JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 18, Number 5, 1247–1283, 2020

Elliptic diffeomorphisms of symplectic 4-manifolds

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We show that symplectically embedded (-1)-tori give rise to certain elements in the symplectic mapping class group of 4-manifolds. An example is given where such elements are proved to be of infinite order.

0. Introduction

Let (X, ω) be a closed symplectic 4-manifold. Denote by $\pi_0\mathscr{S}\!ymp(X, \omega)$ the group of symplectic diffeomorphisms of X modulo symplectic isotopy. Let us consider the forgetful homomorphism

$$\pi_0\mathscr{S}\!ymp(X, \omega) \to \pi_0\mathscr{D}\!i\!f\!f(X).$$

Here $\pi_0\mathscr{D}\!i\!f\!f(X)$ denotes the smooth mapping class group for X. It is known this homomorphism is not necessary injective. If Σ is a smooth Lagrangian sphere in X, then there exists a symplectomorphism $T_\Sigma : X \to X$, called *symplectic Dehn twist along* Σ, such that T_Σ^2 is smoothly isotopic to the identity. In his thesis [SeiTh], Seidel proved that in many cases T_Σ^2 is not symplectically isotopic to the identity. He than proved that for certain $K3$ surfaces containing two Lagrangian spheres Σ_1 and Σ_2, the element $T_{\Sigma_1}^2$

has infinite order, and hence the forgetful homomorphism has infinite kernel. The reader is invited to look at [Sei2] for a detailed description of symplectic Dehn twists.

Somewhat later, Biran and Giroux introduced different symplectomorphisms, namely the fibered Dehn twists, among which one can find smoothly yet not symplectically trivial maps. In fact, Seidel's Dehn twist is a particular case of a fibered Dehn twist. Suppose that that X admits a separating contact type hypersurface P carrying a free S^1-action in $P \times [0,1]$ that preserves the contact form on P. Then one can define the *fibered Dehn twist* as

$$T_P\colon P \times [0,1] \to P \times [0,1], \quad (x,t) \to (x \cdot [f(t) \bmod 2\pi], t),$$

where a function $f\colon [0,1] \to \mathbb{R}$ equals 2π near $t=0$ and 0 near $t=1$. As T_P is a symplectomorphism of $P \times [0,1]$ that is the identity near the boundary of $P \times [0,1]$, it can be extended to be a symplectomorphism of the whole X. We refer the reader to [R-D-O, U] for an extensive study of fibered Dehn twists.

Given that it is easy to find a separating contact hypersurface, fibered Dehn twists make an effective tool to construct symplectomorphisms of a given 4-manifold (and of a higher-dimensional manifold, for that matter.) But even though a plethora of results has been obtained in symplectic mapping class groups (see e.g. [Ab-McD, Bu, Anj, Anj-Gr, Anj-Lec, Ev, La-Pin, LiJ-LiT-Wu, Ton, Sei1, Sei3, Wen]), it remains hard to detect nontriviality of symplectomorphisms.

In this paper we introduce and study a new type of symplectomorphisms for 4-manifolds. In short, our construction is as follows. Let (X, ω_0) be a symplectic 4-manifold which contains a symplectically embedded torus $C \subset X$ of self-intersection (-1). In particular, $\mu := \int_C \omega_0 > 0$. We construct a family of symplectic forms ω_t on X in the cohomology class $[\omega_t]$ such that $\int_C \omega_t = \mu - t$. We show that such a family exists for t large enough for C to have a negative symplectic area.

For each t we construct an ω_t-symplectomorphism $T_C : X \to X$, called the *elliptic twist along* C. As smooth maps, those symplectomorphisms T_C for different t are isotopic, so we can think of T_C as a single diffeomorphism defined up to isotopy.

We then study whether or not these elliptic twist are symplectically isotopic to the identity. It appears that it is so in the case when $\int_C \omega_t > 0$. In particular, T_C is always smoothly isotopic to the identity. As we shall see below, it is not so in the case $\int_C \omega_t \leqslant 0$, and T_C could be non-trivial.

Let Y be a tubular neighbourhood of C in X. Then ∂Y is a separating contact hypersurface in X, which carries a free S^1-action. One can pick a symplectic form $\widetilde{\omega}_0$ on X such that $\omega_0|_{X-Y} = \widetilde{\omega}_0|_{X-Y}$ and $\int_C \widetilde{\omega}_0 \leqslant 0$. We conjecture that, for $(X, \widetilde{\omega}_0)$, the fibered Dehn twist associated to ∂Y is symplectically isotopic to T_C.

Our first result is an example of a 4-manifold X and a (-1)-torus C in it, where the elliptic twist T_C turns out to be always symplectically trivial.

Theorem 0.1. *Let (X, ω) be a symplectic ruled 4-manifold diffeomorphic to the total space of the non-trivial S^2-bundle over T^2; we denote it by $S^2 \tilde{\times} T^2$ for short. Then*

i) there is a symplectic form ω_0 on X which admits an ω_0-symplectic (-1)-torus $C \subset X$, and the elliptic twist T_C is well-defined.

ii) the forgetful homomorphism $\pi_0 \mathscr{S}ymp(X, \omega) \to \pi_0 \mathscr{D}iff(X)$ is injective for any symplectic form ω. In particular, the elliptic twist T_C is always symplectically isotopic to the identity.

The injectivity property claimed in part **ii**) was proved previously by McDuff for $S^2 \times T^2$, see [McD-B]. We thus cover the remaining non-spin case and, therefore, prove the so-called symplectic isotopy conjecture for elliptic ruled surfaces, see Problem 14 in [McD-Sa-1].

The main result of this note shows that it is possible for an elliptic twist to contribute nontrivially to a symplectic mapping class group.

Theorem 0.2. *Let Z be $S^2 \tilde{\times} T^2 \# \overline{\mathbb{CP}^2}$. There exist a symplectic form ω on Z and three ω-symplectic (-1)-tori C_1, C_2, and C_3 in Z such that the elliptic twists T_{C_i} are well-defined and none of them is symplectically isotopic to the identity; each T_{C_i} has infinite order in the symplectic mapping class group.*

Our proof follows closely to the ideas introduced by Abreu-McDuff in [Ab-McD] and McDuff in [McD-B].

The main technique we use in the proof is Gromov's theory of pseudoholomorphic curves. This theory involves various Banach manifolds and constructions with them. Dealing with them we often pretend to be in the finite-dimensional case. We refer the reader to the book [Iv-Sh-1] and articles [Iv-Sh-2, Iv-Sh-3] for a comprehensive analytic setup to Gromov's theory of pseudoholomorphic curves. Of course, reader is free to address to any of numerous alternative sources and expositions of the theory such as [McD-Sa-3] or the seminal paper [Gro].

Acknowledgements. We are deeply indebted to Boris Dubrovin, Yakov Eliashberg, and Viatcheslav Kharlamov for a number of useful suggestions which were crucial for the present exposition of this paper. Part of this note was significantly improved during our stay at the University of Pisa and the Humboldt University of Berlin, and we are very grateful to Paolo Lisca and Klaus Mohnke for numerous discussions and for the wonderful research environment they provided. We also would like to thank Rafael Torres for reading the manuscript and pointing out certain inconsistencies. Special thanks to Dasha Alexeeva for sending us a preprint of her thesis. Finally, we are grateful to the referee for a positive comment on our paper, for her/his constructive and thorough criticism, and for the tremendous amount of work he/she did reviewing the manuscript. The second author was supported by an ETH Fellowship.

1. Construction of the elliptic twist

1.1. Elliptic twist

Let (X, ω) be a symplectic 4-manifold, and let C be an embedded symplectic (-1)-torus in X. We let $\Omega(X, \omega)$ to denote the space of symplectic forms on X that are isotopic to ω, and let $\mathcal{J}(X, \Omega)$ to denote the space of almost-complex structures for which there exists a taming form in $\Omega(X, \omega)$.

Pick an almost-complex structure $J_0 \in \mathcal{J}(X, \Omega)$ for which C is pseudoholomorphic. One thinks of J_0 as a point of the subspace $\mathcal{D}_{[C]} \subset \mathcal{J}(X, \Omega)$ of those almost-complex structures which admit a smooth pseudoholomorphic curve in the class $[C]$. In what follows, we refer to $\mathcal{D}_{[C]}$ as the *elliptic divisorial locus* for the class $[C]$. The term *divisorial locus* is taken from the fact that in some neighbourhood of J_0 the subspace $\mathcal{D}_{[C]}$ locally behaves as a submanifold of $\mathcal{J}(X, \Omega)$ of real codimension 2, see e.g. [Iv-Sh-1].

Let $\Delta \subset \mathcal{J}(X, \Omega)$ be a small disc transverally intersecting $\mathcal{D}_{[C]}$ precisely at J_0, and let $J \colon [0, 1] \to \Delta$ be the boundary of Δ. We will make the following assumption:

(A) There exists a class $\xi \in \mathsf{H}_2(X; \mathbb{R})$, $\xi \cdot [C] \leqslant 0$ such that every $J(t) \in \partial\Delta$ is tamed by some symplectic form θ_t, $[\theta_t] = \xi$.

One can arrange θ_t so that they depend smoothly on t. Moser isotopy then gives us a path of diffeomorphisms $f_t \colon X \to X$, $f_t^* \theta_t = \theta_0$. Now f_1 is a symplectomorphism of (X, θ_0). We call f_1 the *elliptic twist along* C and use the notation T_C for it.

As we will explain below (see §1.2), for T_C to give a non-trivial element in $\pi_0\mathscr{S}ymp(X,\theta_0)$, it is necessary that $[J(t)] \in \pi_1(\mathcal{J}(X,\Theta))$ is non-trivial; here Θ stands for the space of symplectic forms on X that are isotopic to θ_0. We emphasize that $J(t)$ is contractible in $\mathcal{J}(X,\Omega)$; thus, T_C is trivial for (X,ω).

Assumption (A) always holds, though we do not prove it in the full generality. But we shall consider a series of 4-manifolds for which the assumption is easy to verify. Let (X,ω) be a symplectic 4-manifold, and let C be a symplectic torus of self-intersection number 0. Take an ω-tamed almost-complex structure on X for which C becomes pseudoholomorphic, and then perturb this structure slightly to make it integrable in some tubular neighbourhood of C. More precisely, we want a sufficiently small neighbourhood of C to admit an elliptic fibration with C being a multiple fiber of multiplity $m > 1$.

Let T^2 be an elliptic curve $\mathbb{C}/\mathbb{Z}\tau_1 \oplus \mathbb{Z}\tau 2$, where (τ_1,τ_2) form a basis for $\mathbb{C}(u)$ as a real vector space, and let Δ be a complex disc with a local parameter z. The neighbourhood of C in X is biholomorphic to the quotient $\Delta \times T^2/\sim$, where $(z,u) \sim (ze^{2\pi i/m}, u+\tau_1/m)$. Here C is given by the equation $\{z=0\}$.

Blowing-up X at a point $(0,u_0)$, we get a manifold Z which contains a smooth elliptic curve in the class $[C]-\boldsymbol{E}$ (the strict transform of C.) Here $\boldsymbol{E}$ stands for the homology class of the exceptional line. Unless $z_0 = 0$, the blow-up of X at (z_0,u_0) does not contain such a curve, since it contains one in the class $m[C]-\boldsymbol{E}$ (the strict transform of $\{z=z_0\}$, which we denote by C_m.)

Pick a taming symplectic form ω_0 on Z. Clearly, the form satisfies

$$\int_{[C]-\boldsymbol{E}} \omega > 0.$$

Let $Z(t)$ be the blow-up of X at $(R\,e^{it},u_0)$. Observe that the complex structures on $Z(t)$ are ω-tamed for R sufficiently small. Using the deflation (see §1.3) along C_m, we deform ω on $Z(t)$ into a symplectic form θ_t for which

$$\int_{[C_m]} \theta_t = \varepsilon$$

for ε positive arbitrary small. Implementing deflation does not violate the taming condition for Z_t. Being performed in a small neighbourhood of C_m, the deflation does not affect the symplectic area of C, see [Bu]. Since

$$\int_{[C]-\boldsymbol{E}} \theta_t = \varepsilon - (m-1)\int_{[C]} \omega,$$

one may take m sufficiently large to make the area of $[C] - \boldsymbol{E}$ as negative as desired. We have now verified (A) for the family $Z(t)$.

1.2. The Abreu-McDuff framework

Let $\mathscr{D}\!\mathit{iff}_0(X)$ be the identity component of the diffeomorphism group of X and $\Omega(X,\omega)$ be the space of all symplectic forms on X that are isotopic to ω. We have a natural transitive action of $\mathscr{D}\!\mathit{iff}_0(X)$ on $\Omega(X,\omega)$. So we get a principle fiber bundle

$$\mathscr{S}\!\mathit{ymp}(X,\omega) \cap \mathscr{D}\!\mathit{iff}_0(X) \to \mathscr{D}\!\mathit{iff}_0(X) \to \Omega(X,\omega), \tag{1.1}$$

where the last arrow stands for the map

$$\varphi : \mathscr{D}\!\mathit{iff}_0(X) \to \Omega(X,\omega) \qquad \text{with} \qquad \varphi : f \mapsto f_*\omega.$$

To shorten notation, we put:

$$\mathscr{S}\!\mathit{ymp}^*(X,\omega) := \mathscr{S}\!\mathit{ymp}(X,\omega) \cap \mathscr{D}\!\mathit{iff}_0(X)$$

Following [Kh], we consider an exact sequence of homotopy groups

$$\begin{aligned}\cdots \to \pi_1(\mathscr{D}\!\mathit{iff}_0(X)) &\xrightarrow{\varphi_*} \pi_1(\Omega(X,\omega))\\ &\xrightarrow{\partial} \pi_0(\mathscr{S}\!\mathit{ymp}^*(X,\omega)) \to 1 = \pi_0(\mathscr{D}\!\mathit{iff}_0(X)).\end{aligned}$$

Let $\mathcal{J}(X,\Omega)$ be the space of those almost-complex structures J on X for which there exists a taming symplectic form $\omega_J \in \Omega(X,\omega)$. It is easy to see that $\mathcal{J}(X,\Omega)$ is connected. Let J_0 be some ω-tamed almost-complex structure. It was shown by McDuff, see Lemma 2.1 in [McD-B], that there exists a homotopy equivalence $\psi\colon \Omega(X,\omega) \to \mathcal{J}(X,\Omega)$ for which the diagram

$$\begin{tikzcd} \mathscr{D}\!\mathit{iff}_0(X) \arrow[r, "\varphi"] \arrow[dr, "\nu"'] & \Omega(X,\omega) \arrow[d, "\psi"] \\ & \mathcal{J}(X,\Omega) \end{tikzcd} \tag{1.2}$$

commutes. Here $\nu\colon \mathscr{D}\!\mathit{iff}_0(X) \to \mathcal{J}(X,\Omega)$ is given by $\nu : f \mapsto f_*J_0$.

Following the fundamental idea of Gromov's theory [Gro] we study the space $\mathcal{J}(X,\Omega)$ rather than $\Omega(X,\omega)$. We see from the following diagram (1.3)

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \arrow[r, "\varphi_*"] \arrow[d, "id"'] & \pi_1(\Omega(X,\omega)) \arrow[r, "\partial"] \arrow[d, "\psi_*"'] & \pi_0(\mathscr{S}\!\mathit{ymp}^*(X,\omega)) \arrow[r] & 0 \\
\cdots \arrow[r] & \pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \arrow[r, "\nu_*"] & \pi_1(\mathcal{J}(X,\Omega)),
\end{tikzcd}$$

that each loop in $\mathcal{J}(X,\Omega)$ contributes non-trivially to the symplectic mapping class group of X, provided this loop does not come from $\mathscr{D}\!\mathit{iff}_0(X)$. We will use diagram (1.3) to prove both Theorem 0.1 and Theorem 0.2. The reader is referred to [McD-B] for more extensive discussion of the topic.

In what follows we work with a slightly bigger space $\mathcal{J}^k(X,\omega)$ of C^k-smooth almost-complex structures. Here and below "C^k-smoothness" means some $C^{k,\alpha}$-smoothness with $0<\alpha<1$ and k natural sufficiently large. The reason to do this is that the space $\mathcal{J}^k(X,\omega)$ is a Banach manifold, while the space of C^∞-smooth structures $\mathcal{J}(X,\Omega)$ is merely Fréchet. What we prove for $\pi_i(\mathcal{J}^k(X,\omega))$ works perfectly for $\pi_i(\mathcal{J}(X,\Omega))$ because the inclusion $\mathcal{J}(X,\Omega)\hookrightarrow\mathcal{J}^k(X,\omega)$ induces the weak homotopy equivalence $\pi_i(\mathcal{J}(X,\Omega)) \to \pi_i(\mathcal{J}^k(X,\omega))$.

1.3. Symplectic economics

Here we give a brief description of the inflation technique developed by Lalonde-McDuff [La-McD, McD-B], and a generalization of this procedure given by Buşe, see [Bu].

Theorem 1.1 (Inflation). *Let J be an ω_0-tamed almost complex structure on a symplectic 4-manifold (X,ω_0) that admits an embedded J-holomorphic curve C with $[C]\cdot[C]\geqslant 0$. Then there is a family $\omega_s, s\geqslant 0$, of symplectic forms that all tame J and have cohomology class*

$$[\omega_s]=[\omega_0]+s\,\mathsf{PD}([C]),$$

where $\mathsf{PD}([C])$ is Poincaré dual to $[C]$.

For negative curves a somewhat reverse procedure exists, called negative inflation or deflation.

Theorem 1.2 (Deflation). *Let J be an ω_0-tamed almost complex structure on a symplectic 4-manifold (X,ω_0) that admits an embedded J-holomorphic curve C with $[C]\cdot[C]=-m$. Then there is a family ω_s of symplectic forms that all tame J and have cohomology class*

$$[\omega_s]=[\omega_0]+s\,\mathsf{PD}([C])$$

for all $0\leqslant s<\dfrac{\omega_0([C])}{m}$.

2. Elliptic geometrically ruled surfaces

2.1. General remarks

A complex surface X is called ruled if there exists a holomorphic map $\pi : X\to Y$ to a Riemann surface Y such that each fiber $\pi^{-1}(y)$ is a rational curve; if, in addition, each fiber is irreducible, then X is called geometrically ruled. A ruled surface is obtained by blowing up a geometrically ruled surface. Note however that a geometrically ruled surface need not be minimal (the blow up of $\mathbb{CP}^2$, denoted by $\mathbb{CP}^2\,\#\,\overline{\mathbb{CP}^2}$, is the unique example of a geometrically ruled surface that is not a minimal one). Unless otherwise noted, all ruled surfaces are assumed to be geometrically ruled. One can speak of the genus of the ruled surface X, meaning thereby the genus of Y. We thus have rational ruled surfaces, elliptic ruled surfaces and so on.

Up to diffeomorphism, there are two total spaces of orientable S^2-bundles over a Riemann surface: the product $S^2\times Y$ and the non-trivial bundle $S^2\tilde{\times}Y$. The product bundle admits sections $Y_{2\,k}$ of even self-intersection number $[Y_{2\,k}]^2=2\,k$, and the non-trivial bundle admits sections $Y_{2\,k+1}$ of odd self-intersection number $[Y_{2\,k+1}]^2=2\,k+1$. We will choose the basis $\boldsymbol{Y}=[Y_0], \boldsymbol{S}=[\mathrm{pt}\times S^2]$ for $\mathsf{H}_2(S^2\times Y;\mathbb{Z})$, and use the basis $\boldsymbol{Y}_-=[Y_{-1}], \boldsymbol{Y}_+=[Y_1]$ for $\mathsf{H}_2(S^2\tilde{\times}Y;\mathbb{Z})$. To simplify notations, we denote both the classes $\boldsymbol{S}$ and $\boldsymbol{Y}_+-\boldsymbol{Y}_-$, which are the fiber classes of the ruling, by $\boldsymbol{F}$. Further, the class $\boldsymbol{Y}_++\boldsymbol{Y}_-$, which is a class for a bisection of X, will be of particular interest for us, and will be widely used in forthcoming computations; we denote this class by $\boldsymbol{B}$. Throughout this paper we will freely identify homology and cohomology by Poincaré duality.

Clearly, we have $[Y_{2\,k}]=\boldsymbol{Y}+k\,\boldsymbol{F}$ and $[Y_{2\,k+1}]=\boldsymbol{Y}_++k\,(\boldsymbol{Y}_++\boldsymbol{Y}_-)$. This can be seen by evaluating the intersection forms for these 4-manifolds on the given basis:

$$\mathcal{Q}_{S^2\times Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{Q}_{S^2\tilde{\times}Y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that these forms are non-isomorphic. That is why the manifolds $S^2 \times Y$ and $S^2\tilde{\times}Y$ are non-diffeomorphic. One more way to express the difference between them is to note that the product $S^2 \times Y$ is a spin 4-manifold, but $S^2\tilde{\times}Y$ is not spin. Note that after blowing up one point, they become diffeomorphic: $S^2 \times Y \# \overline{\mathbb{CP}^2} \simeq S^2\tilde{\times}Y \# \overline{\mathbb{CP}^2}$.

This section is mainly about the non-spin elliptic ruled surface $S^2\tilde{\times}T^2$. When studying this manifold we sometimes use the notations $\boldsymbol{T}_+$ and $\boldsymbol{T}_-$ instead of $\boldsymbol{Y}_+$ and $\boldsymbol{Y}_-$ for the standard homology basis in $\mathsf{H}^2(S^2\tilde{\times}T^2;\mathbb{Z})$.

From the viewpoint of complex geometry every such X is a holomorphic $\mathbb{CP}^1$-bundle over a Riemann surface Y whose structure group is $\boldsymbol{PGl}(2,\mathbb{C})$. Biholomorphic classification of ruled surfaces is well understood, at least for low values of the genus. Below we recall a part of the classification of elliptic ruled surfaces given by Atiyah in [At-2]; this being the first step towards understanding the almost-complex geometry of these surfaces. We also provide a short summary of Suwa's results: i) an explicit construction of a complex analytic family of ruled surfaces, where one can see the jump phenomenon of complex structures, see §2.3, ii) an examination of those complex surfaces which are both ruled and admit an elliptic pencil, see Theorem 2.3.

In what follows we will use a formula for the first Chern class of a geometrically ruled surface. In terms of $\boldsymbol{Y}, \boldsymbol{S}, \boldsymbol{Y}_\pm$, it becomes

$$\begin{aligned} c_1(S^2 \times Y) &= 2\,\boldsymbol{Y} + \chi(Y)\,\boldsymbol{S}, \\ c_1(S^2\tilde{\times}Y) &= (1+\chi(Y))\,\boldsymbol{Y}_+ + (1-\chi(Y))\,\boldsymbol{Y}_-. \end{aligned} \tag{2.1}$$

The symplectic geometry of ruled surfaces has been extensively studied by many authors [Li-Li, Li-Liu-1, Li-Liu-2, AGK, Sh-4, H-Iv]. Ruled surfaces are of great interest from the symplectic point of view mainly because of the following significant result due to Lalonde-McDuff, see [La-McD, McD-6].

Theorem 2.1 (The classification of ruled 4-manifolds). *Let X be oriented diffeomorphic to a minimal rational or ruled surface, and let $\xi \in \mathsf{H}^2(X)$. Then there is a symplectic form (even a Kähler one) on X in the class ξ iff $\xi^2 > 0$. Moreover, any two symplectic forms in the class ξ are diffeomorphic.*

Thus all symplectic properties of ruled surfaces depend only on the cohomology class of a symplectic form.

Our main interest is to study symplectic (-1)-tori in X and the corresponding elliptic twists. It is easy to prove that, except for $S^2\tilde{\times}T^2$, there are no symplectic (-1)-tori in ruled surfaces. For a suitable symplectic form the homology class $\boldsymbol{T}_- \in \mathsf{H}_2(S^2\tilde{\times}T^2;\mathbb{Z})$ can be represented by a symplectic (-1)-torus, but none of the other classes of $\mathsf{H}_2(S^2\tilde{\times}T^2;\mathbb{Z})$ can.

Let (X,ω_μ) be a symplectic ruled 4-manifold $(S^2\tilde{\times}T^2,\omega_\mu)$, where ω_μ is a symplectic structure of the cohomology class $[\omega_\mu] = \boldsymbol{T}_+ - \mu\boldsymbol{T}_-$, $\mu \in (-1,1)$. By Theorem 2.1 (X,ω_μ) is well-defined up to symplectomorphism. As promised in the introduction, we will prove that $\pi_0\mathscr{S}\!ymp^*(X,\omega_\mu)$ is trivial. Here and in §2.7 we abbreviate $\Omega(X,\omega_\mu)$ to Ω_μ.

Given $\mu > 0$, the elliptic divisorial locus is contained in $\mathcal{J}(X,\Omega_\mu)$. Thus, each loop linked to the locus is contractible in $\mathcal{J}(X,\Omega_\mu)$. As such, we do not expect any non-trivial elliptic twists in this case. Following McDuff [McD-B], we will show that the group $\mathscr{S}\!ymp^*(X,\omega_\mu)$ coincides with a group of certain diffeomorphisms, see Lemma 2.15; the latter group can be proved to be connected by standard topological techniques, see Proposition 2.11.

When $\mu \leqslant 0$, the elliptic divisorial locus $\mathcal{D}_{\boldsymbol{T}_-}$ is no longer included in $\mathcal{J}(X,\Omega_\mu)$. The geometry of this divisorial locus is studied below in §2.7, and particularly it is proved that: i) Assumption (A) is satisfied for each loop linked to $\mathcal{D}_{\boldsymbol{T}_-}$; hence, (X,ω_μ) admits certain elliptic twists, see Lemma 2.13. ii) The symplectic mapping class group $\mathscr{S}\!ymp^*(X,\omega_\mu)$ is generated by elliptic twists coming from $\mathcal{D}_{\boldsymbol{T}_-}$. iii) Each of them is symplectically isotopic to the identity, see Lemma 2.18.

2.2. Classification of complex surfaces ruled over elliptic curves

Here we very briefly describe possible complex structures on elliptic ruled surfaces and study some of their properties.

Let X be diffeomorphic to either $S^2 \times Y^2$ or $S^2\tilde{\times}Y^2$. The Enriques-Kodaira classification of complex surfaces (see e.g.[BHPV]) ensures the following:

1) Every complex surface X of this diffeomorphism type is algebraic and hence Kähler.

2) Every such complex surface X is ruled, i.e. there exists a holomorphic map $\pi : X \to Y$ such that Y is a complex curve, and each fiber $\pi^{-1}(y)$ is an irreducible rational curve. Note that, with the single exception of $\mathbb{CP}^1 \times \mathbb{CP}^1$, a ruled surface admits at most one ruling.

It was shown by Atiyah [At-2] that every holomorphic $\mathbb{CP}^1$-bundle over a curve Y with structure group the projective group $\boldsymbol{PGl}(2,\mathbb{C})$ admits a holomorphic section, and hence the structure group of such bundle can be reduced to the affine group $\boldsymbol{Aff}(1,\mathbb{C}) \subset \boldsymbol{PGl}(2,\mathbb{C})$.

All of what was said so far applied for any ruled surface, irrespective of genus. Keep in mind, however, that everything below is for genus one surfaces. It was Atiyah who gave a classification of ruled surfaces with base an elliptic curve. The description presented here is taken from [Sw].

Theorem 2.2 (Atiyah). *Every holomorphic $\mathbb{CP}^1$-bundle with structure group $\boldsymbol{PGl}(2,\mathbb{C})$ over an elliptic curve is isomorphic to preciesly one of the following:*

i) a bundle associated to a principal $\mathbb{C}^*$-bundle of nonpositive degree,

ii) a bundle $\mathcal{A}$, defined below, having structure group $\boldsymbol{Aff}(1,\mathbb{C})$, and

iii) a bundle $\mathcal{A}^{Spin}$, having structure group $\boldsymbol{Aff}(1,\mathbb{C})$.

We shall proceed with a little discussion of these bundles:

i) We first describe those $\boldsymbol{PGl}(2,\mathbb{C})$-bundles whose structure group reduces to $\mathbb{C}^*$. Let $y \in Y$ be a point on the curve Y, and let $\{V_0, V_1\}$ be an open cover of Y such that $V_0 = Y \setminus \{y\}$ and V_1 is a small neighbourhood of y, so the domain $V_0 \cap V_1 =: \hat{V}$ is a punctured disc. We choose a multivalued coordinate u on Y centered at y.

A surface X_k associated to the line bundle $\mathscr{O}(k\,y)$ (or if desired, a $\mathbb{C}^*$-bundle) can be described as follows:

$$X_k := \left(V_0 \times \mathbb{CP}^1\right) \cup \left(V_1 \times \mathbb{CP}^1\right) / \sim,$$

where $(u, z_0) \in V_0 \times \mathbb{CP}^1$ and $(u, z_1) \in V_1 \times \mathbb{CP}^1$ are identified iff $u \in \hat{V}$, $z_1 = z_0 u^k$. Here z_0, z_1 are inhomogeneous coordinates on the two copies of $\mathbb{CP}^1$. Clearly, the biholomorphism $(u, z_0) \to (u, z_0^{-1})$, $(u, z_1) \to (u, z_1^{-1})$ maps X_k to X_{-k}. Thus it is sufficient to consider only values of k that are nonpositive.

There is a natural $\mathbb{C}^*$-action on X_k via $g \cdot (z_0, u) := (gz_0, u)$, $g \cdot (z_1, u) := (gz_1, u)$ for each $g \in \mathbb{C}^*$. The fixed point set of this action consists of two mutually disjoint sections Y_k and Y_{-k} defined respectively by $z_0 = z_1 = 0$ and $z_0 = z_1 = \infty$. We have $[Y_k]^2 = k$ and $[Y_{-k}]^2 = -k$.

It is very well known that any line bundle L of degree $\mathsf{deg}\,(L) = k \neq 0$ is isomorphic to $\mathscr{O}(k\,y)$ for some $y \in Y$. Thus all the ruled surfaces associated with line bundles of non-zero degree k are biholomorphic to one and the same surface X_k. On the other hand, the parity of the degree of the underlying line bundle is a topological invariant of a ruled surface. More precisely, a ruled surface X associated with a line bundle L is diffeomorphic to $Y \times S^2$ for $\mathsf{deg}\,(L)$ even, and to $Y \tilde{\times} S^2$ for $\mathsf{deg}\,(L)$ odd.

ii) Again, we start with an explicit description of the ruled surface $X_{\mathcal{A}}$ associated with the affine bundle $\mathcal{A}$. Let $\{V_0, V_1, \hat{V}\}$ be the open cover of Y as before, u be a coordinate on Y centered at y, and z_0, z_1 be fiber coordinates. Define

$$X_{\mathcal{A}} := \big(V_0 \times \mathbb{CP}^1\big) \cup \big(V_1 \times \mathbb{CP}^1\big) \,/ \sim,$$

where $(z_0, u) \sim (z_0, u)$ for $u \in \hat{V}$ and $z_0 = z_1 u + u^{-1}$.

There is an obvious section Y_1 defined by the equation $z_0 = z_1 = \infty$, but in contrast to $\mathbb{C}^*$-bundles, the surface $X_{\mathcal{A}}$ contains no section disjoint from that one. This can be shown by means of direct computation in local coordinates, but one easily deduce this from Theorem 2.3 below.

We will make repeated use of the following geometric characterization of $X_{\mathcal{A}}$, whose proof is given in [Sw], see Theorem 5.

Theorem 2.3. *The surface $X_{\mathcal{A}}$ associated with the affine bundle $\mathcal{A}$ admits an elliptic fibration over $\mathbb{CP}^1$; the general fiber is a smooth elliptic curve in the class $2\boldsymbol{T}_+ + 2\boldsymbol{T}_-$ and there are three multiple fibers each of which is a smooth elliptic curve in the class $\boldsymbol{T}_+ + \boldsymbol{T}_-$. There are no other singular fibers.*

The following corollary will be used later. The reader is invited to look at [McD-D] for the definition of the Gromov invariants and some examples of their computation.

Corollary 2.4. $\mathsf{Gr}(\boldsymbol{Y}_+ + \boldsymbol{Y}_-) = 3$.

Proof. There are no smooth curves in $X_{\mathcal{A}}$, other than the multiple fibers, which are in the class $\boldsymbol{T}_+ + \boldsymbol{T}_-$. Each multiple fiber contributes ± 1 to $\mathsf{Gr}(\boldsymbol{T}_+ + \boldsymbol{T}_-)$, for their normal bundles are holomorphically non-trivial, see § 1.7 in [McD-D]. If the complex structure is integrable, then each *non-multiple-covered* torus should appear with sign $(+1)$, see [Tb]. □

Based on this theorem, Suwa then gives another construction of $X_{\mathcal{A}}$. We mention this construction here because it appears to have interest for the sequel.

Let $Y \cong \mathbb{C}/\mathbb{Z}\tau_1 \oplus \mathbb{Z}\tau_2$ be an elliptic curve, and u be a multivalued coordinate on Y. Define $X_{\mathcal{A}}$ to be a quotient space of $\mathbb{CP}^1 \times Y/\mathcal{G}$, where $\mathcal{G} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by the following involutions

$$(z,u) \to \left(-z, u + \frac{\tau_1}{2}\right), \quad (z,u) \to \left(\frac{1}{z}, u + \frac{\tau_2}{2}\right).$$

The surface obtained is elliptic ruled and is non-spin; see [Sw], where the latter is proved by constructing a section for $X_{\mathcal{A}}$ of odd self-intersection number, see also Exercises 6.13 and 6.14 in [McD-Sa-1].

The elliptic fibration of $X_{\mathcal{A}}$ mentioned in Theorem 2.3 comes from the $\mathcal{G}$-invariant function

$$f(z,u) = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right),$$

whose values are regular for all but three points of $\mathbb{CP}^1$. For a regular point, when $z \neq \{-1, 1, \infty\}$, the fiber $f^{-1}(z)$ is an elliptic curve in the class $2(\boldsymbol{T}_+ + \boldsymbol{T}_-)$, whereas each of the three singular fibers is a curve in the class $\boldsymbol{T}_+ + \boldsymbol{T}_-$.

There is an obvious action of the complex torus $\mathcal{T} \cong Y$ on $\mathbb{CP}^1 \times Y$ by translations. This action commutes with that of $\mathcal{G}$. Hence, $\mathcal{T}$ acts also on $X_{\mathcal{A}}$. As the function f is $\mathcal{T}$-invariant, so are the fibers $f^{-1}(z)$, $z \in \mathbb{CP}^1$ of our elliptic fibration; they are, in fact, simply the orbits of the action. Although $\mathcal{T}$ acts effectively on $X_{\mathcal{A}}$, it does not act freely; the isotropy groups of the singular fibers correspond to the three pairwise different order two subgroups of $\mathcal{T}$. For instance, for $(z,u) \in f^{-1}(\infty)$, the stabilizer is $z \to -z$.

As each fiber $f^{-1}(z)$ is the torus, it gives a homomorphism $\mathsf{H}_1(f^{-1}(z);\mathbb{Z}) \to \mathsf{H}_1(X_{\mathcal{A}};\mathbb{Z})$ between the two copies of $\mathbb{Z}^2$. To see what this homomorphism is for the multiple fibers, we regard $X_{\mathcal{A}}$ as a ruled surface over $Y' \cong \mathbb{C}/\mathbb{Z}(\tau_1/2) \oplus \mathbb{Z}(\tau_2/2)$. Then the multiple fibers appear as bisections, double covering of Y'. Note that there is a one-to-one correspondence between the double covering of Y' and the index 2 subgroups of $\mathsf{H}_1(Y';\mathbb{Z})$. This implies that, for the singular fibers $f^{-1}(z), z = \{-1, 1, \infty\}$, the images of $\mathsf{H}_1(f^{-1}(z);\mathbb{Z}) \to \mathsf{H}_1(X_{\mathcal{A}};\mathbb{Z})$ correspond to three pairwise different index 2 subgroups of $\mathsf{H}_1(X_{\mathcal{A}};\mathbb{Z}) \cong \mathsf{H}_1(Y';\mathbb{Z})$.

iii) The ruled surface associated to $\mathcal{A}^{Spin}$ is diffeomorphic to $S^2 \times T^2$, thus it will not be discussed in this note, but see [Sw].

Summarizing our above observations, we see that $X \cong S^2 \tilde{\times} T^2$ admits countably many diffeomorphism classes of complex structures. These structures are as follows:

- The structures $J \in \mathscr{J}_{1-2k}$, $k > 0$, such that the ruled surface (X, J) contains a section of self-intersection number $1 - 2k$; these are all biholomorphic to X_{1-2k}.
- The type $\mathcal{A}$ structures $J \in \mathscr{J}_{\mathcal{A}}$ such that the ruled surface (X, J) contains no sections of negative self-intersection number but does contain a triple of smooth bisections; these are all biholomorphic to $X_{\mathcal{A}}$.

2.3. One family of ruled surfaces over elliptic base

Here is a construction of a one-parametric complex-analytic family $p : \mathscr{X} \to \mathbb{C}$ of non-spin elliptic ruled surfaces, such that the surfaces $p^{-1}(t)$, $t \neq 0$, are biholomorphic to $X_{\mathcal{A}}$ and $p^{-1}(0) \cong X_{-1}$.

As before, we take a point y on Y, let u be a coordinate of the center y, and put $\{V_0, V_1, \hat{V}\}$ to be an open cover for Y such that $V_0 := Y \setminus \{y\}$, V_1 is a small neighbourhood of y, and $\hat{V} := V_0 \cap V_1$. Further, let Δ be a complex plane, and let t be a coordinate on it.

We construct the complex 3-manifold $\mathscr{X}$ by patching $\Delta \times V_0 \times \mathbb{CP}^1$ and $\Delta \times V_1 \times \mathbb{CP}^1$ in such a way that $(t, z_0, u) \sim (t, z_1, u)$ for $u \in \hat{V}$ and $z_0 = z_1 u + t u^{-1}$.

The preimage of 0 and 1 under the natural projection $p : \mathscr{X} \to \Delta$ are biholomorphic respectively to X_{-1} and $X_{\mathcal{A}}$. In fact, it is not hard to see that for each $t \neq 0$, the surface $p^{-1}(t)$ is biholomorphic to $X_{\mathcal{A}}$ as well. One way to prove this is to use the $\mathbb{C}^*$-action on $\mathscr{X}$

$$g \cdot (t, z_0, u) := (tg, g z_0, u), \quad g \cdot (t, u z_1, u) := (tg, g z_1, u) \quad \text{for each } g \in \mathbb{C}.$$

This proves even more than we desired, namely, that there exists a $\mathbb{C}^*$-action on $\mathscr{X}$ such that for each $g \in \mathbb{C}^*$ we get a commutative diagram

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\cdot g} & \mathscr{X} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow[\cdot g]{} & \mathbb{C}, \end{array}$$

where $\mathscr{X} \xrightarrow{\cdot g} \mathscr{X}$ denotes the biholomorphism induced by $g \in \mathbb{C}^*$.

The construction of the complex-analytic family $\mathscr{X}$ is due to Suwa, see [Sw], though the existence of the $\mathbb{C}^*$-action was not mentioned in Suwa's paper. Let us summarize his result in a theorem.

2.4. Embedded curves and almost-complex structures

In Section 2.2 the classification for non-spin elliptic ruled surfaces was given. It turns out that this classification can be extended to the almost-complex geometry of $S^2\tilde{\times}T^2$.

Let X be diffeomorphic to $S^2\tilde{\times}T^2$, and let $\mathcal{J}(X)$ be the space of almost-complex structures on X that are tamed by some symplectic form; the symplectic forms need not be the same. Here we use the short notation $\mathcal{J}$ for $\mathcal{J}(X)$.

Given $k>0$, let $\mathcal{J}_{1-2k}(X)$ (we will abbreviate it to $\mathcal{J}_{1-2k}$) be the subset of $J\in\mathcal{J}$ consisting of elements that admit a smooth irreducible J-holomorphic elliptic curve in the class $\boldsymbol{T}_+ - k\,\boldsymbol{F}$. It is well known that $\mathcal{J}_{1-2k}$ forms a subvariety of $\mathcal{J}$ of real codimension $2\cdot(2k-1)$, see e.g. Corollary 8.2.3 in [Iv-Sh-1].

Further, define $\mathcal{J}_{\mathcal{A}}(X)$ (or $\mathcal{J}_{\mathcal{A}}$, for short) be the subset $J\in\mathcal{J}$ of those element for which there exists a smooth irreducible J-holomorphic elliptic curve in the class $\boldsymbol{B}$.

By straightforward computations one can show that the sets $\mathcal{J}_{1-2k}$ are mutually disjoint, and each $\mathcal{J}_{1-2k}$ is disjoint from $\mathcal{J}_{\mathcal{A}}$. Further, it is not hard to see that $\mathcal{J}_{-1}\subset\overline{\mathcal{J}}_{\mathcal{A}}$ and $\mathcal{J}_{1-2(k+1)}\subset\overline{\mathcal{J}}_{1-2k}$, where $\overline{\mathcal{J}}_{1-2k}$ is for the closure of $\mathcal{J}_{1-2k}$. A less trivial fact is that

$$\mathcal{J}=\mathcal{J}_{\mathcal{A}}\bigsqcup\mathcal{J}_{-1}\bigsqcup\mathcal{J}_{-3}\bigsqcup\mathcal{J}_{-5}\cdots, \tag{2.2}$$

which can be also stated as follows.

Proposition 2.5 (cf. Lemma 4.2 in [McD-B]). *Let (X,ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2\tilde{\times}T^2$. Then every ω-tamed almost-complex structure J admits a smooth irreducible J-holomorphic representative in either $\boldsymbol{B}$ or $\boldsymbol{T}_+ - k\,\boldsymbol{F}$ for some $k>0$.*

Proof. The proof is analogous to that of Lemma 4.2 in [McD-B]. Observe that the expected codimension for the class $\boldsymbol{B}$ is zero. By Lemma 2.4 we have $\mathsf{Gr}(\boldsymbol{T}_+ + \boldsymbol{T}_-)>0$. Hence, $\mathcal{J}_{\mathcal{A}}$ is an open dense subset of $\mathcal{J}$, and, thanks to the Gromov compactness theorem, for each $J\in\mathcal{J}$ the class $\boldsymbol{B}$ has at least one J-holomorphic representative, possibly singular, reducible or having multiple components.

By virtue of Theorem 2.7, no matter what J was chosen, our manifold X admits the smooth J-holomorphic ruling π by rational curves in the class $\boldsymbol{F}$.

Since $\boldsymbol{B} \cdot \boldsymbol{F} > 0$, it follows from positivity of intersections that any J-holomorphic representative B of the class $\boldsymbol{B}$ must either intersect a J-holomorphic fiber of π or must contain this fiber completely.

a) First assume that B is irreducible. Then it is of genus not greater than 1 because of the adjunction formula. This curve is of genus 1 because every spherical homology class of X is proportional to $\boldsymbol{F}$. We now can apply the adjunction formula one more time to conclude that B is smooth, i.e. $J \in \mathcal{J}_{\mathcal{A}}$.

b) The curve B is reducible but contains no irreducible components which are the fibers of π. Then it contains precisely two components B_1 and B_2, since $\boldsymbol{B} \cdot \boldsymbol{F} = 2$. Both the curves B_1 and B_2 are smooth sections of π, and hence $[B_i] = \boldsymbol{T}_+ + k_i \boldsymbol{F}$, $i = 1, 2$. Since $[B_1] + [B_2] = \boldsymbol{B}$, it follows that $k_1 + k_2 = -1$, and hence either k_1 or k_2 is negative. Thus we have that either B_1 or B_2 is a smooth J-holomorphic section of negative self-intersection index.

c) If some of the irreducible components of B are in the fibers class $\boldsymbol{F}$, then one can apply arguments similar to that used in *a)* and *b)* to prove that the part B' of B which contains no fiber components has a section of negative self-intersection index as a component. □

2.5. Rulings and almost-complex structures

Let X be a ruled surface equipped with a ruling $\pi : X \to Y$, and let J be an almost-complex structure on X. We shall say that J *is compatible with the ruling* $\pi : X \to Y$ if each fiber $\pi^{-1}(y)$ is J-holomorphic.

We wish to express our thanks to D. Alexeeva [Al] for sharing her proof of the following statement.

Proposition 2.6. *Let $\mathcal{J}(X, \pi)$ be the space of almost-complex structures on X compatible with π.*

i) $\mathcal{J}(X, \pi)$ is contractible.

ii) Any structure $J \in \mathcal{J}(X, \pi)$, as well as any compact family $J_t \in \mathcal{J}(X, \pi)$, is tamed by some symplectic form.

Proof. i) Let be $\mathbb{J}(\mathbb{R}^4, \mathbb{R}^2)$ be the space of linear maps $J : \mathbb{R}^4 \to \mathbb{R}^2$ such that $J^2 = -\mathit{id}$ and $J(\mathbb{R}^2) = \mathbb{R}^2$, i.e. it is the space of linear complex structures preserving $\mathbb{R}^2$. In addition, we assume $\mathbb{R}^4$ and $\mathbb{R}^2$ are both oriented and each $J \in \mathbb{J}(\mathbb{R}^4, \mathbb{R}^2)$ induces the given orientations for both $\mathbb{R}^4$ and $\mathbb{R}^2$. We now prove the space $\mathbb{J}(\mathbb{R}^4, \mathbb{R}^2)$ is contractible.

Indeed, let us take $J \in \mathbb{J}(\mathbb{R}^4, \mathbb{R}^2)$. Fix two vectors $e_1 \in \mathbb{R}^2$ and $e_2 \in \mathbb{R}^4 \setminus \mathbb{R}^2$. The vectors e_1 and Je_1 form a positively oriented basis for $\mathbb{R}^2$. Therefore Je_1 is in the upper half-plane for e_1. Further, the vectors e_1, Je_1, e_2, Je_2 form a positively oriented basis for $\mathbb{R}^4$. Therefore Je_2 is in the upper half-space for the hyperplane spanned on e_1, Je_1, e_2.

We see that the space $\mathbb{J}(\mathbb{R}^4, \mathbb{R}^2)$ is homeomorphic to the direct product of two half-spaces, and hence it is for sure contractible.

To finish the proof of i) we consider the subbundle $V_x := \mathsf{Ker}\, \mathrm{d}\,\pi(x) \subset T_xX$, $x \in X$, of the tangent bundle TX of X. Every $J \in \mathcal{J}(X, \pi)$ is a section of the bundle $\mathbb{J}(TX, V) \to X$ whose fiber over $x \in X$ is the space $\mathbb{J}(T_xX, V_x)$. Since the fibers of $\mathbb{J}(TX, V)$ are contractible; it follows that the space of section for $\mathbb{J}(TX, V)$ is contractible as well.

ii) Again, we start with some linear algebra. Let V be a 2-subspace of $W \cong \mathbb{R}^4$, and let $J \in \mathbb{J}(W, V)$. Choose a 2-form $\tau \in \Lambda^2(W)$ such that the restriction $\tau|_V \in \Lambda^2(V)$ of τ to V is positive with respect to the J-orientation of V, i.e. $\tau(\xi, J\xi) > 0$. Clearly, the subspace $H := \mathsf{Ker}\, \tau \subset W$ is a complement to V. Further, let $\sigma \in \Lambda^2(V)$ be any 2-form such that $\sigma|_V$ vanishes, but $\sigma|_H$ does not. If H is given the orientation induced by σ, then the J-orientation of W agrees with that defined by the direct sum decomposition $W \cong V \oplus H$. We now prove that J is tamed by $\tau + K\,\sigma$ for $K > 0$ sufficiently large.

It is easy to show that there exists a basis $e_1, e_2 \in V$, $e_3, e_4 \in H$ for W such that J takes the form

$$J = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix Ω of $\tau + K\,\sigma$ with respect to this basis is block-diagonal, say

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & K\,\sigma + \cdots \\ 0 & 0 & -K\,\sigma + \cdots & 0 \end{pmatrix} \text{ for } \sigma > 0.$$

It remains to check that the matrix ΩJ is positive definite, i.e. $(\xi, \Omega J\xi) > 0$. A matrix is positive definite iff its symmetrization is positive definite. It is straightforward to check that $\Omega J + (\Omega J)^t$ is of that kind for K large enough.

Let us go back to the ruled surface X. The theorem of Thurston [Th] (see also Theorem 6.3 in [McD-Sa-1]) ensures the existence of a closed 2-form τ on X such that the restrictions of τ to each fiber $\pi^{-1}(y)$ is non-degenerate. Choose an area form σ on Y. By the same reasoning as before, any $J \in \mathcal{J}(X, \pi)$ is tamed by $\tau + K\,\pi^*\sigma$ for K large enough. $\square$

The following theorem by McDuff motivates the study of compatible almost-complex structures, see Lemma 4.1 in [McD-B].

Theorem 2.7. *Let X be an irrational ruled surface, and let $J \in \mathcal{J}(X)$. Then there exists a unique ruling $\pi : X \to Y$ such that $J \in \mathcal{J}(X, \pi)$.*

2.6. Diffeomorphisms

Let X be diffeomorphic to either $T^2 \times S^2$ or $T^2 \tilde{\times} S^2$, and let $\pi : X \to Y$ be a smooth ruling. Further, let $\mathsf{Fol}(X)$ be the space of all smooth foliations of X by spheres in the fiber class $\boldsymbol{F}$ (the class $\boldsymbol{F}$ generates $\pi_2(X)$ and, therefore, it is the only class that can be the fiber class of an S^2-fibration.)

The group $\mathscr{D}\!\mathit{iff}(X)$ acts transitively on $\mathsf{Fol}(X)$ as well as the group $\mathscr{D}\!\mathit{iff}_0(X)$ acts transitively on a connected component $\mathsf{Fol}_0(X)$ of $\mathsf{Fol}(X)$. This gives rise to a fibration sequence

$$\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X) \to \mathscr{D}\!\mathit{iff}_0(X) \to \mathsf{Fol}_0(X),$$

where $\mathcal{D}$ is the group of fiberwise diffeomorphisms of X. By the definition of $\mathcal{D}$ there exists a projection homomorphism $\tau : \mathcal{D} \to \mathscr{D}\!\mathit{iff}(T^2)$ such that for every $F \in \mathcal{D}$ we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ T^2 & \xrightarrow[\tau(F)]{} & T^2, \end{array}$$

which induces the corresponding commutative diagram for homology

$$\begin{array}{ccc} \mathsf{H}_1(X;\mathbb{Z}) & \xrightarrow{F_*} & \mathsf{H}_1(X;\mathbb{Z}) \\ {\scriptstyle\pi_*}\big\downarrow & & \big\downarrow{\scriptstyle\pi_*} \\ \mathsf{H}_1(T^2;\mathbb{Z}) & \xrightarrow[\tau(F)_*]{} & \mathsf{H}_1(T^2;\mathbb{Z}). \end{array}$$

Notice that $\tau(F)$ is isotopic to the identity if only if $\tau(F)_* = \mathit{id}$. Since π_* is an isomorphism, it follows that the subgroup $\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$ of $\mathcal{D}$ is mapped by τ to $\mathscr{D}\!\mathit{iff}_0(T^2)$, so we end up with the restricted projection homomorphism

$$\tau : \mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X) \to \mathscr{D}\!\mathit{iff}_0(T^2).$$

Since we shall exclusively be considering this restricted homomorphism, we use the same notation τ for this.

Given an isotopy $f_t \in \mathscr{D}\!\mathit{iff}_0(T^2)$, $f_0 = \mathit{id}$, one can lift it to an isotopy $F_t \in \mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$, $F_0 = \mathit{id}$ such that $\tau(F_t) = f_t$. This immediately implies that the inclusion $\mathsf{Ker}\,\tau \subset \mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$ induces an epimorphism

$$\pi_0(\mathsf{Ker}\,\tau) \to \pi_0(\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)). \tag{2.3}$$

Because of this property we would like to look at the group $\mathsf{Ker}\,\tau$ in more detail, but first introduce some useful notion.

Let X be a smooth manifold, and let f be a self-diffeomorphism X. Define the *mapping torus* $T(X, f)$ as the quotient of $X \times [0, 1]$ by the identification $(x, 1) \sim (f(x), 0)$. For the diffeomorphism f to be isotopic to identity it is necessary to have the mapping torus diffeomorphic to $T(X, \mathit{id}) \cong X \times S^1$.

Let us go back to the group $\mathsf{Ker}\,\tau$ that consists of bundle automorphisms of $\pi : X \to T^2$. Let $F \in \mathsf{Ker}\,\tau$ be a bundle automorphism of π, and let γ be a simple closed curve on T^2. By F_γ denote the restriction of F to $\pi^{-1}(\gamma) \cong S^1 \times S^2$. The mapping torus $T(\pi^{-1}(\gamma), F_\gamma)$ is either diffeomorphic to $S^2 \times T$ or $S^2 \tilde{\times} T$. In the later case we shall say that the automorphism F is *twisted* along γ.

Lemma 2.8. *Let X be diffeomorphic to either $T^2 \times S^2$ or $T^2 \tilde{\times} S^2$, and let $F \in \mathsf{Ker}\,\tau$. Then F is isotopic to the identity through $\mathsf{Ker}\,\tau$ iff T^2 contains no curve for F to be twisted along.*

Proof. The 2-torus T^2 has a cell structure with one cell, 2 1-cells, and one 2-cell. Clearly, F can be isotopically deformed to *id* over the 0-skeleton of T^2. The obstruction for extending this isotopy to the 1-skeleton of T^2 is a well-defined cohomology class $c(F) \in \mathsf{H}^1(T^2; \mathbb{Z}_2)$; the obstruction cochain $c(F)$ is the cochain whose value on a 1-cell e equals 1 if F is twisted along e and 0 otherwise. It is evident that $c(F)$ is a cocycle.

By assumption $c(F) = 0$. Consequently there is an extension of our isotopy to an isotopy over a neighbourhood of the 1-skeleton of T^2, but such an isotopy always can be extended to the rest of T^2. $\square$

A short way of represent the issue algebraically is by means of the *obstruction homomorphism*

$$c : \mathsf{Ker}\,\tau \to \mathsf{H}^1(X;\mathbb{Z}_2)$$

defined in the lemma; any two elements $F, G \in \mathsf{Ker}\,\tau$ are isotopic to each other through $\mathsf{Ker}\,\tau$ iff $c(F) = c(G)$.

Lemma 2.9. *Let X be diffeomorphic to $S^2 \times T^2$, and let $F \in \mathsf{Ker}\,\tau$, then T^2 contains no curve for F to be twisted along. This means that the obstruction homomorphism vanishes.*

Proof. The converse would imply that the mapping torus $T(X, F)$ is not spin, but $T(X, \mathit{id}) \cong S^2 \times T^2 \times S^1$ is a spin 5-manifold. □

The following result is due to McDuff [McD-B], but the proof follows by combining Lemma 2.9 with Lemma 2.8.

Proposition 2.10. *Let X be diffeomorphic to $S^2 \times T^2$, then the group $\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$ is connected.*

In what follows we need a non-spin analogue of this Proposition for the case of elliptic ruled surfaces.

Proposition 2.11. *Let X be diffeomorphic to $S^2 \tilde{\times} T^2$, then the group $\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$ is connected.*

Proof. Fix any cocycle $c \in \mathsf{H}^1(X;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then we claim there exists $F \in \mathsf{Ker}\,\tau$ such that $c(F) = c$ and, moreover, F is isotopic to id through diffeomorphisms $\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$. It follows from Suwa's model, see §2.2, that the automorphism group for the complex ruled surface $X_{\mathcal{A}}$ contains the complex torus $\mathcal{T}$ as a subgroup. By construction, it is clear that $\mathcal{T}$ is a subgroup of $\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$. Besides that, the 2-torsion subgroup $\mathcal{T}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of $\mathcal{T}$ is a subgroup of $\mathsf{Ker}\,\tau$. We trust the reader to check $\mathcal{T}_2$ is mapped isomorphically by the obstruction homomorphism to $\mathsf{H}^1(T^2;\mathbb{Z}_2)$.

The algebra behind this argument is expressed by a commutative diagram

$$\begin{array}{ccccc}
\mathcal{T}_2 & \xrightarrow{i} & \mathsf{Ker}\,\tau & \xrightarrow{j} & \mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_0(\mathcal{T}_2) & \xrightarrow{i_*} & \pi_0(\mathsf{Ker}\,\tau) & \xrightarrow{j_*} & \pi_0(\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)),
\end{array}$$

where i_* is an isomorphism, $j_* \circ i_*$ is zero, and therefore j_* is zero as well. But we already know that j_* is an isomorphism, and hence $\pi_0(\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X))$ is trivial. $\square$

2.7. Vanishing of elliptic twists

Here is the part where a proof of Theorem 0.1 comes. We split it into a few pieces. Let X be the symplectic ruled 4-manifold $(S^2 \tilde{\times} T^2, \omega_\mu)$, $[\omega_\mu] = \boldsymbol{T}_+ - \mu \boldsymbol{T}_-$, and let Ω_μ be the space of symplectic forms on X that are isotopic to ω_μ. Here we work with the connected component of $\mathcal{J}(X)$ that contains $\mathcal{J}(X, \Omega_\mu)$; the same applies to $\mathcal{J}_{\mathcal{A}}$ and $\mathcal{J}_{1-2k}$.

Lemma 2.12. *$\mathcal{J}_{\mathcal{A}} \subset \mathcal{J}(X, \Omega_\mu)$ for every $\mu \in (-1, 1)$.*

Proof. For every $J \in \mathcal{J}_{\mathcal{A}}$ we take any symplectic form ω such that J is ω-tamed. Then inflate ω along the classes $\boldsymbol{Y}_+ - \boldsymbol{Y}_-$ and $\boldsymbol{Y}_+ + \boldsymbol{Y}_-$, and then rescale it. $\square$

Lemma 2.13. *$\mathcal{J}_{\mathcal{A}} = \mathcal{J}(X, \Omega_\mu)$ for every $\mu \in (-1, 0]$.*

Proof. It is clear that $\mathcal{J}(X, \Omega_\mu)$ does not contain the structures $\mathcal{J}_{1-2k}$ for $\mu \in (-1, 0]$, and hence by (2.2) and Lemma 2.12 the proof follows. $\square$

This means that there is no topology change for the space $\mathcal{J}(X, \Omega_\mu)$ when μ is being varied in $(-1, 0]$. In particular,

$$\pi_1(\mathcal{J}(X, \Omega_\mu)) = \pi_1(\mathcal{J}_{\mathcal{A}}(X)) \quad \text{for } \mu \in (-1, 0].$$

Lemma 2.14. *$\mathcal{J}_{1-2\,k} \subset \mathcal{J}(X, \Omega_\mu)$ iff $\mu \in \left(1 - \dfrac{1}{k}, 1\right)$.*

Proof. The "only if" part is obvious, while the "if" can be proved by deflating along $\boldsymbol{Y}_+ - k\,(\boldsymbol{Y}_+ - \boldsymbol{Y}_-)$ and inflating along $\boldsymbol{Y}_+ - \boldsymbol{Y}_-$. $\square$

Combining Lemma 2.12 with Lemma 2.14, as well as the fact that the higher codimension submanifolds $\mathcal{J}_{1-2\,k}$, $k \geqslant 2$ do not affect the fundamental group of $\mathcal{J}(X, \Omega_\mu)$, we see that there is no topology change in $\pi_1(\mathcal{J}(X, \Omega_\mu))$ as μ increased within $(0, 1)$, i.e. we have

$$\pi_1(\mathcal{J}(X, \Omega_\mu)) = \pi_1(\mathcal{J}(X)) \quad \text{for } \mu \in (0, 1). \tag{2.4}$$

Diagram (1.3) implies that the symplectic mapping class group is the cokernel of $\nu_* : \pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \to \pi_1(\mathcal{J}(X))$ which we now show is trivial.

Lemma 2.15. *ν_* is an epimorphism.*

Proof. Though the map $\nu : \mathscr{D}\!\mathit{iff}_0(X) \to \mathcal{J}(X)$ is not a fibration, it can be extended to one; namely, to

$$\mathscr{D}\!\mathit{iff}_0(X) \to \mathcal{J}(X) \to \mathsf{Fol}_0(X),$$

where the last arrow is a homotopy equivalence, see Theorem 2.7 and Proposition 2.6. Thus, we end up with the homotopy exact sequence

$$\cdots \to \pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \to \pi_1(\mathcal{J}(X)) \to \pi_0(\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)).$$

If X is of genus 1, the group $\pi_0(\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X))$ is trivial by Propositions 2.10 and 2.11. This finishes the proof. □

The following corollary will not be used in the remainder of the paper, but it is a very natural application of Lemma 2.15.

Corollary 2.16. *The space $\mathcal{J}(X)$ is homotopy simple. In other words, $\pi_1(\mathcal{J}(X))$ is abelian and acts trivially on $\pi_n(\mathcal{J}(X))$.*

By virtue of (1.3) and (2.4), Lemma 2.15 immediately implies

Proposition 2.17. *$\pi_0(\mathscr{S}\!\mathit{ymp}^*(X, \omega_\mu)) = 0$ for every $\mu \in (0, 1)$.*

In order to compute the group $\pi_0(\mathscr{S}\!\mathit{ymp}^*(X, \omega_\mu))$ for $\mu \in (-1, 0)$ it is necessary to know better the fundamental group of $\mathcal{J}_\mathcal{A}$. The space $\mathcal{J}_\mathcal{A}$ is the complement to (the closure of) the elliptic divisorial locus $\mathcal{D}_{\boldsymbol{T}_-}$ in the ambient space $\mathcal{J}(X)$. We denote by i the inclusion

$$i : \mathcal{J}_\mathcal{A}(X) \to \mathcal{J}(X).$$

By Lemma 2.15 every loop $J(t) \in \pi_1(\mathcal{J}_\mathcal{A})$ can be decomposed into a product $J(t) = J_0(t) \cdot J_1(t)$, where $J_0(t) \in \mathsf{Im}\, \nu_*$, and $J_1(t) \in \mathsf{Ker}\, i_*$.

By Lemma 2.13, for $\mu \in (-1, 0]$, $\mathcal{J}_\mathcal{A} = \mathcal{J}(X, \Omega_\mu)$. In particular, assumption (A) is satisfied for each loop in $\mathcal{J}_\mathcal{A}$. Thus, each loop in $\mathcal{J}_\mathcal{A}$, that lies in $\mathsf{Ker}\, i_*$, could contribute drastically to $\pi_0(\mathscr{S}\!\mathit{ymp}^*(X, \omega_\mu))$ via the corresponding elliptic twists. But this will not happen, because the following holds.

Lemma 2.18. $\mathsf{Ker}\, i_* \subset \mathsf{Im}\, \nu_*$.

Proof. Choose some $J_* \in \mathcal{D}_{\boldsymbol{T}_-}$, and let Δ be a 2-disc which intersects $\mathcal{D}_{T_-}$ transversally at the single point J_*. Denote by $J(t)$ the boundary of Δ. By Lemma 2.19 one simply needs to show that the homotopy class of $J(t)$ comes from the natural action of $\mathscr{D}\!\mathit{iff}_0(X)$ on $\mathcal{J}_A$, and the lemma will follow.

If J_* is integrable, then one can choose Δ such that $J(t)$ is indeed an orbit of the action of a certain loop in $\mathscr{D}\!\mathit{iff}_0(X)$, see the description of the complex-analytic family constructed in §2.3. Thus it remains to check that every structure $\mathcal{J}_* \in \mathcal{D}_{\boldsymbol{T}_-}$ can be deformed to be integrable through structures on $\mathcal{D}_{\boldsymbol{T}_-}$. This will be proved by Lemma 2.20 below. $\square$

Lemma 2.19. *Let $x, y \in \mathcal{J}_{\mathcal{A}}$, and let $H(t) \in \mathcal{J}_{\mathcal{A}}$, $t \in [0,1]$ be a path joining them such that $H(0) = x$, $H(1) = y$. If a loop $J(t) \in \pi_1(\mathcal{J}_{\mathcal{A}}, y)$, $t \in [0,1]$ lies in the image of $\pi_1(\mathscr{D}\!\mathit{iff}_0(X), \mathsf{id}) \to \pi_1(\mathcal{J}_{\mathcal{A}}, y)$, then $H^{-1} \cdot J \cdot H \in \pi_1(\mathcal{J}_{\mathcal{A}}, x)$ lies in the image of $\pi_1(\mathscr{D}\!\mathit{iff}_0(X), \mathsf{id}) \to \pi_1(\mathcal{J}_{\mathcal{A}}, x)$.*

Proof. Without loss of generality we assume that there exists a loop $f(t) \in \pi_1(\mathscr{D}\!\mathit{iff}_0, \mathsf{id})$ such that $J(t) = f_*(t)J(0)$. Let H_s be the piece of the path H that joins the points $H(0) = x$ and $H(s)$. To prove the lemma it remains to consider the homotopy

$$J(s,t) := H_s^{-1} \cdot f_*(t)H(s) \cdot H_s,$$

where $J(1,t) = H^{-1} \cdot J \cdot H$ and $J(0,t) = f_*(t)H(0)$. $\square$

Lemma 2.20. *Every connected component of $\mathcal{J}_{-1}$ contains at least one integrable structure.*

Proof. Take a structure $J \in \mathcal{J}_{-1}$, and denote by C the corresponding smooth elliptic curve in the class $[C] = \boldsymbol{T}_-$. Let $\pi\colon X \to C$ be the ruling such that $J \in \mathcal{J}(X, \pi)$, see Theorem 2.7. Apart from the section given by C, we now choose one more smooth section C_1 of π such that C_1 is disjoint from C; the section C_1 need not be holomorphic, but be smooth. We claim that there exists a unique $\mathbb{C}^*$-action on X such that

(a) it is fiberwise, i.e. this diagram

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\cdot g} & \mathscr{X} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow[\cdot g]{} & \mathbb{C}, \end{array} \tag{2.5}$$

commute for each $g \in \mathbb{C}^*$,

(b) it acts on the fibers of π by means of biholomorphisms, and

(c) it fixes both C and C_1.

The complement $X - C_1$ is a $\mathbb{C}$-bundle with C being the zero-section; we keep the notation π for the projection $X - C_1 \to C$. This bundle inherits the $\mathbb{C}^*$-action described above. Let $\boldsymbol{U}(1)$ be the unitary subgroup of $\mathbb{C}^*$. The $(0,1)$-part of a $\boldsymbol{U}(1)$-invariant connection on the $\mathbb{C}$-bundle $\pi\colon X - C_1 \to C$ defines a $\bar\partial$-operator which associated to some holomorphic structure J_1 on $X - C_1$, see Chapter 0, §5 in [Gr-Ha]. As a complex structure, J_1 agrees with J on the fibers of π.

To every holomorphic $\mathbb{C}^*$-bundle one canonically associates a $\mathbb{CP}^1$-bundle. Hence, there is a unique extension J_1 to a complex structure $J_1 \in \mathcal{J}(X,\pi)$ such that C_1 becomes holomorphic.

When restricted to the bundle $TX|_C$, J_1 coincides with J. By Proposition 2.6 there is a symplectic form ω taming both structures J and J_1. Given a symplectic curve, say C, in X, and an almost-complex structure, say J, defined along C (i.e. on $TX|_C$) and tamed by ω. There exists an ω-tamed almost-complex structure on X which extends the given one. Moreover, such an extension is homotopically unique. In particular, one can always construct a family J_t joining J and J_1 such that C stays J_t-holomorphic, and the lemma is proved. □

Summarizing the results of Lemma 2.15 and Lemma 2.18 we obtain

Lemma 2.21. *$\pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \to \pi_1(\mathcal{J}_{\mathcal{A}}(X))$ is epimorphic.*

Again, it is implied by diagram 1.3 that the following holds.

Proposition 2.22. *$\pi_0(\mathscr{S}\!\mathit{ymp}^*(X,\omega_\mu)) = 0$ for every $\mu \in (-1,0]$.*

Together with Proposition 2.17, this statement covers what is claimed in Theorem 0.1.

3. Blow up once

3.1. Rational (−1)-curves

Let (Z,ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2\tilde{\times}Y^2\#\overline{\mathbb{CP}}^2$. Here we study homology classes in $\mathsf{H}_2(Z;\mathbb{Z})$ that can be represented by a symplectically embedded (-1)-sphere. Given a symplectically embedded

(-1)-sphere A, it satisfies

$$[A]^2 = -1, \quad c_1([A]) = 1. \tag{3.1}$$

A simple computation shows that there are only two homology classes satisfying (3.1), namely, $[A] = \boldsymbol{E}$ and $[A] = \boldsymbol{F} - \boldsymbol{E}$.

The following lemma will be used in the sequel, often without any specific reference.

Lemma 3.1. *Let (Z, ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2 \tilde{\times} Y^2 \# \overline{\mathbb{CP}}^2$. Then for every choice of ω-tamed almost-complex structure J, both the classes $\boldsymbol{E}$ and $\boldsymbol{F} - \boldsymbol{E}$ are represented by smooth rational J-holomorphic curves.*

Proof. Given an arbitrary ω-tamed almost-complex structure J, the exceptional class $\boldsymbol{F} - \boldsymbol{E}$ is represented by either a smooth J-holomorphic curve or by a J-holomorphic cusp-curve A of the form $A = \sum m_i A_i$ where each A_i stands for a rational curve occuring with the multiplicity $m_i \geqslant 1$. Clearly, we have

$$0 < \int_{A_i} \omega < \int_A \omega. \tag{3.2}$$

Because $c_1(\boldsymbol{F} - \boldsymbol{E}) = 1$, there exists at least one irreducible component of the curve A, say A_1, with $c_1([A_1]) \geqslant 1$.

Note that spherical homology classes in $\mathsf{H}_2(Z; \mathbb{Z})$ are generated by $\boldsymbol{F}$ and $\boldsymbol{E}$. Hence, we have $[A_1] = p\boldsymbol{F} - q\boldsymbol{E}$, which implies in particular that $[A_1]^2 = -q^2 \leqslant 0$, with equality iff $[A_1] = p\boldsymbol{F}$. But the latter is prohibited by (3.2) because

$$\int_{\boldsymbol{E}} \omega > 0.$$

Therefore, we have $[A_1]^2 \leqslant -1$. Further, one may use the adjunction formula to obtain that A_1 is a smooth rational curve with $[A_1]^2 = -1$ and $c_1(A_1) = 1$. Note that it is not possible for A_1 to be in the class $\boldsymbol{F} - \boldsymbol{E}$ because of (3.2). Hence, we have $[A_1] = \boldsymbol{E}$.

Take another irreducible component, say A_2. If A_2 does not intersect A_1, then $[A_2] = p\boldsymbol{F}$, which contradicts (3.2). Thus A_2 intersects A_1, positively. Hence, $[A_2] = p\boldsymbol{F} - q\boldsymbol{E}$ for q positive. The same argument works for the other irreducible components $A_2, A_3, \ldots$ of the curve A. But note that $[A_2] \cdot [A_3] < 0$, and hence there are no other components of A, except A_1 and A_2. We thus have $m_2[A_2] = \boldsymbol{F} - (m_1 + 1)\boldsymbol{E}$ for $m_1, m_2 \geqslant 1$. The class

$\boldsymbol{F}-(m_1+1)\boldsymbol{E}$ is primitive, and hence $m_2=1$. Further, this class cannot be represented by a rational curve, which can be easily checked using the adjunction formula. We thus proved the lemma for the class $\boldsymbol{F}-\boldsymbol{E}$; the case of $\boldsymbol{E}$ is analogous. $\square$

This lemma leads to the following generalization of Theorem 2.7 for ruled but not geometrically ruled symplectic 4-manifolds.

Lemma 3.2. *Let (Z,ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2\tilde{\times}Y^2\#\overline{\mathbb{CP}}^2$, and let J be an ω-tamed almost-complex structure. Then Z admits a* singular ruling *given by a proper projection $\pi: Z\to Y$ onto Y such that*

i) there is a singular value $y^*\in Y$ such that π is a spherical fiber bundle over $Y-y^*$, and each fiber $\pi^{-1}(y)$, $y\in Y-y^*$, is a J-holomorphic smooth rational curve in the class $\boldsymbol{F}$;

ii) the fiber $\pi^{-1}(y^*)$ consists of the two exceptional J-holomorphic smooth rational curves in the classes $\boldsymbol{F}-\boldsymbol{E}$ and $\boldsymbol{E}$.

Proof. It follows from Lemma 3.1 that Z admits J-holomorphic (-1)-curves E and E' in the classes $\boldsymbol{E}$ and $\boldsymbol{F}-\boldsymbol{E}$, respectively. We have to show that for each point $p\in Z$, except for those on E and E', there exists a smooth J-holomorphic sphere in the class $\boldsymbol{F}$ that passes through p. Such a sphere would necessarily be unique due to positivity of intersections. To get such a curve for a generic (by Gromov compactness, for every) point $p\in Z$ it suffices to show $\mathsf{Gr}(\boldsymbol{F})\neq 0$. But this follows from the Seiberg-Witten theory, see [McD-Sa-2].

Having a J-holomorphic curve passing through $p\in Z$, we have to prove it is smooth. Along the same lines as Lemma 3.1, one shows that the only non-smooth J-holomorphic curve in the class $\boldsymbol{F}$ is $E\cup E'$. $\square$

3.2. Straight structures

Let $Z\cong S^2\tilde{\times}T^2\#\overline{\mathbb{CP}}^2$ be a complex ruled surface, and let E be a smooth rational (-1)-curve in $\boldsymbol{E}\in\mathsf{H}_2(Z;\mathbb{Z})$. The blow-down of E from Z, which is a non-spin geometrically ruled genus one surface, will be denoted by X. The surface Z is said to be a *type $\mathcal{A}$ surface* if X is biholomorphic to the surface $X_{\mathcal{A}}$, see §2.2.

Let $p\in X$ be the image of E under the contraction map. Recall that X_A contains the triple of bisections, which are smooth elliptic curves in the class $\boldsymbol{B}\in\mathsf{H}_2(X;\mathbb{Z})$. The surface Z is called *straight type $\mathcal{A}$ surface* if there is no bisection passing through p in X. In other words, a straight type $\mathcal{A}$ surface contains a triple of smooth curves in the homology class $\boldsymbol{B}$, while a non-straight type $\mathcal{A}$ surface contains a smooth elliptic (-1)-curve in the class $\boldsymbol{B}-\boldsymbol{E} \in \mathsf{H}_2(Z;\mathbb{Z})$. We remark that it follows from Theorem 2.3 that straight type $\mathcal{A}$ surfaces can be characterized as those for which there exists a smooth elliptic (-1)-curve in the homology class $2\boldsymbol{B}-\boldsymbol{E} \in \mathsf{H}_2(Z;\mathbb{Z})$.

Let π be the ruling of X, and let S be the fiber of π that passes through p. When Z is type $\mathcal{A}$, there are three bisections $B_i \subset X$, each of which intersects S at precisely two distinct points. The following result was established in §2.2, see the construction of Suwa's model.

Lemma 3.3. *There exists a complex coordinate s on S such that the intersection points $B_i \cap S$ are as follows:*

$$B_1 \cap S = \{0, \infty\}\,, \quad B_2 \cap S = \{-1, 1\}\,, \quad B_3 \cap S = \{-i, i\}\,. \tag{3.3}$$

We then claim

Lemma 3.4. *There exists a complex-analytic family $\mathscr{X} \to \mathbb{CP}^1$ of type $\mathcal{A}$ surfaces Z_s parametrized by $s \in \mathbb{CP}^1$. When s equals one of the exceptional values*

$$\{0, \infty\}\,, \quad \{-1, 1\}\,, \quad \{-i, i\}\,,$$

the surface Z_s is not a straight type $\mathcal{A}$ surface, while for other parameter values, Z_s is straight type $\mathcal{A}$.

Proof. Pick a fiber F of the ruling of $X \cong X_{\mathcal{A}}$. Consider the complex submanifold $F \times \mathbb{CP}^1 \subset X \times \mathbb{CP}^1$, and denote by S the diagonal in $F \times \mathbb{CP}^1$. We construct $\mathscr{X}$ as the blow-up of $X \times \mathbb{CP}^1$ along S. The 3-fold $\mathscr{X}$ forms the complex-analytic family $\mathscr{X} \to S$ that was claimed to exist in the lemma. □

The notion of the straight type $\mathcal{A}$ complex structure can be generalized to almost-complex geometry as follows. Choose a tamed almost-complex structure $J \in \mathcal{J}(Z)$. We will call J straight type $\mathcal{A}$, or simply *straight*, if each J-holomorphic representative in the class $\boldsymbol{B} \in \mathsf{H}_2(Z;\mathbb{Z})$ is smooth. Clearly, the space of straight structures $\mathcal{J}_{\mathrm{st}}(Z)$ is an open dense submanifold in $\mathcal{J}(Z)$. Instead of $\mathcal{J}(Z)$ or $\mathcal{J}_{\mathrm{st}}(Z)$ we write $\mathcal{J}$ and $\mathcal{J}_{\mathrm{st}}$ for short. This definition of straightness is motivated by the following lemma the proof of which is left to the reader because it is similar to the proof of Proposition 2.5, (but the modified version of Theorem 2.7 given by Lemma 3.2 should be used).

Let $\boldsymbol{s}\colon [0,1] \to S$ be a loop in S, and let $Z(t) \to \boldsymbol{s}(t)$ be the restriction of $\mathscr{Z} \to S$ to $\boldsymbol{s}(t)$. Because $\boldsymbol{s}(t)$ is contractible inside the sphere S, we can think of $Z(t)$ as a family of type $\mathcal{A}$ complex structures on Z. Each structure $J(t)$ is straight iff $\boldsymbol{s}(t)$ does not pass through any of points (3.3). The following choice of $\boldsymbol{s}(t)$ will be used in §3.4

$$\boldsymbol{s}(t) = \varepsilon e^{2\pi i t}. \tag{3.4}$$

Lemma 3.5. *Let (Z,ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2\tilde{\times}T^2\#\overline{\mathbb{CP}}^2$, and let J be an ω-tamed almost-complex structure. Then every J-holomorphic representative in the class $\boldsymbol{B}$ is either irreducible smooth or contains a smooth component in one of the classes $\boldsymbol{B}-\boldsymbol{E}$, $\boldsymbol{T}_+ - k\boldsymbol{F}$, $k>0$.*

Similarly to Proposition 2.5 this lemma leads to a natural stratification of the space $\mathcal{J}$ of tamed almost-complex structures. Namely, this space can be presented as the disjoint union

$$\mathcal{J} = \mathcal{J}_{\rm st} \bigsqcup \mathcal{D}_{\boldsymbol{T}_-} \bigsqcup \mathcal{D}_{\boldsymbol{B}-\boldsymbol{E}} + \cdots ,$$

where $\mathcal{D}_{\boldsymbol{T}_-}$ and $\mathcal{D}_{\boldsymbol{B}-\boldsymbol{E}}$, which are submanifolds of real codimension 2 in $\mathcal{J}$, are the elliptic divisorial locus for respectively the classes $\boldsymbol{T}_-$ and $\boldsymbol{B}-\boldsymbol{E}$. Here we omitted the terms of real codimension greater than 2, because they do not affect the fundamental group of $\mathcal{J}$.

Coming to the symplectic side of straightness, we claim that if a symplectic form ω on Z satisfies the *period conditions*

$$\int_{\boldsymbol{T}_-} \omega < 0, \quad \int_{\boldsymbol{B}-\boldsymbol{E}} \omega < 0,$$

then $\mathcal{J}(Z,\omega) \subset \mathcal{J}_{\rm st}$. Moreover, a somewhat inverse statement holds, at least for integrable structures.

Lemma 3.6 (cf. §1.1). *Every complex straight type $\mathcal{A}$ structure is tamed by a symplectic form satisfying the period conditions. Moreover, every compact family of straight type $\mathcal{A}$ structures is tamed by a family of cohomologous forms satisfying the period conditions.*

Proof. We first check that a complex straight type $\mathcal{A}$ surface Z has a taming symplectic form θ such that θ satisfies the period conditions.

If Z is type $\mathcal{A}$ then it is the surface $X_{\mathcal{A}} \cong S^2\tilde{\times}T^2$ blown-up once. Since $X_{\mathcal{A}}$ admits a symplectic structure which satisfies the first period condition, then so does Z. Further, the second period condition can be achieved by means of deflation along a smooth elliptic curve in the class $2\boldsymbol{B} - \boldsymbol{E}$; such a curve indeed exists thanks to the straightness of Z.

We let K to denote the parameter space for our family Z_t, and let θ_t, $t \in K$ be a taming symplectic form on Z_t that satisfies the period condition. For every point $t' \in K$, let $U_{t'} \in K$ be a sufficiently small neighbourhood of $t' \in K$ such that for each $t \in U_{t'}$

$$\theta_{t'} \text{ tames the complex structure in } Z_t.$$

As K is compact, one may take a finite subcover U_{t_i}, $t_i \in I$ of K. The forms θ_I are not necessarily cohomologous because they may have different integrals on the homology class $\boldsymbol{E}$. Set $\varepsilon_{t_i} := \int_{\boldsymbol{E}} \theta_{t_i}$, $t_i \in I$, and $\varepsilon := \mathsf{min}\, \varepsilon_{t_i}$. We now deflate (Z_{t_i}, θ_{t_i}) along the homology class $\boldsymbol{E}$ to get $\int_{\boldsymbol{E}} \theta_{t_i} = \varepsilon$. Thanks to this deflation the forms θ_I become cohomologous and still do satisfy the period conditions.

Finally, set $\hat{\theta}(t) := \sum_{I'} \rho_{t_i}(t)\theta_{t_i}$, where the functions $\rho_{t_i} = \rho_{t_i}(t)$ is a partition of unity for the finite open cover $U_{t_i}, t_i \in I$ of K. What remains is to verify that Z_t is tamed by $\hat{\theta}(t)$ for every $t \in K$. Pick some $t^* \in K$, then there are but finitely many charts $U_{t_1}, \dots, U_{t_p}$ that contains the point $t^* \in K$. Then $\hat{\theta}(t^*) = \rho_{t_1}(t^*)\theta_{t_1} + \cdots + \rho_{t_p}(t^*)\theta_{t_p}$. Since each of $\theta_{t_1}, \dots, \theta_{t_p}$ tames Z_{t^*}, then so does $\hat{\theta}(t^*)$. $\square$

From this we have verified assumption (A) for the family of straight structures given by (3.4).

3.3. Refined Gromov invariants

In this subsection, we work with an almost-complex manifold (Z, J) equipped with a straight structure $J \in \mathcal{J}_{\mathrm{st}}$, i.e. every J-holomorphic curve of class $\boldsymbol{B} \in \mathsf{H}_2(Z; \mathbb{Z})$ in Z is smooth. We also note that such a curve is not multiply-covered, because the homology class $\boldsymbol{B}$ is primitive. The universal moduli space $\mathcal{M}(\boldsymbol{B}; \mathcal{J}_{\mathrm{st}})$ of embedded non-parametrized pseudoholomorphic curves of class $\boldsymbol{B}$ is a smooth manifold, and the natural projection $\mathsf{pr} : \mathcal{M}(\boldsymbol{B}; \mathcal{J}_{\mathrm{st}}) \to \mathcal{J}_{\mathrm{st}}$ is a Fredholm map, see [Iv-Sh-1, McD-Sa-3]. Given a generic $J \in \mathcal{J}_{\mathrm{st}}$, the preimage $\mathsf{pr}^{-1}(J)$ is canonically oriented zero-dimensional manifold, see [Tb] where it is explained how this orientation is chosen. The cobordism class of $\mathsf{pr}^{-1}(J)$ is independent of a generic J, thus giving us a well-defined element of $\Omega_0^{\boldsymbol{SO}} = \mathbb{Z}$; the number is equal to $\mathsf{Gr}(\boldsymbol{B})$.

Corollary 2.4 states that $\mathsf{Gr}(\boldsymbol{B}) = 3$, and hence Z contains not one but several curves in the class $\boldsymbol{B}$. Once we restricted almost-complex structures to those with the straightness property, the following modification of Gromov invariants can be proposed: given the image G of a certain homomorphism $\mathbb{Z}^2 \to \mathsf{H}_1(Z;\mathbb{Z})$, instead of counting pseudoholomorphic curves C such that $[C] = \boldsymbol{B}$, we will count curves C such that $[C] = \boldsymbol{B}$ and the embedding $i : C \hookrightarrow Z$ satisfies $\mathsf{Im}\, i = G$. The definitions of Gromov invariants $\mathsf{Gr}(\boldsymbol{B}, G)$, moduli space $\mathcal{M}(\boldsymbol{B}, G; \mathcal{J}_{\mathrm{st}})$, and so forth are completely analogous to those in "usual" theory of Gromov invariants.

Suppose J is an integrable straight type $\mathcal{A}$ structure, then the complex surface (Z, J) contains precisely 3 smooth elliptic curves C_1, C_2, and C_3 in the homology class $\boldsymbol{B}$. We denote by G_k the subgroup of $\mathsf{H}_1(Z;\mathbb{Z})$ generated by cycles on C_k; these subgroups G_k are pairwise distinct, see §2.2.

It is clear now that the space $\mathcal{M}(\boldsymbol{B}; \mathcal{J}_{\mathrm{st}})$ is disconnected and can be presented as the union

$$\mathcal{M}(\boldsymbol{B}; \mathcal{J}_{\mathrm{st}}) = \bigsqcup_{k=1}^{3} \mathcal{M}(\boldsymbol{B}, G_k; \mathcal{J}_{\mathrm{st}}).$$

We define the *moduli space of bisections* to be the fiber product

$$\mathcal{M}_{3B} = \{(x_1, x_2, x_3) \mid x_k \in \mathcal{M}(\boldsymbol{B}, G_k; \mathcal{J}_{\mathrm{st}}),\ \mathsf{pr}(x_1) = \mathsf{pr}(x_2) = \mathsf{pr}(x_3))\}\,.$$

Similarly to $\mathcal{M}(\boldsymbol{B}; \mathcal{J}_{\mathrm{st}})$, the moduli space $\mathcal{M}_{3B}$ is a smooth manifold and the projection $\mathsf{pr} : \mathcal{M}_{3B} \to \mathcal{J}_{\mathrm{st}}$ is a smooth map. We close this section by stating an obvious property of the projection map that we shall use in the sequel.

Lemma 3.7. *The projection map* $\mathsf{pr} : \mathcal{M}_{3B} \to \mathcal{J}_{st}$ *is a diffeomorphism, when is restricted to the subset of integrable straight type* $\mathcal{A}$ *complex structures.*

3.4. Loops in $\mathcal{M}_{3B}$

The map $\nu : \mathscr{D}\!\mathit{iff}_0(Z) \to \mathcal{J}_{\mathrm{st}}$ defined by

$$\mathscr{D}\!\mathit{iff}_0(Z) \xrightarrow{\nu} \mathcal{J}(Z, \omega) : f \to f_* J,$$

can be naturally lifted to a map $\mathscr{D}\!\mathit{iff}_0(Z) \to \mathcal{M}_{3B}$. Indeed, take a point $\boldsymbol{s} \in \mathcal{M}_{3B}$, which is a quadruple $[J, B_1, B_2, B_3](\boldsymbol{s})$ consisting of an almost-complex structure $J(\boldsymbol{s}) \in \mathcal{J}_{\mathrm{st}}$ on Z and a triple of smooth $J(\boldsymbol{s})$-holomorphic elliptic curves $B_1(\boldsymbol{s})$, $B_2(\boldsymbol{s})$, and $B_3(\boldsymbol{s})$ in Z. Then one can define

$$\mathscr{Diff}_0(Z) \xrightarrow{\nu} \mathcal{M}_{3B} : f \to [f_*J, f(B_1), f(B_2), f(B_3)].$$

Here we construct an element of $\pi_1(\mathcal{M}_{3B})$ that does not lie in the image of $\nu_*\colon \pi_1(\mathscr{Diff}_0(Z)) \to \pi_1(\mathcal{M}_{3B})$.

To start, we consider the tautological bundle $\mathcal{Z} \cong \mathcal{M}_{3B} \times Z$ over $\mathcal{M}_{3B}$ whose fiber over a point $x \in \mathcal{M}_{3B}$ is the almost-complex manifold $(Z, J(\boldsymbol{s}))$. By Lemma 3.1 every almost-complex manifold $(Z, J(\boldsymbol{s}))$ contains a unique smooth rational (-1)-curve $S(\boldsymbol{s})$ in the class $\boldsymbol{F} - \boldsymbol{E}$. Thus, one associates to $\mathcal{Z}$ an auxiliary bundle $\mathcal{S}$ whose fiber over $x \in \mathcal{M}_{3B}$ is the rational curve $S(\boldsymbol{s})$. Note that each $B_i(\boldsymbol{s})$ intersects $S(\boldsymbol{s})$ at precisely 2 distinct points denoted by $P_{i,1}$ and $P_{i,2}$. Hence we can mark out 3 distinct pairs of points $(P_{i,1}, P_{i,2})$, $i = 1, 2, 3$ on each fiber $S(\boldsymbol{s})$ of $\mathcal{S}$. Besides that, every $(Z, J(\boldsymbol{s}))$ contains a unique smooth rational curve $E(\boldsymbol{s})$ in the class $\boldsymbol{E}$. The curve $E(\boldsymbol{s})$ intersects $S(\boldsymbol{s})$ at precisely one point, say $Q(\boldsymbol{s})$. This point Q does not coincide with any of the point $P_{i,1}, P_{i,2}$, because $J(x)$ is assumed to be a straight one. Therefore $\mathcal{S}$ can be considered as a fiber bundle over $\mathcal{M}_{3B}$ whose fiber is the rational curve $S(\boldsymbol{s})$ with 7 distinct marked points, partially ordered as

$$(\{P_{1,1}, P_{1,2}\}, \{P_{2,1}, P_{2,2}\}, \{P_{3,1}, P_{3,2}\}, Q) \tag{3.5}$$

As such, there is an obvious map

$$\lambda\colon \mathcal{M}_{3B} \to \mathscr{M}$$

sending $S(\boldsymbol{s})$ to the corresponding point in the moduli space of 7 points in $\mathbb{CP}^1$, partially ordered as (3.5). Notice that the space $\mathscr{M}$ is also the moduli space of 6 points on $\mathbb{C}$, partially ordered as $(\{P_{1,1}, P_{1,2}\}, \{P_{2,1}, P_{2,2}\}, \{P_{3,1}, P_{3,2}\})$. One considers $\mathscr{M}$ as a quotient $\mathsf{Conf}_6(\mathbb{C})/\boldsymbol{\mathit{Aff}}(1, \mathbb{C})$, where $\mathsf{Conf}_6(\mathbb{C})$ is the configuration space of sextuples $(z_1, \ldots, z_6) \in \mathbb{C}^6$, $z_i \neq z_j$ with the identifications

$$\begin{aligned}(z_1, z_2, \ldots) \sim (z_2, z_1, \ldots),\ (\ldots, z_3, z_4, \ldots)\\ \sim (\ldots, z_4, z_3, \ldots),\ (\ldots, z_5, z_6) \sim (\ldots, z_6, z_5).\end{aligned}$$

The homotopy exact sequence for $\mathsf{Conf}_6(\mathbb{C}) \to \mathscr{M}$ reads

$$\mathbb{Z} \cong \pi_1(\boldsymbol{\mathit{Aff}}(1, \mathbb{C})) \to \pi_1(\mathsf{Conf}_6(\mathbb{C})) \to \pi_1(\mathscr{M}) \to 1 = \pi_0(\boldsymbol{\mathit{Aff}}(1, \mathbb{C})).$$

Let δ^2 be the element of $\pi_1(\mathsf{Conf}_6(\mathbb{C}))$ coming from $\pi_1(\boldsymbol{Aff}(1,\mathbb{C}))$. It is known that δ^2 generates the center of $\pi_1(\mathsf{Conf}_6(\mathbb{C}))$ (and even the center of a larger group, the braid group on 6 strands.)

Let $\boldsymbol{\gamma}\colon [0,1] \to \mathscr{M}$ be the loop given by

$$\begin{aligned} &(P_{1,1}(t), P_{1,2}(t),\, P_{2,1}(t), P_{2,2}(t),\, P_{3,1}(t), P_{3,2}(t),\, Q(t)) \\ &= (0, \infty,\, 1, -1,\, i, -i,\, \varepsilon\, e^{2\pi i t}) \end{aligned}$$

with respect to some inhomogeneous coordinate on $\mathbb{CP}^1$. Introducing the transformation

$$z \to \frac{\varepsilon e^{2\pi i t} z - 1}{\varepsilon e^{2\pi i t} - z},$$

in which $Q(t) = \infty$, one lifts $\boldsymbol{\gamma}$ to $\mathsf{Conf}_6(\mathbb{C})$ as

$$z_1(t) = -\frac{1}{\varepsilon e^{2\pi i t}}, \quad z_2(t) = -\varepsilon e^{2\pi i t}, \quad z_3(t) = 1, \quad z_4(t) = -1,$$
$$z_5(t) = \frac{\varepsilon e^{2\pi i t} i - 1}{\varepsilon e^{2\pi i t} - i}, \quad z_6(t) = -\frac{\varepsilon e^{2\pi i t} i + 1}{\varepsilon e^{2\pi i t} + i}.$$

It is not hard to show that the homology class of this loop is non-zero in $\mathsf{H}_1(\mathsf{Conf}_6(\mathbb{C}));\mathbb{R})$ and not a multiple of δ^2. Following [Ar], one can prove this by integrating the differential form

$$\boldsymbol{\alpha} := \frac{1}{2\pi i}\frac{\mathrm{d}\, z_1 - \mathrm{d}\, z_2}{z_1 - z_2} - \frac{1}{2\pi i}\frac{\mathrm{d}\, z_3 - \mathrm{d}\, z_4}{z_3 - z_4},$$

for which $\int_{\delta^2} \boldsymbol{\alpha} = 0$ yet $\int_\gamma \boldsymbol{\alpha} = -1$. As such, one obtains: $[\gamma] \neq 0$ in $\mathsf{H}_1(\mathscr{M};\mathbb{R})$.

Using the family $\mathscr{Z} \to \mathbb{CP}^1$ from Lemma 3.4, we get a loop $\boldsymbol{s}\colon [0,1] \to \mathcal{M}_{3B}$ with $\lambda(\boldsymbol{s}(t)) = \boldsymbol{\gamma}(t)$. The class $[\boldsymbol{s}] \in \pi_1(\mathcal{M}_{3B})$ does not lie in $\mathsf{Im}\,\nu_*$, as for it were, that would imply that $[\boldsymbol{s}] \in \mathsf{Ker}\,\lambda_*$. (Here we used the inclusion $\mathsf{Im}\,\nu_* \subset \mathsf{Ker}\,\lambda_*$ following from the fact that λ is $\mathscr{D}\!i\!f\!f_0(Z)$-invariant.)

If $\boldsymbol{\gamma}$ was given either by

$$\begin{aligned} &(P_{1,1}(t), P_{1,2}(t),\, P_{2,1}(t), P_{2,2}(t),\, P_{3,1}(t), P_{3,2}(t),\, Q(t)) \\ &= (0, \infty,\, 1 + \varepsilon e^{2\pi i t}, -1,\, i, -i,\, 0), \quad \text{or} \\ &(P_{1,1}(t), P_{1,2}(t),\, P_{2,1}(t), P_{2,2}(t),\, P_{3,1}(t), P_{3,2}(t),\, Q(t)) \\ &= (0, \infty,\, 1, -1,\, i + \varepsilon e^{2\pi i t}, -i,\, 0), \end{aligned}$$

then a similar argument would work to get another non-contractible loop in $\mathcal{M}_{3B}$.

3.5. Loops in $\mathcal{J}_{\mathrm{st}}$

Here we construct an element of $\pi_1(\mathcal{J}_{\mathrm{st}})$ that does not lie in the image of $\nu_*\colon \pi_1(\mathscr{Diff}_0(Z)) \to \pi_1(\mathcal{J}_{\mathrm{st}})$.

Let $J(t)$ be the loop of integrable structures with $\mathsf{pr}(\boldsymbol{s}(t)) = J(t)$ for the loop $\boldsymbol{s}(t)$ constructed in §3.4.

Lemma 3.8. *The class $[J] \in \pi_1(\mathcal{J}_{st})$ does not lie in $\mathsf{Im}\,\nu_*$.*

Proof. Assume the contrary, i.e. that there exists a family $f\colon [0,1] \to \mathscr{Diff}_0(Z)$, $f(0) = f(1) = \mathsf{id}$ such that $J(t)$ is homotopic to $\widetilde{J}(t) := f(t)_* J(0)$. We join $J(t)$ and $\widetilde{J}(t)$ with a tube $T \subset \mathcal{J}_{\mathrm{st}}$. By Sard-Smale theorem we can arrange that T is transverse to pr. Thus, the preimage $\mathsf{pr}^{-1}(T)$ is a smooth orientable surface that bounds $\boldsymbol{s}(t) \cup \widetilde{\boldsymbol{s}}(t)$. Note that $\widetilde{\boldsymbol{s}}(t) := \mathsf{pr}^{-1}(\widetilde{J}(t))$ is connected thanks to Lemma 3.7. It follows that $[\widetilde{\boldsymbol{s}}] = [\boldsymbol{s}]$ in $\mathsf{H}_1(\mathcal{M}_{3B};\mathbb{Z})$. This is a contradiction, as $[\boldsymbol{s}]$ does not lie in $\mathsf{Ker}\,\lambda_*$, whereas λ itself is constant on $\widetilde{\boldsymbol{s}}(t)$. □

3.6. Let's twist again

Here we outline the proof of Theorem 0.2, referring the reader to the previous subsections for details.

Let X be a type $\mathcal{A}$ surface, see §2.2. It follows from Theorem 2.3 hat X contains a triple of smooth elliptic curves C_1, C_2, and C_3 in the homology class $\boldsymbol{B} \in \mathsf{H}_2(X;\mathbb{Z})$, $[\boldsymbol{B}]^2 = 0$. The procedure given in §1.1 shows that there are three elliptic twists for $X\#\overline{\mathbb{CP}}^2$, see also Lemma 3.6. Denote the corresponding loops by J_{C_1}, J_{C_2}, and J_{C_3}; they are contained in the space $\mathcal{J}_{\mathrm{st}}$ of the straight almost-complex structures, see §3.2. We prove these loops do not lie in the image of $\nu_* : \pi_1(\mathscr{Diff}_0(X\#\overline{\mathbb{CP}}^2)) \to \pi_1(\mathcal{J}_{\mathrm{st}})$, see §3.5.

Using Lemma 3.6, we find a symplectic form θ with $\int_{\boldsymbol{B}-\boldsymbol{E}} \theta \leqslant 0$ such that $J_{C_i} \in \mathcal{J}(X\#\overline{\mathbb{CP}}^2, \Theta)$, $i = 1, 2, 3$. Here Θ stands for $\Omega(X.\theta)$. Since the inclusion $\mathcal{J}(X\#\overline{\mathbb{CP}}^2, \Theta) \subset \mathcal{J}_{\mathrm{st}}$ is equivariant w.r.t. to the natural action of $\mathscr{Diff}_0(X\#\overline{\mathbb{CP}}^2)$ on these spaces, it follows that J_{C_i} do not lie in the image of $\nu_* : \pi_1(\mathscr{Diff}_0(X\#\overline{\mathbb{CP}}^2)) \to \mathcal{J}(X\#\overline{\mathbb{CP}}^2, \Theta)$, and the theorem follows.

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Received September 26, 2018
Accepted July 29, 2019