}
$$

Thus the question of looking for a solution of (2.1) without level-crossings about $y(t)$ is equivalent to that of looking for a solution of (4.23) without zero-crossing. Our proof is based on an application of the well known Schauder-Tychonoff fixed point theorem. From (4.22), there exists a $\delta > 0$ small enough such that

$$
b^o(M_\alpha + \delta)\int_0^\infty K_\alpha(s)e^{\beta s}ds \leq \beta.
$$

Using the almost-periodicity of a, b, and y and following the arguments in the proof of Lemma 2 in Murakami [21], we can show from Theorem 2.1 that for $\delta > 0$,

$$y(t) \le M_\alpha + \delta \qquad \text{for } t \in \mathbb{R}.$$

Let $C(R)$ denote the set of all continuous bounded functions on $(-\infty, \infty)$. Define a set S as follows:

$$S = \left\{ x \in C(R) \,\middle|\, \begin{array}{ll} x \text{ is nondecreasing on } [0,\infty) & \\ -(1-\epsilon) \le x(t) \le -(1-\epsilon)e^{-\beta t}, & \text{for } t \ge 0, \\ x(t) \equiv -(1-\epsilon), & \text{for } t \le 0, \\ x(t)\beta \le b^o(M_\alpha+\delta)\displaystyle\int_0^\infty K_\alpha(s)\left[e^{x(t-s)}-1\right]ds, & \text{for } t \ge 0 \end{array} \right\}, \tag{4.24}$$

in which $\epsilon > 0$ is a fixed number such that $1 - \epsilon > 0$ and M_α is defined by (2.5). We first show that the set S is nonempty. This will follow if we can verify that the function $z(t)$ defined by

$$z(t) = \begin{cases} -(1-\epsilon), & t \le 0 \\ -(1-\epsilon)e^{-\beta t}, & t \ge 0 \end{cases}$$

belongs to the set S. It is sufficient to verify the last requirement of (4.24) about z. In fact,

$$\begin{aligned} &b^o(M_\alpha+\delta)\int_0^\infty K_\alpha(s)\left[e^{z(t-s)}-1\right]ds \\ &\quad = b^o(M_\alpha+\delta)\left\{\int_0^t K_\alpha(s)\left[e^{z(t-s)}-1\right]ds + \int_t^\infty K_\alpha(s)\left[e^{z(t-s)}-1\right]ds\right\} \\ &\quad = b^o(M_\alpha+\delta)\left\{\int_0^t K_\alpha(s)\left[e^{-(1-\epsilon)e^{-\beta(t-s)}}-1\right]ds + \int_t^\infty K_\alpha(s)\left[e^{-(1-\epsilon)}-1\right]ds\right\}. \end{aligned}$$

Using the fact that $e^{-x} - 1 > -x$ for $x > 0$, we derive

$$\begin{aligned} &b^o(M_\alpha+\delta)\int_0^\infty K_\alpha(s)\left[e^{z(t-s)}-1\right]ds \\ &\quad \ge b^o(M_\alpha+\delta)\left\{\int_0^t K_\alpha(s)\left[-(1-\epsilon)e^{-\beta(t-s)}\right]ds + \int_t^\infty K_\alpha(s)\left[-(1-\epsilon)\right]ds\right\} \\ &\quad = -b^o(M_\alpha+\delta)(1-\epsilon)\left(\int_0^t K_\alpha(s)e^{-\beta(t-s)}ds + \int_t^\infty K_\alpha(s)ds\right) \\ &\quad \ge -b^o(M_\alpha+\delta)(1-\epsilon)\int_0^\infty K_\alpha(s)e^{-\beta(t-s)}ds \\ &\quad = -(1-\epsilon)e^{-\beta t}b^o(M_\alpha+\delta)\int_0^\infty K_\alpha(s)e^{\beta s}ds \\ &\quad \ge -(1-\epsilon)\beta e^{-\beta t} = \beta z(t) \qquad t \ge 0, \end{aligned}$$

and hence $z(t) \in S$. Therefore S is nonempty. Next we define a mapping $F: S \to C(R)$ by

$$F(x)(t) = \begin{cases} -(1-\epsilon), & t \le 0, \\ -(1-\epsilon)\exp\Big[-\int_0^t b(s) \\ \qquad \Big(\int_0^\infty K_\alpha(r) y(s-r) \dfrac{e^{x(s-r)}-1}{x(s)}\, dr\Big)\, ds\Big], & t > 0, \end{cases} \tag{4.25}$$

in which $y(t)$ is the almost-periodic solution of (2.1). We now proceed to verify that $FS \subset S$. From the definition (4.25), we have

$$F(x)(t) = -(1-\epsilon) \qquad \text{for } t \le 0.$$

Since $x \in S$ and $x < 0$, by the positivity of the almost-periodic solution $y(t)$,

$$\int_0^t b(s)\Big(\int_0^\infty K_\alpha(r) y(s-r) \frac{e^{x(s-r)}-1}{x(s)} dr\Big) ds > 0;$$

hence

$$F(x)(t) \ge -(1-\epsilon) \qquad \text{for } t \ge 0.$$

It follows from $x \in S$ that

$$x(t)\beta \le b^o(M_\alpha + \delta) \int_0^\infty K_\alpha(s)\Big[e^{x(t-s)} - 1\Big] ds,$$

which implies

$$b^o(M_\alpha + \delta) \int_0^\infty K_\alpha(s) \frac{e^{x(t-s)}-1}{x(t)} ds \le \beta \qquad \text{for } t \ge 0. \tag{4.26}$$

It is found from (4.26) that

$$\begin{aligned} \int_0^t b(s) \int_0^\infty & K_\alpha(r) y(s-r) \frac{e^{x(s-r)}-1}{x(s)} duds \\ &\le b^o(M_\alpha+\delta) \int_0^t \int_0^\infty K_\alpha(r) \frac{e^{x(s-r)}-1}{x(s)} drds \\ &\le \int_0^t \beta ds = \beta t \qquad \text{for } t \ge 0 \end{aligned} \tag{4.27}$$

which implies

$$\begin{aligned} -(1-\epsilon)\exp&\Big[-\int_0^t b(s)\Big(\int_0^\infty K_\alpha(r) y(s-r) \frac{e^{x(s-r)}-1}{x(s)} dr\Big) ds\Big] \\ &\le -(1-\epsilon)e^{-\beta t} \quad \text{for } t \ge 0, \end{aligned}$$

and therefore

$$-(1-\epsilon) \le F(x)(t) \le -(1-\epsilon)e^{-\beta t} \qquad \text{for } t \ge 0.$$

To show $FS \subset S$, we still need to show that

$$\beta F(x)(t) \le b^o(M_\alpha + \delta) \int_0^\infty \big[e^{F(x)(t-s)} - 1\big] ds \quad \text{for } t \ge 0. \tag{4.28}$$

By the definition of F and the fact $e^x - 1 > x$ for $x < 0$,

$$
\begin{aligned}
&b^o(M_\alpha + \delta)\int_0^\infty K_\alpha(s)\big[e^{F(x)(t-s)} - 1\big]ds \\
&\quad \geq b^0(M_\alpha + \delta)\int_0^\infty K_\alpha(s)F(x)(t-s)ds \\
&\quad = b^o(M_\alpha + \delta)\left\{\int_0^t K_\alpha(s)F(x)(t-s)ds + \int_t^\infty K_\alpha(s)F(x)(t-s)ds\right\} \\
&\quad = b^o(M_\alpha + \delta)\bigg\{\int_0^t K_\alpha(s)\bigg[-(1-\epsilon) \\
&\qquad\qquad \times \exp\bigg(-\int_0^{t-s} b(r)\int_0^\infty K_\alpha(q)y(r-q)\frac{e^{x(r-q)}-1}{x(r)}dqdr\bigg)\bigg]ds \\
&\qquad + \int_t^\infty K_\alpha(s)[-(1-\epsilon)]ds\bigg\} \\
&\quad = -(1-\epsilon)b^o(M_\alpha + \delta)\int_t^\infty K_\alpha(s)ds \\
&\qquad - (1-\epsilon)b^o(M_\alpha + \delta)\int_0^t K_\alpha(s) \\
&\qquad\qquad \times \exp\bigg[-\int_0^{t-s} b(r)\int_0^\infty K_\alpha(q)y(r-q)\frac{e^{x(r-q)}-1}{x(r)}dqdr\bigg]ds.
\end{aligned} \tag{4.29}
$$

We know that $F(x)(t) \leq -(1-\epsilon)e^{-\beta t}$ and hence $-(1-\epsilon) \geq F(x)(t)e^{\beta t}$. Noting that $F(x)(t)$ is nondecreasing, one can see that

$$
\begin{aligned}
&-(1-\epsilon)b^o(M_\alpha + \delta)\int_t^\infty K_\alpha(s)ds \\
&\quad = b^o(M_\alpha + \delta)\int_t^\infty -(1-\epsilon)K_\alpha(s)ds \\
&\quad \geq b^o(M_\alpha + \delta)\int_t^\infty F(x)(s)e^{\beta s}K_\alpha(s)ds \\
&\quad \geq b^o(M_\alpha + \delta)F(x)(t)\int_t^\infty K_\alpha(s)e^{\beta s}ds.
\end{aligned} \tag{4.30}
$$

And for the last term in (4.29), we derive that

$$
\begin{aligned}
&-(1-\epsilon)b^o(M_\alpha + \delta)\int_0^t K_\alpha(s) \\
&\qquad \times \exp\bigg[-\int_0^{t-s} b(r)\int_0^\infty K_\alpha(q)y(r-q)\frac{e^{x(r-q)}-1}{x(r)}dqdr\bigg]ds
\end{aligned}
$$

$$
\begin{aligned}
&= -(1-\epsilon)b^o(M_\alpha+\delta)\int_0^t K_\alpha(s) \\
&\quad \times \exp\left\{-\int_0^t b(r)\int_0^\infty K_\alpha(q)y(r-q)\frac{e^{x(r-q)}-1}{x(r)}dqdr\right. \\
&\qquad \left.+\int_{t-s}^t b(r)\int_0^\infty K_\alpha(q)y(r-q)\frac{e^{x(r-q)}-1}{x(r)}dqdr\right\}ds \\
&= b^o(M_\alpha+\delta)\int_0^t K_\alpha(s)F(x)(t) \\
&\quad \times \exp\left[\int_{t-s}^t b(r)\int_0^\infty K_\alpha(q)y(r-q)\frac{e^{x(r-q)}-1}{x(r)}dqdr\right]ds.
\end{aligned}
\tag{4.31}
$$

From (4.26),

$$
\begin{aligned}
&\int_{t-s}^t b(r)\int_0^\infty K_\alpha(q)y(r-q)\frac{e^{x(r-q)}-1}{x(r)}dqdr \\
&\quad \le b^o(M_\alpha+\delta)\int_{t-s}^t\int_0^\infty K_\alpha(q)\frac{e^{x(r-q)}-1}{x(r)}dqdr \\
&\quad \le \int_{t-s}^t \beta dr = \beta s.
\end{aligned}
\tag{4.32}
$$

Using (4.32), (4.31) leads to (since $F(x)(t) < 0$)

$$
\begin{aligned}
&-(1-\epsilon)b^o(M_\alpha+\delta)\int_0^t K_\alpha(s) \\
&\quad \times \exp\left[-\int_0^{t-s} b(r)\int_0^\infty K_\alpha(q)y(r-q)\frac{e^{x(r-q)}-1}{x(r)}dqdr\right]ds \\
&\ge b_o(M_\alpha+\delta)\int_0^t K_\alpha(s)F(x)(t)e^{\beta s}ds \\
&= F(x)(t)b^o(M_\alpha+\delta)\int_0^t K_\alpha(s)e^{\beta s}ds.
\end{aligned}
\tag{4.33}
$$

We can now conclude from (4.29), (4.30), and (4.33) that

$$
\begin{aligned}
&b^o(M_\alpha+\delta)\int_0^\infty K_\alpha(s)\left[e^{F(x)(t-s)}-1\right]ds \\
&\quad \ge b^o(M_\alpha+\delta)F(x)(t)\int_t^\infty K_\alpha(s)e^{\beta s}ds \\
&\qquad + b^o(M_\alpha+\delta)F(x)(t)\int_0^t K_\alpha(s)e^{\beta s}ds \\
&\quad = b^o(M_\alpha+\delta)F(x)(t)\int_0^\infty K_\alpha(s)e^{\beta s}ds \ge \beta F(x)(t).
\end{aligned}
$$

That is, (4.28) follows, and consequently we have shown that $FS \subset S$. Then, similar to the arguments in the proof of Theorem 4.1 in Gopalsamy and Lalli [16], one can show that FS is also equicontinuous. Based on the above analysis and the Schauder-Tychonoff fixed point theorem, we conclude that there exists $x^* \in S$ such that

$$F(x^*)(t) = x^*(t). \tag{4.34}$$

Since x^* satisfies

$$\begin{aligned}
\frac{d}{dt}x^*(t) &= \frac{d}{dt}(F(x^*)(t)) \\
&= F(x^*)(t)\left[-b(t)\int_0^\infty K_\alpha(r)y(t-r)\frac{e^{x^*(t-r)}-1}{x^*(t)}dr\right] \\
&= x^*(t)\left[-b(t)\int_0^\infty K_\alpha(r)y(t-r)\frac{e^{x^*(t-r)}-1}{x^*(t)}dr\right] \\
&= -b(t)\int_0^\infty K_\alpha(s)y(t-s)\left[e^{x^*(t-s)}-1\right]ds,
\end{aligned}$$

it follows that $x^*(t)$ is a solution of (4.23). Obviously, $x^*(t) < 0$, and hence (2.1) has a solution with no level-crossings about the almost-periodic solution of (2.1). This completes the proof.

5. Two examples

In this section, we illustrate the calculation of the asymptotic upper and lower estimates M_α and m_α for the two kernels

$$K_\alpha^{(1)}(s) = \alpha e^{-\alpha s}, \tag{5.1}$$

$$K_\alpha^{(2)}(s) = \alpha^2 s e^{-\alpha s}. \tag{5.2}$$

Corresponding to $K_\alpha^{(1)}$ in (5.1), we have

$$M_\alpha^{(1)} = \frac{a^o}{b_o}\frac{1}{\int_0^\infty K_\alpha^{(1)}(s)ve^{-a^o s}ds} = \frac{a^o}{b_o}\frac{1}{\int_0^\infty \alpha e^{-\alpha s}e^{-a^o s}ds} = \frac{a^o}{b_o}\left(\frac{\alpha + a^o}{\alpha}\right), \tag{5.3}$$

$$\begin{aligned}
m_\alpha^{(1)} &= \frac{a_o}{b^o}\frac{1}{\int_0^\infty K_\alpha^{(1)}(s)e^{-(a_o - b^o M_\alpha^{(1)})s}ds} \\
&= \frac{a_o}{b^o}\frac{\alpha + a_o - b^o M_\alpha^{(1)}}{\alpha} = \frac{a_o}{b^o}\frac{\alpha + a_o}{\alpha}\left\{1 - \frac{b^o a^o}{\alpha b_o}\right\}.
\end{aligned} \tag{5.4}$$

The requirement that $m_\alpha^{(1)} > 0$ is satisfied when $\alpha > b^o a^o / b_o$. Next, corresponding to $K_\alpha^{(2)}$ in (5.2), we have

$$M_\alpha^{(2)} = \frac{a^o}{b_o}\frac{1}{\int_0^\infty K_\alpha^{(2)}(s)e^{-a^o s}ds} = \frac{a^o}{b_o}\frac{1}{\int_0^\infty \alpha^2 s e^{-\alpha s}e^{-a^o s}ds}, \tag{5.5}$$

in which

$$\alpha^2\int_0^\infty se^{-\alpha s}e^{-a^o s}ds = \frac{\alpha^2}{(\alpha + a^o)^2}\left(\int_0^\infty ye^{-y}dy\right) = \frac{\alpha^2}{(\alpha + a^o)^2},$$

and hence

$$M_\alpha^{(2)} = \frac{a^o}{b_o}\left(\frac{\alpha + a^o}{\alpha}\right)^2, \tag{5.6}$$

$$m_\alpha^{(2)} = \frac{a_o}{b^o}\frac{1}{\int_0^\infty K_\alpha^{(2)}(s)e^{-(a_o - b^o M_\alpha^{(2)})s}ds}. \tag{5.7}$$

By (5.6),

$$\int_0^\infty K_\alpha^{(2)}(s)e^{-(a_o-b^oM_\alpha^{(2)})s}ds = \alpha^2\int_0^\infty se^{-(\alpha+a_o-b^oM_\alpha^{(2)})s}ds$$
$$= \Big(\frac{\alpha}{\alpha + a_o - b^o M_\alpha^{(2)}}\Big)^2\int_0^\infty ye^{-y}dy = \Big(\frac{\alpha}{\alpha + a_o - b^o M_\alpha^{(2)}}\Big)^2. \tag{5.8}$$

It follows from (5.7), (5.8), and (5.6) that

$$m_\alpha^{(2)} = \frac{a_o}{b^o}\Big(\frac{\alpha + a_o - b^o M_\alpha^{(2)}}{\alpha}\Big)^2 = \frac{a_o}{b^o}\Big[\frac{\alpha + a_o}{\alpha} - \frac{b^o}{\alpha}\frac{a^o}{b_o}\left(\frac{\alpha + a^o}{\alpha}\right)^2\Big]^2. \tag{5.9}$$

It is easy to see that we can choose $\alpha > 0$ satisfying $\alpha^2 > a^o b^o(\alpha + a^o)/b_o$ such that the integrals in (5.8) exist. Furthermore for such α, one can see that the assumption of Theorem 3.1 holds for some $\epsilon_o > 0$.

We conclude with the following remarks. It is our opinion that we have obtained, for the first time, a uniform lower bound (independent of initial values) for the integrodifferential system, and this guarantees uniform persistence. Furthermore, our analysis of (2.1) includes autonomous, periodic, and almost-periodic parameters. For biological and ecological significance of the variability in the coefficients 'a' and 'b', we refer to the discussion in [14].

6. Numerical simulations

A few computer simulations of a system of illustrative examples are given below. Our simulations are based on the technique (due to Fargue; for details, see Gopalsamy [8]) of converting the scalar integrodifferential equations into a system of ordinary differential equations and then numerically solving them using Maple and its built in graphical output routine. We first consider

$$\frac{dN(t)}{dt} = N(t)\Big[1 - \int_0^\infty \alpha e^{-\alpha s}N(t-s)ds\Big]. \tag{6.1}$$

This integrodifferential equation satisfies the assumptions of Theorem 3.1 and can be converted into a system of ordinary differential equations by the introduction of an auxiliary variable U, where

$$U(t) = \int_0^\infty \alpha e^{-\alpha s}N(t-s)ds = \alpha\int_{-\infty}^t e^{-\alpha(t-s)}N(s)ds.$$

The scalar integrodifferential equation (6.1) becomes

$$\begin{aligned}\frac{dN(t)}{dt} &= N(t)[1 - U(t)],\\ \frac{dU(t)}{dt} &= -\alpha[U(t) - N(t)].\end{aligned} \tag{6.2}$$

The convergence of solutions of (6.1) corresponding to $\alpha = 1.189$ and four initial values are displayed in Figure 2(a).

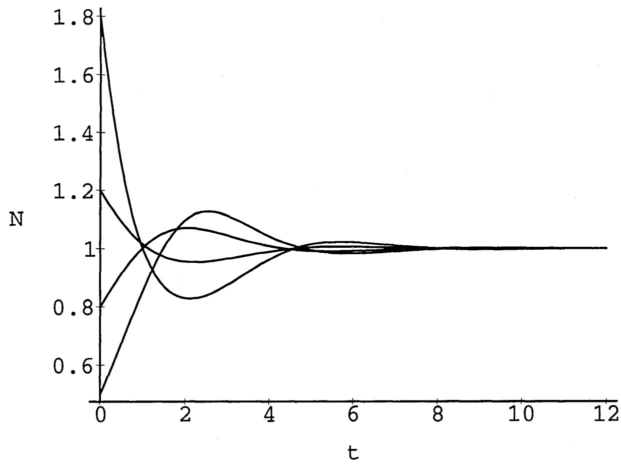

FIGURE 2(a). Graphs of the solution $N(t)$ of (6.3) with $\alpha = 1.189$ and the initial values $(N(0), U(0)) = (0.5, 0.35), (0.8, 0.7), (1.2, 1.2), (1.8, 1.7)$.

Our next example is the periodic integrodifferential equation

$$\frac{dN(t)}{dt} = N(t)\left[(2+\sin(t)) - (2-\cos(t))\int_0^\infty \alpha e^{-\alpha s} N(t-s)ds\right]. \tag{6.3}$$

We let again

$$U(t) = \int_0^\infty \alpha e^{-\alpha s} N(t-s)\, ds$$

and obtain the nonautonomous ordinary differential system

$$\begin{aligned} \frac{dN(t)}{dt} &= N(t)[(2+\sin(t)) - (2-\cos(t))U(t)], \\ \frac{dU(t)}{dt} &= -\alpha[U(t) - N(t)]. \end{aligned} \tag{6.4}$$

The convergence of solutions of (6.3) corresponding to $\alpha = 12.69$ and four initial values are displayed in Figure 2(b).

The following integrodifferential equation has periodic coefficients with rationally independent periods:

$$\frac{dN(t)}{dt} = N(t)\left[(2+\sin(t)) - (2-\cos(\pi t))\int_0^\infty \alpha e^{-\alpha s} N(t-s)ds\right]. \tag{6.5}$$

We use again the auxiliary variable

$$U(t) = \int_0^\infty \alpha e^{-\alpha s} N(t-s)ds,$$

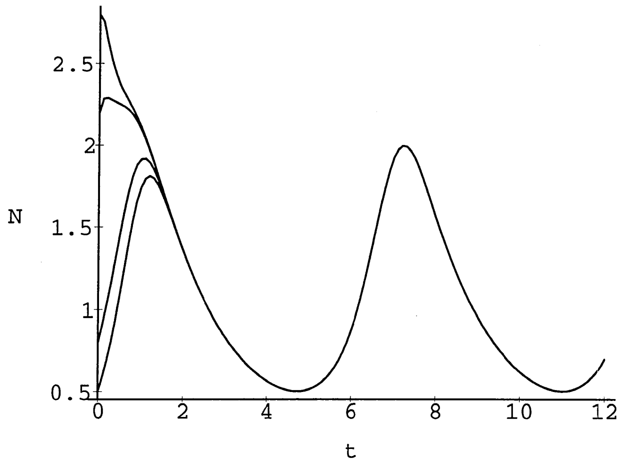

FIGURE 2(b). Graphs of the solution $N(t)$ of (6.5) with $\alpha = 12.69$ and the initial values $(N(0), U(0)) = (0.5, 0.35), (0.8, 0.7), (2.2, 1.2), (2.8, 1.7)$.

and convert (6.5) into the almost-periodic ordinary differential system

$$\begin{aligned} \frac{dN(t)}{dt} &= N(t)[(2+\sin(t)) - (2-\cos(\pi t))U(t)], \\ \frac{dU(t)}{dt} &= -\alpha[U(t) - N(t)]. \end{aligned} \tag{6.6}$$

The convergence of the almost-periodic solutions of (6.5) with $\alpha = 12.69$ and four initial values are displayed in Figure 2(c).

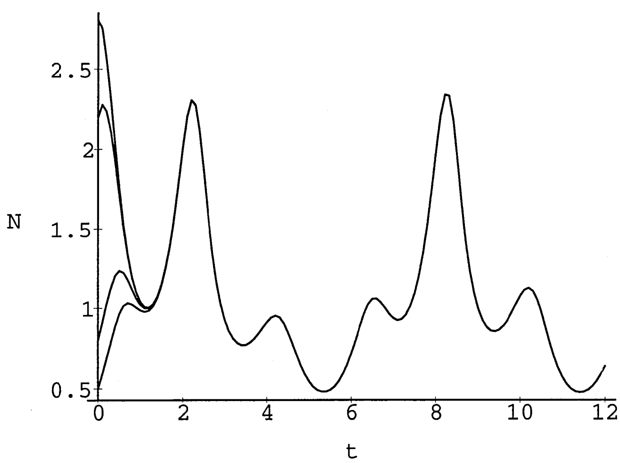

FIGURE 2(c). Graphs of the solution $N(t)$ of (6.7) with $\alpha = 12.69$ and the initial values $(N(0), U(0)) = (0.5, 0.35), (0.8, 0.7), (2.2, 1.2), (2.8, 1.7)$.

The first of the second category of examples is the integrodifferential equation with constant coefficients

$$\frac{dN(t)}{dt} = N(t)\left[1 - \int_0^\infty \alpha^2 s e^{-\alpha s} N(t-s)ds\right]. \tag{6.7}$$

We introduce the auxiliary variables U and V by

$$\left.\begin{aligned} U(t) &= \int_0^\infty \alpha^2 s e^{-\alpha s} N(t-s)ds \\ V(t) &= \int_0^\infty \alpha e^{-\alpha s} N(t-s)ds \end{aligned}\right\}, \qquad \alpha = 2.$$

The integrodifferential equation leads to the coupled system of autonomous ordinary differential equations:

$$\begin{aligned} \frac{dN(t)}{dt} &= N(t)[1 - U(t)], \\ \frac{dU(t)}{dt} &= \alpha[V(t) - U(t)], \\ \frac{dV(t)}{dt} &= \alpha[N(t) - V(t)]. \end{aligned} \tag{6.8}$$

A set of solutions of (6.7) are illustrated in Figure 3(a).

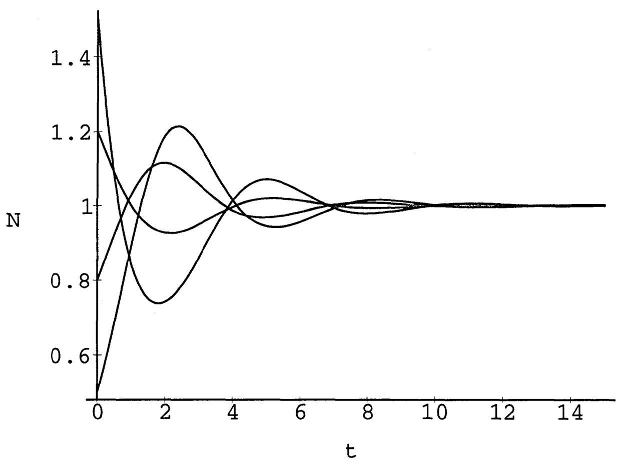

FIGURE 3(a). Graphs of the solution $N(t)$ of (6.10) with $\alpha = 2$ and the initial values $(N(0), U(0), V(0)) = (0.5, 0.35, 0.35)$, $(0.8, 0.7, 0.7)$, $(1.2, 1.2, 1.2)$, $(1.5, 1.7, 1.7)$.

The periodic integrodifferential equation

$$\frac{dN(t)}{dt} = N(t)\left[(2+\sin(t)) - (2-\cos(t))\int_0^\infty \alpha^2 s e^{-\alpha s} N(t-s)ds\right] \tag{6.9}$$

is converted by the introduction of the auxiliary variables

$$\left.\begin{aligned} U(t) &= \int_0^\infty \alpha^2 s e^{-\alpha s} N(t-s)ds \\ V(t) &= \int_0^\infty \alpha e^{-\alpha s} N(t-s)ds \end{aligned}\right\}, \qquad \alpha = 15.6$$

into a system of periodic ordinary differential equations,

$$\begin{aligned} \frac{dN(t)}{dt} &= N(t)[(2+\sin(t)) - (2-\cos(t))U(t)], \\ \frac{dU(t)}{dt} &= \alpha[V(t) - U(t)], \\ \frac{dV(t)}{dt} &= \alpha[N(t) - V(t)]. \end{aligned} \tag{6.10}$$

Solutions of (6.9) corresponding to four different initial values are displayed in Figure 3(b).

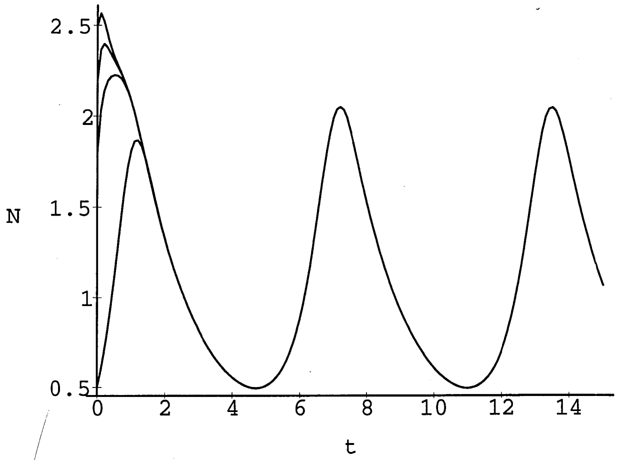

FIGURE 3(b). Graphs of the solution $N(t)$ of (6.12) with $\alpha = 15.6$ and the initial values $(N(0), U(0), V(0)) = (0.5, 0.35, 0.35)$, $(1.8, 0.7, 0.7)$, $(2.2, 1.2, 1.2)$, $(2.5, 1.7, 1.7)$.

Our final example is the almost-periodic integrodifferential equation

$$\frac{dN(t)}{dt} = N(t)\left[(2+\sin(t)) - (2-\cos(\pi t))\int_0^\infty \alpha^2 s e^{-\alpha s} N(t-s)ds\right]. \tag{6.11}$$

As before we let

$$\left.\begin{aligned} U(t) &= \int_0^\infty \alpha^2 s e^{-\alpha s} N(t-s)ds \\ V(t) &= \int_0^\infty \alpha e^{-\alpha s} N(t-s)ds \end{aligned}\right\}, \qquad \alpha = 15.6$$

and obtain the system

$$\begin{aligned}\frac{dN(t)}{dt} &= N(t)[(2+\sin(t)) - (2-\cos(\pi t))U(t)],\\ \frac{dU(t)}{dt} &= \alpha[V(t) - U(t)],\\ \frac{dV(t)}{dt} &= \alpha[N(t) - V(t)].\end{aligned} \tag{6.12}$$

Solutions of (6.11) are displayed in Figure 3(c).

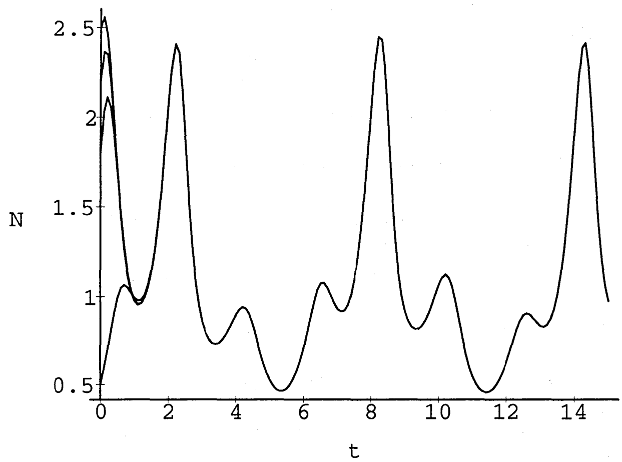

FIGURE 3(c). Graphs of the solution $N(t)$ of (6.14) with $\alpha = 15.6$ and the initial values $(N(0), U(0), V(0)) = (0.5, 0.35, 0.35)$, $(1.8, 0.7, 0.7)$, $(2.2, 1.2, 1.2)$, $(2.5, 1.7, 1.7)$.

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