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Tight neighborhoods of contact submanifolds

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We prove that any small enough neighborhood of a closed contact submanifold is always tight provided its normal bundle has a nowhere vanishing section. The non-existence of $\mathcal{C}^0$–small positive loops of contactomorphisms in general overtwisted manifolds is shown as a corollary.

1. Introduction

A contact manifold (M, ξ) is an $(2n+1)$–dimensional manifold equipped with a maximally non–integrable codimension 1 distribution $\xi \subset TM$. If we assume that ξ is coorientable, as will be the case in the article, the hyperplane distribution can be written as the kernel of a global 1–form α, $\xi = \ker(\alpha)$, and the maximal non–integrable condition reads as $\alpha \wedge (d\alpha)^n \neq 0$. These conditions imply that $(\xi, d\alpha)$ is a symplectic vector bundle over M. However, in general a hyperplane distribution ξ with a symplectic structure ω on its fibers does not yield a contact structure. The data (ξ, ω) is called almost contact structure or formal contact structure. We will employ the second alternative throughout the paper as it is closer to other terminology related to the h–principle.

Let $\mathfrak{Cont}(M)$ and $\mathfrak{FCont}(M)$ denote the set of contact and formal contact structures, respectively. Gromov proved that if M is open the natural inclusion is a homotopy equivalence [13]. This equivalence does not hold for closed manifolds. However, in dimension 3 Eliashberg introduced a subclass $\mathfrak{Cont}_{OT}(M)$ of $\mathfrak{Cont}(M)$, the so–called overtwisted contact structures, and proved that any formal contact homotopy class contains a unique, up to isotopy, overtwisted contact structure. Recently, this result has been extended to arbitrary dimension in [2] so the notion of overtwisted contact structure has been settled in general.

Prior to [2], different proposals for the definition of the overtwisting phenomenum appeared in the literature. The plastikstufe, introduced in [19], resembled the overtwisted disk in the sense that it provides an obstruction to symplectic fillability. The presence of a plastikstufe has been shown to be equivalent to the contact structure being overtwisted (check [3, Theorem 1.1] and [16] for a list of disguises of an overtwisted structure). One of the corollaries obtained in [3] is a stability property for overtwisted structures: if $(M, \ker\alpha)$ is overtwisted then $(M \times \mathbb{D}^2(R), \ker(\alpha + r^2 d\theta))$ is also overtwisted provided $R > 0$ is large enough, where $\mathbb{D}^2(R)$ denotes the open 2–disk of radius R and $r^2 d\theta$ denotes the standard radial Liouville form in $\mathbb{R}^2$. It is conjectured that tight structures (by definition, tight means not overtwisted) are also preserved by stabilization, i.e. $(M, \ker\alpha)$ tight implies $(M \times \mathbb{R}^2, \ker(\alpha + r^2 d\theta)$ is tight. In fact, our article can be interpreted as a first step towards the conjecture.

1.1. Statements of the results

This paper explores the other end of the previous discussion, can small neighborhoods of contact submanifolds be overtwisted? We provide a negative answer to the question in several instances. The main result presented in the article is the following:

Theorem 1. *Let $(M, \ker\alpha)$ be a closed contact manifold. Then there exists $\varepsilon > 0$ such that $(M \times \mathbb{D}^2(\varepsilon), \ker(\alpha + r^2\, d\theta))$ is tight.*

This theorem was previously obtained by Casals, Presas and Sandon in the case of overtwisted 3–manifolds [5] and later by Gironella [11, Corollary H] for general 3–manifolds with a completely different approach. An interesting consequence of Theorem 1 is stated in the next corollary:

Corollary 2. *Given any overtwisted closed contact manifold (M, α), there exists a radius $R_0 \in \mathbb{R}^+ \setminus \{0\}$ such that $(M \times \mathbb{D}^2(R), \alpha + r^2\, d\theta)$ is tight if $R \in (0, R_0]$ and is overtwisted if $R \in (R_0, +\infty)$.*

As stated above, the property for large radii was proven in [3]. Note that a statement similar to the part of Corollary 2 that concerns large radii was already proven in [20] for GPS (generalized plastikstufe) instead of overtwisted disks.

The size threshold pointed out by Corollary 2 is similar to the case of Legendrians studied by Murphy in [18]. In brief, any Legendrian has a chart in which it can be expressed as the product of a zig-zag and a disk but the size of the disk can only be greater than 1 if the Legendrian is loose.

Theorem 1 can be extended to arbitrary neighborhoods of contact submanifolds M whose normal bundle has a nowhere vanishing section:

Theorem 3. *Suppose M is a closed contact submanifold of the contact manifold (N, ξ). Assume that the normal bundle of M has a nowhere vanishing section. Then, there is a neighborhood of M in N that is tight.*

The nowhere vanishing section provides a term $\mathbb{D}^2(\varepsilon)$ in a splitting of a neighborhood of M in N but some additional arguments that involve h–principle are needed to be able to apply Theorem 1.

Let us briefly outline how the results and ideas are presented in the article. The proof of Theorem 1, and thus also of 3, is based on Theorem 10. This result studies the contact manifold $M \times P^{2m}(\varepsilon)$, where $P^{2m}(\varepsilon)$ denotes the $2m$-dimensional polydisk $\mathbb{D}^2(\varepsilon) \times \cdots \times \mathbb{D}^2(\varepsilon)$. Theorem 10 states that for m large enough and ε small $(M \times P^{2m}(\varepsilon), \ker(\alpha + \sum_{i=1}^m r_i^2\, d\theta_i))$ admits a contact embedding in a closed contact manifold of the same dimension that is Stein fillable, therefore $M \times P^{2m}(\varepsilon)$ is automatically tight. However, Theorems 1 and 3 do not prove such a strong result. Their proof uses [3, Theorem 1.1.(ii)] and some packing lemmas to obtain a contradiction by stabilizing and reducing to Theorem 10.

The case in which the contact distribution $\xi = \ker \alpha$ is trivial as a vector bundle over M is adequate to introduce the basic ideas needed to prove the tightness of $M \times P^{2m}(\varepsilon)$ for large m. Choose a metric on M and extend it to T^*M in a way that α has norm 1 everywhere. Then M contact embeds in the unit cotangent bundle ST^*M via $e\colon p \mapsto \alpha_p$ for every $p \in M$. The normal bundle to the embedding is equal to ξ^* so it is trivial. Thus, if $\varepsilon > 0$ is small $M \times P^{2n}(\varepsilon)$, where $2n + 1 = \dim M$, is contactomorphic to a neighborhood of $e(M)$ in the Weinstein fillable manifold ST^*M (a filling is given by $\mathbb{D}T^*M$). Since ST^*M is tight we conclude that $M \times P^{2n}(\varepsilon)$, being contactomorphic to a neighborhood of $e(M)$, is also tight. The general case is addressed in Section 2. Theorem 10 shows that we can always embed M in the boundary of a Stein fillable manifold W with trivial normal bundle, the tightness of $M \times P^{2m}(\varepsilon)$ then follows automatically. The idea is to “add” a bundle τ that trivializes ξ and consider the pullback of the previous construction. Our arguments use h–principle crucially. The minimum codimension of the embedding found in the proof is $2n + \dim \tau$.

Theorems 1 and 3 do not prove tightness as a consequence of the stronger fact of the existence of an embedding into a Stein fillable manifold. Section 3 presents the proof of Theorem 1. The first step deals with the stabilization of the previous construction. Suppose for the moment that W is the

Weinstein filling of a contact manifold $Y = \partial W$ and $R > 0$ is arbitrary. Inside $W \times \mathbb{R}^2$ we can find a Weinstein domain such that $Y \times \mathbb{D}^2(R)$ contact embeds in its boundary and we deduce that $Y \times \mathbb{D}^2(R)$ is tight. Now, in our setting, after composing with the embedding that led to Theorem 10 we obtain that $M \times P^{2m+2}(\varepsilon, \ldots, \varepsilon, R)$ is tight for any $R > 0$. The notation $P^{2m+2}(\varepsilon, \ldots, \varepsilon, R)$ simply means $\mathbb{D}^2(\varepsilon) \times \cdots \times \mathbb{D}^2(\varepsilon) \times \mathbb{D}^2(R)$. On the other hand, if $M \times \mathbb{D}^2(\varepsilon)$ were overtwisted after stabilization (cf. [3, Theorem 1.1.(ii)]) we would produce an overtwisted manifold that would contact embed (using a symplectic packing result [14]) in $M \times P^{2m+2}(\varepsilon, \ldots, \varepsilon, R)$ if R is large enough. This is the contradiction that concludes Theorem 1.

Finally, Section 4 discusses the extension of the previous results to contact submanifolds. The nowhere vanishing section provides a splitting of a neighborhood of M in N of the form $U \times \mathbb{D}^2(\varepsilon)$. We use h–principle to embed U into a closed contact manifold of the same dimension to apply Theorem 1 and conclude Theorem 3.

1.2. Applications

1.2.1. Remarks about contact submanifolds. We are assuming an arbitrary but fixed choice of contact forms whenever the measure of a radius of the tubular neighborhoods of a contact submanifold is required.

1. Assume that (M, ξ) contact embeds into an overtwisted contact manifold (N, ξ_{OT}) as a codimension 2 submanifold with trivial normal bundle. By Theorem 1, it is clear that the overtwisted disk cannot be localized on arbitrary small neighborhoods of M, even assuming that M itself is overtwisted. This stands in sharp contrast with some related questions addressed in the literature. For instance, in dimension 3 Giroux criterium [12, Theorem 4.5a] tells that whenever you have a convex surface with a homotopically trivial dividing curve then there is an overtwisted disk in an arbitrary neighborhood of the surface. Also, examples of overtwisted convex hypersurfaces have been found in higher dimensions [15]. Another related work is [16] in which it is shown that the overtwisted disk can be localized around a very special kind of codimension n submanifold: a plastikstufe [19].

2. Assume now that (M, ξ_{OT}) is overtwisted and contact embeds into a tight contact manifold (N, ξ) as a codimension 2 submanifold with trivial normal bundle. Then we can perform a fibered connected sum of (N, ξ) with itself along $(M, \xi_{OT} = \ker \alpha_{OT})$. Briefly, the construction of the connected sum goes as follows (cf. [10, Section 7.4]). Given a sufficiently small neighborhood $\mathcal{U}$ of M, There is a contactomorphism between the open domain $\mathcal{U} \setminus M$ in N and $(M \times (0, \varepsilon) \times \mathbb{S}^1, \alpha_{OT} + t\, d\theta)$. Form the fibered connected sum $N\#_M N$ by identifying the two copies of M inside N. After changing coordinates $(t,\theta)\mapsto(-t,-\theta)$ in a neighborhood of M in one of the copies of N, $N\#_M N$ can be given a contact structure that agrees with the original one in the complementary of M and a neighborhood of the gluing region in the connected sum is contactomorphic to $(M\times(-\varepsilon,\varepsilon)\times\mathbb{S}^1, \alpha_{OT}+t\,d\theta)$.

It is clear that the contact connection associated to the contact fibration $M\times(-\varepsilon,\varepsilon)\times\mathbb{S}^1\to(-\varepsilon,\varepsilon)\times\mathbb{S}^1$ [21] induces the identity map when we lift by parallel transport the loop $\{0\}\times\mathbb{S}^1$. The parallel transport of an overtwisted disk of the fiber generates a plastikstufe, see [21] for more details. By [16], the manifold is overtwisted.

Call $R_M>0$ the biggest radius for which $M\times D^2(R_M)$ contact embeds in N. We claim that the connected sum $N\#_M N$ increases the biggest radius at least to $R_{N\#_M N}\geq\sqrt{2}R_M$. Indeed, the annulus has twice the area of the original disk and therefore you can embed symplectically a disk of radius $r=\sqrt{2}R_M-\delta$ in $(\varepsilon,\varepsilon)\times\mathbb{S}^1$ for any $\delta>0$. This (exact) symplectic embedding can be lifted to a contact embedding $M\times\mathbb{D}^2(r)\to M\times(\varepsilon,\varepsilon)\times\mathbb{S}^1$. However, we get much more, since we actually obtain $R_{N\#_M N}=\infty$. This is because we can always formally contact embed $M\times\mathbb{R}^2$ into $N\#_M N$. Moreover, we can assume that the embedding restricted to a very small neighborhood U of the fiber $M\times\{0\}$ provides a honest fibered contact embedding into $M\times(-\varepsilon,\varepsilon)\times\mathbb{S}^1$. Indeed, applying [2, Corollary 1.4] relative to the domain U we obtain a contact embedding of $M\times\mathbb{R}^2$ thanks to the fact that $N\#_M N$ is overtwisted. This just means that the contact embedding of the tubular neighborhood can be really sophisticated and its explicit construction is far from obvious.

1.2.2. Small loops of contactomorphisms. Theorem 1 allows to extend the result of non–existence of small positive loops of contactomorphisms in overtwisted 3–manifolds contained in [5] to arbitrary dimension. A loop of contactomorphisms or, more generally, a contact isotopy is said to be positive if it moves every point in a direction positively transverse to the contact distribution. The notion of positivity induces for certain manifolds, called orderable, a partial order on the universal cover of the contactomorphism group and it is related with non–squeezing and rigidity in contact geometry, see [7, 9]. As explained in [9], orderability is equivalent to the non–existence of a positive contractible loop of contactomorphisms.

Any contact isotopy is generated by a contact Hamiltonian $H_t\colon M\to\mathbb{R}$ that takes only positive values in case the isotopy is positive. The main result of [5] states that if $(M,\ker\alpha)$ is an overtwisted 3–manifold there exists a constant $C(\alpha)$ such that any positive loop of contactomorphisms generated by a Hamiltonian $H\colon M \times S^1 \to \mathbb{R}^+$ satisfies $||H||_{\mathcal{C}^0} \geq C(\alpha)$. The result has been recently extended to arbitrary hypertight or Liouville (exact symplectically) fillable contact manifolds in [1]. As a consequence of Theorem 1, we can eliminate the restriction on the dimension in the overtwisted case:

Theorem 4. *Let $(M, \ker\alpha)$ be an overtwisted contact manifold. There exists a constant $C(\alpha)$ such that the norm of a Hamiltonian $H\colon M \times S^1 \to \mathbb{R}^+$ that generates a positive loop $\{\phi_\theta\}$ of contactomorphisms on M satisfies*

$$||H||_{\mathcal{C}^0} \geq C(\alpha)$$

The strategy of the proof copies that of [5]. The first step is to prove that $M \times \mathbb{D}^2(\varepsilon)$ is tight, this is provided by Theorem 1. The second step shows that a small positive loop provides a way to lift a plastikstufe in M (whose existence is equivalent to overtwistedness as discussed above [16]) to a plastikstufe in $M \times \mathbb{D}^2(\varepsilon)$. This is exactly Proposition 9 in [5]. This provides a contradiction that forbids the existence of the small positive loop.

It is worth mentioning that the argument forbids the existence of (possibly non–contractible) small positive loops as in [1, 5]. This is in contrast with the work in progress by S. Sandon [22] in which they rule out the existence of contractible small positive loops. It is also important to notice that overtwisted manifolds are not orderable in general, positive loops in certain overtwisted 3-manifolds were found in [4].

Remark 5. *The hypothesis in Theorem 4 can be changed by the probably weaker notion of GPS-overtwisted, see [20]. Indeed, assume that the manifold (M, ξ) is GPS-overtwisted. This means that there is an immersed GPS in the manifold. The positive loop produce a GPS in $M \times \mathbb{D}^2(\varepsilon)$ by parallel transport of the GPS around a closed loop in the base $\mathbb{D}^2(\varepsilon)$. In this case, we need to iterate the process k times to produce a GPS in $M \times P^{2k}(\varepsilon)$. Now, Theorem 10 concludes that this manifold embeds into a Stein fillable one providing a contradiction with the main result in [20].*

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2. $M \times P^{2m}(\varepsilon)$ admits a contact embedding into a Stein fillable contact manifold

2.1. Construction of a formal contact embedding $M \to \partial W$ with trivial normal bundle

Recall that $(\xi, d\alpha)$ defines a symplectic vector bundle over M, thus it is equipped with a complex bundle structure unique up to homotopy. Denote ξ^* the dual complex vector bundle of ξ. The theory of vector bundles (check, for instance [17, Proposition 5.8]) guarantees the existence of a complex vector bundle $\tau \to M$ such that $\xi^* \oplus \tau \to M$ is trivial, that is, there is an isomorphism of complex vector bundles over M between $\xi^* \oplus \tau$ and $M \times \mathbb{C}^k = \underline{\mathbb{C}}^k$, where k is a positive integer large enough.

Denote $\pi\colon T^*M \to M$ the cotangent bundle projection and denote $\mathrm{pr}: \pi^*\tau \to T^*M$ the bundle projection. Define $\tilde{\pi} = \pi \circ \mathrm{pr}$. Let us understand $\widehat{W} = \pi^*\tau$ as a smooth almost complex manifold. Choosing a ξ–compatible contact form α, i.e $\xi = \ker \alpha$, it is clear that

$$\begin{aligned} T\widehat{W} &\cong \tilde{\pi}^*\tau \oplus \mathrm{pr}^*T(T^*M) \cong \tilde{\pi}^*\tau \oplus \tilde{\pi}^*T^*M \oplus \tilde{\pi}^*TM \\ &\cong \tilde{\pi}^*\tau \oplus \tilde{\pi}^*(\xi^* \oplus \langle\alpha\rangle) \oplus \tilde{\pi}^*TM \\ &\cong \tilde{\pi}^*(\tau \oplus \xi^*) \oplus \tilde{\pi}^*\langle\alpha\rangle \oplus \tilde{\pi}^*TM \\ &\cong \tilde{\pi}^*\underline{\mathbb{C}}^k \oplus \tilde{\pi}^*\langle\alpha\rangle \oplus \tilde{\pi}^*TM \end{aligned}$$

In particular, the vector bundle $\pi^*\tau \xrightarrow{\tilde{\pi}} M$ is isomorphic to $\underline{\mathbb{C}}^k \oplus \langle\alpha\rangle$. Fix a direct sum bundle metric h in $\pi^*\tau$ such that $h(\alpha, \alpha) = 1$. Now define

$$W = \{(v, p) \in \widehat{W} : h(v, v) \leq 1\}.$$

We are going to fix a compatible almost complex structure on T^*M. Compatible in the sense that it is defined by choosing a Riemannian metric in TM that automatically induces a Riemannian metric in T^*M and this obviously induces a metric in $TT^*M = \pi^*TM \oplus \pi^*T^*M$. It is well known that this choice makes the almost complex structure compatible with the Liouville form in T^*M. The choice that we make for the Riemannian metric in the tangent bundle is the usual compatible metric for a contact form;

it is defined by fixing a $d\alpha$–compatible j in the distribution ξ and setting $g(u,v) = d\alpha(u, jv)$ and declaring the Reeb vector field R_α in M unitary and orthogonal to ξ. The almost complex structure pairs the lifts of the Reeb vector field R_α and the contact form α to TT^*M.

By a direct sum with a complex structure in τ we obtain a complex structure J in $T\widehat{W}$. Then, (W, J) is an almost complex manifold with boundary ∂W that has a natural formal contact structure $\xi_0 = T\partial W \cap J(T\partial W)$. The reason to choose this adapted complex structure J is that we will later have to check that the ensuing embedding is formal contact. Consider the embedding

$$e_0 \colon M \to \partial W = \mathbb{S}(\underline{\mathbb{C}}^k \oplus \langle\alpha\rangle) : p \mapsto (0, 1) \tag{1}$$

that sends each point $p \in M$ to $\alpha(p)$. In particular, $e_0 = z_0 \circ e$ where $e : M \to \mathbb{S}(T^*M)$ is the natural section provided by the choice of contact form for ξ and $z_0 : \mathbb{S}(T^*M) \to \pi^*\tau$ is the embedding defined by the zero section of the vector bundle. We claim that the normal bundle to e_0 is trivial because it is equal to $\tilde{\pi}^*\underline{\mathbb{C}}^k$.

By the choice of metric, the tangent to the embedding is $(\tilde{\pi}^*TM)_{|im(e_0)}$ and its normal bundle inside $\widehat{W} = \pi^*\tau$ is $(\tilde{\pi}^*\underline{\mathbb{C}}^k \oplus \tilde{\pi}^*\langle\alpha\rangle)_{|im(e_0)}$. It follows that the normal bundle to e_0 is $(T\partial W \cap (\tilde{\pi}^*\underline{\mathbb{C}}^k \oplus \tilde{\pi}^*\langle\alpha\rangle))_{|im(e_0)}$ and, in view of the definition of e_0, this simplifies to $(\tilde{\pi}^*\underline{\mathbb{C}}^k)_{|im(e_0)}$.

2.2. W admits a Stein structure and thus ∂W is contact

The distribution $T\partial W \cap J(T\partial W)$ is not necessarily a contact structure in ∂W. However, we will deform this distribution to a genuine contact structure using the following result.

Theorem 6 (Eliashberg [6]). *Let (V^{2n}, J) be an almost complex manifold with boundary of dimension $2n > 4$ and suppose that $f \colon V \to [0, 1]$ is a Morse function constant on ∂V such that $\mathrm{ind}_p(f) \leq n$ for every $p \in \mathrm{Crit}(f)$. Then, there exists a homotopy of almost complex structures $\{J_t\}_{t=0}^1$ such that $J_0 = J$, J_1 is integrable and f is J_1–convex.*

We are clearly in the hypothesis since our manifold W is almost complex, has dimension $2k + 1 + \dim M > 4$ (because $2k \geq \dim \xi = \dim M - 1$) and deformation retracts to M.

From Theorem 6 we obtain a homotopy of almost complex structures $\{J_t\}$ in W such that $J_0 = J$ and, J_1 is integrable. Moreover (W, J_1) is a

Stein domain and ∂W inherits a contact structure given by $\xi_1 = J_1(T\partial W) \cap T\partial W$. In fact, there is a homotopy of formal contact structures between ξ_0 and ξ_1 provided by $\xi_t = J_t(T\partial W) \cap T\partial W$.

2.3. Properties of the embedding $e_0\colon (M, \xi) \to (\partial W, \xi_1)$

Recall the following definition:

Definition 7. *An embedding* $e\colon (M_0, \xi_0, J_0) \to (M_1, \xi_1, J_1)$ *is called* formal contact *if there exists an homotopy of monomorphisms* $\{\Psi_t\colon TM_0 \to TM_1\}_{t=0}^1$ *such that* $\Psi_0 = de$, $\xi_0 = \Psi_1^{-1}(\xi_1)$ *and* $\Psi_1\colon (\xi_0, J_0) \to (\xi_1, J_1)$ *is complex.*

So far we have produced an embedding $e_0\colon (M, \xi, j) \to (\partial W, \xi_0, J_0)$ that is formal contact with the constant homotopy equal to de_0. Indeed, we have that $de_0^{-1}(\xi_0) = \xi$. Since by the definition (1), we have that $e_0 = z_0 \circ e$ and from this composition formula the claim is clear.

Also $de_0(\xi)$ is a complex subbundle of ξ_0. There is a family of complex isomorphisms $\Phi_t\colon \xi_0 \to \xi_t$ such that $\Phi_0 = \mathrm{id}$. Fix a Reeb vector field R associated to ξ and define $\widehat{R}_0 = de_0(R)$. Build a family $\{R_t\}$ of vector fields in $T\partial W$ satisfying $R_0|_{\mathrm{im}\, e_0} = \widehat{R}_0$ and $\langle R_t\rangle \oplus \xi_t = T\partial W$. We take a family of metrics g_t in ∂W defined in the following way: its restriction to ξ_t is Hermitian for the complex bundle (ξ_t, J_t) and R_t is unitary and orthogonal to ξ_t.

Extend Φ_t to an isomorphism of $T\partial W|_{\mathrm{im}\, e_0}$ in such a way that $\Phi_t(R_0) = R_t$. Define

$$E_t = \Phi_t \circ de_0\colon TM \to T\partial W.$$

The family $\{E_t\}_{t=0}^1$ is composed of bundle monomorphisms and clearly satisfies that $E_t^{-1}(\xi_t) = \xi$ and $E_1(\xi)$ is a complex subbundle of ξ_1. Therefore, (e_0, E_t) is a formal contact embedding.

Define $\mathcal{N}_t = E_t(TM)^{\perp g_t}$ that is a bundle over $\mathrm{im}\, e_0$ which is complex by construction. $\mathcal{N}_0$ is isomorphic to $\underline{\mathbb{C}}^k$ and therefore all the bundles $\mathcal{N}_t$ are trivial complex bundles.

2.4. Obtaining a contact embedding via h–principle

The only missing piece to complete the puzzle is to prove that the embedding e_0 can be made contact.

Using h–principle it is possible to deform (e_0, E_t) to a contact embedding thanks to the following theorem (cf. [8, Theorem 12.3.1]):

Theorem 8. *Let* (e, E_t)*,* $e\colon (M_0, \xi_0 = \ker\alpha_0) \to (M_1, \xi_1 = \ker\alpha_1)$*, be a formal contact embedding between closed contact manifolds such that* $\dim M_0 + 2 < \dim M_1$*. Then, there exists an isotopy of embeddings* $\widetilde{e}_s\colon M_0 \to M_1$ *such that:*

- $\widetilde{e}_0 = e$ *and* $\widetilde{e}_1$ *is contact,*
- $d\widetilde{e}_1$ *is homotopic to* E_1 *through monomorphisms* $G_s\colon TM_0 \to TM_1$*, lifting the embeddings* $\widetilde{e}_s$*, such that* $G_s(\xi_0) \subset \xi_1$ *and the restrictions* $G_s|_{\xi_0}\colon (\xi_0, d\alpha_0) \to (\xi_1, d\alpha_1)$ *are symplectic.*

Theorem 8 applied to (e_0, E_t) provides a family of embeddings $\{e_s\}$ in which $e_1\colon (M,\xi) \to (\partial W, \xi_1)$ is a contact embedding and a family of monomorphisms $G_s\colon TM \to T\partial W$ that lift e_s such that $G_0 = E_1$, $G_1 = de_1$ and $G_s(\xi) \subset \xi_1$ is a complex subbundle.

Lemma 9. *The normal bundle of* $\operatorname{im}(e_1)$ *in* $(\partial W, \xi_1)$ *is trivial.*

Proof. Recall that $\mathcal{N}_1 = E_1(TM)^{\perp g_1} = G_0(TM)^{\perp g_1}$ is a trivial complex vector bundle. Define, for $s \in [1,2]$, $\mathcal{N}_s = G_{s-1}(TM)^{\perp g_1}$. Clearly, $\mathcal{N}_2$ is the normal bundle of the contact embedding e_1. Since $\mathcal{N}_1$ is a trivial vector bundle so is $\mathcal{N}_2$. □

Denote the $2m$–dimensional polydisk by

$$P^{2m}(r_1, \ldots, r_m) = \mathbb{D}^2(r_1) \times \cdots \times \mathbb{D}^2(r_m)$$

and abbreviate it as $P^{2m}(r)$ when $r_1 = \ldots = r_m = r$. The following result summarizes the work completed in this section and an important consequence (namely, the title of the section): $M \times P^{2m}(\varepsilon)$ admits a contact embedding into a Stein fillable contact manifold.

Theorem 10. *Any closed contact manifold* $(M, \ker\alpha)$ *contact embeds in the boundary of a Stein fillable manifold with trivial normal bundle. Furthermore, there exists* $k \geq 1$ *such that for any* $m \geq k$

$$\left(M \times P^{2m}(\varepsilon), \ker\left(\alpha + \sum_{i=1}^{m} r_i^2\, d\theta_i\right)\right)$$

is tight with $\varepsilon > 0$ *small enough depending only on* α *and* m*.*

Proof. The map e_1 proves the first part because by Lemma 9 the normal bundle of the contact embedding $e_1\colon (M,\xi) \to (\partial W, \xi_1)$ is trivial. Notice that the codimension of the embedding is equal to $2k = \dim \tau$ and by replacing τ with $\tau' = \tau \oplus \underline{\mathbb{C}}^{m-k}$ we obtain embeddings of arbitrary codimension $2m \geq 2k$.

Suppose henceforth that $m \geq k$. By an standard neighborhood theorem in contact geometry it follows that there is a contactomorphism between a neighborhood of $\mathrm{im}(e_1)$ in $(\partial W, \xi_1)$ and a neighborhood of $M \times \{0\}$ in $(M \times \mathbb{R}^{2m}, \ker(\alpha + \sum_{i=1}^{k} r_i^2 d\theta_i))$. Therefore, for some $\varepsilon_0 > 0$, the previous contactomorphism provides an embedding from $M \times P^{2m}(\varepsilon_0)$ into ∂W.

Finally, since $(\partial W, \xi_1)$ is Stein fillable, it is tight. Thus, any of its open subsets is also tight and the conclusion follows. □

3. $M \times \mathbb{D}^2(\varepsilon)$ is tight if ε is small

The argument leading to Theorem 10 provided no bound on the first positive integer k such that $M \times P^{2k}(\varepsilon)$ is tight. Indeed, k was fixed at the beginning of Section 2, depending on the rank of $\tau \to M$, the bundle constructed to make the sum $\xi^* \oplus \tau$ trivial.

The insight needed to prove Theorem 1 is supplied by the understanding of overtwisted contact manifolds briefly discussed in the introduction. To be more concrete, the precise statement we will use in this section is the following:

Theorem 11 (Casals, Murphy and Presas [3, Theorem 3.2]). *Suppose that $(M, \ker \alpha)$ is an overtwisted contact manifold. Then, if R is large enough, $(M \times \mathbb{D}^2(R), \ker(\alpha + r^2 d\theta))$ is also overtwisted.*

The idea is to embed $\partial W \times \mathbb{D}^2(R)$ for arbitrary R in the boundary ∂V of a Weinstein manifold that depends on R. Using the embedding constructed in the previous section we obtain then an embedding $M \times \mathbb{D}^2(R) \to \partial V$ that has trivial normal bundle. This leads to the proof of a statement similar to Theorem 10 in which we replace $(M, \ker \alpha)$ by $(M \times \mathbb{D}^2(R), \ker(\alpha + r^2 d\theta))$. Note that it is key to construct the embedding for arbitrary (but fixed) $R > 0$.

The Stein fillable manifold W supplied by Theorem 10 is naturally equipped with a Weinstein structure $(W = f^{-1}(0,1], \omega, f, Y)$ that satisfies $\xi_1 = \ker(i_Y \omega|_{\partial W})$. Recall that a Weinstein structure in W is composed of a symplectic form ω, a Liouville vector field Y and a Morse function $f\colon W \to \mathbb{R}$ that is pseudo-gradient for Y. The product $W \times \mathbb{R}^2(r, \theta)$ can be equipped with the Weinstein structure $(\omega + r\, dr \wedge d\theta, X = Y + r\frac{\partial}{\partial r}, f_q)$, where $q \geq 1$

and $f_q\colon W \times \mathbb{R}^2 \to (0,\infty)$ is defined by:

$$(f_q)^q = f^q + (r/2R)^{2q}$$

Note that (check for instance [10, Lemma 2.1.5]) the flow along a Liouville vector field induces contactomorphisms between transversal hypersurfaces (equipped with the contact form $\iota_X(\omega + r\,dr \wedge d\theta)$). The ensuing proposition follows from the observation that $f_q^{-1}(1)$ gets C^∞–close to $\partial W \times \mathbb{R}^2$ on compact sets as $q \to \infty$ (this explains the choice of f_q above) and X is transverse to $\partial W \times \mathbb{R}^2$.

Proposition 12. *There exists $q > 1$ large enough and a function $\mu\colon \partial W \times \mathbb{D}^2(R) \to \mathbb{R}^-$ such that $\phi_\mu\colon \partial W \times \mathbb{D}^2(R) \to W \times \mathbb{D}^2(R)$ satisfies $\phi_\mu(\partial W \times \mathbb{D}^2(R)) \subset f_q^{-1}(1)$, where ϕ_μ denotes the Liouville flow up to time μ.*

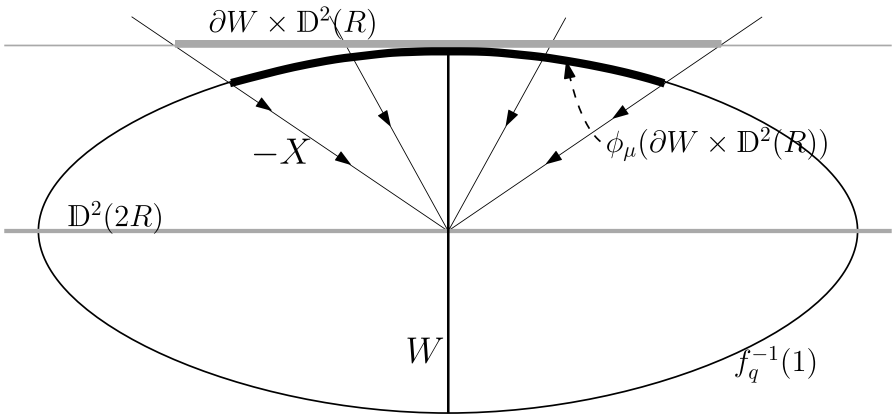

Figure 1: Contact embedding of $\partial W \times \mathbb{D}^2(R)$ into $f_q^{-1}(1)$.

Thus, for the integer q supplied by Proposition 12 the manifold $V = f_q^{-1}(0,1]$ is Weinstein. Denote $\alpha' = \iota_X(\omega + r\,dr \wedge d\theta)$. The following proposition summarizes the previous reasoning:

Proposition 13. *For any $R > 0$, the contact manifold*

$$(\partial W \times \mathbb{D}^2(R), \ker(\alpha'|_{\partial W \times \mathbb{D}^2(R)}))$$

admits a contact embedding into the boundary of a Weinstein manifold.

Combining the last proposition and the results from the previous section we obtain:

Corollary 14. *Given a contact manifold* (M, α) *there exists* $k \in \mathbb{N}$ *and* $\varepsilon_0 > 0$ *such that for every* $R > 0$ *the contact manifold*

$$\left(M \times P^{2k+2}(\varepsilon_0, \dots, \varepsilon_0, R), \ker\left(\alpha + \sum_{i=1}^{k+1} r_i^2 d\theta_i\right)\right)$$

is tight.

Let us emphasize that ε_0 does not depend on R: for any $R > 0$, $M \times P^{2k+2}(\varepsilon_0, \dots, \varepsilon_0, R)$ is tight.

Proof. The integer k and the number ε_0 both come from Theorem 10. Denote by e' the contact embedding from $(M \times P^{2k}(\varepsilon_0), \ker(\alpha + \sum_{i=1}^{k} r_i^2 d\theta_i))$ into $(\partial W, \xi_1 = \ker(i_Y \omega))$ and let $\exp(\eta)$ be the conformal factor of e', $(e')^* i_Y \omega = \exp(\eta)(\alpha + \sum_{i=1}^{k} r_i^2 d\theta_i))$. If necessary, decrease the value of ε_0 to guarantee that $\sup \eta$ is finite.

Proposition 13 supplies a Weinstein manifold $(V = f_q^{-1}(0, 1], \omega + r\, dr \wedge d\theta, f_q, X)$ and a contact embedding

$$\varphi \colon (\partial W \times \mathbb{D}^2(\exp(\sup \eta/2)R), \alpha') \hookrightarrow (\partial V, \alpha').$$

Therefore, the map $\widetilde{\varphi} \colon M \times P^{2k+2}(\varepsilon_0, \dots, \varepsilon_0, R) \to \partial V$ given by

$$\widetilde{\varphi}(p, r_i, \theta_i, r, \theta) = \varphi(e'(p, r_i, \theta_i), \exp(\eta/2) r, \theta)$$

is a contact embedding. Since ∂V is exact symplectically fillable the conclusion follows. ☐

We are ready now to prove Theorem 1. To ease the notation, we shall understand the contact form is equal to $\alpha + \sum r_i^2 d\theta_i$ in case it is omitted.

Let us proceed by contradiction. Suppose that $M \times \mathbb{D}^2(\varepsilon)$ is overtwisted for ε smaller than ε_0. Applying Theorem 11 k times consecutively we obtain a radius $R_\varepsilon > 0$ such that $M \times P^{2k+2}(\varepsilon, R_\varepsilon, \dots, R_\varepsilon)$ is overtwisted. As we will show below, this manifold contact embeds into $M \times P^{2k+2}(\varepsilon_0, \dots, \varepsilon_0, R)$ provided R is large enough. From Corollary 14 we know that the latter manifold is tight so we reach a contradiction. Therefore, $M \times \mathbb{D}^2(\varepsilon)$ is tight.

The only missing ingredient is the announced contact embedding:

$$M \times P^{2k+2}(\varepsilon, R_\varepsilon, \dots, R_\varepsilon) \to M \times P^{2k+2}(\varepsilon_0, \dots, \varepsilon_0, R) \tag{2}$$

Its existence, subject to the conditions $\varepsilon < \varepsilon_0$ and R large enough, is a consequence of the following packing theorem in symplectic geometry proved by Guth [14, Theorem 1].

Theorem 15. *For every $m \in \mathbb{N}$ there is a constant $C(m) \geq 1$ such that for any pair of ordered m–tuples of positive numbers $R_1 \leq \dots \leq R_m$ and $R'_1 \leq \dots \leq R'_m$ that satisfy*

- $C(m)R_1 \leq R'_1$ *and*
- $C(m)R_1 \cdot \ldots \cdot R_m \leq R'_1 \cdot \ldots \cdot R'_m$.

there is a symplectic embedding

$$P^{2m}(R_1, \dots, R_m) \hookrightarrow P^{2m}(R'_1, \dots, R'_m)$$

The symplectic embedding supplied by Theorem 15 is automatically extended to our desired contact embedding (2) thanks to the following lemma. Notice that the Reeb flow is complete in M because M is closed.

Lemma 16. *Let $\Psi \colon (D_1, d\lambda_1) \to (D_2, d\lambda_2)$ be an exact symplectic embedding. For any contact manifold $(M, \ker\alpha)$ with a choice of contact form α that makes the associated Reeb flow complete, Ψ induces a (strict) contact embedding*

$$(M \times D_1, \alpha + \lambda_1) \to (M \times D_2, \alpha + \lambda_2).$$

Proof. Since Ψ is exact, there exists a smooth function $H \colon D_1 \to \mathbb{R}$ such that $dH = \Psi^*\lambda_2 - \lambda_1$. If we denote the Reeb flow in M by Φ,

$$\varphi \colon (M \times D_1, \alpha + \lambda_1) \to (M \times D_2, \alpha + \lambda_2), \quad \varphi(p, x) = (\Phi_{-H(x)}(p), \Psi(x))$$

is a strict contact embedding because $\varphi^*(\alpha + \lambda_2) = \varphi^*\alpha + \Psi^*\lambda_2 = (\Phi^*\alpha - dH) + (\lambda_1 + dH) = \alpha + \lambda_1$. □

4. Extension to contact submanifolds

The results from the previous sections can be extended to a more general setting: contact submanifolds with arbitrary normal bundle. In the presence of a nowhere vanishing section of the normal bundle we will prove that the contact submanifold has a tight neighborhood. This is the content of Theorem 3.

Let $\pi\colon E \to M$ be a complex vector bundle over a contact manifold equipped with an Hermitian metric and a unitary connection ∇. The associated vertical bundle is denoted by $\mathcal{V} = \ker(d\pi)$. The standard Liouville form in $\mathbb{R}^{2n}$ is $U(n)$–invariant and induces a global 1–form in $\mathcal{V}$ that will be denoted $\widetilde{\lambda}$. This real 1–form can be extended to TE by the expression $\lambda = \widetilde{\lambda} \circ \pi_{\mathcal{V}}$ after we choose a projection onto the vertical direction $\pi_{\mathcal{V}}\colon TE \to \mathcal{V}$. The map π_V is determined by the choice of unitary connection so it is not canonical. The 1–form in TE associated to the connection ∇ is $\widetilde{\alpha} = \pi^*\alpha + \lambda$.

Even though the 1-form $\widetilde{\alpha}$ can be seen as the lift of the contact form α to E, it may not satisfy the contact condition everywhere in E. However, the contact condition is verified around the zero section E_0 of the vector bundle so $\widetilde{\alpha}$ defines a contact form in a neighborhood of E_0.

Lemma 17. *$\widetilde{\alpha}$ is a contact form in a neighborhood of E_0. The restriction $(E_0, \ker(\widetilde{\alpha}|_{E_0}))$ is contactomorphic to $(M, \xi = \ker\alpha)$. Moreover, given any other contact structure $\ker\beta$ that coincides with $\ker(\widetilde{\alpha})$ in E_0 and with the same complex structure in the normal bundle, there exist neighborhoods U, V of E_0 such that $(U, \ker(\beta|_U))$ and $(V, \ker(\widetilde{\alpha}|_V))$ are contactomorphic.*

Suppose henceforth that π has a global nowhere vanishing section $s\colon M \to E$. The section s creates a complex line subbundle $\pi|_L\colon L \to M$. Then, using the Hermitian metric on the fibers, the bundle E splits as $E = F \oplus L$ and L is trivial, i.e. there is an isomorphism $\phi\colon L \to \underline{\mathbb{C}}$ that sends $s(p)$ to $1_p \in \mathbb{C}$ in the fiber above every point $p \in M$.

A suitable choice of unitary connection on $\pi\colon E \to M$ ensures that the associated contact form on a neighborhood of E_0 can be written as $\widetilde{\alpha} = \alpha' + \lambda$, where α' is a contact form in F and λ is the radial Liouville form in $\mathbb{R}^2$.

Proposition 18. *There exists U, a neighborhood of the zero section F_0 of F, and $\varepsilon > 0$ such that $(U \times \mathbb{D}^2(\varepsilon), \ker(\alpha' + \lambda))$ is tight.*

Note that this statement is exactly Theorem 1 except from the fact that U is not closed.

Proof. The proof of Proposition 18 follows by embedding $(U, \ker\alpha')$ in a closed contact manifold $(\widetilde{F}, \ker\widetilde{\alpha}')$ of the same dimension and then applying

Theorem 1 to this manifold to deduce that $(\widetilde{F} \times \mathbb{D}^2(\varepsilon'), \ker(\widetilde{\alpha}' + \lambda))$ is tight if $\varepsilon' > 0$ is small. This result implies that $(U \times \mathbb{D}^2(\varepsilon), \ker(\alpha' + \lambda))$ is also tight if ε is small enough. More precisely, it suffices to ask for $(\varepsilon'/\varepsilon)^2$ to be greater than the supremum conformal factor of the embedding as in Corollary 14.

The aforementioned embedding is defined by the natural inclusion of F in the projectivization of $F \oplus \mathbb{C}$:

$$F \hookrightarrow Q = \mathbb{P}(F \oplus \mathbb{C})$$

The complex bundle $\pi_Q \colon Q \to M$ carries a natural formal contact structure $\xi' = (d\pi_Q)^{-1}(\xi)$ Indeed, an almost complex structure in ξ' is obtained as the sum of the pullback of a complex structure in ξ compatible with $d\alpha$ and a complex structure on the fibers of π_Q. This formal contact structure is genuine (i.e., it is a true contact structure) in a neighborhood U of F_0 by Lemma 17. The h–principle for closed manifolds proved in [2, Theorem 1.1] provides a homotopy from any formal contact structure to a contact structure. Furthermore, the homotopy can be made relative to a closed set in which the formal contact structure is already genuine. Applying this theorem we obtain a contact structure $\widetilde{\xi}'$ on Q that agrees with $\ker \alpha'$ in U. □

We can reformulate Proposition 18 in the following way:

Theorem 19. *Let $\pi \colon E \to M$ be a complex vector bundle over a closed contact manifold (M, ξ). Suppose that π has a global nowhere vanishing section. Then, there exists a neighborhood U of the zero section of the bundle such that $(U, \widetilde{\xi})$ is tight for any contact structure $\widetilde{\xi}$ extending ξ and preserving the complex structure of E.*

An immediate application of Theorem 19 to the case in which M is a contact submanifold and π is its normal bundle yields Theorem 3.

References

[1] P. Albers, U. Fuchs, and W. J. Merry, *Positive loops and L^∞-contact systolic inequalities*, Selecta Math. (N.S.) **23** (2017), no. 4, 2491–2521.

[2] M. S. Borman, Y. Eliashberg, and E. Murphy, *Existence and classification of overtwisted contact structures in all dimensions*, Acta Math. **215** (2015), no. 2, 281–361.

[3] R. Casals, E. Murphy, and F. Presas, *Geometric criteria for overtwistedness*, J. Amer. Math. Soc. **32** (2019) 563–604.

[4] R. Casals and F. Presas, *On the strong orderability of overtwisted* 3-*folds*, Comment. Math. Helv. **91** (2016), no. 2, 305–316.

[5] R. Casals, F. Presas, and S. Sandon, *Small positive loops on overtwisted manifolds*, J. Symplectic Geom. **14** (2016), no. 4, 1013–1031.

[6] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, Internat. J. Math. **1** (1990), no. 1, 29–46.

[7] Y. Eliashberg, S. S. Kim, and L. Polterovich, *Geometry of contact transformations and domains: orderability versus squeezing*, Geom. Topol. **10** (2006), 1635–1747.

[8] Y. Eliashberg and N. Mishachev, Introduction to The h-Principle, Vol. 48 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI (2002), ISBN 0-8218-3227-1.

[9] Y. Eliashberg and L. Polterovich, *Partially ordered groups and geometry of contact transformations*, Geom. Funct. Anal. **10** (2000), no. 6, 1448–1476.

[10] H. Geiges, An Introduction to Contact Topology, Vol. 109 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (2008), ISBN 978-0-521-86585-2.

[11] F. Gironella, *On some examples and constructions of contact manifolds*, Math. Ann. **376** (2020), no. 3-4, 957–1008.

[12] E. Giroux, *Structures de contact sur les variétés fibrées en cercles au-dessus d'une surface*, Comment. Math. Helv. **76** (2001), no. 2, 218–262.

[13] M. L. Gromov, *Stable mappings of foliations into manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969) 707–734.

[14] L. Guth, *Symplectic embeddings of polydisks*, Invent. Math. **172** (2008), no. 3, 477–489.

[15] K. Honda and Y. Huang, *Bypass attachments in higher-dimensional contact topology*, arXiv:1803.09142, (2018).

[16] Y. Huang, *On plastikstufe, bordered Legendrian open book and overtwisted contact structures*, J. Topol. **10** (2017), no. 3, 720–743.

[17] D. Husemoller, Fibre Bundles, Vol. 20 of Graduate Texts in Mathematics, Springer-Verlag, New York, third edition (1994), ISBN 0-387-94087-1.

[18] E. Murphy, *Loose Legendrian embeddings in high dimensional contact manifolds*, arXiv:1201.2245, (2019).

[19] K. Niederkrüger, *The plastikstufe—a generalization of the overtwisted disk to higher dimensions*, Algebr. Geom. Topol. **6** (2006), 2473–2508.

[20] K. Niederkrüger and F. Presas, *Some remarks on the size of tubular neighborhoods in contact topology and fillability*, Geom. Topol. **14** (2010), no. 2, 719–754.

[21] F. Presas, *A class of non-fillable contact structures*, Geometry & Topology **11** (2007), no. 4, 2203–2225.

[22] S. Sandon, *Floer homology for traslated points*, (2016).

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