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THE DOUBLE CONFLUENT HEUN EQUATION: CHARACTERISTIC EXPONENT AND CONNECTION FORMULAE

Wolfgang Bühring

ABSTRACT. The connection relations between the solutions of different type are obtained for the double confluent Heun equation, a second-order linear differential equation with two irregular singular points of unit rank. For the relevant quantity which determines the characteristic exponent and also enters the connection coefficients, two entirely different representations are given. One is essentially a finite determinant (of size 4 by 4 or 3 by 3 or 2 by 2, depending on details of the derivation) the elements of which are Taylor series at half the convergence radius with recursively available coefficients. The other one is an asymptotic expansion in terms of the recursively known coefficients of the formal power series solutions of the differential equation at one of the irregular singular points. In terms of the same coefficients, a series representation converging like a power series at half the convergence radius is obtained for the other relevant quantity which enters the connection coefficients. Of the same type is a numerically stable explicit representation of the coefficients of the Floquet solutions.

1. Introduction

Heun's differential equation [19], which is a Fuchsian differential equation with four regular singular points, has received renewed attention recently [44], together with its various confluent forms [13], [14]. The present paper is concerned with the double confluent Heun equation, a linear second-order differential equation with two irregular singular points of unit rank. If they are located at zero and infinity, the equation contains four parameters B, D, L, κ and may be written

$$z^2 f'' + zf + (-\kappa^2 z^2 + Bz - L^2 + Dz^{-1} - \kappa^2 z^{-2}) f(z) = 0. \tag{1.1}$$

The other standard forms proposed in the literature [13] would be less convenient than (1.1) for our investigation, which reviews and further develops methods for computing connection coefficients. A treatment based on the symmetric canonical form, with more emphasis on the other aspects which are not covered here, may be found in the contribution of Schmidt and Wolf to a forthcoming monograph [43].

Equation (1.1) is more complicated than its special case where $B = D = 0$, which has been treated by several authors, either directly [11, 16, 17, 35] or by transformation into the Mathieu equation [2, 3, 21, 39, 45, 46], the properties of which are well-known from [1, 4, 15, 30, 32, 41, 47], to mention a few of the numerous references.

In order to avoid unnecessary complications of presentation, we assume that the parameters B, D, L, κ^2 are real. Then it suffices to consider $L \geq 0$.

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Relative to the singular point ∞, there are formal power series solutions of (1.1),

$$\begin{aligned}\bar{f}_{\infty 1}(z) &= \exp(\kappa z) z^{-\frac{1}{2}-\frac{1}{2}(B/\kappa)} \sum_{n=0}^{\infty} a_n(B, D, \kappa) n!\, (2\kappa z)^{-n},\\ \bar{f}_{\infty 2}(z) &= \exp(-\kappa z) z^{-\frac{1}{2}+\frac{1}{2}(B/\kappa)} \sum_{n=0}^{\infty} a_n(B, D, -\kappa) n!\, (-2\kappa z)^{-n},\end{aligned} \tag{1.2}$$

where the coefficients are given by a four-term recurrence relation,

$$\begin{aligned} a_n &= a_n(B, D, \kappa),\\ a_0 &= 1, \ a_{-2} = a_{-1} = 0,\\ a_n &= \frac{1}{n^2}\Big\{\Big(-L - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + n\Big)\Big(L - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + n\Big) a_{n-1}\\ &\quad + \frac{2\kappa D}{n-1} a_{n-2} - \frac{4\kappa^4}{(n-2)(n-1)} a_{n-3}\Big\}, \quad n = 1, 2, \ldots .\end{aligned} \tag{1.3}$$

The formal solutions (1.2) are asymptotic expansions as $z \to \infty$ in suitable sectors of the complex plane.

Since a replacement of z by $1/z$ leaves $z^2 f'' + z f'$ unchanged and so the whole differential equation (1.1) is transformed into itself if simultaneously the parameters B and D are interchanged, we may write down immediately the formal power series solutions relative to the singular point $z = 0$,

$$\begin{aligned}\bar{f}_{01}(z) &= \exp\Big(\frac{\kappa}{z}\Big) z^{\frac{1}{2}+\frac{1}{2}(D/\kappa)} \sum_{n=0}^{\infty} a_n(D, B, \kappa) n!\, \Big(\frac{z}{2\kappa}\Big)^n,\\ \bar{f}_{02}(z) &= \exp\Big(-\frac{\kappa}{z}\Big) z^{\frac{1}{2}-\frac{1}{2}(D/\kappa)} \sum_{n=0}^{\infty} a_n(D, B, -\kappa) n!\, \Big(-\frac{z}{2\kappa}\Big)^n.\end{aligned} \tag{1.4}$$

These formal solutions are asymptotic expansions as $z \to 0$ in suitable sectors of the complex plane.

In the ring-shaped region $0 < |z| < \infty$, there are Floquet solutions

$$f_\mu(z) = z^\mu \sum_{n=-\infty}^{\infty} c_n^\mu z^n, \quad \mu \in \{-\nu, \nu\} \quad \text{(if } 2\nu \text{ is not equal to an integer)}, \tag{1.5}$$

where the coefficients obey the five-term recurrence relation

$$-\kappa^2 c_{n+2}^\mu + D c_{n+1}^\mu + (\mu + n - L)(\mu + n + L) c_n^\mu + B c_{n-1}^\mu - \kappa^2 c_{n-2}^\mu = 0. \tag{1.6}$$

There are several equivalent definitions of the characteristic exponent μ as well as several different methods for computing it.

Looking at (1.6) as an infinite homogeneous system of linear equations, the task may be reduced [51] (after the equations have been divided by appropriate n-dependent factors to ensure convergence) to evaluating its determinant for $\mu = 0$. Such infinite determinants, which are known as Hill determinants [51], may be evaluated numerically as the limit $N \to \infty$ of cut-off N by N determinants after, by analytical means, the speed of convergence has been improved [33, 34, 48].

Another method for computing the characteristic exponent uses numerical integration of the differential equation [49] and the requirement that analytic continuation along a path which surrounds the origin once reproduces each of the Floquet solutions apart from a constant factor.

Numerical values of the characteristic exponent and of the coefficients of the Floquet solutions may be computed iteratively from (1.6) viewed as a non-linear eigenvalue problem [36] for an eigenvalue μ such that the sum over n of the absolute squares of the coefficients remains finite.

It is the main purpose of this work to obtain, for the differential equation (1.1), the linear relations between the solutions of different type, in particular between the asymptotic solutions at the two irregular singular points. For the relevant quantities which enter these connection formulas, including the characteristic exponent, we want to obtain explicit expressions which are not only of theoretical interest but are also suitable for computing numerical values.

As suggested by the classical theory of linear differential equations with an irregular singular point of unit rank [23], [50], we consider contour integral solutions with the Laplace kernel. Our treatment may be viewed as a generalization and further development of the special, considerably simpler case investigated earlier [11] combined with our recent new method [12] for computing the characteristic exponent, which is quite different from all the methods mentioned above.

For one of the relevant quantities, which determines the characteristic exponent and enters the connection coefficients, we obtain an explicit asymptotic formula in terms of the recursively available coefficients $a_n(B, D, \kappa)$ of the formal solutions (1.2). For the same quantity, we get also other expressions in terms of a finite determinant the elements of which are Taylor series at half the convergence radius with recursively available coefficients. Here the size of the determinant is four by four, three by three, or two by two, depending on some details of the derivation. The other relevant quantity which enters the connection coefficients (but is not needed for the characteristic exponent) is obtained in terms of convergent series which converge like a power series at half the convergence radius, involving the characteristic exponent and the coefficients of the formal solutions. Of the same type is a numerically stable explicit representation of all the coefficients of the Floquet solutions, which follows as an interesting by-product of our investigation.

2. Contour integral solutions

2.1. Integral representation. In order to introduce the various quantities needed in this investigation, we closely follow our earlier work [12]. There are, however, some differences due to the different forms of the underlying differential equation. In particular, the quantity κ here is a parameter of the differential equation rather than standing for $+1$ or -1.

Extracting first an arbitrary power of z for later flexibility, we consider solutions

$$f(z) = z^{\lambda}(2\pi i)^{-1} \int_C \exp(\kappa z t) V(t)\, dt. \tag{2.1}$$

Then the weight function $V(t)$ is seen, by standard techniques [23], to be a solution of the t-equation

$$\begin{aligned}(t^2 - 1)V^{(4)} + \Big\{(7 + 2\lambda)t - \frac{B}{\kappa}\Big\}V^{(3)} + (3 - \lambda - L)(3 - \lambda + L)V'' \\ - \kappa D V' - \kappa^4 V(t) = 0,\end{aligned} \tag{2.2}$$

and the possible contours C are such that the integrand term (or bilinear concomitant [23])

$$\begin{aligned}\exp(\kappa z t)\Big\{&\frac{1}{\kappa}DV+\frac{1}{\kappa}(\lambda^2-L^2)\Big(zV-\frac{1}{\kappa}V'\Big)+z^2\Big[(2\lambda+1)t+\frac{B}{\kappa}\Big]V\\&-\frac{1}{\kappa}z\Big(\Big[(2\lambda+1)t+\frac{B}{\kappa}\Big]V\Big)'+\frac{1}{\kappa^2}\Big(\Big[(2\lambda+1)t+\frac{B}{\kappa}\Big]V\Big)''\\&+\kappa z^3(t^2-1)V-z^2[(t^2-1)V]'\\&+\frac{1}{\kappa}z[(t^2-1)V]''-\frac{1}{\kappa^2}[(t^2-1)V]'''\Big\}\end{aligned}\tag{2.3}$$

has the same value, identically in z, at both termini of the contour.

At infinity the t-equation has an irregular singular point. It can be shown that the associated four independent solutions behave asymptotically, when $t\to\infty$, as

$$\exp(2\varepsilon\kappa\sqrt{t})t^{-1+\frac{1}{2}\lambda+\frac{1}{4}\varepsilon^2(D/\kappa)}$$

with $\varepsilon=1,\ i,\ -1,\ -i$, respectively. Due to the common factor $\exp(\kappa z t)$, (2.3) tends to zero when $t\to\infty$ in a certain sector of the t-plane. So there are permissible contours which start at and return to infinity in appropriate directions depending on $\arg(\kappa z)$.

2.2. Floquet solutions of the t-equation. Besides the irregular singular point at infinity, the t-equation (2.2) has two regular singular points at $t=-1$ and $t=1$. Outside the unit circle, we have Floquet solutions

$$V_\mu(t)=t^{\lambda-\mu-1}\Phi_\mu(t),\tag{2.4}$$

$$\Phi_\mu(t)=\sum_{n=-\infty}^{+\infty}d_n^\mu t^{-n},\tag{2.5}$$

where the coefficients obey the recurrence relation

$$\begin{aligned}-\kappa^4 d_{n+2}^\mu&+\kappa D(\mu-\lambda+n+2)d_{n+1}^\mu\\&+(\mu+n-L)(\mu+n+L)(\mu-\lambda+n+1)(\mu-\lambda+n+2)\,d_n^\mu\\&+\frac{B}{\kappa}(\mu-\lambda+n)(\mu-\lambda+n+1)(\mu-\lambda+n+2)\,d_{n-1}^\mu\\&-(\mu-\lambda+n-1)(\mu-\lambda+n)(\mu-\lambda+n+1)(\mu-\lambda+n+2)\,d_{n-2}^\mu\\=0&\end{aligned}\tag{2.6}$$

which, by comparison with (1.6), is satisfied if

$$d_n^\mu=\frac{\Gamma(\mu-\lambda+1+n)}{\Gamma(\mu-\lambda+1)}\kappa^{-n}c_n^\mu.\tag{2.7}$$

Possible values of μ are therefore $\mu=-\nu$ or $\mu=+\nu$, where ν is the characteristic exponent as before. There are two further solutions which, however, are entire functions of t and so do not contribute to the contour integrals we will consider.

2.3. Solutions relative to the regular singular points of the t-equation. The exponents of the t-equation relative to the regular singular points $t = \pm 1$ are 0, 1, 2, $\lambda - 1/2 \pm B/(2\kappa)$. Provided that $\lambda \pm B/(2\kappa)$ is not equal to half an odd integer, the solutions can be written,

$$V^+(t) = F(\kappa, 1-t), \quad |t-1| < 2, \tag{2.8}$$

$$U_j^+(t) = G_j(\kappa, 1-t), \quad |t-1| < 2,\ j = 0, 1, 2, \tag{2.9}$$

and

$$V^-(t) = F(-\kappa, 1+t), \quad |t+1| < 2, \tag{2.10}$$

$$U_j^-(t) = G_j(-\kappa, 1+t), \quad |t+1| < 2,\ j = 0, 1, 2, \tag{2.11}$$

where

$$G_j(\kappa, x) = x^j \sum_{n=0}^{\infty} A_n(\kappa, j) x^n, \quad |x| < 2, \tag{2.12}$$

$$F(\kappa, x) = x^{\lambda - \frac{1}{2} + \frac{1}{2}(B/\kappa)} H(\kappa, x), \tag{2.13}$$

$$H(\kappa, x) = \sum_{n=0}^{\infty} A_n\Big(\kappa, \lambda - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}\Big) x^n, \quad |x| < 2, \tag{2.14}$$

with the initial coefficients chosen arbitrarily as

$$A_0(\kappa, q) = 1 \quad (\text{for } q = 0, 1, 2, \lambda - \tfrac{1}{2} + \tfrac{1}{2}\tfrac{B}{\kappa}), \tag{2.15}$$

$$A_1(\kappa, 0) = 0, \quad A_2(\kappa, 0) = 0, \quad A_1(\kappa, 1) = 0. \tag{2.16}$$

The other coefficients then are determined by the recurrence relation

$$\begin{aligned} A_n(\kappa, q) = {} & \frac{(q+n-\lambda-L)(q+n-\lambda+L)}{2(q+n)(q+n-\lambda+\frac{1}{2}-\frac{1}{2}B/\kappa)} A_{n-1}(\kappa, q) \\ & + \frac{\kappa D}{2(q+n)(q+n-1)(q+n-\lambda+\frac{1}{2}-\frac{1}{2}B/\kappa)} A_{n-2}(\kappa, q) \\ & - \frac{\kappa^4}{2(q+n)(q+n-1)(q+n-2)(q+n-\lambda+\frac{1}{2}-\frac{1}{2}B/\kappa)} A_{n-3}(\kappa, q), \end{aligned} \tag{2.17}$$

where $A_{-1}(\kappa, q) = A_{-2}(\kappa, q) = 0$; $n > 0$ if $q = 2, \lambda - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}$; $n > 1$ if $q = 1$; $n > 2$ if $q = 0$.

2.4. Analytic continuation in the t-plane. By equations (2.8)–(2.11), we have two fundamental sets of solutions, valid in different but overlapping domains. Any solution of one set may be expressed as a linear combination of the solutions of the other set, in particular

$$V^+(t) = E(-\kappa) V^-(t) + \sum_{j=0}^{2} B_j(-\kappa) U_j^-(t). \tag{2.18}$$

The coefficients E and B_j may be determined by evaluating this equation and its first three derivatives at $t = 0$. They therefore appear as the solution of the system of linear equations

$$\begin{pmatrix} F(-\kappa,1) & G_0(-\kappa,1) & G_1(-\kappa,1) & G_2(-\kappa,1) \\ F'(-\kappa,1) & G_0'(-\kappa,1) & G_1'(-\kappa,1) & G_2'(-\kappa,1) \\ F''(-\kappa,1) & G_0''(-\kappa,1) & G_1''(-\kappa,1) & G_2''(-\kappa,1) \\ F'''(-\kappa,1) & G_0'''(-\kappa,1) & G_1'''(-\kappa,1) & G_2'''(-\kappa,1) \end{pmatrix} \begin{pmatrix} E(-\kappa) \\ B_0(-\kappa) \\ B_1(-\kappa) \\ B_2(-\kappa) \end{pmatrix} = \begin{pmatrix} F(\kappa,1) \\ -F'(\kappa,1) \\ F''(\kappa,1) \\ -F'''(\kappa,1) \end{pmatrix} \tag{2.19}$$

and may be considered as known numbers. Introducing

$$G(-\kappa,x) = \sum_{j=0}^{2} B_j(-\kappa)G_j(-\kappa,x), \tag{2.20}$$

$$U^+(t) = G(\kappa,1-t), \tag{2.21}$$

$$U^-(t) = G(-\kappa,1+t), \tag{2.22}$$

we have

$$V^+(t) = E(-\kappa)V^-(t) + U^-(t). \tag{2.23}$$

Similarly, we have

$$V^-(t) = E(\kappa)V^+(t) + U^+(t). \tag{2.24}$$

It then follows, for consistency of (2.23), (2.24), that

$$U^+(t) = \{1 - E(-\kappa)E(\kappa)\}V^-(t) - E(\kappa)U^-(t), \tag{2.25}$$

$$U^-(t) = \{1 - E(-\kappa)E(\kappa)\}V^+(t) - E(-\kappa)U^+(t). \tag{2.26}$$

2.5. Multiplicative solutions in the t-plane. Near the origin, let us start with the solution

$$W(t) = \alpha V^+(t) + \gamma U^+(t) \tag{2.27}$$

or, by means of (2.23), (2.25),

$$W(t) = \big\{\alpha E(-\kappa) + \gamma[1 - E(-\kappa)E(\kappa)]\big\}V^-(t) + \{\alpha - \gamma E(\kappa)\}U^-(t), \tag{2.28}$$

and consider analytic continuation along a path in the form of a simple closed loop surrounding the two regular singular points 1, −1 in the positive sense. We want (2.27)–(2.28) to become a multiplicative solution [5], that is, a solution which, after the loop has been traversed, is reproduced apart from a constant factor. It is convenient to write this factor in the form

$$p\exp(2i\pi\lambda). \tag{2.29}$$

In the same way as in [12], we then can find that α and γ must satisfy the homogeneous system of linear equations

$$\begin{aligned} E(-\kappa)\{p-\exp(2\pi i\lambda)\}\alpha+[1-E(-\kappa)E(\kappa)]\Big\{p+\exp\Big(-\pi i\frac{B}{\kappa}\Big)\Big\}\gamma=0,\\ -\Big\{p+\exp\Big(\pi i\frac{B}{\kappa}\Big)\Big\}\alpha+E(\kappa)\{p-\exp(-2\pi i\lambda)\}\gamma=0 \end{aligned} \tag{2.30}$$

and that, as a consequence,

$$p=\exp(-2i\pi\mu), \tag{2.31}$$

where $\mu\in\{-\nu,\nu\}$ is the characteristic exponent as before, which is determined by

$$\cos\Big(\pi\Big[\mu+\frac{1}{2}\frac{B}{\kappa}\Big]\Big)\cos\Big(\pi\Big[\mu-\frac{1}{2}\frac{B}{\kappa}\Big]\Big)=e(-\kappa)e(\kappa) \tag{2.32}$$

or

$$[\cos(\pi\mu)]^2=\Big[\sin\Big(\frac{1}{2}\frac{\pi B}{\kappa}\Big)\Big]^2+e(-\kappa)e(\kappa). \tag{2.33}$$

Here we have introduced

$$e(\kappa)=\frac{\pi}{\Gamma(\lambda+\frac{1}{2}-\frac{1}{2}\frac{B}{\kappa})\Gamma(-\lambda+\frac{1}{2}-\frac{1}{2}\frac{B}{\kappa})}E(\kappa), \tag{2.34}$$

a quantity which is independent of λ, as will be shown below. In view of (2.31), all the quantities depending on p will carry an index μ.

Now (2.28) can be simplified by means of (2.30), so that we have

$$W_\mu(t)=\begin{cases}\alpha_\mu V^+(t)+\gamma_\mu U^+(t), & \text{if } |t-1|<2\\ \exp(-i\pi[\lambda-\mu-1])\Big\{\exp(2i\pi[\lambda-\frac{1}{2}])\dfrac{\cos(\pi[\lambda+\frac{1}{2}\frac{B}{\kappa}])}{\cos(\pi[\mu-\frac{1}{2}\frac{B}{\kappa}])}E(-\kappa)\alpha_\mu V^-(t) & \\ \quad+\dfrac{\cos(\pi[\lambda+\frac{1}{2}\frac{B}{\kappa}])}{\cos(\pi[\mu+\frac{1}{2}\frac{B}{\kappa}])}E(\kappa)\gamma_\mu U^-(t)\Big\}, & \text{if } |t+1|<2,\end{cases} \tag{2.35}$$

with

$$\gamma_\mu=\exp\Big(i\pi\Big[\lambda-\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa}\Big]\Big)\frac{\cos(\pi[\mu+\frac{1}{2}\frac{B}{\kappa}])}{E(\kappa)\sin(\pi[\lambda-\mu])}\alpha_\mu \tag{2.36}$$

from (2.30). We may choose the arbitrary normalization to be

$$\alpha_\mu = \exp\Big(-i\pi\Big[\lambda - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}\Big]\Big)\Gamma\Big(-\lambda + \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa}\Big)\sqrt{\frac{e(\kappa)}{\cos(\pi[\mu + \frac{1}{2}\frac{B}{\kappa}])}}, \tag{2.37}$$

so as to get the two parts of (2.35) looking similar and differing in some signs and a common phase factor only. Furthermore, according to (2.29), (2.31), we are constructing a multiplicative solution which obeys the same circuit relation as (2.4) and therefore is proportional to (2.4) outside the unit circle. We thus obtain finally

$$W_\mu(t) = \begin{cases} \exp(-i\pi[\lambda - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}])\Gamma(-\lambda + \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa})\sqrt{\frac{e(\kappa)}{\cos(\pi[\mu + \frac{1}{2}\frac{B}{\kappa}])}}V^+(t) \\ \quad + \Gamma(-\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa})\frac{\cos(\pi[\lambda - \frac{1}{2}\frac{B}{\kappa}])}{\sin(\pi[\lambda - \mu])}\sqrt{\frac{e(-\kappa)}{\cos(\pi[\mu - \frac{1}{2}\frac{B}{\kappa}])}}U^+(t), \\ \quad \text{if } |t-1| < 2, \\ \exp(-i\pi[\lambda - \mu - 1])\Big\{\exp(i\pi[\lambda - \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa}])\Gamma(-\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}) \\ \qquad \times \sqrt{\frac{e(-\kappa)}{\cos(\pi[\mu - \frac{1}{2}\frac{B}{\kappa}])}}V^-(t) \\ \quad + \Gamma(-\lambda + \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa})\frac{\cos(\pi[\lambda + \frac{1}{2}\frac{B}{\kappa}])}{\sin(\pi[\lambda - \mu])}\sqrt{\frac{e(\kappa)}{\cos(\pi[\mu + \frac{1}{2}\frac{B}{\kappa}])}}U^-(t)\Big\}, \\ \quad \text{if } |t+1| < 2, \\ \Delta_\mu \Gamma(\mu - \lambda + 1)V_\mu(t), \qquad \text{if } |t| > 1. \end{cases} \tag{2.38}$$

Here the constant Δ_μ may be determined by comparison of the first and third part of (2.38) evaluated at $t = 2$ or by comparison of the second and third part of (2.38) evaluated at $t = -2$. We do not display these formulas, but shall obtain below another, more attractive formula for Δ_μ.

The phases of the different square roots in (2.37)–(2.38) should be chosen so that their sum is zero, which is possible and necessary because of (2.32). In any situation such that the denominator vanishes, the square root should be rewritten by means of (2.32).

3. Special contour integral solutions

We consider contours which start at and return to infinity near the positive imaginary axis, surrounding in the positive sense one or both of the finite singular points, respectively. With the phase convention $\arg(1-t) = \arg(1+t) = 0$ on the interval $(-1, 1)$ of the real axis and $\arg(t) = 0$ on the positive real axis, we then may define the solutions

$$f_{\infty 1}(z) = \exp\Big(i\pi\Big[-\lambda + \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa}\Big]\Big)\Gamma\Big(-\lambda + \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa}\Big)\kappa^{\lambda + \frac{1}{2} + \frac{1}{2}(B/\kappa)} z^\lambda$$
$$\times (2\pi i)^{-1} \int_{i\infty}^{(1+)} \exp(\kappa z t) V^+(t)\, dt, \quad 0 < \arg(\kappa z) < \pi, \tag{3.1}$$

$$f_{\infty 2}(z) = \Gamma\Big(-\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}\Big)\kappa^{\lambda+\frac{1}{2}-\frac{1}{2}(B/\kappa)} z^{\lambda}$$
$$\times (2\pi i)^{-1} \int_{i\infty}^{(-1+)} \exp(\kappa z t) V^{-}(t)\, dt, \quad 0 < \arg(\kappa z) < \pi, \tag{3.2}$$
$$f_{\mu}(z) = \Gamma(\mu - \lambda + 1)\kappa^{\lambda-\mu} z^{\lambda}$$
$$\times (2\pi i)^{-1} \int_{i\infty}^{(-1+,1+)} \exp(\kappa z t) V_{\mu}(t)\, dt, \quad 0 < \arg(\kappa z) < \pi. \tag{3.3}$$

In addition, we may consider such integrals with contours which are rotated by an angle ψ and so yield the analytic continuation for $0 < \arg(\kappa z) - \psi < \pi$. In the case of (3.1), we have $-3\pi/2 < \psi < \pi/2$, and in the case of (3.2), we have $-\pi/2 < \psi < 3\pi/2$, since in each case the presence of the other singular point inhibits larger rotations. We thus have extended the definition of the solution (3.1) to the sector $-3\pi/2 < \arg(\kappa z) < 3\pi/2$ and of the solution (3.2) to the sector $-\pi/2 < \arg(\kappa z) < 5\pi/2$. In these sectors, the solutions are represented asymptotically by the respective formal solutions (1.2), which follow if the series in the integrals are integrated term-by-term and use is made of the fact that

$$A_n\Big(\kappa, \lambda - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}\Big) = \frac{\Gamma(\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa})n!}{\Gamma(\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + n)} 2^{-n} a_n(B, D, \kappa) \tag{3.4}$$

by comparison of the recurrence relation (2.17) for $q = \lambda - 1/2 + B/(2\kappa)$ with the recurrence relation (1.3).

In the case of (3.3), the contour surrounds both singular points and so the angle of rotation ψ is not restricted. Term-by-term integration of the series yields the Floquet solutions (1.5), which therefore are valid for arbitrary values of $\arg(\kappa z)$.

4. Linear relations between the solutions

4.1. Connection between the Floquet solutions and the formal solutions. Next we consider the solution

$$\Delta_{\mu} f_{\mu}(z) = \kappa^{\lambda-\mu} z^{\lambda} (2\pi i)^{-1} \int_{i\infty}^{(-1+,1+)} \exp(\kappa z t) W_{\mu}(t)\, dt,$$
$$0 < \arg(\kappa z) < \pi. \tag{4.1}$$

As it stands, it essentially represents the Floquet solutions, by (3.3) and the last line of (2.38). The contour is equivalent to the sum of the contours which surround only one of the singular points, each of which yields one of the local solutions at infinity by (3.1) or (3.2), respectively. Accounting for the various constant factors, we obtain

$$\Delta_{\mu}\kappa^{\mu} f_{\mu}(z) = \kappa^{-\frac{1}{2}-\frac{1}{2}(B/\kappa)} \sqrt{\frac{e(B, D, \kappa)}{\cos(\pi[\mu + \frac{1}{2}\frac{B}{\kappa}])}} f_{\infty 1}(z)$$
$$+ \exp\Big(i\pi\Big[\mu + \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa}\Big]\Big)\kappa^{-\frac{1}{2}+\frac{1}{2}(B/\kappa)}$$
$$\times \sqrt{\frac{e(B, D, -\kappa)}{\cos(\pi[\mu - \frac{1}{2}\frac{B}{\kappa}])}} f_{\infty 2}(z) \tag{4.2}$$

for $\mu \in \{\nu, -\nu\}$ using the more detailed notation $e(\kappa) = e(B, D, \kappa)$. It then follows that

$$f_{\infty 1}(z) = \exp\Big(-i\pi\Big[\frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}\Big]\Big)\kappa^{\frac{1}{2}+\frac{1}{2}(B/\kappa)}\frac{e(B,D,-\kappa)}{\sin(2\pi\nu)}$$
$$\times\left\{\exp(i\pi\nu)\sqrt{\frac{e(B,D,\kappa)}{\cos(\pi[-\nu+\frac{1}{2}\frac{B}{\kappa}])}}\Delta_{-\nu}\kappa^{-\nu}f_{-\nu}(z)\right.$$
$$\left.-\exp(-i\pi\nu)\sqrt{\frac{e(B,D,\kappa)}{\cos(\pi[\nu+\frac{1}{2}\frac{B}{\kappa}])}}\Delta_{\nu}\kappa^{\nu}f_{\nu}(z)\right\}, \tag{4.3}$$

$$f_{\infty 2}(z) = \kappa^{\frac{1}{2}-\frac{1}{2}(B/\kappa)}\frac{e(B,D,\kappa)}{\sin(2\pi\nu)}$$
$$\times\left\{\sqrt{\frac{e(B,D,-\kappa)}{\cos(\pi[-\nu-\frac{1}{2}\frac{B}{\kappa}])}}\Delta_{-\nu}\kappa^{-\nu}f_{-\nu}(z)\right.$$
$$\left.-\sqrt{\frac{e(B,D,-\kappa)}{\cos(\pi[\nu-\frac{1}{2}\frac{B}{\kappa}])}}\Delta_{\nu}\kappa^{\nu}f_{\nu}(z)\right\}, \tag{4.4}$$

provided that the characteristic exponent is not equal to an integer or half an odd integer; otherwise each of (4.3), (4.4) is the starting point from which the relevant formula follows by an appropriate limiting process.

4.2. Circuit relations. The circuit relations for the Floquet solutions are

$$f_\mu(e^{2m\pi i}z) = e^{2m\mu\pi i}f_\mu(z), \tag{4.5}$$

where m is any integer. By means of (4.3), (4.4), the circuit relations for the local solutions at infinity may then be found as

$$\begin{aligned} f_{\infty 1}(e^{2m\pi i}z) &= T_{11}f_{\infty 1}(z) + T_{12}f_{\infty 2}(z),\\ f_{\infty 2}(e^{2m\pi i}z) &= T_{21}f_{\infty 1}(z) + T_{22}f_{\infty 2}(z), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} T_{11} &= \frac{\sin([1-2m]\pi\nu)}{\sin(\pi\nu)} + 2i\exp\Big(-i\pi\frac{B}{2\kappa}\Big)\sin\Big(\pi\frac{B}{2\kappa}\Big)\frac{\sin(2m\pi\nu)}{\sin(2\pi\nu)},\\ T_{12} &= -i\kappa^{B/\kappa}\exp\Big(-i\pi\frac{B}{\kappa}\Big)\frac{e(B,D,-\kappa)}{\cos(\pi\nu)}\frac{\sin(2m\pi\nu)}{\sin(\pi\nu)},\\ T_{21} &= -i\kappa^{-B/\kappa}\frac{e(B,D,\kappa)}{\cos(\pi\nu)}\frac{\sin(2m\pi\nu)}{\sin(\pi\nu)},\\ T_{22} &= \frac{\sin([1+2m]\pi\nu)}{\sin(\pi\nu)} - 2i\exp\Big(-i\pi\frac{B}{2\kappa}\Big)\sin\Big(\pi\frac{B}{2\kappa}\Big)\frac{\sin(2m\pi\nu)}{\sin(2\pi\nu)}. \end{aligned} \tag{4.7}$$

These circuit relations give the solution to the so-called lateral connection problem [8]. They also provide a simple and convenient description of the Stokes phenomenon, which has again received attention by many authors, for example, by Kohno [27], Braaksma [10], Immink [22], Martinet and Ramis [31], Balser et al. [6], and by the authors mentioned below in the context of the limit formulas. The entirely new aspect of smoothing the Stokes discontinuities, raised by Berry [9] and further investigated by Olver [38] and others (the references may be found in [38]), is beyond the scope of the present work.

4.3. Connection between the local solutions at the irregular singular points. By the substitution $(B, D, z) \to (D, B, 1/z)$, we obtain from (4.2)

$$\begin{aligned}\Delta_\mu(D,B)\kappa^\mu f_{-\mu}(z) = \kappa^{-\frac{1}{2}-\frac{1}{2}(D/\kappa)}&\sqrt{\frac{e(D,B,\kappa)}{\cos(\pi[\mu+\frac{1}{2}\frac{D}{\kappa}])}}f_{01}(z)\\ &+\exp\Big(i\pi\Big[\mu+\frac{1}{2}-\frac{1}{2}\frac{D}{\kappa}\Big]\Big)\kappa^{-\frac{1}{2}+\frac{1}{2}(D/\kappa)}\\ &\times\sqrt{\frac{e(D,B,-\kappa)}{\cos(\pi[\mu-\frac{1}{2}\frac{D}{\kappa}])}}f_{02}(z)\end{aligned} \tag{4.8}$$

for $\mu \in \{\nu, -\nu\}$. It then follows that

$$\begin{aligned}f_{01}(z) &= \exp\Big(-i\pi\Big[\frac{1}{2}+\frac{1}{2}\frac{D}{\kappa}\Big]\Big)\kappa^{\frac{1}{2}+\frac{1}{2}(D/\kappa)}\frac{e(D,B,-\kappa)}{\sin(2\pi\nu)}\\ &\times\Bigg\{\exp(i\pi\nu)\sqrt{\frac{e(D,B,\kappa)}{\cos(\pi[-\nu+\frac{1}{2}\frac{D}{\kappa}])}}\Delta_{-\nu}(D,B)\kappa^{-\nu}f_\nu(z)\\ &\quad-\exp(-i\pi\nu)\sqrt{\frac{e(D,B,\kappa)}{\cos(\pi[\nu+\frac{1}{2}\frac{D}{\kappa}])}}\Delta_\nu(D,B)\kappa^\nu f_{-\nu}(z)\Bigg\},\end{aligned} \tag{4.9}$$

$$\begin{aligned}f_{02}(z) &= \kappa^{\frac{1}{2}-\frac{1}{2}(D/\kappa)}\frac{e(D,B,\kappa)}{\sin(2\pi\nu)}\\ &\times\Bigg\{\sqrt{\frac{e(D,B,-\kappa)}{\cos(\pi[-\nu-\frac{1}{2}\frac{D}{\kappa}])}}\Delta_{-\nu}(D,B)\kappa^{-\nu}f_\nu(z)\\ &\quad-\sqrt{\frac{e(D,B,-\kappa)}{\cos(\pi[\nu-\frac{1}{2}\frac{D}{\kappa}])}}\Delta_\nu(D,B)\kappa^\nu f_{-\nu}(z)\Bigg\}.\end{aligned} \tag{4.10}$$

From (4.9)–(4.10) and (4.2), the coefficients in the linear relations

$$\begin{aligned}f_{01}(z) &= Q_{11}f_{\infty 1}(z) + Q_{12}f_{\infty 2}(z),\\ f_{02}(z) &= Q_{21}f_{\infty 1}(z) + Q_{22}f_{\infty 2}(z)\end{aligned} \tag{4.11}$$

may then be found to be

$$
\begin{aligned}
Q_{11} &= \kappa^{\frac{1}{2}[(D-B)/\kappa]} \exp\Big(i\pi\Big[\frac{1}{2}-\frac{1}{2}\frac{D}{\kappa}\Big]\Big)\frac{e(D,B,-\kappa)}{\cos(\pi\nu)}\frac{1}{2\sin(\pi\nu)} \\
&\times\Bigg\{-\sqrt{\frac{e(B,D,\kappa)}{\cos(\pi[\nu+\frac{1}{2}\frac{B}{\kappa}])}}\sqrt{\frac{e(D,B,\kappa)}{\cos(\pi[-\nu+\frac{1}{2}\frac{D}{\kappa}])}}r_\nu\exp(i\pi\nu) \\
&+\sqrt{\frac{e(B,D,\kappa)}{\cos(\pi[-\nu+\frac{1}{2}\frac{B}{\kappa}])}}\sqrt{\frac{e(D,B,\kappa)}{\cos(\pi[\nu+\frac{1}{2}\frac{D}{\kappa}])}}r_{-\nu}\exp(-i\pi\nu)\Bigg\}, \\
Q_{12} &= \kappa^{\frac{1}{2}[(B+D)/\kappa]} \exp(-i\pi[\frac{B+D}{2\kappa}])\frac{e(D,B,-\kappa)}{\cos(\pi\nu)}\frac{1}{2\sin(\pi\nu)} \\
&\times\Bigg\{\sqrt{\frac{e(B,D,-\kappa)}{\cos(\pi[\nu-\frac{1}{2}\frac{B}{\kappa}])}}\sqrt{\frac{e(D,B,\kappa)}{\cos(\pi[-\nu+\frac{1}{2}\frac{D}{\kappa}])}}r_\nu\exp(2i\pi\nu) \\
&-\sqrt{\frac{e(B,D,-\kappa)}{\cos(\pi[-\nu-\frac{1}{2}\frac{B}{\kappa}])}}\sqrt{\frac{e(D,B,\kappa)}{\cos(\pi[\nu+\frac{1}{2}\frac{D}{\kappa}])}}r_{-\nu}\exp(-2i\pi\nu)\Bigg\}, \\
Q_{21} &= \kappa^{-\frac{1}{2}[(B+D)/\kappa]}\frac{e(D,B,\kappa)}{\cos(\pi\nu)}\frac{1}{2\sin(\pi\nu)} \\
&\times\Bigg\{\sqrt{\frac{e(B,D,\kappa)}{\cos(\pi[\nu+\frac{1}{2}\frac{B}{\kappa}])}}\sqrt{\frac{e(D,B,-\kappa)}{\cos(\pi[-\nu-\frac{1}{2}\frac{D}{\kappa}])}}r_\nu \\
&-\sqrt{\frac{e(B,D,\kappa)}{\cos(\pi[-\nu+\frac{1}{2}\frac{B}{\kappa}])}}\sqrt{\frac{e(D,B,-\kappa)}{\cos(\pi[\nu-\frac{1}{2}\frac{D}{\kappa}])}}r_{-\nu}\Bigg\}, \\
Q_{22} &= \kappa^{\frac{1}{2}[(B-D)/\kappa]} \exp\Big(i\pi\Big[\frac{1}{2}-\frac{1}{2}\frac{B}{\kappa}\Big]\Big)\frac{e(D,B,\kappa)}{\cos(\pi\nu)}\frac{1}{2\sin(\pi\nu)} \\
&\times\Bigg\{\sqrt{\frac{e(B,D,-\kappa)}{\cos(\pi[\nu-\frac{1}{2}\frac{B}{\kappa}])}}\sqrt{\frac{e(D,B,-\kappa)}{\cos(\pi[-\nu-\frac{1}{2}\frac{D}{\kappa}])}}r_\nu\exp(i\pi\nu) \\
&-\sqrt{\frac{e(B,D,-\kappa)}{\cos(\pi[-\nu-\frac{1}{2}\frac{B}{\kappa}])}}\sqrt{\frac{e(D,B,-\kappa)}{\cos(\pi[\nu-\frac{1}{2}\frac{D}{\kappa}])}}r_{-\nu}\exp(-i\pi\nu)\Bigg\},
\end{aligned}
\tag{4.12}
$$

where

$$r_\mu = \frac{\Delta_{-\mu}(D,B)}{\Delta_\mu(B,D)}\kappa^{-2\mu}, \quad \mu\in\{-\nu,\nu\}. \tag{4.13}$$

The connection relations between the various solutions are given by (4.2)–(4.4), (4.6)–(4.7), (4.8)–(4.10), and (4.11)–(4.13). Two functions of the parameters, $e(\kappa) = e(B, D, \kappa)$ and $\Delta_\mu(B, D)$, enter, besides the characteristic exponent μ which is also determined by $e(\kappa)$ according to (2.33). We are going to obtain explicit expressions for these quantities.

5. Expressions for $e(\kappa)$

5.1. Asymptotic expansion for $e(\kappa)$. It is convenient to use Pochhammer symbols

$$(x)_n = x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{5.1}$$

in order to state the following result.

Theorem 1. *With*

$$C_n(\kappa) = \frac{n!\,n!}{\Gamma(-\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + n)\Gamma(\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + n)} 2^{B/\kappa} a_n(B, D, \kappa) \tag{5.2}$$

and

$$h_m(-\kappa) = \frac{(\lambda - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa})_m}{m!} \sum_{l=0}^{m} \frac{(-m)_l l!}{(-\lambda + \frac{3}{2} - \frac{1}{2}\frac{B}{\kappa} - m)_l (\lambda + \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa})_l} a_l(B, D, -\kappa) \tag{5.3}$$

we have

$$e(-\kappa) = \pi C_n(\kappa) \left\{ 1 + \sum_{m=1}^{M} \frac{(\lambda + \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa})_m}{(\lambda + \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa} - n)_m} h_m(-\kappa) + O(n^{-M-1}) \right\}^{-1} \tag{5.4}$$

as $n \to \infty$.

This theorem, which gives the quantity $e(\kappa)$ in terms of the coefficients $a_n(B, D, \kappa)$ of the formal solutions (1.2), may be viewed as a special case of the corresponding theorem for a more general differential equation [12]. It is based on the work of Schäfke and Schmidt [42] and may conveniently be derived by means of Darboux's method [37] as explained in detail in [12]. There is much freedom in choosing appropriate values of n and M in order to get accurate numerical results. Also, the parameter λ is quite arbitrary except that $\lambda - B/(2\kappa)$ or $\lambda + B/(2\kappa)$ must not be equal to or should not be close to half an odd integer. If not conflicting with this restriction, the choice $\lambda = L$ (or $\lambda = -L$) is most advantageous, as shown below. As long as n is finite, the right-hand side of (5.4) depends on λ, but this dependence disappears asymptotically as $n \to \infty$. For we have

$$\frac{n!\,n!}{\Gamma(-\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + n)\Gamma(\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + n)} = n^{1-B/\kappa}[1 + O(n^{-1})] \tag{5.5}$$

as $n \to \infty$. So, comparison of corresponding results for different values of the computational parameter λ may give an idea of the achieved accuracy.

Theorem 1 implies the limit formula

$$e(-\kappa) = \pi \lim_{n\to\infty} \frac{n!\,n!}{\Gamma(-\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + n)\Gamma(\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + n)} 2^{B/\kappa} a_n(B, D, \kappa). \tag{5.6}$$

Such limit formulas have been obtained, apart from factors which tend to 1 as $n \to \infty$, by Jurkat, Lutz and Peyerimhoff [24, 25, 26], Hinton [20], Kurth and Schmidt [28], and, more recently, Balser, Jurkat, and Lutz [8]. In addition, Balser et al. [8] give an equivalent infinite series representation. Proceeding in the same way, we may get from (5.6), using (5.5) and (1.3),

$$e(-\kappa) = \pi 2^{B/\kappa} \frac{1}{\Gamma(-L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})\Gamma(L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})}$$
$$\times \Bigg\{ 1 + \frac{2\kappa D}{(-L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})_2 (L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})_2}$$
$$+ \sum_{j=3}^{\infty} \frac{(j-1)!\,(j-1)!}{(-L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})_j (L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})_j}$$
$$\times \left[\frac{2\kappa D}{j-1} a_{j-2}(B,D,\kappa) - \frac{4\kappa^4}{(j-2)(j-1)} a_{j-3}(B,D,\kappa) \right] \Bigg\}. \tag{5.7}$$

Also, some of the authors mentioned above in the context of the Stokes phenomenon give limit formulas for the Stokes multipliers.

From a computational point of view the applicability of both the series representation (5.7) and the limit formulas such as (5.6) is questionable, however, because the rate of convergence is slow. Thus Theorem 1 implies significant progress.

5.2. Another expression for $e(\kappa)$. Alternatively, we may compute the related quantity $E(\kappa)$ by solving the system of linear equations (2.19) by means of Cramer's rule and the fact that the determinant D_0 of the system is a Wronskian of the t-equation (2.2) equal to

$$D_0 = -\left(\lambda - \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa}\right)\left(\lambda - \frac{3}{2} - \frac{1}{2}\frac{B}{\kappa}\right)\left(\lambda - \frac{5}{2} - \frac{1}{2}\frac{B}{\kappa}\right) 2^{-\lambda+\frac{9}{2}-\frac{1}{2}(B/\kappa)}. \tag{5.8}$$

We then have

$$E(-\kappa) = \frac{-D_1}{(\lambda-\frac{1}{2}-\frac{1}{2}\frac{B}{\kappa})(\lambda-\frac{3}{2}-\frac{1}{2}\frac{B}{\kappa})(\lambda-\frac{5}{2}-\frac{1}{2}\frac{B}{\kappa})} 2^{\lambda-\frac{9}{2}+\frac{1}{2}(B/\kappa)}, \tag{5.9}$$

where

$$D_1 = \begin{vmatrix} F(\kappa,1) & G_0(-\kappa,1) & G_1(-\kappa,1) & G_2(-\kappa,1) \\ -F'(\kappa,1) & G_0'(-\kappa,1) & G_1'(-\kappa,1) & G_2'(-\kappa,1) \\ F''(\kappa,1) & G_0''(-\kappa,1) & G_1''(-\kappa,1) & G_2''(-\kappa,1) \\ -F'''(\kappa,1) & G_0'''(-\kappa,1) & G_1'''(-\kappa,1) & G_2'''(-\kappa,1) \end{vmatrix}. \tag{5.10}$$

The elements of this determinant are Taylor series at half the convergence radius with recursively known coefficients according to (2.12)–(2.17). Using (2.34), we then have finally

$$e(-\kappa) = \frac{\pi D_1}{\Gamma(\lambda+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})\Gamma(-\lambda+\frac{7}{2}+\frac{1}{2}\frac{B}{\kappa})} 2^{\lambda-\frac{9}{2}+\frac{1}{2}(B/\kappa)}. \tag{5.11}$$

Again λ is a computational parameter on which the value of (5.11) does not depend. Inspection of (2.17) shows that here, notably if L is not small, it is advantageous to choose λ equal or near to L or $-L$, but with the reservation that $\lambda \pm B/(2\kappa)$ must not be equal or should not be near to half an odd integer.

It will be shown in Section 7 below that a similar expression can be found with a 3 by 3 determinant in place of D_1 if λ is kept at our disposal, or even with a 2 by 2 determinant if a special value of λ is used which, however, is different from the recommended value.

The method of evaluating $e(\kappa)$ by means of (5.10)–(5.11) or by the corresponding modified equations in Section 7 below is well suited for computing accurate numerical values and in so far competes with the method of Theorem 1. Depending on the values of the parameters of the differential equation, it may be that one or the other of the methods is more advantageous.

6. The constant Δ_μ and the coefficients of the Floquet solutions

An expression for $\Delta_\mu = \Delta_\mu(B, D)$ can be obtained, according to Naundorf [35], if use is made of the asymptotic representation of the exponential function by Heaviside's exponential series [18],

$$\exp(x) \sim \sum_{n=-\infty}^{\infty} \frac{1}{\Gamma(n+\delta+1)} x^{n+\delta} \qquad \text{as } x \to \infty, \; |\arg(x)| < \pi. \tag{6.1}$$

By means of (6.1) with $\delta = \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa} + \mu$ for the first or $\delta = \frac{1}{2} - \frac{1}{2}\frac{B}{\kappa} + \mu$ for the second equation, the formal solutions (1.2) get the same analytical structure as the Floquet solutions (1.5). Inserting the so modified formal solutions in (4.2) and then comparing the coefficients of the power series on the left and right-hand side, we obtain

$$\begin{aligned} \Delta_\mu(B, D) c_n^\mu = \kappa^n \sqrt{\frac{e(B, D, \kappa)}{\cos(\pi[\mu + \frac{1}{2}\frac{B}{\kappa}])}} \sum_{l=0}^{\infty} \frac{l!}{\Gamma(\mu + \frac{3}{2} + \frac{1}{2}\frac{B}{\kappa} + n + l)} 2^{-l} a_l(B, D, \kappa) \\ + (-\kappa)^n \sqrt{\frac{e(B, D, -\kappa)}{\cos(\pi[\mu - \frac{1}{2}\frac{B}{\kappa}])}} \\ \times \sum_{l=0}^{\infty} \frac{l!}{\Gamma(\mu + \frac{3}{2} - \frac{1}{2}\frac{B}{\kappa} + n + l)} 2^{-l} a_l(B, D, -\kappa), \end{aligned} \tag{6.2}$$

where the series converge like a power series at half the convergence radius.

Balser et al. [8] obtain a relation such as (6.2) in a different way using the "associated functions" introduced by them [7] or by Schäfke 40], but our method of deriving (6.2) on the basis of Naundorf [35] seems to be simpler and more convenient.

Each of the two terms on the right-hand side of (6.2) alone satisfies the recurrence relation (1.6) for the coefficients c_n^μ of the Floquet solutions and tends to zero as $n \to \infty$, even if μ is different from the characteristic exponent, but generally increases without limit as $n \to -\infty$. It is only with the correct value of μ that the sum of the two terms tends to zero as $n \to -\infty$ too and so has the appropriate behavior required for the c_n^μ.

If we normalize the Floquet solutions by $c_0^\mu = 1$, we may use (6.2) with $n = 0$ to get

$$\begin{aligned} \Delta_\mu(B, D) = \sqrt{\frac{e(B, D, \kappa)}{\cos(\pi[\mu + \frac{1}{2}\frac{B}{\kappa}])}} \sum_{l=0}^{\infty} \frac{l!}{\Gamma(\mu + \frac{3}{2} + \frac{1}{2}\frac{B}{\kappa} + l)} 2^{-l} a_l(B, D, \kappa) \\ + \sqrt{\frac{e(B, D, -\kappa)}{\cos(\pi[\mu - \frac{1}{2}\frac{B}{\kappa}])}} \sum_{l=0}^{\infty} \frac{l!}{\Gamma(\mu + \frac{3}{2} - \frac{1}{2}\frac{B}{\kappa} + l)} 2^{-l} a_l(B, D, -\kappa). \end{aligned} \tag{6.3}$$

After Δ_μ has been determined in this way, (6.2) with other values of n, positive or negative, gives an explicit representation of the normalized coefficients c_n^μ of the Floquet solutions. However, for negative values of n the representation (6.2) becomes more and more unstable numerically, each small value being the result of heavy cancellation of large terms. We therefore want to derive another representation, which is stable just for negative values of n while it becomes increasingly unstable if n is positive.

The substitution $(B, D, z) \to (D, B, 1/z)$ leaves the differential equation unaltered, and the coefficients $c_n^\mu = c_n^\mu(B, D)$ of the Floquet solutions satisfy

$$c_n^\mu(D, B) = c_{-n}^{-\mu}(B, D), \tag{6.4}$$

provided that the normalization is always chosen in the same way,

$$c_0^\mu(B, D) = c_0^\mu(D, B) = c_0^{-\mu}(B, D) = c_0^{-\mu}(D, B) = 1. \tag{6.5}$$

Corresponding to (6.2), we therefore also have

$$\begin{aligned} \Delta_\mu(D, B) c_{-n}^{-\mu} = \kappa^n &\sqrt{\frac{e(D, B, \kappa)}{\cos(\pi[\mu + \frac{1}{2}\frac{D}{\kappa}])}} \sum_{l=0}^\infty \frac{l!}{\Gamma(\mu + \frac{3}{2} + \frac{1}{2}\frac{D}{\kappa} + n + l)} 2^{-l} a_l(D, B, \kappa) \\ &+ (-\kappa)^n \sqrt{\frac{e(D, B, -\kappa)}{\cos(\pi[\mu - \frac{1}{2}\frac{D}{\kappa}])}} \\ &\times \sum_{l=0}^\infty \frac{l!}{\Gamma(\mu + \frac{3}{2} - \frac{1}{2}\frac{D}{\kappa} + n + l)} 2^{-l} a_l(D, B, -\kappa), \end{aligned} \tag{6.6}$$

which may be rewritten with (n, μ) replaced by $(-n, -\mu)$. For $n = 0$, this yields, with the normalization (6.5),

$$\begin{aligned} \Delta_{-\mu}(D, B) = &\sqrt{\frac{e(D, B, \kappa)}{\cos(\pi[-\mu + \frac{1}{2}\frac{D}{\kappa}])}} \sum_{l=0}^\infty \frac{l!}{\Gamma(-\mu + \frac{3}{2} + \frac{1}{2}\frac{D}{\kappa} + l)} 2^{-l} a_l(D, B, \kappa) \\ &+ \sqrt{\frac{e(D, B, -\kappa)}{\cos(\pi[-\mu - \frac{1}{2}\frac{D}{\kappa}])}} \sum_{l=0}^\infty \frac{l!}{\Gamma(-\mu + \frac{3}{2} - \frac{1}{2}\frac{D}{\kappa} + l)} 2^{-l} a_l(D, B, -\kappa). \end{aligned} \tag{6.7}$$

By means of (6.3) and (6.7), the quantities r_μ appearing in the coefficients (4.12) of the connection relation (4.11) can be computed from (4.13).

So far we have needed (6.2) and (6.6) for $n = 0$ only, but it might be interesting to consider them for all n. Using each result only in the stable region of the n-values we may state the following result.

Theorem 2. *With the quantities* (6.3) *and* (6.7), *the coefficients of the Floquet solutions* (1.5) *are*

$$\begin{aligned} c_n^\mu = \frac{1}{\Delta_\mu(B, D)} \Bigg\{ &\kappa^n \sqrt{\frac{e(B, D, \kappa)}{\cos(\pi[\mu + \frac{1}{2}\frac{B}{\kappa}])}} \sum_{l=0}^\infty \frac{l!}{\Gamma(\mu + \frac{3}{2} + \frac{1}{2}\frac{B}{\kappa} + n + l)} 2^{-l} a_l(B, D, \kappa) \\ &+ (-\kappa)^n \sqrt{\frac{e(B, D, -\kappa)}{\cos(\pi[\mu - \frac{1}{2}\frac{B}{\kappa}])}} \sum_{l=0}^\infty \frac{l!}{\Gamma(\mu + \frac{3}{2} - \frac{1}{2}\frac{B}{\kappa} + n + l)} 2^{-l} a_l(B, D, -\kappa) \Bigg\} \end{aligned}$$

for $n = 0, 1, 2, \ldots,$ *and*

$$c_n^\mu = \frac{1}{\Delta_{-\mu}(D,B)}\left\{\kappa^n\sqrt{\frac{e(D,B,\kappa)}{\cos(\pi[-\mu+\frac{1}{2}\frac{D}{\kappa}])}}\sum_{l=0}^{\infty}\frac{l!}{\Gamma(-\mu+\frac{3}{2}+\frac{1}{2}\frac{D}{\kappa}-n+l)}2^{-l}a_l(D,B,\kappa)\right.$$
$$\left.+(-\kappa)^n\sqrt{\frac{e(D,B,-\kappa)}{\cos(\pi[-\mu-\frac{1}{2}\frac{D}{\kappa}])}}\sum_{l=0}^{\infty}\frac{l!}{\Gamma(-\mu+\frac{3}{2}-\frac{1}{2}\frac{D}{\kappa}-n+l)}2^{-l}a_l(D,B,-\kappa)\right\} \tag{6.8}$$

for $n = 0, -1, -2, \ldots.$

The upper expression tends to zero for $n \to \infty$ and satisfies the recurrence relation (1.6) for $n = 2, 3, \ldots$, the lower expression tends to zero for $n \to -\infty$ and satisfies the recurrence relation for $n = -2, -3, \ldots$, and this all would be true even if the value of the characteristic exponent is incorrect. It is only for every correct value of μ that the two expressions match and the recurrence relation is satisfied for $n = -1, 0, 1$, too.

7. Further representations

There are different representations for the various solutions of the differential equation (1.1) considered above.

If in the local solutions at each of the irregular singular points an exponential factor appropriate to the other singular point is extracted, we may find

$$\begin{aligned}\bar{f}_{\infty 1}(z) &= \exp(\kappa z) z^{-\frac{1}{2}-\frac{1}{2}(B/\kappa)}\exp\Big(\frac{s}{z}\Big)\sum_{n=0}^{\infty} b_n(B,D,\kappa,s)n!\,(2\kappa z)^{-n},\\ \bar{f}_{\infty 2}(z) &= \exp(-\kappa z) z^{-\frac{1}{2}+\frac{1}{2}(B/\kappa)}\exp\Big(\frac{s}{z}\Big)\sum_{n=0}^{\infty} b_n(B,D,-\kappa,s)n!\,(-2\kappa z)^{-n},\end{aligned} \tag{7.1}$$

$s \in \{-\kappa, \kappa\}$, where the new coefficients satisfy a three-term recurrence relation,

$$\begin{aligned} b_n &= b_n(B,D,\kappa,s),\\ b_0 &= 1,\ b_{-2} = b_{-1} = 0,\\ b_n &= \frac{1}{n^2}\Big\{\Big[\Big(-L-\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa}+n\Big)\Big(L-\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa}+n\Big)-2\kappa s\Big]b_{n-1}\\ &\quad+\frac{2[\kappa D+Bs+2\kappa s(n-1)]}{n-1}b_{n-2}\Big\}, \qquad n = 1, 2, \ldots. \end{aligned} \tag{7.2}$$

The old and new coefficients are related to each other by

$$\begin{aligned} a_n &= \sum_{m=0}^{n}\frac{m!}{n!\,(n-m)!}(2\kappa s)^{n-m}b_m,\\ b_n &= \sum_{m=0}^{n}\frac{m!}{n!\,(n-m)!}(-2\kappa s)^{n-m}a_m. \end{aligned} \tag{7.3}$$

Again, if an exponential factor appropriate to the other singular point is extracted, we also have

$$\bar{f}_{01}(z) = \exp\Big(\frac{\kappa}{z}\Big) z^{\frac{1}{2}+\frac{1}{2}(D/\kappa)} \exp(sz) \sum_{n=0}^{\infty} b_n(D,B,\kappa,s) n! \Big(\frac{z}{2\kappa}\Big)^n,$$
$$\bar{f}_{02}(z) = \exp\Big(-\frac{\kappa}{z}\Big) z^{\frac{1}{2}-\frac{1}{2}(D/\kappa)} \exp(sz) \sum_{n=0}^{\infty} b_n(D,B,-\kappa,s) n! \Big(-\frac{z}{2\kappa}\Big)^n, \tag{7.4}$$

$s \in \{-\kappa, \kappa\}$. Different representations of the Floquet solutions, with coefficients obeying a four-term or three-term recurrence relation, may be obtained if appropriate exponential factors are extracted. We have

$$f_\mu(z) = z^\mu \exp\Big(\frac{s}{z}\Big) \exp(rz) \sum_{n=-\infty}^{\infty} e_n^\mu z^n, \quad \mu \in \{-\nu, \nu\}, \tag{7.5}$$

$$(s^2-\kappa^2) e_{n+2}^\mu + \Big\{D - 2s\Big(\mu + n + \frac{1}{2}\Big)\Big\} e_{n+1}^\mu + \{(\mu+n-L)(\mu+n+L) - 2rs\} e_n^\mu$$
$$+ \Big\{B + 2r\Big(\mu + n - \frac{1}{2}\Big)\Big\} e_{n-1}^\mu - (r^2-\kappa^2) e_{n-2}^\mu = 0. \tag{7.6}$$

So if $r = 0$ and $s \in \{-\kappa, \kappa\}$, the coefficients satisfy a four-term recurrence relation. If $s = 0$ and $r \in \{-\kappa, \kappa\}$, they satisfy a four-term recurrence relation. If $r, s \in \{-\kappa, \kappa\}$, they satisfy a three-term recurrence relation. The choice $r = -s = \kappa$ would essentially correspond to a solution of the canonic equation [13, 43]. The Floquet solutions of the canonic equation therefore have coefficients which satisfy a three-term recurrence relation. This is interesting in so far as (if the characteristic exponent is already known) the coefficients can then be computed also by means of continued fractions, a method which has been investigated extensively in the context of the Mathieu equation.

It should be noted that some quantities which appear here in the following part of Section 7 have a local meaning for this section only and their definitions here are different from those of the corresponding quantities in the other sections. Moreover, inside this section, the same symbols are used for different but analogical quantities in the case of the third-order or the second-order t-equation, respectively.

In place of the integral representation (2.1), we now might consider

$$f(z) = \exp\Big(\frac{s}{z}\Big) z^\lambda (2\pi i)^{-1} \int_C \exp(\kappa z t) V(t)\, dt \tag{7.7}$$

with $s = \kappa$ or $s = -\kappa$ and obtain a third-order t-equation,

$$(t^2-1)V''' + \Big[(5-2\lambda)t - \frac{B}{\kappa}\Big] V'' + [-2\kappa s t + (2-\lambda-L)(2-\lambda+L)] V'$$
$$+ [-\kappa D + (2\lambda - 3)\kappa s] V(t) = 0. \tag{7.8}$$

With the appropriate contours, this integral representation essentially yields the Floquet solutions in the form (7.5) with $r = 0$ and the formal solutions at infinity in the form (7.1)–(7.2). As a consequence, a modified version of Theorem 1 is also true in which all the coefficients $a_n(B,D,\kappa)$ are replaced by the corresponding coefficients $b_n(B,D,\kappa,s)$ with $s = \kappa$ or $s = -\kappa$, but we cannot see any advantage in this modified representation of $e(\kappa)$. The other method for computing $e(\kappa)$ by means of a finite determinant according to (5.10)–(5.11), however, becomes more attractive by the reduction of the order of the t-equation. For we now have

$$e(-\kappa) = \frac{\pi D_1}{\Gamma(\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa})\Gamma(-\lambda + \frac{5}{2} + \frac{1}{2}\frac{B}{\kappa})} 2^{\lambda - \frac{5}{2} + \frac{1}{2}(B/\kappa)} \tag{7.9}$$

with the 3 by 3 (rather than 4 by 4) determinant

$$D_1 = \begin{vmatrix} F(\kappa, 1) & G_0(-\kappa, 1) & G_1(-\kappa, 1) \\ -F'(\kappa, 1) & G_0'(-\kappa, 1) & G_1'(-\kappa, 1) \\ F''(\kappa, 1) & G_0''(-\kappa, 1) & G_1''(-\kappa, 1) \end{vmatrix}, \tag{7.10}$$

where F, G_0, G_1 are formally the same as above in (2.12)–(2.14), but in terms of new coefficients defined here by the starting values

$$A_0(\kappa, q) = 1 \quad (\text{for } q = 0, 1, \lambda - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}), \tag{7.11}$$

$$A_1(\kappa, 0) = 0, \tag{7.12}$$

and the new recurrence relation

$$A_n(\kappa, q) = \frac{(q + n - \lambda - L)(q + n - \lambda + L) - 2\kappa s}{2(q + n)(q + n - \lambda + \frac{1}{2} - \frac{1}{2}B/\kappa)} A_{n-1}(\kappa, q) + \frac{\kappa D + 2\left(q + n - \lambda - \frac{1}{2}\right)\kappa s}{2(q + n)(q + n - 1)\left(q + n - \lambda + \frac{1}{2} - \frac{1}{2}B/\kappa\right)} A_{n-2}(\kappa, q), \tag{7.13}$$

where $A_{-1}(\kappa, q) = 0$; $n > 0$ if $q = 1, \lambda - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}$; $n > 1$ if $q = 0$. The new formula (7.9) contains the computational parameter λ, which is still at our disposal except for the obvious restrictions, but its value is independent of λ.

With the special choice [43]

$$\lambda = \frac{1}{2}\left(1 + \frac{D}{s}\right), \tag{7.14}$$

it is even possible to find a second-order t-equation, which is

$$(t^2 - 1)V'' + \left[(3 - 2\lambda)t - \frac{B}{\kappa}\right]V' + [(1 - \lambda - L)(1 - \lambda + L) - 2\kappa s t]V(t) = 0, \tag{7.15}$$

and the formula

$$e(-\kappa) = \frac{\pi D_1}{\Gamma(\lambda + \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa})\Gamma(-\lambda + \frac{3}{2} + \frac{1}{2}\frac{B}{\kappa})} 2^{\lambda - \frac{3}{2} + \frac{1}{2}(B/\kappa)} \tag{7.16}$$

with the 2 by 2 determinant

$$D_1 = \begin{vmatrix} F(\kappa, 1) & G_0(-\kappa, 1) \\ -F'(\kappa, 1) & G_0'(-\kappa, 1) \end{vmatrix}, \tag{7.17}$$

where F, G_0 are formally the same as above in (2.12)–(2.14), but in terms of new coefficients defined here by the starting values

$$A_0(\kappa, q) = 1 \quad (\text{for } q = 0, \lambda - \frac{1}{2} + \frac{1}{2}\frac{B}{\kappa}), \tag{7.18}$$

and the new recurrence relation

$$A_n(\kappa,q) = \frac{(q+n-\lambda-L)(q+n-\lambda+L)-2\kappa s}{2(q+n)(q+n-\lambda+\frac{1}{2}-\frac{1}{2}B/\kappa)}A_{n-1}(\kappa,q) + \frac{\kappa s}{(q+n)(q+n-\lambda+\frac{1}{2}-\frac{1}{2}B/\kappa)}A_{n-2}(\kappa,q), \tag{7.19}$$

where $A_{-1}(\kappa,q) = 0$ and $n > 0$. It should be noted that λ in (7.15)–(7.19), which has been kept for convenience of presentation, is fixed and always given by (7.14). So the new formula (7.16) breaks down in the exceptional cases when $(D/s \pm B/\kappa)/2$ is equal to an integer.

Although generally the representation of $e(\kappa)$ in terms of the 2 by 2 determinant is more attractive than that with the 3 by 3 determinant, the latter has the advantage that here the parameter λ is still at our disposal and so the breakdown in the exceptional cases may easily be avoided by a suitable choice of λ.

8. The special case where $B = D = 0$

The results of this work simplify considerably when $B = D = 0$. Then the dependence on the sign of κ disappears for various quantities, so that we have

$$a_n(-\kappa) = a_n(\kappa), \tag{8.1}$$

$$E(-\kappa) = E(\kappa), \tag{8.2}$$

$$e(-\kappa) = e(\kappa) = e. \tag{8.3}$$

If we impose the condition that $c_0^\mu = 1$ (that is different from zero), then the characteristic exponent is determined modulo 2 by

$$\cos(\pi\mu) = e, \tag{8.4}$$

and so the various square roots in (2.37), (2.38) etc. become equal to 1 and disappear from the results. Also, we then have

$$c_{2n+1}^\mu = 0, \tag{8.5}$$

$$c_{-2n}^{-\mu} = c_{2n}^\mu. \tag{8.6}$$

Further aspects of this special case are treated in [11, 17, 35, 8].

9. A simple example

It might be interesting to see what happens in the case of the simpler differential equation

$$z^2 f'' + zf' + [-\kappa^2 z^2 + Bz - L^2]f(z) = 0, \tag{9.1}$$

for which the origin is a regular singular point with indices $-L$, L. Here the coefficients a_n of the formal solutions can be given explicitly and then the h_m essentially become hypergeometric series which can be summed by means of Saalschütz's formula [29]. So, as in [12], we obtain from Theorem 1

$$e(-\kappa) = \frac{\pi 2^{B/\kappa}}{\Gamma(-\lambda+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})\Gamma(\lambda+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})} \frac{(-L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})_n(L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})_n}{(-\lambda+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})_n(\lambda+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})_n}$$
$$\times \left\{1+\sum_{k=1}^{K}\frac{(\lambda+L)_k(\lambda-L)_k}{k!\,(\lambda+\frac{1}{2}-\frac{1}{2}\frac{B}{\kappa}-n)_k}+O(n^{-K-1})\right\}^{-1}. \tag{9.2}$$

The n-dependence of this expression disappears for the choice $\lambda = L$ (or $\lambda = -L$), and this is the reason why this choice of the computational parameter λ, recommended above, is so advantageous. Then (9.2) simplifies and yields

$$e(-\kappa) = \frac{\pi 2^{B/\kappa}}{\Gamma(-L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})\Gamma(L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})}, \tag{9.3}$$

and so

$$e(-\kappa)e(\kappa) = \cos\Big(\pi\Big[L-\frac{1}{2}\frac{B}{\kappa}\Big]\Big)\cos\Big(\pi\Big[L+\frac{1}{2}\frac{B}{\kappa}\Big]\Big) \tag{9.4}$$

and, by (2.32) or (2.33),

$$\mu = -L, L \mod 1, \tag{9.5}$$

as expected.

For $\mu = L$, (6.2) now becomes

$$\Delta_L c_n^L = \kappa^n\sqrt{2^{-B/\kappa}\frac{\Gamma(L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})}{\Gamma(L+\frac{1}{2}-\frac{1}{2}\frac{B}{\kappa})}}\frac{1}{\Gamma(L+\frac{3}{2}+\frac{1}{2}\frac{B}{\kappa}+n)}$$
$$\times {}_2F_1\left(\begin{matrix}-L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa},\ L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa}\\ L+\frac{3}{2}+\frac{1}{2}\frac{B}{\kappa}+n\end{matrix}\;\middle|\;\frac{1}{2}\right)$$
$$+(-\kappa)^n\sqrt{2^{B/\kappa}\frac{\Gamma(L+\frac{1}{2}-\frac{1}{2}\frac{B}{\kappa})}{\Gamma(L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa})}}\frac{1}{\Gamma(L+\frac{3}{2}-\frac{1}{2}\frac{B}{\kappa}+n)}$$
$$\times {}_2F_1\left(\begin{matrix}-L+\frac{1}{2}-\frac{1}{2}\frac{B}{\kappa},\ L+\frac{1}{2}-\frac{1}{2}\frac{B}{\kappa}\\ L+\frac{3}{2}-\frac{1}{2}\frac{B}{\kappa}+n\end{matrix}\;\middle|\;\frac{1}{2}\right). \tag{9.6}$$

The hypergeometric series in (9.6) can be evaluated analytically in the special cases when $B = 0$ or when $n = -1$.

For $B = 0$ we have, using (15.1.26) of [1],

$$\Delta_L c_n^L = 2^{-L-n-1/2}\frac{\sqrt{\pi}}{\Gamma(1+\frac{1}{2}n)\Gamma(L+1+\frac{1}{2}n)}\{\kappa^n+(-\kappa)^n\}, \tag{9.7}$$

an expression which vanishes if n is odd or if n is even and negative, as expected for the (modified) Bessel functions.

For $B \neq 0$, we can evaluate (9.6) immediately when $n = -1$, since then the (upper) hypergeometric function becomes equal to

$${}_1F_0\Big(-L+\frac{1}{2}+\frac{1}{2}\frac{B}{\kappa};;\frac{1}{2}\Big) = \Big(1-\frac{1}{2}\Big)^{L-\frac{1}{2}-\frac{1}{2}(B/\kappa)} = 2^{-L+\frac{1}{2}+\frac{1}{2}(B/\kappa)}. \tag{9.8}$$

By means of the appropriate Gaussian recurrence relation (15.2.27) of [1], the values of the hypergeometric series for all the other negative n may be obtained. Then the two terms in (9.6) are seen to cancel for all the negative n, as expected.

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PHYSIKALISCHES INSTITUT, UNIVERSITÄT HEIDELBERG, D-69120 HEIDELBERG, GERMANY