

Multiplicative gray stability

CAMILO ANGULO, MARÍA AMELIA SALAZAR, AND DANIELE SEPE

In this paper we prove Gray stability for compact contact groupoids and we use it to prove stability results for deformations of the induced Jacobi bundles.

Introduction

Lie groupoids endowed with multiplicative geometries have been used successfully to study singular geometric structures on their bases (see, e.g., [5, 7, 9, 11, 13, 14, 16–19]). Recently, particular attention has been given to **contact groupoids** and the induced Jacobi bundles on their bases (see, e.g., [3, 16, 19, 25, 32, 33]). Contact geometry is the odd-dimensional analog of symplectic geometry and it arises naturally in the study of geometric PDEs (see, e.g., [2]). A contact structure is a maximally non-integrable hyperplane distribution (see Definition 1.1). A foundational result in contact geometry is **Gray stability**: there are no non-trivial deformations of a contact structure on a compact manifold (see [23, Theorem 5.2.1]). The main result of this paper is a multiplicative version of Gray stability, stated informally below (see Theorem 3.1).

Main Result. *A smooth 1-parameter family $\{H_\tau\}$ of multiplicative contact structures on a compact Lie groupoid G over a connected manifold is trivial, i.e., there exists a smooth 1-parameter family $\{\Phi_\tau\}$ of automorphisms of G with $\Phi_0 = \text{id}$ such that $d\Phi_\tau(H_\tau) = H_0$ for all τ .*

The above should be compared with the very recent [6, Theorem 1.2], which describes the moduli space of deformations of multiplicative symplectic forms on a compact symplectic groupoid.

Combining our Main Result with the stability result for deformations of compact Lie groupoids proved in [15, 20], the following holds (see Definitions 2.13 and 2.16, and Corollary 3.2).

Corollary. *Any deformation of a compact contact groupoid over a connected manifold is trivial.*

Motivated by understanding stability phenomena in Poisson geometry (see [10]), recently Crainic, Fernandes and Martínez-Torres have started an ambitious program to study Poisson manifolds of compact types, i.e., those induced by proper symplectic groupoids. These enjoy special properties and are, in some sense, rare (see [13, 14, 30, 36]). Contact groupoids induce *Jacobi bundles* on their base manifolds. Jacobi bundles can be thought of as infinite dimensional Lie algebras of ‘geometric type’ and include symplectic, contact and Poisson structures as examples (see, e.g., [26]). Formally, a Jacobi bundle is a line bundle endowed with a local Lie bracket on its space of sections (see Definition 4.1). The present work suggests that studying Jacobi bundles that are induced by proper contact groupoids is not only analogous, but also complementary, to the above program. To illustrate this, we begin with the following remark (see Corollary 4.15 for a precise statement).

Observation. *A Jacobi bundle induced by a proper co-orientable contact groupoid comes from a Poisson bivector.*

We intend to undertake a systematic study of Jacobi bundles induced by proper contact groupoids in future papers. By the above Observation, a proper deformation of a proper co-orientable contact groupoid induces a deformation of a Jacobi bundle that can be seen as a deformation of a Poisson bivector (see Definition 2.3 and Section 4.3). We use our Main Result to obtain the following characterization of such deformations (see Theorem 4.19 for a precise statement).

Theorem. *Let $\{\pi_\tau\}$ be a smooth 1-parameter family of Poisson bivectors on M induced by a smooth 1-parameter family of co-orientable contact structures $\{H_\tau\}$ on a compact Lie groupoid G . Then there exists a diffeotopy $\{\phi_\tau\}$ of M and a smooth 1-parameter family of positive Casimirs $\{a_\tau\}$ of (M, π) such that $(\phi_\tau)_*\pi_\tau = a_\tau\pi$ for all τ .*

The above Theorem applied to a Lie-Poisson sphere in the dual of a compact Lie algebra should be compared with [29, Part (a) of Theorem 1], which is stronger but uses infinite dimensional techniques (see Remark 5.4).

Our strategy to prove the Main Result is to consider first the case in which the contact structures are co-orientable, i.e., given by the kernel of 1-forms, and then the case in which they are not (see Theorems 3.3 and 3.4). In the former, we use the Moser trick, in analogy with the well-known proof of Gray stability (see, e.g., [21, Theorem 2.2.2], and Theorem 3.3 below). We look for the desired smooth 1-parameter of diffeomorphisms $\{\Phi_\tau\}$ of G as

the flow of a time-dependent vector field X_τ . Moreover, since we want $\{\Phi_\tau\}$ to be a family of Lie groupoid automorphisms, X_τ must be *multiplicative*, i.e., $X_\tau : G \rightarrow TG$ need be a Lie groupoid homomorphism for all τ (see Section 1.2 for the Lie groupoid structure on TG). We achieve this by finding a ‘good’ smooth 1-parameter family of multiplicative contact forms for $\{H_\tau\}$ (see Definition 1.16 and Corollary 2.24 for a precise statement). Crucially, this uses compactness of G , which implies that its differentiable cohomology vanishes in all positive degrees by [8, Proposition 1]. The above contact forms are multiplicative with values in a representation of G on the trivial line bundle that is codified by a Lie groupoid homomorphism $\sigma : G \rightarrow \{\pm 1\}$ (see Definition 1.13). This brings in some small technicalities to prove that X_τ is multiplicative (see Appendix A).

In order to deal with the case in which the contact structures are not co-orientable, in Section 1.4 we introduce a simple, seemingly new construction for such contact groupoids that we call the *co-orientable finite cover*. It is analogous to the co-orientable double cover of a contact manifold (see Remark 1.7). Given a smooth 1-parameter family of contact structures on a compact Lie groupoid G , we consider the smooth 1-parameter family of co-orientable contact structures given by their co-orientable finite covers. This allows us to argue as above, making sure that the smooth 1-parameter family of automorphisms of co-orientable finite covers descends to G (see the proof of Theorem 3.4). Aside from its use in this paper, we expect that the co-orientable finite cover of contact groupoids will be useful in the study of multiplicative contact structures that are not co-orientable.

Outline

In Section 1 we discuss properties of contact groupoids. In Sections 1.1–1.3, we recall the basics of contact structures and of (co-orientable) contact groupoids. In Section 1.4, we define and study the co-orientable finite cover of a contact groupoid, which we use in the proof of our main result. Section 2 formalizes the notion of ‘smooth 1-parameter families of contact groupoids’ using deformations as in [15, 20] (see also [6]). In Section 2.1 we recall the basics of deformations of Lie groupoids, while in Sections 2.2 and 2.3 we introduce deformations of (co-orientable) contact groupoids. Section 3 proves our main result, Theorem 3.1. Sections 1 – 3 can be read independently of the remaining sections. In Section 4 we consider multiplicative Gray stability at the level of objects. Sections 4.1 and 4.2 introduce Jacobi bundles and recall how they are induced by contact groupoids. In Section 4.3, we explain how our main result can be used to study deformations of Jacobi bundles

(see Theorems 4.18 and 4.19). Three families of examples of compact contact groupoids and their induced Jacobi bundles are given in Section 5. Finally, Appendix A deals with technical lemmas that we use in the proof of our main result.

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Conventions

Throughout the paper, $I \subseteq \mathbb{R}$ is an open interval containing 0 and all vector bundles are real. Moreover, if G_b denotes the fiber of a surjective submersion $G \rightarrow B$ over $b \in B$, we use $i_b : G_b \hookrightarrow G$ for the natural inclusion. This is used extensively in Sections 2 and 3.

1. Contact groupoids

In this section we recall some notions and results regarding contact groupoids, most of which are well-known except for Proposition 1.19 and Corollary 1.20 in Section 1.3, and Section 1.4. The main references for Lie groupoids are [12, 27], for multiplicative distributions is [17], and for contact groupoids are [16, 18, 19, 25, 35].

1.1. Contact manifolds

Definition 1.1. A **contact structure** on a manifold N is a hyperplane distribution $H \subset TN$ such that the vector bundle morphism $c_H : H \times H \rightarrow$

TN/H given at the level of sections by $c_H(X, Y) = [X, Y] \bmod H$ is non-degenerate. The pair (N, H) is a **contact manifold**.

There is a dual approach to contact structures using 1-forms taking values in line bundles. Given a contact manifold (N, H) , setting $\alpha_{\text{can}} : TN \rightarrow L := TN/H$, we have that $H = \ker \alpha_{\text{can}}$ and $\alpha_{\text{can}} \in \Omega^1(N; L)$. Following [31, Note 2.3], we refer to α_{can} as the **generalized contact form** of (N, H) .

Let (N, H) be **co-orientable**, i.e., the line bundle $L \rightarrow N$ is trivializable, and let $\psi : L \rightarrow N \times \mathbb{R}$ be a trivialization. If α_{can} is the generalized contact form of (N, H) , then $\alpha := \psi \circ \alpha_{\text{can}} \in \Omega^1(N)$ and $H = \ker \alpha$. If ψ' is another trivialization of L , set $\alpha' := \psi' \circ \alpha_{\text{can}}$. If we identify $\psi' \circ \psi^{-1}$ with a nowhere vanishing $a \in C^\infty(N)$, then we have that $\alpha' = a\alpha_\psi$.

Definition 1.2. Let (N, H) be a co-orientable contact manifold. Any $\alpha \in \Omega^1(N)$ with $H = \ker \alpha$ is a **contact form** (for (N, H)). The pair (N, α) is a **co-oriented contact manifold**.

Remark 1.3. If (N, α) is a co-oriented contact manifold, then $\alpha : TN \rightarrow N \times \mathbb{R}$ is onto and, if $H = \ker \alpha$, then $(H, d\alpha) \rightarrow N$ is a symplectic vector bundle. Conversely, if $\alpha \in \Omega^1(N)$ has the above properties, then $(N, H = \ker \alpha)$ is a contact manifold.

A choice of contact form specifies the following vector field.

Definition 1.4. Let (N, α) be a co-oriented contact manifold. The **Reeb vector field of (N, α)** is the unique $R^\alpha \in \mathfrak{X}(N)$ such that

$$(1.1) \quad \alpha(R^\alpha) = 1 \quad \text{and} \quad d\alpha(R^\alpha, -) = 0.$$

A simple, but computationally useful property of co-oriented contact manifolds is that they admit a natural complement to the contact distribution. More precisely, if (N, α) is a co-oriented contact manifold, setting $H = \ker \alpha$, then

$$(1.2) \quad \begin{aligned} TN &\rightarrow \mathbb{R}\langle R^\alpha \rangle \oplus H \\ v &\mapsto (\alpha(v)R^\alpha, v - \alpha(v)R^\alpha) \end{aligned}$$

and

$$(1.3) \quad \begin{aligned} T^*N &\rightarrow \mathbb{R}\langle \alpha \rangle \oplus \text{Ann}(\mathbb{R}\langle R^\alpha \rangle) \\ \beta &\mapsto (\beta(R^\alpha)\alpha, \beta - \beta(R^\alpha)\alpha) \end{aligned}$$

are vector bundle isomorphisms.

Next we discuss diffeomorphisms that preserve contact structures.

Definition 1.5. Let (N_j, H_j) be a contact manifold for $j = 1, 2$. A **contactomorphism between (N_1, H_1) and (N_2, H_2)** is a diffeomorphism $\Phi : N_1 \rightarrow N_2$ such that $d\Phi(H_1) = H_2$.

Remark 1.6.

- Any contactomorphism Φ between (N_1, H_1) and (N_2, H_2) induces a vector bundle isomorphism $B : \Phi^*L_2 \xrightarrow{\cong} L_1$ covering the identity.
- Suppose that (N_j, α_j) is a co-oriented contact manifold for $j = 1, 2$. Set $H_j = \ker \alpha_j$. Given a contactomorphism Φ between (N_1, H_1) and (N_2, H_2) , there exists a nowhere vanishing $a \in C^\infty(N_1)$ such that $\Phi^*\alpha_2 = a\alpha_1$. Moreover, using α_j to identify L_j with $N_j \times \mathbb{R}$ and using the canonical isomorphism $\Phi^*(N_2 \times \mathbb{R}) \cong N_1 \times \mathbb{R}$, the above isomorphism B is given by $(x, \lambda) \mapsto (x, a^{-1}\lambda)$.

To conclude this section, we show that, up to taking double covers, every connected contact manifold is co-orientable.

Remark 1.7. Let (N, H) be a connected contact manifold such that $L \rightarrow N$ is not trivializable. Since $L \rightarrow N$ is a line bundle, its structure group can be reduced to $\{\pm 1\}$. Hence, there exists a double cover $q : \hat{N} \rightarrow N$ such that $\hat{L} := q^*L$ is trivializable and \hat{N} is connected. Since q is a local diffeomorphism, $T\hat{N}$ is isomorphic to q^*TN . Setting $\hat{H} := q^*H$, we have that (\hat{N}, \hat{H}) is a contact manifold such that $T\hat{N}/\hat{H}$ is isomorphic to \hat{L} and, hence, trivializable. We call (\hat{N}, \hat{H}) the **co-orientable double cover** of (N, H) .

1.2. Contact groupoids

Throughout this paper, $G \rightrightarrows M$ and G denote a **Lie groupoid** over a manifold M , where G and M are the spaces of **arrows** and **objects** respectively. We refer to M as the **base** of G . The structure maps of $G \rightrightarrows M$ are denoted as follows: $s, t : G \rightarrow M$ are the **source** and **target** maps respectively, $u : M \rightarrow G$ is the **unit** map, $m : G^{(2)} \rightarrow G$, $m(g, h) = gh$ is the **multiplication** map, where $G^{(2)} := \{(g, h) \in G \times G \mid s(g) = t(h)\}$, and $i : G \rightarrow G$, $i(g) = g^{-1}$ is the **inversion** map. Since $u : M \rightarrow G$ is an embedding, we often identify a point $x \in M$ with the unit $u(x) = 1_x$ and view M as an embedded submanifold of G . Given $x \in M$, the **orbit of x** is $S_x := t(s^{-1}(x)) \subset M$. If $\Phi : G_1 \rightarrow G_2$ is a Lie groupoid homomorphism, we say that it **covers** the induced map $\phi : M_1 \rightarrow M_2$ on the bases.

Remark 1.8 (On Hausdorffness). In general, the base and the source fibers of a Lie groupoid are assumed to be Hausdorff, while the space of arrows is not. In this paper, we assume that the space of arrows is also Hausdorff. This is primarily because we deal with proper maps and uniqueness of flows of vector fields.

In this paper we are mostly interested in Lie groupoids that possess some degree of ‘compactness’.

Definition 1.9. A Lie groupoid $G \rightrightarrows M$ is

- **proper** if the map $(s, t) : G \rightarrow M \times M$ is proper, i.e., the preimage of a compact set is compact, and
- **compact** if G is compact.

We are interested in Lie groupoids equipped with contact structures on the spaces of arrows that are, in some sense, ‘compatible’ with the multiplication. To this end, we recall that, given a Lie groupoid $G \rightrightarrows M$, its *tangent Lie groupoid* $TG \rightrightarrows TM$ is the Lie groupoid with structure maps given by taking derivatives of those of $G \rightrightarrows M$.

Definition 1.10. A **multiplicative distribution** on a Lie groupoid $G \rightrightarrows M$ is a distribution $H \subseteq TG$ that is a Lie subgroupoid of TG with base TM .

Multiplicative distributions satisfy several properties that are immediate consequences of Definition 1.10. If H is a multiplicative distribution on G , then

- (i) if $X_g \in H_g$ and $Y_h \in H_h$ are composable, i.e., $ds(X_g) = dt(Y_h)$, then $dm(X_g, Y_h) \in H_{gh}$,
- (ii) for all $x \in M$, $T_x M \subset H_x$, and
- (iii) H is both s -transversal and t -transversal, i.e.,

$$TG = H + \ker ds \quad \text{and} \quad TG = H + \ker dt.$$

Definition 1.11. A **contact groupoid** is a pair (G, H) , where G is a Lie groupoid and H is a multiplicative contact structure on G .

Remark 1.12. Contact groupoids are also known as *conformal contact groupoids* (see [35, Appendix I, Definition 1.1]), or *locally conformal contact groupoids* (see [18, Example 7.6]).

In what follows we discuss the 1-form approach to contact groupoids. A **representation** of a Lie groupoid $G \rightrightarrows M$ is a vector bundle $E \rightarrow M$ together with a linear action of G , i.e., a smooth assignment of a linear isomorphism $E_{s(g)} \rightarrow E_{t(g)}$, $v \mapsto g \cdot v$ to each arrow $g \in G$ that satisfies the usual axioms of an action.

Definition 1.13. Let E be a representation of a Lie groupoid G . A **multiplicative form (with values in E)** is a form $\alpha \in \Omega^k(G; t^*E)$ such that

$$(1.4) \quad (m^*\alpha)_{(g,h)} = (\mathrm{pr}_1^*\alpha)_{(g,h)} + g \cdot (\mathrm{pr}_2^*\alpha)_{(g,h)},$$

for all $(g, h) \in G^{(2)}$, where $\mathrm{pr}_j : G^{(2)} \rightarrow G$ is the projection onto the j th component for $j = 1, 2$.

Given a multiplicative distribution H on G , set $H^s := H \cap \ker ds$. Property (iii) implies that TG/H is canonically isomorphic to $\ker ds/H^s$. Hence the vector bundle

$$E := TG/H|_M \rightarrow M$$

is canonically isomorphic to $(\ker ds/H^s)|_M$. For any $(g, h) \in G^{(2)}$, right translation by h

$$(1.5) \quad \mathbf{r}_h : \ker d_g s \rightarrow \ker d_{gh} s, \quad \mathbf{r}_h(X) = dm(X, 0_h)$$

is an isomorphism that maps H_g^s to H_{gh}^s . Hence, the maps (1.5) induce a vector bundle isomorphism

$$(1.6) \quad t^*E \stackrel{\mathbf{r}}{\cong} TG/H$$

covering the identity over G .

Proposition 1.14 (Lemmas 3.6 and 3.7 in [17]). *Let H be a multiplicative distribution on $G \rightrightarrows M$. Then $E := TG/H|_M$ inherits the structure of a representation of G . Moreover, the canonical projection $\alpha_{\mathrm{can}} : TG \rightarrow TG/H \cong t^*E$ is a multiplicative 1-form with values in E . Conversely, any multiplicative distribution is the kernel of a pointwise surjective multiplicative 1-form.*

By Proposition 1.14, if (G, H) is a contact groupoid, then the line bundle $L := TG/H|_M$ inherits the structure of a representation of G and the generalized contact form $\alpha_{\mathrm{can}} : TG \rightarrow TG/H \cong t^*L$ is a multiplicative

1-form with values in L . Conversely, any multiplicative contact structure is the kernel of a multiplicative generalized contact form.

To conclude this section, we introduce the multiplicative analog of Definition 1.5.

Definition 1.15. An **isomorphism of contact groupoids** between (G_1, H_1) and (G_2, H_2) is a Lie groupoid isomorphism $\Phi : G_1 \rightarrow G_2$ that is a contactomorphism. We use the notation $\Phi : (G_1, H_1) \rightarrow (G_2, H_2)$.

1.3. Co-orientable contact groupoids

In what follows we fix a co-orientable contact groupoid (G, H) over M unless otherwise stated. Then $L := TG/H|_M$ is trivializable. In fact, a choice of trivialization $\psi : L \rightarrow M \times \mathbb{R}$ induces a trivialization of $t^*L \cong TG/H$ (see equation (1.6)). Consequently, by Proposition 1.14,

- $M \times \mathbb{R}$ inherits the structure of representation of G , and
- the contact form $\alpha = \psi \circ \alpha_{\text{can}}$ is multiplicative with respect to the above representation.

The above representation of G on $M \times \mathbb{R}$ is given by fiberwise multiplication by a Lie groupoid homomorphism $F : G \rightarrow \mathbb{R}^*$, i.e., a nowhere vanishing function F such that $F(gh) = F(g)F(h)$ for all $(g, h) \in G^{(2)}$. Multiplicativity of α becomes

$$(1.7) \quad m^*\alpha = \text{pr}_1^*\alpha + \text{pr}_1^*(F)\text{pr}_2^*\alpha,$$

(cf. equation (1.4)).

Definition 1.16. Let (G, H) be a co-orientable contact groupoid. A pair (α, F) as above is a **multiplicative contact form** (for (G, H)). We also say that α is **F -multiplicative** and that (G, α, F) is a **co-oriented contact groupoid**.

The following result, stated without proof, shows the degrees of freedom in choosing multiplicative contact forms for a given co-orientable contact groupoid (see [35, Appendix I, Lemma 1.5, Part (ii)]¹).

¹In this paper we use the ‘opposite’ convention to the one in [35, Definition 2.1], cf. Definition 1.13.

Lemma 1.17. *Let (α, F) be a multiplicative contact form for a co-orientable contact groupoid (G, H) over M . Then (α', F') is a multiplicative contact form for (G, H) if and only if there exists a nowhere vanishing $a \in C^\infty(M)$ such that*

$$\alpha' = (t^*a)\alpha \quad \text{and} \quad F' = \frac{t^*a}{s^*a}F.$$

Given a co-oriented contact groupoid (G, α, F) , set

$$(1.8) \quad r := \ln(|F|) \quad \text{and} \quad \sigma := \operatorname{sgn}(F).$$

Then $r(gh) = r(g) + r(h)$ for all $(g, h) \in G^{(2)}$, i.e., r is a 1-cocycle in the differentiable cohomology of G (see [8] for a definition). Moreover, $\sigma : G \rightarrow \{\pm 1\}$ is a Lie groupoid homomorphism and $F = \sigma e^r$.

Definition 1.18. Let (G, α, F) be a co-oriented contact groupoid. We call r the **Reeb cocycle (of (G, α, F))** (see [18, Definition 1.3]), and σ the **sign of F** .

Suppose that the Reeb cocycle r of a co-oriented contact groupoid (G, α, F) is a coboundary, i.e., there exists $\kappa \in C^\infty(M)$ such that

$$(1.9) \quad r = s^*\kappa - t^*\kappa.$$

Then, applying Lemma 1.17 with $a := e^\kappa$, we have that $(e^{t^*\kappa}\alpha, \sigma)$ is a multiplicative contact form for $(G, H = \ker \alpha)$. This proves the following result.

Proposition 1.19. *Let (G, α, F) be a co-oriented contact groupoid and let $\sigma = \operatorname{sgn}(F)$. If the Reeb cocycle r of (G, α, F) is a coboundary, then, for all $\kappa \in C^\infty(M)$ satisfying (1.9), $(e^{t^*\kappa}\alpha, \sigma)$ is a multiplicative contact form for $(G, H = \ker \alpha)$.*

Since the differentiable cohomology of proper Lie groupoids vanishes in all positive degrees (see [8, Proposition 1]), Proposition 1.19 immediately implies the following result.

Corollary 1.20. *Let (G, H) be a proper co-orientable contact groupoid. Then there exist a Lie groupoid homomorphism $\sigma : G \rightarrow \{\pm 1\}$ and a contact form α for (G, H) such that (α, σ) is a multiplicative contact form for (G, H) .*

Remark 1.21. Let H be a multiplicative distribution on $G \rightrightarrows M$ such that TG/H has rank one and is trivializable. Then

- upon fixing a trivialization of $TG/H|_M$, H is encoded by a multiplicative 1-form (α, F) ,
- the space of such multiplicative 1-forms is given by Lemma 1.17,
- equation (1.8) gives a 1-cocycle r in the differentiable cohomology of G and a Lie groupoid homomorphism $\sigma : G \rightarrow \{\pm 1\}$, and
- the analogs of Proposition 1.19 and Corollary 1.20 hold.

In particular, we say that (α, F) is a **multiplicative 1-form** (for (G, H)).

Let (G, α, F) be a co-oriented contact groupoid. In some works in the literature the sign of F is assumed to be positive (see [18]). While this holds if the groupoid or its s -fibers are connected, in general it need not be, as shown below (see Section 5.2 for the notation).

Example 1.22. Throughout this example, $n \geq 3$ is an odd natural number. The map $O(n) \rightarrow \{\pm 1\} \times SO(n)$, $A \mapsto (\det A, (\det A)^{-1}A)$ is an isomorphism of Lie groups. Hence, upon identifying $\mathfrak{o}(n)^*$ with $\mathfrak{so}(n)^*$, the coadjoint action of $O(n)$ factors through that of $SO(n)$. The negative of the dual of the Killing form on $\mathfrak{o}(n)^*$ is a Riemannian metric that is bi-invariant with respect to both $O(n)$ and $SO(n)$. We use this metric to identify the oriented projectivization $\mathbb{S}(T^*O(n))$ of $T^*O(n)$ with the unit sphere bundle $U(T^*O(n))$ of $T^*O(n)$. Upon using right trivializations and the above isomorphism, there is a diffeomorphism

$$U(T^*O(n)) \cong U(\mathfrak{o}(n)^*) \times (\{\pm 1\} \times SO(n)),$$

where $U(\mathfrak{o}(n)^*) \subset \mathfrak{o}(n)^*$ denotes the unit sphere. We denote by $\alpha \in \Omega^1(U(\mathfrak{o}(n)^*) \times (\{\pm 1\} \times SO(n)))$ the pullback of the restriction of the Liouville 1-form to $U(T^*O(n))$ along the above diffeomorphism. Then $\ker \alpha$ is a contact structure. In what follows we define a structure of Lie groupoid on $U(\mathfrak{o}(n)^*) \times (\{\pm 1\} \times SO(n))$ over $U(\mathfrak{o}(n)^*)$ with the property that α is F -multiplicative for some function F that takes both positive and negative values.

Consider the right $\{\pm 1\} \times SO(n)$ -action on $U(\mathfrak{o}(n)^*)$ given by

$$\xi \cdot (\pm 1, A) := \pm \text{Ad}_A^*(\xi).$$

(This is *not* the coadjoint action of $O(n)$!) Endow $G := U(\mathfrak{o}(n)^*) \times (\{\pm 1\} \times SO(n))$ with the structure of an action Lie groupoid and define a Lie groupoid homomorphism $F : G \rightarrow \mathbb{R}^*$ by $F(\xi; \pm 1, A) = \pm 1$. Then a direct calculation (very similar to that in [35, Example 2.3]), shows that α is F -multiplicative. Hence, by Proposition 1.14, (G, α, F) is a co-oriented contact groupoid. By construction, G is compact, the base is connected and F takes both positive and negative values.

To conclude this section, we discuss properties of the Reeb vector field of a multiplicative contact form. Recall that if G is a Lie groupoid and $X \in \mathfrak{X}(G)$, then the vector field $X^L \in \mathfrak{X}(G)$ given by

$$(1.10) \quad X_g^L := dm(0_g, X_{1_{s(g)}} - dt(X_{1_{s(g)}}))$$

is *left-invariant*, i.e., $X^L \in \Gamma(\ker dt)$, and $\mathbf{l}_g(X_h^L) = X_{gh}^L$ for all $(g, h) \in G^{(2)}$, where \mathbf{l}_g is left translation by g (cf. equation (1.5)).

Lemma 1.23. *Let (G, α, F) be a co-oriented contact groupoid over M .*

- *The Reeb vector field R^α of (G, α) is right-invariant, i.e., $R^\alpha \in \Gamma(\ker ds)$, and, for all $(g, h) \in G^{(2)}$,*

$$(1.11) \quad \mathbf{r}_h(R_g^\alpha) = R_{gh}^\alpha.$$

- *The left-invariant vector field $R^{\alpha, L}$ induced by R^α is given by*

$$(1.12) \quad R^{\alpha, L} = FR^\alpha + \Lambda^{\alpha, \sharp}(dF),$$

where $\Lambda^\alpha \in \mathfrak{X}^2(G)$ is the unique bivector field such that $\Lambda^\alpha(\alpha, -) = 0$ and $\Lambda^\alpha|_{H^* \times H^*} = (d\alpha|_{H \times H})^{-1}$ with $H = \ker \alpha$ (see Example 4.6 for more details).

Proof. The first statement follows immediately from [16, Corollary 5.2]. To prove the second statement, by equation (1.2), there exists $Y \in \Gamma(\ker \alpha)$ such that

$$R^{\alpha, L} = \alpha(R^{\alpha, L})R^\alpha + Y.$$

We recall that M is a *Legendrian* submanifold of $(G, \ker \alpha)$, i.e., TM is contained in $\ker \alpha$ and is equal to its symplectic orthogonal with respect to $d\alpha|_{\ker \alpha}$ (see Remark 1.3 and [16, Proposition 5.1]). This property, the definition of $R^{\alpha, L}$, F -multiplicativity of α – equation (1.7) –, and the identity

obtained from equation (1.7) by taking exterior derivatives, yield that, for any $X \in TG$,

$$(1.13) \quad \alpha(R^{\alpha, L}) = F, \quad d\alpha(R^{\alpha, L}, X) = -dF(X).$$

Since $d\alpha(R^\alpha, -) = 0$, the definition of Λ^α (see Example 4.6), and equation (1.13), imply the desired result. \square

1.4. Co-orientable finite cover of a contact groupoid

In this section we establish a multiplicative analog of the construction in Remark 1.7 that we use in the proof of our main result (see Theorem 3.1). To the best of our knowledge, this has not appeared elsewhere in the literature.

Let (G, H) be a contact groupoid over a connected manifold M such that $L \rightarrow M$ is not trivializable. As in Remark 1.7, there exists a double cover $q : \hat{M} \rightarrow M$ such that $\hat{L} := q^*L \rightarrow \hat{M}$ is trivializable and \hat{M} is connected. Since q is a local diffeomorphism, the hypotheses of [27, Proposition 2.3.1] are satisfied and we can consider the **pullback Lie groupoid** $\hat{G} := q^!G \rightrightarrows \hat{M}$, where

$$\hat{G} = \{(x, g, y) \in \hat{M} \times G \times \hat{M} \mid s(g) = q(y), t(g) = q(x)\},$$

and the structure maps are:

$$(1.14) \quad \begin{aligned} \hat{s}(x, g, y) &:= y, \quad \hat{t}(x, g, y) := x, \quad \hat{u}(z) := (z, 1_{q(z)}, z), \\ \hat{m}((x, g, y), (y, h, z)) &:= (x, gh, z), \quad \hat{i}(x, g, y) := (y, g^{-1}, x). \end{aligned}$$

We collect a few properties of \hat{G} below.

- (1) The restriction of the projection $\hat{M} \times G \times \hat{M} \rightarrow G$ to \hat{G} , denoted by Q , is a Lie groupoid homomorphism onto G that covers q .
- (2) The free and proper \mathbb{Z}_2 -action on \hat{M} with quotient map $q : \hat{M} \rightarrow M$ lifts to a free and proper $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on \hat{G} with quotient map $Q : \hat{G} \rightarrow G$. The lifted action is the restriction of the free and proper $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on $\hat{M} \times G \times \hat{M}$ defined as follows: the first (respectively second) \mathbb{Z}_2 acts on the first (respectively second) copy of \hat{M} and trivially on G .
- (3) The tangent Lie groupoid $T\hat{G} \rightrightarrows T\hat{M}$ equals the pullback Lie groupoid $(dq)^!TG \rightrightarrows T\hat{M}$.

Remark 1.24. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on \hat{G} is *not* by Lie groupoid isomorphisms, as it does not preserve units. However, the diagonal copy of \mathbb{Z}_2 in $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on \hat{G} by Lie groupoid isomorphisms.

Set $\hat{H} := (dq)^!H \rightrightarrows T\hat{M}$. By property (3), \hat{H} is a multiplicative distribution on \hat{G} . Moreover, by property (2), Q is a local diffeomorphism. Hence, dQ induces an isomorphism $T\hat{G} \cong Q^*TG$ that restricts to an isomorphism $\hat{H} \cong Q^*H$ (all as vector bundles over G). Thus, \hat{H} is a contact structure on \hat{G} . Therefore, (\hat{G}, \hat{H}) is a contact groupoid. If $\hat{L} := T\hat{G}/\hat{H}|_{\hat{M}}$, then \hat{L} is canonically isomorphic to q^*L . Hence, (\hat{G}, \hat{H}) is a co-orientable contact groupoid.

Definition 1.25. Let (G, H) be a contact groupoid over a connected manifold M such that $L \rightarrow M$ is not trivializable. The co-orientable contact groupoid (\hat{G}, \hat{H}) constructed above is called the **co-orientable finite cover of (G, H)** .

Remark 1.26. Let (G, H) be a contact groupoid over a connected manifold M such that $L \rightarrow M$ is not trivializable. Let (\hat{G}, \hat{H}) be the co-orientable finite cover of (G, H) . Then G is compact if and only if \hat{G} is compact.

Remark 1.27. Let H be a corank one multiplicative distribution on $G \rightrightarrows M$ such that $TG/H|_M$ is not trivializable. The above construction holds in this more general setting with the obvious adaptations (cf. Remark 1.21). In particular, it makes sense to consider the co-orientable finite cover of the Lie groupoid G with the distribution H that, by abuse of notation, we also denote by (\hat{G}, \hat{H}) .

To conclude this section, we prove Lemma 1.28, which is a technical result relating the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action and some multiplicative contact forms on the space of arrows of the co-orientable finite cover (\hat{G}, \hat{H}) of (G, H) . (We use this result in the proof of Theorem 3.4.) Let $\gamma \in \text{Diff}(\hat{M})$ be the non-trivial diffeomorphism given by the \mathbb{Z}_2 -action on \hat{M} , and let $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \text{Diff}(\hat{G})$ be the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on \hat{G} . For any $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, $\varphi_{ab} := \varphi(a, b)$ is the restriction of

$$(1.15) \quad (\gamma^a, \text{id}, \gamma^b) \in \text{Diff}(\hat{M} \times G \times \hat{M})$$

to \hat{G} . While φ_{ab} is not a groupoid homomorphism if $a \neq b$ (see Remark 1.24), the multiplication and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on \hat{G} are related as follows. First, observe that the map $(\varphi_{ab}, \varphi_{bc})$ restricts to a diffeomorphism of $\hat{G}^{(2)}$ that

satisfies $\varphi_{ac} \circ \hat{m} = \hat{m} \circ (\varphi_{ab}, \varphi_{bc})$ for all $a, b, c \in \mathbb{Z}_2$. Taking derivatives, we have that

$$(1.16) \quad d\varphi_{ac}(d\hat{m}(X, Y)) = d\hat{m}(d\varphi_{ab}(X), d\varphi_{bc}(Y)),$$

for all $(X, Y) \in T\hat{G}^{(2)}$ and for all $a, b, c \in \mathbb{Z}_2$. By property (2) above, φ_{ab} is a contactomorphism of (\hat{G}, \hat{H}) . If $\hat{\alpha} \in \Omega^1(\hat{G})$ is any contact form for (\hat{G}, \hat{H}) , for each $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, there exists a nowhere vanishing $f_{ab} \in C^\infty(\hat{G})$ satisfying

$$(1.17) \quad \varphi_{ab}^* \hat{\alpha} = f_{ab} \hat{\alpha}.$$

Lemma 1.28. *Let (G, H) be a contact groupoid over a connected manifold M such that $L \rightarrow M$ is not trivializable and let (\hat{G}, \hat{H}) be its co-orientable finite cover. Let $\hat{\sigma} : \hat{G} \rightarrow \{\pm 1\}$ be a Lie groupoid homomorphism and suppose that $(\hat{\alpha}, \hat{\sigma})$ is a multiplicative contact form for (\hat{G}, \hat{H}) . Then, for any $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, the function f_{ab} given by equation (1.17) is constant. In fact, $f_{00} = f_{01} \equiv 1$ and $f_{10} = f_{11} \equiv -1$.*

Proof. Set $\hat{R} := R^{\hat{\alpha}}$ and let $(x, g, y) \in \hat{G}$. Using equations (1.11) and (1.16), we have that, for any $a, b, c \in \mathbb{Z}_2$,

$$(1.18) \quad \begin{aligned} f_{ac}(x, g, y) &= (\varphi_{ac}^* \hat{\alpha})_{(x, g, y)}(\hat{R}_{(x, g, y)}) = \hat{\alpha}_{\varphi_{ac}(x, g, y)}(d\varphi_{ac}(\hat{R}_{(x, g, y)})) \\ &= \hat{\alpha}_{\varphi_{ac}(x, g, y)}(d\varphi_{ac}(d\hat{m}(\hat{R}_{(x, g, y)}, 0_{(y, 1_{q(y)}, y)}))) \\ &= \hat{\alpha}_{\varphi_{ac}(x, g, y)}(d\hat{m}(d\varphi_{ab}(\hat{R}_{(x, g, y)}), d\varphi_{bc}(0_{(y, 1_{q(y)}, y)}))) \\ &= \hat{\alpha}_{\varphi_{ab}(x, g, y)}(d\varphi_{ab}(\hat{R}_{(x, g, y)})) = (\varphi_{ab}^* \hat{\alpha})_{(x, g, y)}(\hat{R}_{(x, g, y)}) \\ &= f_{ab}(x, g, y), \end{aligned}$$

where in the fourth equality we use that $\hat{\alpha}$ is $\hat{\sigma}$ -multiplicative. Hence, for any $a, b, c \in \mathbb{Z}_2$, $f_{ab} \equiv f_{ac}$. Since $f_{00} \equiv 1$, we have that $f_{01} \equiv 1$.

Since $\hat{\alpha}$ is $\hat{\sigma}$ -multiplicative and $d\hat{\sigma} \equiv 0$, Lemma 1.23 implies that

$$\hat{R}_{(x, g, y)} = d\hat{m}(0_{(x, g, y)}, \hat{\sigma}(x, g, y)\hat{R}_{(y, 1_{q(y)}, y)}).$$

Thus arguing as in (1.18), we have that

$$f_{ac}(x, g, y) = \hat{\sigma}(\gamma^a(x), g, \gamma^b(y))\hat{\sigma}(x, g, y)f_{bc}(y, 1_{q(y)}, y),$$

for all $a, b, c \in \mathbb{Z}_2$. Taking $a = c = 1$ and $b = 0$ and using the fact that $f_{01} \equiv 1$, we have that $f_{11}(x, g, y) = \hat{\sigma}(\gamma(x), g, y)\hat{\sigma}(x, g, y)$. We observe that, since

$\hat{\sigma} : \hat{G} \rightarrow \{\pm 1\}$ is a groupoid homomorphism,

$$\hat{\sigma}(\gamma(x), g, y) \hat{\sigma}(x, g, y) = \hat{\sigma}(\gamma(x), 1_{q(x)}, x) (\hat{\sigma}(x, g, y))^2 = \hat{\sigma}(\gamma(x), 1_{q(x)}, x).$$

Hence, $f_{11}(x, g, y) = \hat{\sigma}(\gamma(x), 1_{q(x)}, x)$. This shows that $f_{11} = f_{10}$ is, at the same time, locally constant and the pullback of a function on \hat{M} along \hat{t} . Hence, since \hat{M} is connected, $f_{11} = f_{10}$ is constant; moreover, it has to be equal to 1 or to -1 . Suppose that $f_{11} = f_{10} \equiv 1$. Then the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on \hat{G} preserves the contact form $\hat{\alpha}$ and, therefore, it induces a contact form α for (G, H) . This implies that $L = TG/H|_M \rightarrow M$ is trivializable, which is a contradiction. \square

2. Deformations of contact groupoids

The aim of this section is to define deformations of (co-oriented) contact groupoids and to study some of their properties, with emphasis on deformations of compact contact groupoids. The idea is to formalize the notion of ‘smooth 1-parameter family of (compact) contact groupoids’. The key result of this section is Corollary 2.24, which gives the existence of a ‘good’ smooth 1-parameter family of multiplicative contact forms for deformations of co-orientable compact contact groupoids. We use this result in the proof of Theorem 3.1. The main references for deformations of Lie groupoids are [15, 20].

2.1. Deformations of Lie groupoids

Definition 2.1 (Definition 1.9 in [15]). A family of Lie groupoids (over B) is a Lie groupoid $\tilde{G} \rightrightarrows \tilde{M}$ together with a surjective submersion $p : \tilde{M} \rightarrow B$ such that $p \circ \tilde{s} = p \circ \tilde{t}$. We use the notation $\tilde{G} \rightrightarrows \tilde{M} \rightarrow B$.

Remark 2.2. Given $\tilde{G} \rightrightarrows \tilde{M} \rightarrow B$, for any $b \in B$, set $G_b := (p \circ \tilde{s})^{-1}(b)$ and $M_b := p^{-1}(b)$. There is a unique structure of Lie groupoid on $G_b \rightrightarrows M_b$ making it into a Lie subgroupoid of $\tilde{G} \rightrightarrows \tilde{M}$ that we call the **fiber of $\tilde{G} \rightrightarrows \tilde{M} \rightarrow B$ over b** .

We formalize the notion of ‘a smooth 1-parameter family of Lie groupoids’ as follows.

Definition 2.3. Let $G \rightrightarrows M$ be a Lie groupoid and let $I \subseteq \mathbb{R}$ be an open interval containing 0. A **deformation of G** is a family of Lie groupoids $\tilde{G} \rightrightarrows \tilde{M} \rightarrow I$ such that

- as manifolds, $\tilde{G} = G \times I$ and $\tilde{M} = M \times I$,
- the submersions $p \circ \tilde{s} : G \times I \rightarrow I$ and $p : M \times I \rightarrow I$ are projections onto the second component, and
- as Lie groupoids, $G_0 = G$.

We use the notation $\tilde{G} = \{G_\tau\}$, where $\tau \in I$. A deformation \tilde{G} is **proper** if \tilde{G} is proper. The **constant** deformation of G is the one in which $G_\tau = G$ as Lie groupoids for all τ .

Remark 2.4

- Definition 2.3 is a *strict* deformation in the sense of [15, Definition 1.6]. On the other hand, it is a special case of [20, Definition 5.1.1], which allows for locally trivial submersions over any manifold.
- The Lie groupoid structure on a deformation \tilde{G} of G is determined by the Lie groupoid structures on G_τ for all τ .

Definition 2.5. Let G be a Lie groupoid.

- Two deformations $\tilde{G}_1 = \{G_{1,\tau}\}$, $\tilde{G}_2 = \{G_{2,\tau}\}$ of G are **isomorphic** if there exists a Lie groupoid isomorphism $\tilde{\Phi} : \tilde{G}_1 \rightarrow \tilde{G}_2$ of the form

$$(2.1) \quad \tilde{\Phi}(g, \tau) = (\Phi_\tau(g), \tau)$$

such that $\Phi_0 : G_{1,0} = G \rightarrow G_{2,0} = G$ is the identity. We use the notation $\tilde{\Phi} = \{\Phi_\tau\}$.

- A deformation \tilde{G} of G is **trivial** if it is isomorphic to the constant deformation.

Remark 2.6. By equation (2.1), an isomorphism $\tilde{\Phi}$ of deformations of G is completely determined by the Lie groupoid isomorphisms Φ_τ for all τ (cf. Remark 2.4). This justifies the above notation.

Remark 2.7. Definition 2.5 should be compared with [15, Definitions 1.6 and 1.9]. In *loc. cit.* the authors allow for isomorphisms that are not the identity on the fiber over zero (see [15, Definition 1.9]), and consider a more general equivalence relation on deformations called *equivalence* (see [15, Definition 1.6]).

The following stability result is proved in [15] using vanishing results for deformation cohomology of Lie groupoids (see also [20] for similar stability results proved using Riemannian metrics on Lie groupoids).

Theorem 2.8 (Theorem 1.7 in [15]). *Any deformation of a compact Lie groupoid is trivial.*

Remark 2.9. Strictly speaking, what is proved in [15] is that deformations of compact Lie groupoids are trivial up to equivalence (see Remark 2.7). However, since the desired isomorphism is constructed using the flow of a time-dependent vector field and, in this case, the Lie groupoid is compact, Theorem 2.8 follows immediately from [15, Remark 5.5 and Theorem 7.1].

2.2. Deformations of contact groupoids

If $\tilde{G} \rightrightarrows \tilde{M} \rightarrow B$ is a family of Lie groupoids over B , then $T\tilde{G} \rightrightarrows T\tilde{M} \rightarrow TB$ is a family of Lie groupoids over TB with structure maps obtained by taking derivatives of those of $\tilde{G} \rightrightarrows \tilde{M} \rightarrow B$. Moreover, for any $b \in B$, the fiber of $T\tilde{G} \rightrightarrows T\tilde{M} \rightarrow TB$ over 0_b is the tangent Lie groupoid of $G_b \rightrightarrows M_b$.

Definition 2.10. Let $\tilde{G} \rightrightarrows \tilde{M} \rightarrow B$ be a family of Lie groupoids. A **family of multiplicative distributions on $\tilde{G} \rightrightarrows \tilde{M} \rightarrow B$** is a multiplicative distribution $\tilde{H} \subseteq T\tilde{G}$ on \tilde{G} .

The following result motivates the terminology of Definition 2.10.

Lemma 2.11. *Let \tilde{H} be a family of multiplicative distributions on $\tilde{G} \rightrightarrows \tilde{M} \rightarrow B$. Then $\tilde{H} \rightrightarrows T\tilde{M} \rightarrow TB$ is a family of Lie groupoids over TB . Moreover, for any $b \in B$, if H_b denotes the fiber of $\tilde{H} \rightrightarrows T\tilde{M} \rightarrow TB$ over 0_b , then*

- 1) $H_b \subseteq TG_b$ is a multiplicative distribution on G_b , and
- 2) the vector bundle $E_b := TG_b/H_b|_{M_b}$ is canonically isomorphic to the restriction of $\tilde{E} := T\tilde{G}/\tilde{H}|_{\tilde{M}}$ to M_b .

Proof. Since \tilde{H} is a Lie subgroupoid of $T\tilde{G}$ over $T\tilde{M}$ and since $T\tilde{G} \rightrightarrows T\tilde{M} \rightarrow TB$ is a family of Lie groupoids, it follows that $\tilde{H} \rightrightarrows T\tilde{M} \rightarrow TB$ also is, and that (1) holds for any $b \in B$. It remains to prove (2). For any $b \in B$, the

surjective submersion $p \circ \tilde{s} : \tilde{G} \rightarrow B$ induces the following diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_b & \hookrightarrow & \tilde{H}|_{G_b} & \twoheadrightarrow & G_b \times T_b B \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & TG_b & \hookrightarrow & T\tilde{G}|_{G_b} & \xrightarrow{d(p \circ \tilde{s} \circ i_b)} & G_b \times T_b B \longrightarrow 0.
 \end{array}$$

Hence, TG_b/H_b is canonically isomorphic to $T\tilde{G}/\tilde{H}|_{G_b}$. The result follows by using the fact that G_b is a Lie subgroupoid of \tilde{G} . \square

The following result, stated using the notation of Lemma 2.11, is an immediate consequence of Lemma 2.11 and of Proposition 1.14.

Corollary 2.12. *Let \tilde{H} be a family of multiplicative distributions on $\tilde{G} \rightrightarrows \tilde{M} \rightarrow B$. For any $b \in B$, upon identifying E_b and $\tilde{E}|_{M_b}$ using the canonical isomorphism of Lemma 2.11,*

- *the representation of G_b on E_b equals the restriction of the representation of \tilde{G} on \tilde{E} to G_b , and*
- *the multiplicative 1-forms $i_b^* \tilde{\alpha}_{\text{can}}$ and $\alpha_{b,\text{can}}$ with values in the representation E_b are equal.*

Definition 2.13. A deformation of a contact groupoid (G, H) is a deformation $\tilde{G} = \{G_\tau\}$ of G together with a family of multiplicative distributions \tilde{H} on \tilde{G} , such that

- for all τ , (G_τ, H_τ) is a contact groupoid,
- for all $(g, \tau) \in G \times I$,

$$(2.2) \quad \tilde{H}_{(g,\tau)} = (H_\tau)_g \oplus T_\tau I \subseteq T_g G \oplus T_\tau I, \text{ and}$$

- as contact groupoids, $(G_0, H_0) = (G, H)$.

We use the notation $(\tilde{G}, \tilde{H}) = \{(G_\tau, H_\tau)\}$; if $\{G_\tau\}$ is the constant deformation, we write $(G \times I, \tilde{H}) = \{(G, H_\tau)\}$. The **constant** deformation of (G, H) is the one in which $(G_\tau, H_\tau) = (G, H)$ as contact groupoids for all τ .

Remark 2.14 The second condition in Definition 2.13 can be reformulated as follows: $\{0_g\} \oplus T_\tau I$ is contained in $\tilde{H}_{(g,\tau)}$ for all $(g, \tau) \in G \times I$. Hence, \tilde{H} is completely determined by H_τ for all τ , thus justifying the notation (cf. Remark 2.4). Many of the results below hold without imposing equation (2.2). However, this more general setting goes beyond the scope of this paper.

Let (\tilde{G}, \tilde{H}) be a deformation of a contact groupoid (G, H) . Since $\tilde{M} = M \times I$, setting $\tilde{L} := T\tilde{G}/\tilde{H}|_{\tilde{M}}$ and $L := TG/H|_M$, there exists an isomorphism of vector bundles

$$(2.3) \quad \psi : \tilde{L} \rightarrow \text{pr}^* L,$$

where $\text{pr} : M \times I \rightarrow M$ is projection onto the first component (see, e.g., [24, Chapter 4, Section 1, Theorem 1.5]). Combining this observation with Lemma 2.11, we have proved the following result.

Lemma 2.15. *If $(\tilde{G}, \tilde{H}) = \{(G_\tau, H_\tau)\}$ is a deformation of a contact groupoid (G, H) , then $L_\tau = TG_\tau/H_\tau|_{M_\tau}$ is isomorphic to $L = TG/H|_M$ for all τ .*

Definition 2.16. Let (G, H) be a contact groupoid.

- Two deformations (\tilde{G}, \tilde{H}) and (\tilde{G}', \tilde{H}') of (G, H) are **isomorphic** if there exists an isomorphism $\tilde{\Phi} : \tilde{G} \rightarrow \tilde{G}'$ of deformations of G such that $d\tilde{\Phi}(\tilde{H}) = \tilde{H}'$.
- A deformation (\tilde{G}, \tilde{H}) of (G, H) is **trivial** if it is isomorphic to the constant one.

Remark 2.17. An isomorphism $\tilde{\Phi}$ between two deformations (\tilde{G}, \tilde{H}) and (\tilde{G}', \tilde{H}') of a contact groupoid (G, H) is completely determined by the isomorphisms of contact groupoids $\Phi_\tau : (G_\tau, H_\tau) \rightarrow (G'_\tau, H'_\tau)$ for all τ (cf. Remark 2.6).

The following (partial) stability results for deformations of compact contact groupoids is a consequence of Theorem 2.8.

Corollary 2.18. *Any deformation $(\tilde{G}, \tilde{H}) = \{(G_\tau, H_\tau)\}$ of a compact contact groupoid (G, H) is isomorphic to one of the form $(G \times I, \tilde{H}') = \{(G, H'_\tau)\}$.*

Proof. By Theorem 2.8, there exists an isomorphism $\tilde{\Phi} = \{\Phi_\tau\}$ between \tilde{G} and $G \times I$ that satisfies equation (2.1). Hence, $\tilde{H}' := d\tilde{\Phi}(\tilde{H})$ satisfies equation (2.2) and (G, H'_τ) is a contact groupoid for all τ . This completes the proof. \square

2.3. Deformations of co-oriented contact groupoids

Definition 2.19. A deformation of a co-oriented contact groupoid (G, α, F) is a deformation $\tilde{G} = \{G_\tau\}$ of G together with a Lie groupoid homomorphism $\tilde{F} : \tilde{G} \rightarrow \mathbb{R}^*$ and a surjective \tilde{F} -multiplicative 1-form $\tilde{\alpha} \in \Omega^1(\tilde{G})$ such that

- for all τ , $(G_\tau, \alpha_\tau, F_\tau) := (G_\tau, i_\tau^* \tilde{\alpha}, i_\tau^* \tilde{F})$ is a co-oriented contact groupoid,
- for all $(g, \tau) \in G \times I$,

$$(2.4) \quad \{0_g\} \oplus T_\tau I \subset \ker \tilde{\alpha}_{(g, \tau)}, \text{ and,}$$

- as co-oriented contact groupoids, $(G, \alpha, F) = (G_0, \alpha_0, F_0)$.

We use the notation by $(\tilde{G}, \tilde{\alpha}, \tilde{F}) = \{(G_\tau, \alpha_\tau, F_\tau)\}$; if $\{G_\tau\}$ is the constant deformation, we write $(G \times I, \tilde{\alpha}, \tilde{F}) = \{(G, \alpha_\tau, F_\tau)\}$.

Remark 2.20 A Lie groupoid homomorphism $\tilde{F} : \tilde{G} \rightarrow \mathbb{R}^*$ is completely determined by the Lie groupoid homomorphisms $F_\tau : G_\tau \rightarrow \mathbb{R}^*$ for all τ . Moreover, by equation (2.4), the 1-form $\tilde{\alpha}$ is completely determined by α_τ for all τ . Therefore, the multiplicative 1-form $(\tilde{\alpha}, \tilde{F})$ is completely determined by (α_τ, F_τ) for all τ . This justifies the above notation (cf. Remarks 2.4 and 2.14).

The following result is the ‘smooth 1-parameter’ analog of Proposition 1.14.

Corollary 2.21. *Let (\tilde{G}, \tilde{H}) be a deformation of a co-orientable contact groupoid (G, H) . For any multiplicative 1-form $(\tilde{\alpha}, \tilde{F})$ for (\tilde{G}, \tilde{H}) , $(\tilde{G}, \tilde{\alpha}, \tilde{F})$ is a deformation of the co-oriented contact groupoid (G, α_0, F_0) such that (α_τ, F_τ) is a multiplicative contact form for (G_τ, H_τ) for all τ . Conversely, any deformation $(\tilde{G}, \tilde{\alpha}, \tilde{F})$ of a co-oriented contact groupoid (G, α, F) induces a deformation $(\tilde{G}, \tilde{H} := \ker \tilde{\alpha})$ of the co-orientable contact groupoid $(G, H := \ker \alpha)$ such that (α_τ, F_τ) is a multiplicative contact form for (G_τ, H_τ) .*

Proof. Let (\tilde{G}, \tilde{H}) be a deformation a co-orientable contact groupoid (G, H) and let $(\tilde{\alpha}, \tilde{F})$ be a multiplicative 1-form for \tilde{H} . The latter is equivalent to a trivialization of $\tilde{L} = T\tilde{G}/\tilde{H}|_{\tilde{M}}$. Then Proposition 1.14 and Corollary 2.12 (the latter applied to the canonical 1-forms composed with the above trivialization), imply that (α_τ, F_τ) is a multiplicative contact form for (G_τ, H_τ)

for all τ . Moreover, equation (2.4) holds by equation (2.2). Hence, $(\tilde{G}, \tilde{\alpha}, \tilde{F})$ is a deformation of the co-oriented contact groupoid (G, α_0, F_0) with the desired property.

Conversely, let $(\tilde{G}, \tilde{\alpha}, \tilde{F})$ be a deformation of (G, α, F) and set $\tilde{H} := \ker \tilde{\alpha}$. By Proposition 1.14 and Corollary 2.12, (α_τ, F_τ) is a multiplicative contact form for (G_τ, H_τ) for all τ . Moreover, \tilde{H} satisfies equation (2.2) since $\tilde{\alpha}$ satisfies equation (2.4). Hence, (\tilde{G}, \tilde{H}) is a deformation of (G, H) with the desired property. \square

In a deformation of a co-oriented contact groupoid, the sign of F is constant (see Definition 1.18). More precisely, the following holds.

Lemma 2.22. *Let $(\tilde{G}, \tilde{\alpha}, \tilde{F}) = \{(G_\tau, \alpha_\tau, F_\tau)\}$ be a deformation of a co-oriented contact groupoid (G, α, F) . Let $\text{pr} : G \times I \rightarrow G$ denote projection onto the first component and let $\tilde{\sigma}$ (respectively σ_τ) denote the sign of \tilde{F} (respectively F_τ). Then $\tilde{\sigma} = \text{pr}^* \sigma_0$ and $\sigma_\tau = \sigma_0$ for all τ .*

Proof. Since I is connected, for any $g \in G$, the map $\tau \mapsto \tilde{\sigma}(g, \tau)$ is constant. Hence, $\tilde{\sigma} = \text{pr}^* \sigma_0$ and, by Definition 2.19, $\sigma_\tau = i_\tau^* \text{pr}^* \sigma_0 = \sigma_0$ as desired. \square

To conclude this section, we prove the following results, which are the ‘smooth 1-parameter’ analogs of Proposition 1.19 and of Corollary 1.20.

Proposition 2.23. *Let $(\tilde{G}, \tilde{\alpha}, \tilde{F})$ be a deformation of a co-oriented contact groupoid (G, α, F) and let $\sigma = \text{sgn}(F)$. If the cocycle \tilde{r} associated to $(\tilde{G}, \tilde{\alpha}, \tilde{F})$ is a coboundary, then, for all $\tilde{\kappa} \in C^\infty(\tilde{M})$ satisfying equation (1.9), $(e^{\tilde{t}^* \tilde{\kappa}} \tilde{\alpha}, \text{pr}^* \sigma)$ is a multiplicative 1-form for $(\tilde{G}, \tilde{H} = \ker \tilde{\alpha})$, where $\text{pr} : \tilde{G} = G \times I \rightarrow G$ is projection onto the first component.*

Proof. By Lemma 2.22, the sign of \tilde{F} equals $\text{pr}^* \sigma$. By Remark 1.21 and Proposition 1.19, $(e^{\tilde{t}^* \tilde{\kappa}} \tilde{\alpha}, \text{pr}^* \sigma)$ is a multiplicative 1-form for (\tilde{G}, \tilde{H}) , as desired. \square

Proposition 2.23 and vanishing of differentiable cohomology in positive degrees for proper Lie groupoids (see [8, Proposition 1]), immediately imply the following result.

Corollary 2.24. *Let (\tilde{G}, \tilde{H}) be a proper deformation of a co-orientable contact groupoid (G, H) . Then there exist a Lie groupoid homomorphism $\sigma : G \rightarrow \{\pm 1\}$ and $\tilde{\alpha} \in \Omega^1(\tilde{G})$ such that $(\tilde{\alpha}, \text{pr}^* \sigma)$ is a multiplicative 1-form for (\tilde{G}, \tilde{H}) , where $\text{pr} : \tilde{G} = G \times I \rightarrow G$ is projection onto the first component.*

3. Multiplicative Gray Stability

In this section we prove our main result, Theorem 3.1. Together with Corollary 2.18, it yields stability of compact contact groupoids over connected manifolds (see Corollary 3.2). The proof of Theorem 3.1 is effectively given by combining Theorems 3.3 and 3.4. We present the proof in this fashion to emphasize these other results, which are interesting in their own right, and to simplify the exposition.

Theorem 3.1. *Let (G, H) be a compact contact groupoid over a connected manifold M . Any deformation of (G, H) of the form $(G \times I, \tilde{H}) = \{(G, H_\tau)\}$ is trivial.*

By Corollary 2.18, Theorem 3.1 immediately implies the following stability result.

Corollary 3.2. *Any deformation of a compact contact groupoid over a connected manifold is trivial.*

Proof of Theorem 3.1. We split the proof in two cases, namely whether $L = TG/H|_M$ is trivializable or not.

Suppose that L is trivializable. Since G is compact, $G \times I$ is proper. Hence, by Corollary 2.24, there exist a Lie groupoid homomorphism $\sigma : \tilde{G} \rightarrow \{\pm 1\}$ and $\tilde{\alpha} \in \Omega^1(\tilde{G})$ such that $(\tilde{\alpha}, \text{pr}^*\sigma)$ is a multiplicative 1-form for (\tilde{G}, \tilde{H}) , where $\text{pr} : G \times I \rightarrow G$ is projection onto the first component. For all τ , set $\alpha_\tau := i_\tau^* \tilde{\alpha}$ and observe that $H_\tau = \ker \alpha_\tau$. Then, by Corollary 2.21, $(G \times I, \tilde{\alpha}, \text{pr}^*\sigma) = \{(G, \alpha_\tau, \sigma)\}$ is a deformation of the co-oriented contact groupoid (G, α_0, σ) . By Theorem 3.3 below, there exist an automorphism $\tilde{\Phi} = \{\Phi_\tau\}$ of the constant deformation of G , and $\tilde{f} \in C^\infty(M \times I)$ that is constant along the orbits of $G \times I$, such that $\Phi_\tau^* \alpha_\tau = e^{t^* f_\tau} \alpha$ for all τ , where $f_\tau = i_\tau^* \tilde{f}$. In particular, $\Phi_\tau : (G, H) \rightarrow (G, H_\tau)$ is an isomorphism of contact groupoids for all τ . Hence, by Remark 2.17, $\tilde{\Phi}$ is an isomorphism of deformations of contact groupoids between the constant deformation and (\tilde{G}, \tilde{H}) , i.e., (\tilde{G}, \tilde{H}) is trivial as desired.

The case in which L is not trivializable is precisely Theorem 3.4 below. (This is where we use connectedness of M .) This completes the proof. \square

Theorem 3.3. *Let (G, α, σ) be a compact co-oriented contact groupoid over M . Let $(G \times I, \tilde{\alpha}, \text{pr}^*\sigma) = \{(G, \alpha_\tau, \sigma)\}$ be a deformation of (G, α, σ) . Then there exist an automorphism $\tilde{\Phi} = \{\Phi_\tau\}$ of the constant deformation $G \times I$*

of G , and $\tilde{f} \in C^\infty(M \times I)$ that is constant along the orbits of $G \times I$, such that

$$(3.1) \quad \Phi_\tau^* \alpha_\tau = e^{t^* f_\tau} \alpha$$

for all τ , where $f_\tau = i_\tau^* \tilde{f}$.

Proof. We use the standard proof of Gray stability in contact geometry (see, e.g., [21, Theorem 2.2.2]), making sure that all choices can be made in a multiplicative fashion. We look for an automorphism $\tilde{\Phi} = \{\Phi_\tau\}$ of the constant deformation $G \times I$ and for a positive function $\tilde{a} \in C^\infty(G \times I)$ such that

$$(3.2) \quad \Phi_\tau^* \alpha_\tau = a_\tau \alpha \quad \text{for all } \tau,$$

where $a_\tau = i_\tau^* \tilde{a}$. Moreover, we assume that Φ_τ is the flow of a time dependent vector field $X_\tau \in \mathfrak{X}(G)$. Differentiating (3.2) with respect to τ , we obtain

$$(3.3) \quad \Phi_\tau^* \left(\frac{d\alpha_\tau}{d\tau} + \mathcal{L}_{X_\tau} \alpha_\tau \right) = \frac{da_\tau}{d\tau} \frac{1}{a_\tau} \Phi_\tau^* \alpha_\tau.$$

Setting $\mu_\tau := (\frac{d}{d\tau} \log a_\tau) \circ \Phi_\tau^{-1}$, equation (3.3) is satisfied if and only if

$$(3.4) \quad \frac{d\alpha_\tau}{d\tau} + \mathcal{L}_{X_\tau} \alpha_\tau - \mu_\tau \alpha_\tau = 0.$$

For all τ , we look for a solution of (3.4) of the form $X_\tau \in \Gamma(H_\tau)$, where $H_\tau = \ker \alpha_\tau$. Then equation (3.4) reduces to

$$(3.5) \quad \frac{d\alpha_\tau}{d\tau} + \iota_{X_\tau} d\alpha_\tau = \mu_\tau \alpha_\tau.$$

Due to the defining relations (1.1), evaluating $R_\tau := R^{\alpha_\tau}$ in (3.5) yields

$$\mu_\tau = \frac{d\alpha_\tau}{d\tau}(R_\tau),$$

which determines μ_τ uniquely. Moreover, from the decomposition (1.3), $\mu_\tau \alpha_\tau - \frac{d\alpha_\tau}{d\tau}$ is a section of $\text{Ann}(\langle R_\tau \rangle)$, thus implying that there is a unique solution $X_\tau \in \Gamma(H_\tau)$ of (3.5) depending smoothly on τ .

Suppose that X_τ is a multiplicative vector field and that

$$(3.6) \quad \mu_\tau(g) = \mu_\tau(1_{s(g)}) = \mu_\tau(1_{t(g)})$$

for all τ and for all $g \in G$. Then the flow Φ_τ of X_τ is a Lie groupoid automorphism of G . Moreover, by compactness of G , the flow of X_τ exists for all τ . Hence, we obtain an automorphism $\tilde{\Phi} = \{\Phi_\tau\}$ of the constant deformation of G satisfying equation (3.2) for all τ . Since μ_τ satisfies equation (3.6) for all τ , so does a_τ . This is equivalent to \tilde{a} satisfying the analog of equation (3.6) for the groupoid $G \times I$. Since \tilde{a} is positive by assumption, it follows that there exists $\tilde{f} \in C^\infty(M \times I)$ constant along the orbits of $G \times I$ such that $\tilde{a} = e^{\tilde{t}^* \tilde{f}}$. Hence, equation (3.1) holds for all τ .

To complete the proof, we need to show that X_τ is a multiplicative vector field and that μ_τ satisfies (3.6) for all τ . To this end, we observe that $\frac{d\alpha_\tau}{d\tau} \in \Omega^1(G)$ is σ -multiplicative for all τ . Moreover, since $d\sigma \equiv 0$, $d\alpha_\tau \in \Omega^2(G)$ is also σ -multiplicative. This allows us to prove that, for all τ , μ_τ satisfies (3.6). Indeed, since $\frac{d\alpha_\tau}{d\tau}$ is σ -multiplicative and R_τ is right-invariant (see Lemma 1.23), we have that

$$(3.7) \quad \begin{aligned} \mu_\tau(g) &= \left(\frac{d\alpha_\tau}{d\tau} \right)_g (R_\tau) = \left(\frac{d\alpha_\tau}{d\tau} \right)_{1_{t(g)}g} (dm(R_\tau, 0)) \\ &= \left(\frac{d\alpha_\tau}{d\tau} \right)_{1_{t(g)}} (R_\tau) + \sigma(1_{t(g)}) \left(\frac{d\alpha_\tau}{d\tau} \right)_g (0) = \mu_\tau(1_{t(g)}) \end{aligned}$$

for all τ and for all $g \in G$. Moreover, by Lemma 1.23, $R_{\tau,g} = dm(0_g, \sigma(g)R_{\tau,1_{s(g)}})$ for all τ and for all $g \in G$. Hence, a computation entirely analogous to (3.7) shows that $\mu_\tau(g) = \mu_\tau(1_{s(g)})$ for all τ and for all $g \in G$, as desired.

It remains to prove that X_τ is multiplicative, i.e., the map $X_\tau : G \rightarrow H_\tau$ is a Lie groupoid homomorphism for all τ . Since μ_τ satisfies (3.6) and α_τ is σ -multiplicative for all τ , the 1-form $\mu_\tau \alpha_\tau$ is also σ -multiplicative for all τ . Hence, $\beta_\tau := \mu_\tau \alpha_\tau - \frac{d\alpha_\tau}{d\tau}$ is σ -multiplicative. For each τ , X_τ is the composition

$$(3.8) \quad G \xrightarrow{\beta_\tau} T^*G \longrightarrow H_\tau^* \xrightarrow{\cong} H_\tau,$$

where the middle map is the dual to the inclusion and the last is the inverse of the restriction of $(d\alpha_\tau)^\flat$ to H_τ . By Lemmas A.1 and A.2 in Appendix A

all maps in (3.8) are Lie groupoid homomorphisms² and, therefore, so is X_τ , as desired. \square

Theorem 3.4. *Let (G, H) be a compact contact groupoid over a connected manifold M such that $L = TG/H|_M$ is not trivializable. Any deformation of (G, H) of the form $(G \times I, \tilde{H}) = \{(G, H_\tau)\}$ is trivial.*

Proof. Let $\text{pr} : M \times I \rightarrow M$ be projection onto the first component and let $\tilde{L} = T(G \times I)/\tilde{H}|_{M \times I}$. Fix an isomorphism $\psi : \tilde{L} \rightarrow \text{pr}^*L$ (see equation (2.3)); this identifies L_τ with L for all τ . Consider the co-orientable finite cover (\tilde{G}', \tilde{H}') of (\tilde{G}, \tilde{H}) (see Remark 1.27), and let (\hat{G}, \hat{H}) be the co-orientable finite cover of (G, H) . Then the choice of ψ allows us to identify \tilde{G}' with $\hat{G} \times I$. Under this identification, the natural $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on \hat{G} becomes the following: for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$,

$$(3.9) \quad (a, b) \cdot (\hat{g}, \tau) := (\varphi_{ab}(\hat{g}), \tau),$$

where φ_{ab} is the action of (a, b) on \hat{G} . Moreover, $(\hat{G} \times I, \tilde{H}') = \{(\hat{G}, H'_\tau)\}$ is a deformation of (\hat{G}, \hat{H}) with the property that (\hat{G}, H'_τ) is the co-orientable finite cover (\hat{G}, \hat{H}_τ) of (G, H_τ) for all τ . Since (\hat{G}, \hat{H}) is co-orientable by construction, and compact by Remark 1.26, we can argue as in the proof of Theorem 3.1, i.e., there exists a Lie groupoid homomorphism $\hat{\sigma} : \hat{G} \rightarrow \{\pm 1\}$ and $\tilde{\alpha}' \in \Omega^1(\tilde{G}')$ such that $(\tilde{\alpha}', \text{Pr}^*\hat{\sigma})$ is a multiplicative 1-form for (\tilde{G}', \tilde{H}') , where $\text{Pr} : \hat{G} \times I \rightarrow \hat{G}$ is projection onto the first component. Set $\hat{\alpha}_\tau := \hat{i}_\tau^* \tilde{\alpha}'$, where $\hat{i}_\tau : \hat{G} \rightarrow \hat{G} \times I$, $\hat{i}_\tau(\hat{g}) = (\hat{g}, \tau)$. Then we obtain a deformation $(\hat{G} \times I, \tilde{\alpha}', \text{Pr}^*\hat{\sigma})$ of the co-oriented contact groupoid $(\hat{G}, \hat{\alpha}_0, \hat{\sigma})$. Hence, Theorem 3.3 can be applied. In fact, suppose that the time-dependent vector field constructed in the proof of Theorem 3.3 is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -equivariant. Then the automorphism of the constant deformation $\hat{G} \times I$ in Theorem 3.3 induces an automorphism of the constant deformation $G \times I$. Since the former is an isomorphism of deformations of contact groupoids between the constant deformation of (\hat{G}, \hat{H}) and $(\hat{G} \times I, \tilde{H}')$, it follows that the latter is the desired isomorphism of deformations of contact groupoids between the constant deformation of (G, H) and (\tilde{G}, \tilde{H}) .

It remains to prove the time-dependent vector field constructed in the proof of Theorem 3.3 is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -equivariant. We fix notation as in the proof

²The Lie groupoid structures on T^*G and H_τ^* that we consider in the above argument are not the standard ones. In fact, the ones we consider are obtained by modifying the standard ones appropriately using σ , as explained in Appendix A.

of Theorem 3.3, adding hats. We need to prove that the unique solution $\hat{X}_\tau \in \Gamma(\hat{H}_\tau)$ to

$$(3.10) \quad \frac{d\hat{\alpha}_\tau}{d\tau} + \iota_{\hat{X}_\tau} d\hat{\alpha}_\tau = \hat{\mu}_\tau \hat{\alpha}_\tau$$

is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -equivariant. Since the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on $\hat{G} \times I$ is given as in equation (3.9), this is equivalent to showing that

$$(3.11) \quad \hat{X}_\tau \underset{\varphi_{ab}}{\sim} \hat{X}_\tau \quad \text{for all } \tau \text{ and for all } (a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2.$$

We observe that $d\varphi_{ab}(\hat{H}_\tau) = \hat{H}_\tau$ for all τ and for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, by uniqueness of \hat{X}_τ , in order to prove (3.11), it suffices to show that $d\varphi_{ab}(\hat{X}_\tau)$ solves equation (3.10) for all τ and for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. To this end, let $f_{\tau,ab} \in C^\infty(\hat{G})$ be the nowhere vanishing function satisfying

$$(3.12) \quad \varphi_{ab}^* \hat{\alpha}_\tau = f_{\tau,ab} \hat{\alpha}_\tau.$$

Since \hat{G} is compact, we can apply Lemma 1.28 to (\hat{G}, \hat{H}_τ) for all τ , thus obtaining that

$$(3.13) \quad f_{\tau,00} = f_{\tau,01} \equiv 1 \quad f_{\tau,10} = f_{\tau,11} \equiv -1 \quad \text{for all } \tau.$$

In particular, $f_{\tau,ab}$ does not depend on τ for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. For this reason, we drop the notational dependence on τ . Equation (3.13) implies that $\varphi_{ab}^* \frac{d\alpha_\tau}{d\tau} = f_{ab} \frac{d\alpha_\tau}{d\tau}$ and $\varphi_{ab}^* d\alpha_\tau = f_{ab} d\alpha_\tau$ for all τ and for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, combining equations (3.10), (3.12) and (3.13), we have that

$$(3.14) \quad \iota_{d\varphi_{ab}(\hat{X}_\tau)}(d\hat{\alpha}_\tau) = (\varphi_{ab}^* \hat{\mu}_\tau) \hat{\alpha}_\tau - \frac{d\hat{\alpha}_\tau}{d\tau},$$

for all τ and for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Suppose that $\varphi_{ab}^* \hat{\mu}_\tau = \hat{\mu}_\tau$ for all τ and for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Then equation (3.14) implies the desired result, since both $d\varphi_{ab}(\hat{X}_\tau)$ and \hat{X}_τ are sections of H_τ for all τ .

It remains to show that $\varphi_{ab}^* \hat{\mu}_\tau = \hat{\mu}_\tau$ for all τ and for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. By equation (3.6), for all τ , there exists $\hat{\delta}_\tau \in C^\infty(\hat{M})$ such that $\hat{\mu}_\tau = \hat{s}^* \hat{\delta}_\tau = \hat{t}^* \hat{\delta}_\tau$. Moreover, $\hat{\delta}_\tau$ is constant along the orbits of \hat{G} for all τ . If γ denotes the action of the non-trivial element of \mathbb{Z}_2 on \hat{M} , then by equation (1.15)

we have that

$$(3.15) \quad \varphi_{ab}^* \hat{\mu}_\tau = \hat{t}^*(\gamma^a)^* \hat{\delta}_\tau = \hat{s}^*(\gamma^b)^* \hat{\delta}_\tau$$

for all τ and for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Using equations (1.14) and (1.15), we see that, for all $x \in \hat{M}$, $\gamma^c(x)$ lies in the same orbit of x for all $c \in \mathbb{Z}_2$. Hence, since $\hat{\delta}_\tau$ is constant along the orbits of \hat{G} , $(\gamma^c)^* \hat{\delta}_\tau = \hat{\delta}_\tau$ for all τ and for all $c \in \mathbb{Z}_2$. Therefore, equation (3.15) implies that $\varphi_{ab}^* \hat{\mu}_\tau = \hat{\mu}_\tau$ for all τ and for all $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, as desired. \square

4. Multiplicative Gray stability at the level of objects

This section has three aims: to define Jacobi bundles (see Definition 4.1), to recall that they appear on the base manifolds of contact groupoids (see Theorem 4.10), and to illustrate how Corollary 3.2 can be applied to their deformations (see Theorems 4.18 and 4.19). Moreover, we prove that Jacobi bundles that are induced by proper co-orientable contact groupoids ‘are Poisson’ (see Corollary 4.15). The main references are [16, 18, 19, 25, 35].

4.1. Jacobi bundles

Definition 4.1. Let $L \rightarrow M$ be a line bundle. A **Jacobi bracket on $L \rightarrow M$** is a local Lie bracket on $\Gamma(L)$, i.e., for all $u, v \in \Gamma(L)$, $\text{supp}(\{u, v\}) \subset \text{supp}(u) \cap \text{supp}(v)$. The triple $(M, L, \{\cdot, \cdot\})$ is a **Jacobi bundle**.

Example 4.2 (Contact manifolds). Let (N, H) be a contact manifold. A vector field $X \in \mathfrak{X}(N)$ is *Reeb* if $[X, \Gamma(H)] \subset \Gamma(H)$. By the Jacobi identity, the subspace of Reeb vector fields is a Lie subalgebra of $\mathfrak{X}(N)$ that we denote by $\mathfrak{X}_{\text{Reeb}}(N, H)$. Moreover, the generalized contact form α_{can} induces an isomorphism of vector spaces $\mathfrak{X}_{\text{Reeb}}(N, H) \cong \Gamma(L)$. Since the Lie bracket of vector fields is local, the Lie algebra structure on $\mathfrak{X}_{\text{Reeb}}(N, H)$ induces a Jacobi bracket $\{\cdot, \cdot\}_H$ on $L \rightarrow N$.

Example 4.3 (Poisson manifolds). Let (M, π) be a **Poisson manifold**, i.e., $\pi \in \mathfrak{X}^2(M)$ such that

$$(4.1) \quad \llbracket \pi, \pi \rrbracket = 0,$$

where $\llbracket \cdot, \cdot \rrbracket$ is the Schouten-Nijenhuis bracket. The condition (4.1) is equivalent to $\{f, g\}_\pi := \pi(df, dg)$ being a Lie bracket on $C^\infty(M)$. Since π is a bivector, $\{\cdot, \cdot\}_\pi$ is a Jacobi bracket on $M \times \mathbb{R} \rightarrow M$.

We say that a Jacobi bundle $(M, L, \{\cdot, \cdot\})$ is **trivializable** if $L \rightarrow M$ is trivializable. A trivialization $\psi : L \rightarrow M \times \mathbb{R}$ induces a Jacobi bracket $\{\cdot, \cdot\}$ on $M \times \mathbb{R} \rightarrow M$. In [26] it is shown that there exists a unique pair $(\Lambda, R) \in \mathfrak{X}^2(M) \times \mathfrak{X}(M)$ such that, for all $f, g \in C^\infty(M)$,

$$(4.2) \quad \{f, g\} = \Lambda(df, dg) + fRg - gRf.$$

If ψ' is another trivialization of L , we let $\Lambda' \in \mathfrak{X}^2(M)$, $R' \in \mathfrak{X}(M)$ be as in equation (4.2) for the induced Jacobi bracket on $M \times \mathbb{R} \rightarrow M$. We identify $\psi' \circ \psi^{-1}$ with a nowhere vanishing $a \in C^\infty(M)$. For any pair $(\Lambda, R) \in \mathfrak{X}^2(M) \times \mathfrak{X}(M)$, set

$$(4.3) \quad \Lambda^a := a\Lambda \quad \text{and} \quad R^a := aR + \Lambda^\sharp(da).$$

Then we have that

$$(4.4) \quad (\Lambda', R') = (\Lambda^{a^{-1}}, R^{a^{-1}}).$$

Definition 4.4. Let $(M, L, \{\cdot, \cdot\})$ be a trivializable Jacobi bundle. If $(\Lambda, R) \in \mathfrak{X}^2(M) \times \mathfrak{X}(M)$ satisfies equation (4.2) (for some trivialization ψ of L), then we call (Λ, R) a **Jacobi pair** (for $(M, L, \{\cdot, \cdot\})$). The triple (M, Λ, R) is a **Jacobi manifold**.

Remark 4.5. If (M, Λ, R) is a Jacobi manifold, then

$$(4.5) \quad \llbracket \Lambda, \Lambda \rrbracket = 2R \wedge \Lambda \quad \text{and} \quad \llbracket \Lambda, R \rrbracket = 0.$$

Conversely, any pair $(\Lambda, R) \in \mathfrak{X}^2(M) \times \mathfrak{X}(M)$ satisfying (4.5) yields a Jacobi bundle $(M, M \times \mathbb{R}, \{\cdot, \cdot\})$, where $\{\cdot, \cdot\}$ is defined by the right hand side of (4.2).

Example 4.6 (Co-orientable contact manifolds). Let (N, H) be a co-orientable contact manifold. A contact form α for (N, H) induces a Jacobi pair for the Jacobi bundle $(N, L = TN/H, \{\cdot, \cdot\}_H)$ of Example 4.2 as follows. There exists a unique bivector field³ $\Lambda^\alpha \in \mathfrak{X}^2(N)$ such that $\Lambda^\alpha(\alpha, -) = 0$ and

³The notation used in this Example should not be confused with the notation of equation (4.3). Here we merely wish to indicate that the bivector field and the Reeb vector field depend on α .

$\Lambda^\alpha|_{H^* \times H^*} = (d\alpha|_{H \times H})^{-1}$. If R^α is the Reeb vector field of α , then $(\Lambda^\alpha, R^\alpha)$ is a Jacobi pair for $(N, L, \{\cdot, \cdot\}_H)$.

If α' is another contact form for (N, H) , then there exists a nowhere vanishing $a \in C^\infty(M)$ such that $\alpha' = a\alpha$. Then we have that

$$(\Lambda^{\alpha'}, R^{\alpha'}) = ((\Lambda^\alpha)^{a^{-1}}, (R^\alpha)^{a^{-1}}),$$

(see equations (4.3) and (4.4)).

Example 4.7 (Poisson manifolds as Jacobi manifolds). Every Poisson manifold (M, π) is a Jacobi manifold with $R \equiv 0$. For this reason, we denote a Poisson manifold viewed as a Jacobi manifold simply by (M, π) .

Definition 4.8. Let $(M_j, L_j, \{\cdot, \cdot\}_j)$ be a Jacobi bundle for $j = 1, 2$. A **Jacobi map between $(M_1, L_1, \{\cdot, \cdot\}_1)$ and $(M_2, L_2, \{\cdot, \cdot\}_2)$** is a pair (ϕ, B) , where $\phi : M_1 \rightarrow M_2$ is smooth and $B : \phi^*L_2 \rightarrow L_1$ is an isomorphism of line bundles called the **bundle component** such that, for all $u, v \in \Gamma(L_2)$,

$$B \circ \phi^* \{u, v\}_2 = \{B \circ \phi^*u, B \circ \phi^*v\}_1.$$

An **isomorphism** between Jacobi bundles is a Jacobi map (ϕ, B) with the property that ϕ is a diffeomorphism.

Remark 4.9. A Jacobi map between the Jacobi manifolds (M_1, Λ_1, R_1) and (M_2, Λ_2, R_2) is given by a smooth $\phi : M_1 \rightarrow M_2$ and a nowhere vanishing $a \in C^\infty(M_1)$ (playing the role of the bundle component), such that

$$\phi_*(\Lambda_1^a) = \Lambda_2, \quad \text{and} \quad \phi_*(R_1^a) = R_2,$$

(see equation (4.3)). As in Definition 4.8, we denote this map by (ϕ, a) . In particular, if (ϕ, a) is a Jacobi isomorphism between Poisson manifolds (M_1, π_1) , (M_2, π_2) , then

$$(4.6) \quad \phi_*(a\pi_1) = \pi_2 \quad \text{and} \quad \pi_1^\sharp(da) = 0.$$

4.2. From contact groupoids to Jacobi bundles

In this section we discuss Jacobi bundles that are *integrable* by contact groupoids with particular emphasis on the proper and co-orientable case. We start by recalling one of the main results of [16].

Theorem 4.10 (Theorem 1 in [16]). *Let (G, H) be a contact groupoid over M and let $L := (TG/H)|_M$. There exists a unique Jacobi bracket $\{\cdot, \cdot\}$ on $L \rightarrow M$ such that $t : G \rightarrow M$ is a Jacobi map with bundle component $\mathbf{r} : t^*L \rightarrow TG/H$ (see (1.6)).*

We say that a contact groupoid (G, H) **induces** the Jacobi bracket $\{\cdot, \cdot\}$ on $L \rightarrow M$ of Theorem 4.10, that it **integrates** $(M, L, \{\cdot, \cdot\})$, and that $(M, L, \{\cdot, \cdot\})$ is **integrable** by (G, H) . We use the same terminology for co-oriented contact groupoids integrating Jacobi manifolds (see Remark 4.11 below).

Remark 4.11. When (G, α, F) is a co-oriented contact groupoid, the induced Jacobi pair (Λ, R) on $(M, M \times \mathbb{R}, \{\cdot, \cdot\})$ is given by $\Lambda = t_*(\Lambda^\alpha)$, $R = t_*(R^\alpha)$, where Λ^α, R^α are as in Example 4.6. Hence, $(t, 1)$ is a Jacobi map between $(G, \Lambda^\alpha, R^\alpha)$ and (M, Λ, R) – see Remark 4.9.

Remark 4.12. Let $\Phi : (G_1, H_1) \rightarrow (G_2, H_2)$ be an isomorphism of contact groupoids covering $\phi : M_1 \rightarrow M_2$. Then $(\phi, B) : (M_1, L_1, \{\cdot, \cdot\}_1) \rightarrow (M_2, L_2, \{\cdot, \cdot\}_2)$ is a Jacobi isomorphism, where B is the inverse of

$$TG_1/H_1|_{M_1} \rightarrow \phi^*(TG_2/H_2|_{M_2}), \quad [X] \mapsto (g, [d_g\Phi(X)]).$$

The following result provides a useful characterization of co-oriented contact groupoids integrating Poisson manifolds.

Lemma 4.13. *Let (G, α, F) be a co-oriented contact groupoid integrating the Jacobi manifold (M, Λ, R) . If $dF \equiv 0$ then $R \equiv 0$, i.e., Λ is a Poisson bivector. Conversely, if $R \equiv 0$ then $dF \equiv 0$ on $\ker ds$.*

Proof. Let $R^{\alpha, \mathbf{L}} = FR^\alpha + \Lambda^{\alpha, \sharp}(dF)$ be the left-invariant vector field induced by R^α (see Lemma 1.23 and equation (1.12)). Fix $x \in M$. Then $F(x) = 1$ and $R_x^{\alpha, \mathbf{L}} = R_x^\alpha - dt(R_x^\alpha)$ – see equation (1.10). Hence, $\Lambda^{\alpha, \sharp}(dF)_x = -dt(R_x^\alpha)$ is tangent to M . By Remark 4.11, it follows that

$$(4.7) \quad \Lambda^{\alpha, \sharp}(dF)_x = -R_x.$$

By equation (4.7), if $dF \equiv 0$ then $R \equiv 0$. Conversely, suppose that $R \equiv 0$. Then, by equation (4.7), $\Lambda^{\alpha, \sharp}(dF)|_M \equiv 0$. This means that $dF \equiv 0$ on $(\ker \alpha)|_M$. On the other hand, we have that $0 = d\alpha(\Lambda^{\alpha, \sharp}(dF), R^\alpha) = -dF(R^\alpha)$. Since $TG|_M = \mathbb{R}\langle R^\alpha|_M \rangle \oplus \ker \alpha|_M$ (see equation (1.2)), then dF is zero on $TG|_M$ and, in particular, on $\ker ds|_M$. Since $F : G \rightarrow \mathbb{R}^*$ is a Lie groupoid homomorphism, it follows that $dF \equiv 0$ on $\ker ds$. \square

To conclude this section, we prove the analogs of Proposition 1.19 and Corollary 1.20 for Jacobi bundles that are integrable by proper co-orientable contact groupoids. First, we recall the following facts.

- (a) The base of a Jacobi bundle comes endowed with a *singular* foliation (see [3, 26]). In the case of a Poisson manifold, this coincides with the symplectic foliation. When $(M, L, \{\cdot, \cdot\})$ is integrable by (G, H) , the leaves of the foliation coincide with the connected components of the orbits of G .
- (b) If (M, π) is a Poisson manifold, a function $f \in C^\infty(M)$ is a **Casimir** of (M, π) if $\pi^\sharp(df) \equiv 0$ (cf. the second condition in equation (4.6)). If (M, π) is integrable by a co-oriented contact groupoid, Casimirs of (M, π) are precisely smooth functions that are constant along connected components of the orbits of the groupoid.

Proposition 4.14. *Let (M, Λ, R) be a Jacobi manifold integrable by a co-oriented contact groupoid (G, α, F) . Suppose that the Reeb cocycle r of (G, α, F) is a coboundary. Then*

1) *all leaves of the foliation of (M, Λ, R) are even dimensional.*

Moreover, for all $\kappa \in C^\infty(M)$ satisfying equation (1.9),

- 2) $\Lambda^{e^{-\kappa}} = e^{-\kappa}\Lambda$ *is a Poisson bivector,*
- 3) *if $\sigma = \text{sgn}(F)$, then $(G, e^{t^*\kappa}\alpha, \sigma)$ integrates the Poisson manifold $(M, e^{-\kappa}\Lambda)$, and*
- 4) *if $R \equiv 0$ then κ is a Casimir of the Poisson manifold (M, Λ) .*

Proof. Set $H := \ker \alpha$. By Proposition 1.19, for any $\kappa \in C^\infty(M)$ satisfying equation (1.9), $(e^{t^*\kappa}\alpha, \sigma)$ is a multiplicative contact form for (G, H) . Hence, by Lemma 4.13, the Jacobi bracket on M induced by $(G, e^{t^*\kappa}\alpha, \sigma)$ is, in fact, a Poisson structure. By Remarks 4.9 and 4.11, the corresponding Poisson bivector is given by

$$t_*(\Lambda^{e^{t^*\kappa}\alpha}) = t_*(e^{-t^*\kappa}\Lambda^\alpha) = e^{-\kappa}t_*(\Lambda^\alpha) = e^{-\kappa}\Lambda.$$

This proves parts 2 and 3. Moreover, by (a), the orbits of G are even dimensional and so are the leaves of (M, Λ, R) . This shows part 1. For part 4, if $R \equiv 0$, then by Lemma 4.13, $F = \sigma e^r = \sigma e^{s^*\kappa - t^*\kappa}$ is constant along the connected components of the source fibers. Hence, since σ is locally constant, κ

is constant on the connected components of the orbits $S_x = t(s^{-1}(x)) \subset M$. Hence, by (b) above, κ is a Casimir of Λ . \square

Since the differentiable cohomology of proper Lie groupoids vanishes in all positive degrees (see [8, Proposition 1]), Proposition 4.14 immediately implies the following result.

Corollary 4.15. *Let $(M, L, \{\cdot, \cdot\})$ be a trivializable Jacobi bundle that is integrable by a proper co-orientable contact groupoid (G, H) . Then there exists a Poisson bivector $\pi \in \mathfrak{X}^2(M)$ such that $(\pi, 0)$ is a Jacobi pair for $(M, L, \{\cdot, \cdot\})$.*

4.3. Stability results for integrable deformations of Jacobi bundles and Poisson structures

By Theorem 4.10, given a deformation $(\tilde{G}, \tilde{H}) = \{(G_\tau, H_\tau)\}$ of a contact groupoid (G, H) , (G_τ, H_τ) induces a Jacobi bundle $(M, L_\tau, \{\cdot, \cdot\}_\tau)$ for each τ . This motivates introducing the following notion.

Definition 4.16. An integrable deformation of a Jacobi bundle $(M, L, \{\cdot, \cdot\})$ is

- a contact groupoid (G, H) integrating $(M, L, \{\cdot, \cdot\})$, and
- a deformation $(\tilde{G}, \tilde{H}) = \{(G_\tau, H_\tau)\}$ of (G, H) .

We use the notation $\{(M, L_\tau, \{\cdot, \cdot\}_\tau)\}$, where, for each τ , (G_τ, H_τ) induces the Jacobi bundle $(M, L_\tau, \{\cdot, \cdot\}_\tau)$.

Let (G, H) be a co-orientable contact groupoid integrating a Jacobi bundle $(M, L, \{\cdot, \cdot\})$ and let $(\tilde{G}, \tilde{H}) = \{(G_\tau, H_\tau)\}$ be a *proper* deformation of (G, H) . By Corollary 2.24 and Lemma 4.13, there exists a ‘smooth 1-parameter family’ of Poisson bivectors $\{\pi_\tau\}$ on M such that, for all τ , $(\pi_\tau, 0)$ is a Jacobi pair for $(M, L_\tau, \{\cdot, \cdot\}_\tau)$. This motivates introducing the following notion.

Definition 4.17. A contact integrable deformation of a Poisson manifold (M, π) is

- a co-oriented contact groupoid (G, α, F) integrating (M, π) , and
- a deformation $(\tilde{G}, \tilde{\alpha}, \tilde{F}) = \{(G_\tau, \alpha_\tau, F_\tau)\}$ of (G, α, F) such that, for all τ , $(G_\tau, \alpha_\tau, F_\tau)$ induces a Poisson bivector π_τ on M and $\pi = \pi_0$.

We use the notation $\{(M, \pi_\tau)\}$.

In Definitions 4.16 – 4.17, we say that (a contact) an integrable deformation of a Jacobi bundle (respectively Poisson manifold) is **compact** if the underlying contact groupoid being deformed is compact. In view of Remark 4.12, Corollary 3.2 implies the following stability result for compact integrable deformations of Jacobi bundles.

Theorem 4.18 (Jacobi version). *Let M be a connected manifold. Given any compact integrable deformation $\{(M, L_\tau, \{\cdot, \cdot\}_\tau)\}$ of a Jacobi bundle $(M, L, \{\cdot, \cdot\})$, there exists a smooth 1-parameter family of Jacobi isomorphisms $(\phi_\tau, B_\tau) : (M, L_\tau, \{\cdot, \cdot\}_\tau) \rightarrow (M, L, \{\cdot, \cdot\})$ starting at the identity.*

Corollary 3.2 also provides the following characterization of compact contact integrable deformations of Poisson manifolds.

Theorem 4.19 (Poisson version). *Given a compact contact integrable deformation $\{(M, \pi_\tau)\}$ of a connected Poisson manifold (M, π) , there exist a diffeotopy $\{\phi_\tau\} \subset \text{Diff}(M)$ and a smooth 1-parameter family $\{a_\tau\}$ of positive Casimirs of (M, π) with $a_0 \equiv 1$, such that*

$$(4.8) \quad (\phi_\tau)_* \pi_\tau = a_\tau \pi \quad \text{for all } \tau.$$

Proof. Let $(\tilde{G}, \tilde{\alpha}, \tilde{F}) = \{(G_\tau, \alpha_\tau, F_\tau)\}$ be a deformation of co-orientable groupoids inducing $\{(M, \pi_\tau)\}$ such that $G = G_0$ is compact. Set $\tilde{H} := \ker \tilde{\alpha}$ and $H := \ker \alpha_0$. By Corollary 3.2, there exists an isomorphism $\tilde{\Phi} = \{\Phi_\tau\}$ between (\tilde{G}, \tilde{H}) and the constant deformation of (G, H) . Hence, by Remark 4.12, there exist a diffeotopy $\{\phi_\tau^{-1}\}$ of M and a smooth 1-parameter family $\{a_\tau\}$ of positive functions on M with $a_0 \equiv 1$, such that (ϕ_τ^{-1}, a_τ) is a Jacobi isomorphism between (M, π) and (M, π_τ) for all τ . Therefore, by equation (4.6),

$$(\phi_\tau^{-1})_*(a_\tau \pi) = \pi_\tau \quad \text{and} \quad \pi^\#(da_\tau) = 0.$$

The result follows by applying $(\phi_\tau)_*$ to both sides of the first equation and by observing that the second is precisely the condition that a_τ be a Casimir of π . \square

5. Three families of examples

In this section we give three families of examples of compact contact groupoids and of the Jacobi bundles that they induce. In each case, we

observe how Theorems 4.18 and 4.19 can be applied (see Remarks 5.3, 5.4 and 5.6).

5.1. First jet bundles and integral projective lattices

Let $p : L \rightarrow M$ be a line bundle. The **first jet bundle of L** is the vector bundle $\text{pr} : J^1 L \rightarrow M$, where, for $x \in M$,

$$J^1 L|_x := \{j_x^1 u \mid u \in \Gamma_{\text{loc}}(L)\}.$$

There is a canonical contact structure H_{can} on $J^1 L$ that is the kernel of the **Cartan contact form** $\alpha_{\text{can}} \in \Omega^1(J^1 L; \text{pr}^* L)$. The latter is given by

$$\alpha_{\text{can}, j_x^1 u} := d_{j_x^1 u}(\text{ev} - u \circ \text{pr}) : T_{j_x^1 u} J^1 L \rightarrow L_x,$$

where $\text{ev} : J^1 L \rightarrow L$ is the map $j_x^1 u \mapsto u(x)$, and $\ker dp$ is canonically identified with $p^* L$. In fact, $(J^1 L, H_{\text{can}})$ is a contact groupoid. This is because any vector bundle $E \rightarrow M$ is a Lie groupoid and, in this case, a multiplicative distribution is a vector subbundle of $TE \rightarrow TM$ with base TM . Since $J^1 L$ is a vector bundle, the contact groupoid $(J^1 L, H_{\text{can}})$ integrates $(M, L, 0)$.

While $(J^1 L, H_{\text{can}})$ is never compact, in some special cases there are *quotients* that are compact (if M is compact), and still integrate $(M, L, 0)$. Before defining what we quotient $J^1 L$ by, we recall that the Cartan contact form α_{can} enjoys the following property. A section $\vartheta \in \Gamma_{\text{loc}}(J^1 L)$ satisfies $\vartheta^* \alpha_{\text{can}} = 0$ if and only if it is **holonomic**, i.e., there exists $u \in \Gamma_{\text{loc}}(L)$ such that $\vartheta = j^1 u$.

Definition 5.1 (Definition 4.10 in [31]). An **integral projective lattice on $L \rightarrow M$** is a full rank lattice $\Sigma \subset J^1 L$ such that any local section of $\Sigma \rightarrow M$ is holonomic.

Remark 5.2. Integral projective lattices arise naturally when considering *integral projective structures* (see [31, Definition 4.12]), which are special cases of real projective structures (see, e.g., [22] and references therein). There is a 1 – 1 correspondence between integral projective lattices on line bundles and integral projective structures (see [31, Proposition 4.14]). Moreover, proper contact groupoids and integral projective lattices/structures are related in a way that is analogous to the relation between proper symplectic groupoids and integral affine structures (see [13, Sections 4.2 and 4.3] and [14, Theorem 1.0.1]). We will explore this relation in a separate paper.

If M is compact, an integral projective lattice Σ on $L \rightarrow M$ yields a compact integration of $(M, L, 0)$ as follows. The fiberwise action of Σ on J^1L by translations is free and proper. Its orbit space is a bundle of tori $\text{pr} : J^1L/\Sigma \rightarrow M$ and the quotient map $Q : J^1L \rightarrow J^1L/\Sigma$ is a covering map satisfying $\text{pr} \circ Q = \text{pr}$. Moreover, since (local) sections of $\Sigma \rightarrow L$ are holonomic, the action is by contactomorphisms of (J^1L, H_{can}) , as the action preserves the Cartan contact form α_{can} . Hence, there is a contact structure \bar{H}_{can} on J^1L/Σ satisfying $dQ(H_{\text{can}}) = \bar{H}_{\text{can}}$. Since the fiberwise action of Σ is by Lie groupoid isomorphisms, $(J^1L/\Sigma, \bar{H}_{\text{can}})$ is a contact groupoid. Moreover, since $J^1L/\Sigma \rightrightarrows M$ is a bundle of tori, J^1L/Σ is proper and $(J^1L/\Sigma, \bar{H}_{\text{can}})$ integrates $(M, L, 0)$. Finally, compactness of M implies that J^1L/Σ is compact.

Remark 5.3. If M is compact, the existence of an integral projective lattice on $L \rightarrow M$ allows us to use Theorem 4.18: any compact integrable deformation of $(M, L, 0)$ is trivial. While such deformations are probably very restrictive (as they come from deformations of contact groupoids), the above is a partial stability result for deformations of the *zero* Jacobi bracket in the presence of an integral projective lattice.

5.2. The (oriented) projectivization of the cotangent bundle to a compact Lie group

The **projectivization** of a cotangent bundle T^*M is $\mathbb{P}(T^*M) := (T^*M \setminus \{0\})/\mathbb{R}^*$, where $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ acts by fiberwise scalar multiplication. There is a well-known contact structure⁴ H_{can} on $\mathbb{P}(T^*M)$ that comes from identifying $\mathbb{P}(T^*M)$ with the manifold of contact elements of M (see [1, Appendix 4, Section D]). Explicitly, if $p : \mathbb{P}(T^*M) \rightarrow M$ denotes projection, then, for all $\eta \in T^*M \setminus \{0\}$, $H_{\text{can}, [\eta]} = (dp)^{-1}(\ker \eta)$. Similarly, it is possible to define a contact structure H_{can}^+ on the **oriented projectivization** of T^*M , $\mathbb{S}(T^*M) := (T^*M \setminus \{0\})/\mathbb{R}^+$. Moreover, $(\mathbb{S}(T^*M), H_{\text{can}}^+)$ is co-orientable. One way to construct a contact form for $(\mathbb{S}(T^*M), H_{\text{can}}^+)$ is by choosing a Riemannian metric \mathbf{g} on M . This identifies $\mathbb{S}(T^*M)$ with the unit sphere bundle $U(T^*M)$ with respect to \mathbf{g} . Under this identification, $H_{\text{can}}^+ = \ker \lambda_{\text{can}}|_{U(T^*M)}$, where λ_{can} is the Liouville 1-form.

⁴By an abuse of notation we denote the canonical contact structures on J^1L and on $\mathbb{P}(T^*M)$ with the same symbol. Seeing as it should be clear from the context what the ambient manifold is, we trust that this should not cause confusion.

There is a well-known multiplicative analog of the above construction (see, e.g., [35, Example 2.3], [18, Example 7.11] or [3, Example 3.8]). Let G be a Lie group with Lie algebra \mathfrak{g} . The cotangent bundle T^*G can be seen as a Lie groupoid over \mathfrak{g}^* : once T^*G is trivialized using right translations, this is the action Lie groupoid of the coadjoint action $\mathfrak{g}^* \curvearrowright G$ (see [12, Example 1.15] for details on action Lie groupoids). The projectivization $\mathbb{P}(T^*G)$ (respectively $\mathbb{S}(T^*G)$) is a Lie groupoid over the projectivization $\mathbb{P}(\mathfrak{g}^*)$ of \mathfrak{g}^* (respectively the oriented projectivization $\mathbb{S}(\mathfrak{g}^*)$ of \mathfrak{g}^*), and, using the above identification, it is the action Lie groupoid associated to the induced action $\mathbb{P}(\mathfrak{g}^*) \curvearrowright G$ (respectively $\mathbb{S}(\mathfrak{g}^*) \curvearrowright G$). Moreover, the contact structure H_{can} (respectively H_{can}^+) is multiplicative, so that $(\mathbb{P}(T^*G), H_{\text{can}})$ (respectively $(\mathbb{S}(T^*G), H_{\text{can}}^+)$) is a contact groupoid. These Lie groupoids are compact exactly if G is compact. In this case, it is possible to choose a bi-invariant Riemannian metric on \mathfrak{g}^* . Using this metric to identify $\mathbb{S}(T^*G)$ with $U(T^*G)$, we have that $(\lambda_{\text{can}}|_{U(T^*G)}, 1)$ is a multiplicative contact form for $(U(T^*G), H_{\text{can}}^+)$ (see [35, Example 2.3]).

The contact groupoids $(\mathbb{P}(T^*G), H_{\text{can}})$ and $(\mathbb{S}(T^*G), H_{\text{can}}^+)$ integrate the following families of Jacobi bundles (see [3, Example 3.8]). Let $\pi_{\text{lin}} \in \mathfrak{X}^2(\mathfrak{g}^*)$ denote the Kirillov-Kostant-Souriau Poisson bivector on \mathfrak{g}^* , i.e., for $\xi \in \mathfrak{g}^*$ and $f, g \in C^\infty(\mathfrak{g}^*)$,

$$(5.1) \quad (\pi_{\text{lin}})_\xi(d_\xi f, d_\xi g) := \langle \xi, [d_\xi f, d_\xi g] \rangle,$$

where $d_\xi f, d_\xi g$ are identified with elements of $(\mathfrak{g}^*)^* \cong \mathfrak{g}$, $[\cdot, \cdot]$ is the Lie bracket on \mathfrak{g} , and $\langle \cdot, \cdot \rangle$ is the standard pairing between \mathfrak{g}^* and \mathfrak{g} . By equation (5.1), the subspace of homogeneous functions of degree one on $\mathfrak{g}^* \setminus \{0\}$ is a Lie subalgebra of $C^\infty(\mathfrak{g}^*)$. Identifying such functions with sections of $O(1) \rightarrow \mathbb{P}(\mathfrak{g}^*)$, i.e., the dual of the tautological line bundle, we obtain a Jacobi bundle $(\mathbb{P}(\mathfrak{g}^*), O(1), \{\cdot, \cdot\})$ (for further details, see [3, Example 2.10 and Appendix B], and [31, Example 2.10]). Analogously, if $q : \mathbb{S}(\mathfrak{g}^*) \rightarrow \mathbb{P}(\mathfrak{g}^*)$ is the standard double cover, we obtain a Jacobi bundle $(\mathbb{S}(\mathfrak{g}^*), q^*O(1), \{\cdot, \cdot\}_{\mathbb{S}})$. A choice of inner product on \mathfrak{g}^* identifies $\mathbb{S}(\mathfrak{g}^*)$ with the unit sphere $U(\mathfrak{g}^*)$ and determines a Jacobi pair for $(\mathbb{S}(\mathfrak{g}^*), q^*O(1), \{\cdot, \cdot\}_{\mathbb{S}})$. If, in addition, G is compact, the above inner product can be chosen to be bi-invariant and the resulting Jacobi pair is of the form $(\pi_{U(\mathfrak{g}^*)}, 0)$ for some Poisson bivector $\pi_{U(\mathfrak{g}^*)}$ on $U(\mathfrak{g}^*) \cong \mathbb{S}(\mathfrak{g}^*)$.

Remark 5.4. If G is a compact Lie group, we can use Theorem 4.18: any compact integrable deformation of $(\mathbb{P}(\mathfrak{g}^*), O(1), \{\cdot, \cdot\})$ (respectively $(\mathbb{S}(\mathfrak{g}^*), q^*O(1), \{\cdot, \cdot\}_{\mathbb{S}})$) is trivial. In fact, upon choosing a bi-invariant inner

product on \mathfrak{g}^* , we can use Theorem 4.19: any compact contact integrable deformation $\{(U(\mathfrak{g}^*), \pi_\tau)\}$ of $(U(\mathfrak{g}^*), \pi_{U(\mathfrak{g}^*)})$ satisfies equation (4.8). This last result should be compared with [29, Part (a) of Theorem 1]. Theorem 1 in *loc. cit.* is much stronger as it provides a complete local description of the moduli space of Poisson structures near $(U(\mathfrak{g}^*), \pi_{U(\mathfrak{g}^*)})$, but requires deeper results (Nash-Moser techniques), and semisimplicity of \mathfrak{g} .

5.3. Prequantization of compact symplectic groupoids

A **prequantization** of a symplectic manifold (S, ω) is a principal S^1 -bundle $p : N \rightarrow S$ such that there exists a principal connection $\alpha \in \Omega^1(N)$ satisfying $d\alpha = p^*\omega$. A necessary and sufficient condition for (S, ω) to admit a prequantization is that the cohomology class of ω be *integral*, i.e., it lies in the image of the homomorphism $H^2(S; \mathbb{Z}) \rightarrow H^2(S; \mathbb{R})$ (see, e.g., [21, Theorems 7.2.4 and 7.2.5]). In this case, $H := \ker \alpha$ is a contact structure on N . We abuse terminology and refer to (N, H) as a prequantization of (S, ω) .

In order to discuss the multiplicative analog of the above construction, we recall that a **symplectic groupoid** is a Lie groupoid $S \rightrightarrows M$ endowed with a symplectic form $\omega \in \Omega^2(S)$ that is multiplicative with values in the trivial representation $M \times \mathbb{R}$ (see Definition 1.10). In analogy with Theorem 4.10, a symplectic groupoids induces a unique Poisson structure on its base so that the target map is Poisson (see [7, Theorem 1.1]). Following [18, Definition 5.1], we say that a **prequantization** of a symplectic groupoid (S, ω) over M is a Lie groupoid extension of S by the trivial S^1 -bundle over M ,

$$1 \rightarrow M \times S^1 \rightarrow G \xrightarrow{p} S \rightarrow 1,$$

such that $p : G \rightarrow S$ is a prequantization of (S, ω) with the property that the connection 1-form $\alpha \in \Omega^1(G)$ is multiplicative with values in the trivial representation $M \times \mathbb{R}$. Setting $H := \ker \alpha$, we have that (G, H) is a co-orientable contact groupoid (see [34, Theorem 3.1 and Lemma 3.2] and the remark after [18, Definition 5.1]). Moreover, $(\alpha, 1)$ is a contact form for (G, H) . As above, we refer to (G, H) as a prequantization of (S, ω) .

Remark 5.5. In general, given a symplectic groupoid (S, ω) , integrality of ω is only a necessary condition for the existence of a prequantization (see [34, Theorem 3.1] for necessary and sufficient conditions). However, if S is Hausdorff and source simply connected, then integrality of ω is sufficient (see [18, Theorem 3]).

Let (S, ω) be a symplectic groupoid integrating a Poisson manifold (M, π) and let $(G, H = \ker \alpha)$ be a prequantization of (S, ω) . Then $(G, \alpha, 1)$ also integrates the Poisson manifold (M, π) . Moreover, G is compact if and only if S is compact.

Remark 5.6. If (S, ω) is a prequantizable compact symplectic groupoid integrating a Poisson manifold (M, π) , we can apply Theorem 4.19: any compact contact integrable deformation $\{(M, \pi_\tau)\}$ of (M, π) satisfies equation (4.8). In particular, by Remark 5.5, this result applies to Poisson manifolds admitting a compact s-simply connected symplectic integration with integral symplectic form. These are examples of *Poisson manifolds of strong compact type* (see [13, 14]).

Appendix A. σ -Multiplicative forms

Let $\sigma : G \rightarrow \{\pm 1\}$ be a Lie groupoid homomorphism and let $E \rightrightarrows V$ be a VB-groupoid over $G \rightrightarrows M$ (see [27] for the definition of VB-groupoids). The Lie groupoid homomorphism σ can be used to ‘twist’ the VB-groupoid structure of $E \rightrightarrows V$ over $G \rightrightarrows M$ as follows. For $e \in E_g$ and $(e_1, e_2) \in E_g^\sigma \times_{V_x} E_h^\sigma$, the σ -twisted structure maps are

$$(A.1) \quad \begin{aligned} s_\sigma(e) &:= \sigma(g)s(e), & t_\sigma(e) &:= t(e), & u_\sigma &:= u, \\ m_\sigma(e_1, e_2) &:= m(e_1, \sigma(g)e_2), & i_\sigma(e) &:= \sigma(g)i(e). \end{aligned}$$

Since $E \rightrightarrows V$ is a VB-groupoid over $G \rightrightarrows M$ and since $\sigma : G \rightarrow \{\pm 1\}$ is a Lie groupoid homomorphism, the following result holds at once.

Lemma A.1. *The structure maps of equation (A.1) define a VB-groupoid $E^\sigma \rightrightarrows V$ over $G \rightrightarrows M$. Moreover, if $\Phi : E_1 \rightarrow E_2$ is a fiberwise linear groupoid homomorphism covering the identity on $G \rightrightarrows M$, then $\Phi : E_1^\sigma \rightarrow E_2^\sigma$ is also a fiberwise linear groupoid homomorphism covering the identity.*

Following [4, 28], we use Lemma A.1 to express σ -multiplicativity of a form in terms of Lie groupoid homomorphisms replacing the usual cotangent groupoid by $(T^*G)^\sigma \rightrightarrows A_G^*$ (see [27] for the definition of the cotangent groupoid T^*G). The following result holds by the arguments in [4, Lemma 3.6] with the obvious adaptations.

Lemma A.2. *Let $\sigma : G \rightarrow \{\pm 1\}$ be a groupoid homomorphism, then*

- 1) *a 1-form $\beta \in \Omega^1(G)$ is σ -multiplicative if and only if β defines a Lie groupoid homomorphism $G \rightarrow (T^*G)^\sigma$, and*

- 2) a 2-form $\omega \in \Omega^2(G)$ is σ -multiplicative if and only if ω^\flat defines a Lie groupoid homomorphism

$$\begin{array}{ccc} TG & \xrightarrow{\omega^\flat} & (T^*G)^\sigma \\ \Downarrow & & \Downarrow \\ TM & \longrightarrow & A_G^*. \end{array}$$

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SCHOOL OF MATHEMATICS, JILIN UNIVERSITY
CHANGCHUN CITY, JILIN PROVINCE, 130012, CHINA
E-mail address: `cangulo@jlu.edu.cn`

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
DEPARTAMENTO DE MATEMÁTICA APLICADA (GMA)
UNIVERSIDADE FEDERAL FLUMINENSE, CAMPUS GRAGOATÁ
NITERÓI, RJ, 24210–201, BRAZIL
E-mail address: `mariasalazar@id.uff.br`

ESCUELA DE MATEMÁTICAS
UNIVERSIDAD NACIONAL DE COLOMBIA SEDE MEDELLÍN
EDIFICIO 43, MEDELLÍN, COLOMBIA
AND INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
DEPARTAMENTO DE MATEMÁTICA APLICADA (GMA)
UNIVERSIDADE FEDERAL FLUMINENSE, CAMPUS GRAGOATÁ
NITERÓI, RJ, 24210–201, BRAZIL
E-mail address: `dsepe@unal.edu.co`

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