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Concentration of symplectic volumes on Poisson homogeneous spaces

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For a compact Poisson-Lie group K, the homogeneous space K/T carries a family of symplectic forms ω_ξ^s, where $\xi \in \mathfrak{t}_+^*$ is in the positive Weyl chamber and $s \in \mathbb{R}$. The symplectic form ω_ξ^0 is identified with the natural K-invariant symplectic form on the K coadjoint orbit corresponding to ξ. The cohomology class of ω_ξ^s is independent of s for a fixed value of ξ.

In this paper, we show that as $s \to -\infty$, the symplectic volume of ω_ξ^s concentrates in arbitrarily small neighborhoods of the smallest Schubert cell in $K/T \cong G/B$. This strengthens an earlier result of [10] and is a step towards a conjectured construction of global action-angle coordinates on $\mathrm{Lie}(K)^*$ [4, Conjecture 1.1].

1. Introduction

Let K be a compact connected Lie group with maximal torus T and let $G = K^{\mathbb{C}}$ denote its complexification. Let $\mathfrak{t}$ denote the Lie algebra of T. As our results concern the homogeneous space K/T, we may assume without loss of generality that K is semisimple and simply connected.

The homogeneous space K/T carries an interesting family of symplectic structures ω_ξ^s parameterized by $s \in \mathbb{R}$ and elements of a positive Weyl chamber, $\xi \in \mathfrak{t}_+^*$. Following [13], the Iwasawa decomposition $G = AN_-K$ defines dual Poisson-Lie groups (K, π_K) and (AN_-, π_{AN_-}). The symplectic leaves of π_{AN_-} are the orbits of the so-called dressing action of K on AN_-. Let $\mathscr{D}_\xi \subset AN_-$ denote the dressing orbit through $\exp(\sqrt{-1}\xi)$, where $\xi \in \mathfrak{t}^*$ is identified with an element of $\mathfrak{t}$ via the Killing form. For all $s \neq 0$ and $\xi \in \mathfrak{t}_+^*$, fix the K-equivariant identification of K/T with $\mathscr{D}_{s\xi}$ such that

$eT \mapsto \exp(s\sqrt{-1}\xi)$ and define[1]

$$\pi_\xi^s := s\pi_{AN_-}|_{\mathscr{D}_{s\xi}}, \quad \omega_\xi^s := (\pi_\xi^s)^{-1}. \tag{1}$$

For $s = 0$ and $\xi \in \mathfrak{t}_+^*$, fix the K-equivariant identification of K/T with the coadjoint orbit $\mathcal{O}_\xi$ such that $eT \mapsto \xi$ and define ω_ξ^0 to be the Kostant-Kirillov-Souriau symplectic form.

The family ω_ξ^s was studied in [1, 11] and has several nice properties. First, the action of K on K/T is Poisson: the action map $K \times K/T \to K/T$ is a Poisson map with respect to $s\pi_K$ and π_ξ^s for all s and ξ. In other words, $(K/T, \pi_\xi^s)$ is a *Poisson homogeneous space* for $(K, s\pi_K)$. Poisson homogeneous spaces for (K, π_K) were classified in [9]. Second, for a fixed value of ξ the forms ω_ξ^s are isotopic for all $s \in \mathbb{R}$ [1]. It follows that for fixed ξ and arbitrary s the forms ω_ξ^s are cohomologous. In particular, their symplectic (Liouville) volumes are the same:

$$\mathrm{Vol}(K/T, \omega_\xi^s) = \mathrm{Vol}(K/T, \omega_\xi^0). \tag{2}$$

Let $B \subset G$ be the positive Borel subgroup (corresponding to $\mathfrak{t}_+^*$). The flag variety G/B is isomorphic to K/T and admits a stratification into Schubert cells BwB/B, indexed by elements w of the Weyl group. The smallest Schubert cell is the point $eB \in G/B$ and the biggest Schubert cell, Bw_0B/B, corresponding to the longest element $w_0 \in W$, is dense in G/B.

It follows from [11, Proposition 5.12] that the rescaled family of Poisson structures $s^{-1}\pi_\xi^s$ admits, for all ξ, a common limit π^∞ when $s \to -\infty$. The Poisson structure π^∞ coincides with the pushforward under $K \to K/T$ of $-\pi_K$, and the symplectic leaves of π^∞ are exactly the Schubert cells; see also Remark 2.1. Then Theorem 2.2 in [10] implies the following:

Theorem 1.1. *Let $\overline{U}$ be a compact subset of the big Schubert cell Bw_0B/B. Then, for any $\xi \in \mathfrak{t}_+^*$ and $\varepsilon > 0$, there exists $s_0 \in \mathbb{R}$ such that for $s \leqslant s_0$,*

$$\mathrm{Vol}\left(\overline{U}, \omega_\xi^s\right) < \varepsilon.$$

Proof. Fix $\xi \in \mathfrak{t}_+^*$ and identify K/T with the dressing orbit $\mathscr{D}_{s\xi}$ as above, equipped with $s\pi_{AN_-}$. Let $\mathrm{pr}_A \colon G \to A$ denote projection with respect to

[1]For $s < 0$, ω_ξ^s is the symplectic structure on K/T defined by $-s\pi_\lambda$, $\lambda = -s\sqrt{-1}\xi$, where π_λ is the Poisson structure defined by Lu in [11, Notation 5.11]. See also Remark 2.1.

the Iwasawa decomposition $G = AN_-K$. Identify $\mathfrak{t} \cong \mathfrak{t}^*$ via the Killing form. With these identifications,

$$\Psi_s \colon K/T \to \mathfrak{t}^*, \quad kT \mapsto \frac{1}{s\sqrt{-1}} \log \mathrm{pr}_A(k \exp(s\sqrt{-1}\xi)),$$

is a moment map for the action of T on $(K/T, \omega_\xi^s)$ by left multiplication, for all $s \neq 0$ [12, Theorem 4.13]. The T-fixed points, their weights, and their images under the moment map do not depend on s. Thus the Duistermaat-Heckman measure on the moment polytope (the pushforward under Ψ_s of the Liouville measure of ω_ξ^s) is independent of s.

Fix a compact subset $\overline{U} \subset Bw_0B/B$. By [10, Theorem 2.2], there exists $r > 0$ such that

$$|| \log \mathrm{pr}_A(k \exp(s\sqrt{-1}\xi)) - sw_0\sqrt{-1}\xi|| < r$$

for all $\xi \in \mathfrak{t}_+$, $s < 0$, and $k \in \overline{U}$. The norm $|| \cdot ||$ is taken with respect to the Killing form. It follows that for fixed $\xi \in \mathfrak{t}_+$ and all $s < 0$,

$$||\Psi_s(kT) - w_0\xi|| = \left|\left| \frac{1}{s\sqrt{-1}} \log \mathrm{pr}_A(k \exp(s\sqrt{-1}\xi)) - w_0\xi \right|\right| < \frac{r}{|s|}$$

for all $k \in \overline{U}$. Since the Duistermaat-Heckman measure is independent of s, this implies that $\mathrm{Vol}(\overline{U}, \omega_\xi^s) < \varepsilon$ for all $s < 0$ sufficiently large. □

In other words, any compact subset of the big Schubert cell is depleted of symplectic volume as $s \to -\infty$. Since total volume is constant for fixed ξ, this implies that the volume concentrates in a small neighborhood of the other Schubert cells.

Example 1.2. As an illustration of this phenomenon, consider the example of $K = \mathrm{SU}(2)$. Identify $\mathfrak{t}^* = \mathbb{R}$ and $\xi \in \mathfrak{t}_+^* = \mathbb{R}_{>0}$. Let $(z, \varphi) \in (-1, 1) \times (0, 2\pi)$ be cylindrical coordinates on the unit-sphere $S^2 \subset \mathbb{R}^3$ and fix the K-equivariant identification of K/T with S^2 such that eT is identified with the pole $z = 1$. The family of symplectic forms is

$$\omega_\xi^s = \begin{cases} \dfrac{\sinh(2s\xi)}{2s(\cosh(2s\xi) + z\sinh(2s\xi))}\, dz \wedge d\varphi, & s \neq 0; \\[2ex] \xi dz \wedge d\varphi, & s = 0. \end{cases}$$

One can derive this formula, for instance, from [11, Example 5.4]. Note that $\omega_\xi^0 = \xi dz \wedge d\varphi$ are the rotation-invariant area forms on S^2. We leave it as an exercise to the reader to show that the cohomology class of ω_ξ^s is indeed independent of s and that for $s \ll 0$ the volume concentrates near the pole $z = 1$, which was identified with the smallest Schubert cell, eB.

In general, there are many Schubert cells in G/B of positive codimension and the question of how volume arranges itself on a neighborhood of those Schubert cells when $s \ll 0$ remains. The main result of this paper is an answer to this question (and a strengthening of Theorem 1.1):

Theorem 1.3 (Main Theorem). *Let U be an open neighborhood of the smallest Schubert cell eB. Then for any $\xi \in \mathfrak{t}_+^*$ and $\varepsilon > 0$, there exists $s_0 \in \mathbb{R}$ such that for $s \leqslant s_0$,*

$$\mathrm{Vol}\left(U, \omega_\xi^s\right) > (1-\varepsilon)\,\mathrm{Vol}(K/T, \omega_\xi^s).$$

In other words, any compact subset of G/B not containing eB eventually gets depleted of symplectic volume as $s \to -\infty$.

The remainder of the paper is devoted to setting up the proof of Theorem 1.3, which is given below. Section 2 describes the dual Poisson-Lie group $(K^*, \pi_{K^*}) := (AN_-, \pi_{AN_-})$. There are two important maps defined for $s \neq 0$,

$$\mathfrak{E}_s \colon \mathfrak{k}^* \to K^*$$
$$\mathfrak{L}_s \colon \mathbb{R}^{r+m} \times \mathbb{T}^m \to K^*$$

which are defined in Equations (5) and (9), respectively. Here $r = \dim(T)$, $2m = \dim(K/T)$, and $\mathbb{T}^m$ is a compact torus of dimension m. The map $\mathfrak{E}_s$ is a diffeomorphism. It is K-equivariant with respect to the coadjoint and dressing actions and has the property that $\mathfrak{E}_s(\xi) = \exp(s\sqrt{-1}\xi)$ for all $\xi \in \mathfrak{t}^*$. The map $\mathfrak{L}_s$ is a diffeomorphism onto its image and the image of $\mathfrak{L}_s$ is an open dense subset of K^* that is independent of s. The intersection $\mathfrak{L}_s(\mathbb{R}^{r+m} \times \mathbb{T}^m) \cap \mathfrak{E}_s(\mathcal{O}_\xi)$ is an open dense subset of $\mathfrak{E}_s(\mathcal{O}_\xi)$ for all $\xi \in \mathfrak{t}_+^*$. Moreover, all the maps in the following diagram are Poisson:

$$(3) \qquad (\mathcal{O}_\xi, \pi_\xi^s) \hookrightarrow (\mathfrak{k}^*, \pi^s = \mathfrak{E}_s^*(s\pi_{K^*})) \xrightarrow{\mathfrak{E}_s} (K^*, s\pi_{K^*}) \xleftarrow{\mathfrak{L}_s} (\mathbb{R}^{r+m} \times \mathbb{T}^m, \mathfrak{L}_s^*(s\pi_{K^*})).$$

There is a distinguished open subset $PT(K^*) \subset \mathbb{R}^{r+m} \times \mathbb{T}^m$ called the *partial tropicalization of* K^*, introduced in [2], equipped with a constant Poisson structure π_{PT}. As $s \to -\infty$, the Poisson structure $\mathfrak{L}_s^*(s\pi_{K^*})$ converges to π_{PT} uniformly on certain subsets that exhaust $PT(K^*)$ (Section 2.3). Section 3 shows that the symplectic volume of the leaves of $\mathfrak{L}_s^*(s\pi_{K^*})$

concentrates in $PT(K^*)$ as $s \to -\infty$ (Proposition 3.5). Section 4 contains the proof of Proposition 4.3, which says that, under the maps in (3), points of $PT(K^*)$ correspond to points near $\mathfrak{t}_+^* \subset \mathfrak{k}^*$ when $s \ll 0$. This allows us to translate Proposition 3.5 into a statement about the symplectic volume of $(K/T, \omega_\xi^s)$.

Proof of Theorem 1.3. Let $\mathcal{N}_{s\xi} \subset \mathbb{R}^{r+m} \times \mathbb{T}^m$ denote the preimage $(\mathfrak{E}_s^{-1} \circ \mathfrak{L}_s)^{-1}(\mathcal{O}_\xi)$, which is a symplectic leaf of $\mathfrak{L}_s^*(s\pi_{K^*})$, and denote its symplectic form $\eta_{s\xi} = (\mathfrak{E}_s^{-1} \circ \mathfrak{L}_s)^* \omega_\xi^s$. In Proposition 3.5, we prove that for all $\varepsilon > 0$, there is a compact subset $D_\varepsilon \subset PT(K^*)$ such that

$$\lim_{s \to -\infty} \mathrm{Vol}\,(\mathcal{N}_{s\xi} \cap D_\varepsilon, \eta_{s\xi}) \geqslant (1-\varepsilon)\,\mathrm{Vol}(\mathcal{N}_{s\xi}, \eta_{s\xi}) = (1-\varepsilon)\,\mathrm{Vol}(K/T, \omega_\xi^s).$$

In Proposition 4.3, we show there exists $s_0 < 0$ such that for all $s \leqslant s_0$,

$$\mathfrak{E}_s^{-1} \circ \mathfrak{L}_s(\mathcal{N}_{s\xi} \cap D_\varepsilon) \subseteq U.$$

Since $\mathfrak{E}_s^{-1} \circ \mathfrak{L}_s$ is a Poisson isomorphism, it preserves volumes of the symplectic leaves. Thus

$$\mathrm{Vol}\left(U, \omega_\xi^s\right) \geqslant \mathrm{Vol}\left(\mathfrak{E}_s^{-1} \circ \mathfrak{L}_s(\mathcal{N}_{s\xi} \cap D_\varepsilon), \omega_\xi^s\right) = \mathrm{Vol}\,(\mathcal{N}_{s\xi} \cap D_\varepsilon, \eta_{s\xi})\,.$$

Combining with the limit above completes the proof. □

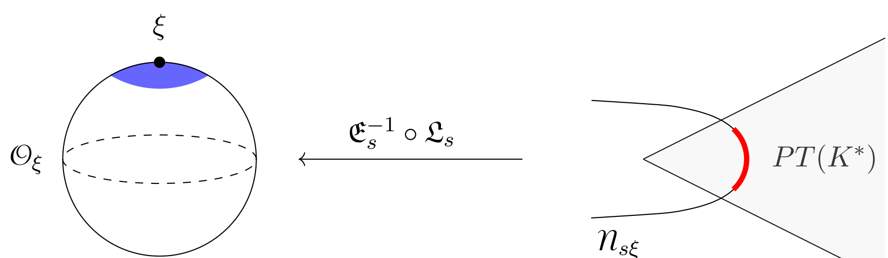

Figure 1. As $s \to -\infty$, volume of the symplectic leaves $\mathcal{N}_{s\xi} = (\mathfrak{E}_s^{-1} \circ \mathfrak{L}_s)^{-1}(\mathcal{O}_\xi)$ concentrates on subsets of $\mathcal{N}_{s\xi} \cap PT(K^*)$, illustrated in red. For s sufficiently large, the image of the red subset is contained in an arbitrarily small neighborhood of ξ, illustrated in blue.

A motivation for our study is provided by the following idea. There exist Poisson isomorphisms between $\mathfrak{k}^*$ and K^* called Ginzburg-Weinstein isomorphisms after the authors of [8]. Given a Ginzburg-Weinstein isomorphism $\gamma : \mathfrak{k}^* \to K^*$, its scaling $\gamma^s(x) := \gamma(sx)$ is a Poisson isomorphism with respect to $\pi_{\mathfrak{k}^*}$ and $s\pi_{K^*}$. Composing γ^s with $\mathfrak{L}_s^{-1}$ defines coordinates on every regular coadjoint orbit which are almost global action-angle coordinates for $s \ll 0$. Conjecturally, the $s \to -\infty$ limit of this composition defines global action-angle coordinates on the regular coadjoint orbits. This has already been shown to be true for $K = \mathrm{U}(n)$, where for a certain choice of Ginzburg-Weinstein diffeomorphism and cluster seed, the limit is the classical Gelfand-Zeitlin system [4].

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2. Background

Fix the following notation. Let G be a connected simply-connected semisimple complex Lie group of rank r. Fix a compact real form $K \subset G$ and a Cartan subgroup $H \subset G$, and let $(\cdot)^* : G \to G$ be the anti-involution of G under which elements $k \in K$ satisfy $k^{-1} = k^*$. Denote the Lie algebras of G, K, and H by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{h}$ respectively. Fix a choice of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Denote the lattice of integral weights by P, and the semigroup of dominant integral weights by P_+. We write $h \mapsto h^\mu \in \mathbb{C}^\times$ for the multiplicative character $H \to \mathbb{C}^\times$ determined by $\mu \in P$. Let $I = \{1, \dots, r\}$ index the simple roots, $\alpha_i \in \mathfrak{h}^*$, the simple coroots, $\alpha_i^\vee \in \mathfrak{h}$, and the fundamental weights, ω_i, which by definition satisfy $\omega_i(\alpha_j^\vee) = \delta_{ij}$. Denote the Weyl group of G by W. Let $s_i \in W$ be the simple reflection generated by α_i and let w_0 be the longest element of W, with length denoted by m.

Let T be the maximal torus of K which has Lie algebra $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Let $\mathfrak{a} = \sqrt{-1}\mathfrak{t}$ and denote the corresponding subgroup of G by A. Corresponding to the choice of positive roots, we have opposite maximal unipotent subgroups N and N_- with Lie algebras $\mathfrak{n}$ and $\mathfrak{n}_-$, as well as opposite Borel subgroups $B = HN$ and $B_- = HN_-$ with Lie algebras $\mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{h} \oplus \mathfrak{n}_-$. Fix a set of Chevalley generators $F_i \in \mathfrak{n}_-$, $\alpha_i^\vee \in \mathfrak{h}$, $E_i \in \mathfrak{n}$, $i \in I$. Recall the Iwasawa decompositions $G = AN_-K$ and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{a} \oplus \mathfrak{k}$.

Fix an invariant non-degenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. The isomorphism $\mathfrak{k} \cong \mathfrak{k}^*$ determined by $(\cdot,\cdot)$ embeds $\mathfrak{t}^* \subseteq \mathfrak{k}^*$, as the image of $\mathfrak{t}$. Let $\mathfrak{t}^*_+ \subseteq \mathfrak{t}^*$ be the open cone such that $\sqrt{-1}\mathfrak{t}^*_+ \subseteq \mathfrak{h}^*$ is the interior of the real cone spanned by P_+. We refer to both $\mathfrak{t}^*_+$ and $\sqrt{-1}\mathfrak{t}^*_+$ as the positive Weyl chamber.

2.1. Dressing orbits and compact Poisson-Lie groups

Recall that a Poisson-Lie group (K,π) is a Lie group K equipped with a Poisson structure π such that the multiplication map $K \times K \to K$ is Poisson (with respect to the product Poisson structure on $K \times K$). For example, the canonical Lie-Poisson structure $\pi_{\mathfrak{k}^*}$ on the dual $\mathfrak{k}^*$ of a Lie algebra $\mathfrak{k}$ is linear, so $(\mathfrak{k}^*, \pi_{\mathfrak{k}^*})$ is a Poisson-Lie group with respect to vector addition.

For G as above, both K and AN_- have natural Poisson-Lie group structures defined as follows (see [13] for details). Let $\Im(\cdot,\cdot)$ be the imaginary part of the fixed G-invariant non-degenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. Then $\mathfrak{k}$ and $\mathfrak{n}_- \oplus \mathfrak{a}$ are isotropic subspaces with respect to $2\Im(\cdot,\cdot)$, and $2\Im(\cdot,\cdot)$ defines an isomorphism $\mathfrak{n}_- \oplus \mathfrak{a} \cong \mathfrak{k}^*$. This identification endows $\mathfrak{k}$ and $\mathfrak{k}^*$ with the structure of dual Lie bialgebras. Since K and AN_- are simply connected, the Lie bialgebra structures on $\mathfrak{k}$ and $\mathfrak{k}^*$ integrate to define Poisson-Lie group structures π_K on K and π_{K^*} on AN_-, respectively. These Poisson-Lie group structures are dual, since they arise by integrating dual Lie bialgebras, thus one denotes $K^* = AN_-$, and refers to (K^*, π_{K^*}) as the *dual Poisson-Lie group* of (K, π_K).

Both $\mathfrak{k}^*$ and K^* have naturally defined K actions. The *coadjoint action* of K on $\mathfrak{k}^*$ is defined in terms of the adjoint action by the equation

$$\langle \mathrm{Ad}^*_k \xi, x\rangle = \langle \xi, \mathrm{Ad}_{k^{-1}} x\rangle, \qquad k \in K,\ \xi \in \mathfrak{k}^*,\ \text{and } x \in \mathfrak{k}.$$

The coadjoint action preserves $\pi_{\mathfrak{k}^*}$, and the symplectic leaves of $\pi_{\mathfrak{k}^*}$ are the coadjoint orbits. The *dressing action* of K action on K^* is defined by re-factorizing $kb \in G$ according to the Iwasawa decomposition. If

$$kb = b'k' \in AN_-K, \qquad k, k' \in K,\ b, b' \in K^*,$$

then the dressing action of k on b is defined as ${}^k b = b'$. The symplectic leaves of π_{K^*} are the dressing orbits. In other words, they are the joint level sets of the Casimir functions [13],

$$C_i(b)^2 := \mathrm{Tr}\left(\rho^{\omega_i}\left(bb^*\right)\right), \qquad b \in K^*, \tag{4}$$

where ρ^{ω_i} is the fundamental irreducible G-representation with highest weight $\omega_i \in P_+$. The map $\varphi\colon b \mapsto bb^*$ is a diffeomorphism of K^* onto the set $S = \{g \in G \mid g^* = g\}$.

There is a family of diffeomorphisms $\mathfrak{E}_s\colon \mathfrak{k}^* \to K^*$ parameterized by $s \neq 0$ [7]. Let $\psi\colon \mathfrak{k}^* \to \mathfrak{k}$ be the K-equivariant isomorphism given by the fixed bilinear form on $\mathfrak{g}$. Then, define

$$\mathfrak{E}_s\colon \mathfrak{k}^* \xrightarrow{\psi} \mathfrak{k} \xrightarrow{\exp\left(2s\sqrt{-1}\cdot\right)} S \xrightarrow{\varphi^{-1}} K^* = AN_-. \tag{5}$$

The map $\mathfrak{E}_s$ is equivariant with respect to the coadjoint and dressing actions of K. Let $\mathcal{O}_\xi$ be the coadjoint orbit through $\xi \in \mathfrak{t}^*_+$. Denote by $\mathcal{D}_{s\xi}$ the dressing orbit through $\mathfrak{E}_s(\xi) = \exp\left(s\sqrt{-1}\psi(\xi)\right)$. Since $\mathfrak{E}_s$ is K-equivariant, $\mathfrak{E}_s(\mathcal{O}_\xi) = \mathcal{D}_{s\xi}$.

Remark 2.1. Most references, such as [13], prefer to use the triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{n} \oplus \mathfrak{a})$ together with the form $2\Im(\cdot,\cdot)$ in their definition of the Lie bialgebra structures on $\mathfrak{k}$ and $\mathfrak{k}^*$. The linearization at the identity of the map $G \to G,\ g \mapsto (g^*)^{-1}$ takes the triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{n}_- \oplus \mathfrak{a})$ together with the form $2\Im(\cdot,\cdot)$, to the triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{n} \oplus \mathfrak{a})$ together with the form $-2\Im(\cdot,\cdot)$. Therefore our Poisson structure π_K on K agrees with the one in [13], up to sign.

2.2. Cluster coordinates on double Bruhat cells

The *double Bruhat cell* determined by a pair of elements $u, v \in W$, is the intersection

$$G^{u,v} := BuB \cap B_-vB_- \subset G.$$

In particular, we will consider $G^{w_0,e} = Bw_0B \cap B_-$, which is an open dense subset of B_-.

Let $G_0 = N_-HN$ be the open dense subset of elements in G that admit a Gaussian decomposition. For a dominant weight $\mu \in P_+$, the *principal minor* $\Delta_{\mu,\mu}$ is a regular function $G \to \mathbb{C}$ uniquely determined by its value on G_0:

$$\Delta_{\mu,\mu}(n_-hn) = h^\mu, \text{ for any } n_- \in N_-, h \in H, n \in N.$$

For any two weights γ and δ of the form $\gamma = w\mu$, $\delta = v\mu$, for some $w, v \in W$, the *generalized minor* $\Delta_{w\mu,v\mu}$ is the regular function on G given by

$$\Delta_{\gamma,\delta}(g) = \Delta_{w\mu,v\mu}(g) = \Delta_{\mu,\mu}(\overline{w}^{\,-1}g\overline{v}), \text{ for } g \in G,$$

where $\overline{w}$ is a specific lift of $w \in W$ to G as in [6, Equation 1.5].

Fix a reduced word $\mathbf{i} = (i_1, \ldots, i_m)$, $i_j \in I$, for $w_0 = s_{i_1} \cdots s_{i_m}$. Let $\boldsymbol{R} = \boldsymbol{R}^- \cup \boldsymbol{R}^+$, where $\boldsymbol{R}^- = [-r, -1]$ and $\boldsymbol{R}^+ = [1, m]$. For $1 < k < m$, let $v_k = s_{i_m} \cdots s_{i_{k+1}}$ and let $v_m = e$. For $k \in \boldsymbol{R}^-$, let $i_k = -k$ and $v_k = w_0$. Consider the functions

$$\Delta_k := \Delta_{v_k \omega_{i_k}, \omega_{i_k}}, \quad k \in \boldsymbol{R}.$$

The functions Δ_k form a seed for the upper cluster algebra structure on $\mathbb{C}[G^{w_0,e}]$ described in [5].

Being an upper cluster algebra implies that any $f \in \mathbb{C}[G^{w_0,e}]$ is a Laurent polynomial in the functions Δ_k. The functions Δ_k then determine an open embedding

$$\sigma(\mathbf{i}) \colon (\mathbb{C}^\times)^{m+r} \to G^{w_0,e}, \tag{6}$$

which is a (birational) inverse to

$$G^{w_0,e} \to \mathbb{C}^{m+r}; \qquad g \mapsto (\Delta_{-r}(g), \ldots, \Delta_m(g)).$$

Note that there is no term Δ_k with index $k = 0$.

We conclude this section by recalling how generalized minors appear in matrix entries of representations of G. A dominant integral weight $\mu \in P_+$ can be written uniquely as

$$\mu = \sum_{i \in I} c_i(\mu) \omega_i, \qquad c_i(\mu) \in \mathbb{Z}_{\geqslant 0}.$$

Then the function $\Delta_{w_0\mu,\mu}$ can be written as

$$\Delta_{w_0\mu,\mu} = \prod_{i \in I} \Delta_{w_0\omega_i,\omega_i}^{c_i(\mu)}. \tag{7}$$

One can check that

$$h \cdot \Delta_{w_0\mu,\mu} \cdot h' = h^{-w_0\mu} h'^{\mu} \Delta_{w_0\mu,\mu},$$
$$E_i \cdot \Delta_{w_0\mu,\mu} = \Delta_{w_0\mu,\mu} \cdot E_i = 0 \text{ for } i \in I,$$

where $h, h' \in H$, and G acts on $\mathbb{C}[G]$ in the standard way

$$(g \cdot f \cdot h)(x) = f(g^{-1} x h) \qquad g, h, x \in G, \;\; f \in \mathbb{C}[G].$$

For a sequence of indices $\mathbf{j} = (j_1, \ldots, j_n)$ in I, write $F_\mathbf{j} = F_{j_1} F_{j_2} \cdots F_{j_n} \in U(\mathfrak{g})$. Recall that the functions $F_\mathbf{j} \cdot \Delta_{w_0\mu,\mu} \cdot F_\mathbf{k}$ arise from representations of G as follows. Let $(V, \rho\colon G \to \mathrm{GL}(V))$ be the irreducible G-module with highest weight μ. Let $v_1, \ldots, v_n$ be a weight basis of V, where H acts on v_j with weight $\mathrm{wt}(v_j) \in \mathfrak{h}^*$, and assume $\mathrm{wt}(v_1) = \mu$ and $\mathrm{wt}(v_n) = w_0\mu$. Let $\rho_{j,k}(g)$ be the (j,k)-entry of the matrix for $\rho(g)$ with respect to the basis $\{v_j\}$. Then $\rho_{n,1} = c\Delta_{w_0\mu,\mu}$, for some $c \in \mathbb{C}^\times$. We may choose the weight basis such that $c = 1$. Each $\rho_{j,k}$ is a linear combination of terms of the form $F_{\mathbf{j}} \cdot \Delta_{w_0\mu,\mu} \cdot F_{\mathbf{k}}$, where $\mathbf{j}$ and $\mathbf{k}$ are such that

$$h \cdot (F_{\mathbf{j}} \cdot \Delta_{w_0\mu,\mu} \cdot F_{\mathbf{k}}) \cdot h' = h^{-\mathrm{wt}(v_j)}(h')^{\mathrm{wt}(v_k)}(F_{\mathbf{j}} \cdot \Delta_{w_0\mu,\mu} \cdot F_{\mathbf{k}}) \tag{8}$$

for all $h, h' \in H$.

2.3. The partial tropicalization and its symplectic leaves

Recall from Section 2.1 that $K^* = AN_-$. Let $\boldsymbol{S} = \{k \in \boldsymbol{R} \mid v_k\omega_{i_k} \neq \omega_{i_k}\}$. Then $|\boldsymbol{R}\backslash\boldsymbol{S}| = r$, and $\Delta_k(K^*) \subset \mathbb{R}_+$ if and only if $k \in \boldsymbol{R}\backslash\boldsymbol{S}$. The collection of functions

$$\{\Delta_k \mid k \in \boldsymbol{R}\} \cup \{\overline{\Delta_k} \mid k \in \boldsymbol{S}\}$$

define a real coordinate system on an open dense subset of K^*. Equip $\mathbb{R}^{r+m} \times \mathbb{T}^m$ with coordinates $(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}})$, where $\lambda_{\boldsymbol{R}} = (\lambda_k)_{k\in\boldsymbol{R}}$ and $\varphi_{\boldsymbol{S}} = (\varphi_k)_{k\in\boldsymbol{S}}$.

There is a Poisson manifold $(PT(K^*), \pi_{PT})$, called the *partial tropicalization of* K^*, which was introduced in [2]. As a manifold, $PT(K^*)$ is defined as

$$PT(K^*) := \mathcal{C} \times \mathbb{T}^m \subset \mathbb{R}^{r+m} \times \mathbb{T}^m,$$

where $\mathcal{C}$ is an open convex polyhedral cone of dimension $r + m$ defined by inequalities described in [6] and [2, Theorem 6.24]. The definition of $\mathcal{C}$ depends on the choice of reduced word $\mathbf{i}$ fixed in Section 2.2. More precisely, $\mathcal{C}$ is the set of points $x \in \mathbb{R}^{m+r}$ satisfying an inequality $\Phi^t(x) > 0$, where $\Phi^t : \mathbb{R}^{m+r} \to \mathbb{R}$ is a certain piecewise-linear function called the tropical Berenstein-Kazhdan potential. The Poisson structure π_{PT} is constant in the coordinates $(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}})$. The symplectic leaves of $PT(K^*)$ are the joint level sets of the coordinates $\lambda_{\boldsymbol{R}^-} = (\lambda_{-r}, \ldots, \lambda_{-1})$ [3, Theorem 6.5].

There is a correspondence between symplectic leaves of $PT(K^*)$ and regular coadjoint orbits of K, which we now describe. To each $\xi \in \mathfrak{t}^*_+$ we associate $\lambda_{\boldsymbol{R}^-} \in \mathbb{R}^r$ with coordinates

$$\lambda_{-i} = (w_0\omega_i, \sqrt{-1}\xi) \text{ for } i = -r, \ldots, -1.$$

Denote the symplectic leaf of $PT(K^*)$ that is the fiber of $\lambda_{\boldsymbol{R}^-}$ by $\mathscr{P}_\xi$. The corresponding coadjoint orbit is $\mathbb{O}_\xi$. For each fixed value of $s \neq 0$, the leaf $\mathscr{P}_\xi$ also corresponds to the dressing orbit $\mathscr{D}_{s\xi}$, defined in Section 2.1,

Each symplectic leaf $\mathscr{P}_\xi \subset PT(K^*)$ inherits a symplectic form from π_{PT} denoted by ν_ξ.

Theorem 2.2. *[3, Theorem 6.11] The symplectic volume of $(\mathscr{P}_\xi, \nu_\xi)$ equals the symplectic volume of the coadjoint orbit $\mathbb{O}_\xi \subset \mathfrak{k}^*$ equipped with the Kirillov-Kostant-Souriau symplectic form:*

$$\mathrm{Vol}\,(\mathscr{P}_\xi, \nu_\xi) = \mathrm{Vol}(\mathbb{O}_\xi, \omega_\xi).$$

Remark 2.3. Although [3, Theorem 6.11] is only stated for leaves parameterized by regular dominant integral weights, the theorem here follows by scaling and continuity.

In order to compare the Poisson structures of $PT(K^*)$ and K^*, we define the *detropicalization map* $\mathfrak{L}_s \colon PT(K^*) \to K^*$ as follows. For $s < 0$, let

$$\mathfrak{L}_s(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}}) = \sigma(\mathbf{i}) \left(e^{s\lambda_{-r} - \sqrt{-1}\varphi_{-r}}, \ldots, e^{s\lambda_m - \sqrt{-1}\varphi_m} \right), \tag{9}$$

where we understand $\varphi_k = 0$ on the right hand side if $k \notin \boldsymbol{S}$. Denote $b_s = \mathfrak{L}_s(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}})$.

Remark 2.4 (Conventions). We follow the conventions of [3, 6] for (partial) tropicalization, which are opposite to those of [2]. We consider $K^* \subset B_-$, as in [3], rather than $K^* \subset B$, as in [2], and take the limit $s \to -\infty$. This accounts for the minus signs in (9).

The Casimir functions for K^* are related to the coordinates $\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}}$ by the detropicalization map via r equations (one for each Casimir function):

$$(10)\quad \begin{aligned} C_i(b_s)^2 &= \mathrm{Tr}(\rho^{\omega_i}(b_s b_s^*)) = \sum_j \rho^{\omega_i}_{j,j}(b_s b_s^*) = \sum_{j,k} |\rho^{\omega_i}_{j,k}(b_s)|^2 \\ &= \sum_{j,k} \left| \sum_{\mathbf{i},\mathbf{j}} c_{\mathbf{i},\mathbf{j}} (F_{\mathbf{i}} \Delta_{w_0\omega_i,\omega_i} F_{\mathbf{j}})(b_s) \right|^2 \\ &= |\Delta_{w_0\omega_i,\omega_i}(b_s)|^2 \left(1 + \sum_{j,k} \left| \sum_{\mathbf{i},\mathbf{j}} c_{\mathbf{i},\mathbf{j}} \frac{(F_{\mathbf{i}} \Delta_{w_0\omega_i,\omega_i} F_{\mathbf{j}})(b_s)}{\Delta_{w_0\omega_i,\omega_i}(b_s)} \right|^2 \right). \end{aligned}$$

Since $b_s = \mathfrak{L}_s(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}})$, the last line on the right side can be rewritten as a Laurent polynomial in the functions $e^{s\lambda_k - \sqrt{-1}\varphi_k}$. The term $|\Delta_{w_0\omega_i,\omega_i}(b_s)|^2 = e^{2s\lambda_{-i}}$ dominates the expression for $s \ll 0$, and the exponents in the other terms are controlled by their distance from the boundary of $\mathcal{C}$, as follows.

Recall that $\mathcal{C}$ is the set of points $x \in \mathbb{R}^{m+r}$ satisfying the inequality $\Phi^t(x) > 0$. For $\delta > 0$, let $\mathcal{C}^\delta \subset \mathcal{C}$ be the set of points $x \in \mathbb{R}^{m+r}$ which satisfy the inequality $\Phi^t > \delta$. Then,

Proposition 2.5. *[2, Theorem 4.13 and Lemma 6.17] For $(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}}) \in \mathcal{C}^\delta \times \mathbb{T}^m$, each term*

$$\left| \sum_{\mathbf{i},\mathbf{j}} c_{\mathbf{i},\mathbf{j}} \frac{(F_{\mathbf{i}} \Delta_{w_0\omega_i,\omega_i} F_{\mathbf{j}})(b_s)}{\Delta_{w_0\omega_i,\omega_i}(b_s)} \right| = O(e^{s\delta}).$$

Here and throughout, a function $f(s)$ is in $O(g(s))$, $g(s) \geqslant 0$, if there exists $c > 0$ such that

$$-cg(s) \leqslant f(s) \leqslant cg(s).$$

As a direct consequence of Proposition 2.5 and Equations (10), we have:

Corollary 2.6. *[3, Remark 6.6] For all $\xi \in \mathfrak{t}^*_+$ and $(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}}) \in \mathcal{P}_\xi$, and for each $i = 1, \dots, r$,*

$$\lim_{s \to -\infty} \frac{1}{s} \log \circ C_i \circ \mathfrak{L}_s(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}}) = \lambda_{-i} = (w_0\omega_i, \sqrt{-1}\xi).$$

Remark 2.7. Corollary 2.6 says that points $\mathfrak{L}_s(\mathcal{P}_\xi)$ in the image of a tropical leaf under the detropicalization map approach the corresponding scaled dressing orbit $\mathcal{D}_{s\xi}$ in the limit $s \to -\infty$. It is useful to note that points in $\mathfrak{L}_s(\mathcal{P}_\xi)$ will concentrate near a certain region of $\mathcal{D}_{s\xi}$, not the entire orbit: there are points in the preimages of the scaled dressing orbits $\mathfrak{L}_s^{-1}(\mathcal{D}_{s\xi})$ that remain far away from $PT(K^*)$, even as $s \to -\infty$ (see Figure 2).

3. Symplectic volumes of the leaves of π_s

In this section we study volumes of the symplectic leaves of the Poisson bivector

$$\pi_s := (\mathfrak{L}_s)^*(s\pi_{K^*}).$$

Note that the pullback of a bivector under a diffeomorphism is by definition the pushforward under the inverse diffeomorphism. The symplectic leaves in question are submanifolds of $\mathbb{R}^{r+m} \times \mathbb{T}^m$. Roughly, for $s \ll 0$ each of these leaves has a piece which lies inside $PT(K^*) = \mathcal{C} \times \mathbb{T}^m$, close to the corresponding leaf of π_{PT} (Section 3.1). For $s \ll 0$, the volume of the symplectic leaves concentrate there (Proposition 3.5). This is illustrated in Figure 2.

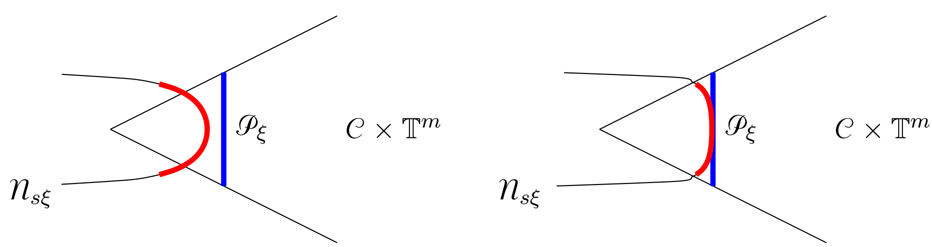
Let us first establish some notation. Each symplectic leaf of π_s is the preimage under $\mathfrak{L}_s$ of a dressing orbit. We denote the leaf and its symplectic form by

$$\mathcal{N}_{s\xi} := \mathfrak{L}_s^{-1}(\mathcal{D}_{s\xi}), \qquad \eta_{s\xi} := (\pi_s)^{-1}.$$

There is a corresponding symplectic leaf $\mathcal{P}_\xi$ of $PT(K^*)$ equipped with ν_ξ, as described in Section 2.3. Recall, for $\xi \in \mathfrak{t}^*_+$,

$$\mathcal{P}_\xi := \left\{(\lambda_{\mathbf{R}}, \varphi_{\mathbf{S}}) \in PT(K^*) \mid \lambda_{-i} = (w_0\omega_i, \sqrt{-1}\xi),\ i = -r, \ldots, -1\right\},$$

which is a product of an open polytope (a fiber in $\mathcal{C}$ of projection to the first r coordinates) times a torus. We will often reference the open subset $\mathcal{P}^\delta_\xi := \mathcal{P}_\xi \cap (\mathcal{C}^\delta \times \mathbb{T}^m)$ and its closure $\overline{\mathcal{P}}^\delta_\xi$.



(a) $s = -1$ (b) $s = -2$

Figure 2. Volume of the symplectic leaves $\mathcal{N}_{s\xi}$ of π_s concentrates on the part of $\mathcal{N}_{s\xi}$ that is close to the corresponding tropical leaf, $\mathcal{P}_\xi$.

3.1. The implicit function theorem argument

Consider the map

$$F_{s\xi} = (f_{-r}, \dots, f_{-1})\colon \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{T}^m \to \mathbb{R}^r \tag{11}$$

with coordinates f_{-i} defined by composing the detropicalization map (9) with the Casimir functions (4) on K^*,

$$f_{-i}(\lambda_{\mathbf{R}}, \varphi_{\mathbf{S}}) = \frac{1}{s}\log \circ C_i \circ \mathfrak{L}_s(\lambda_{\mathbf{R}}, \varphi_{\mathbf{S}}). \tag{12}$$

The fiber $F_{s\xi}^{-1}(\lambda_{\mathbf{R}^-})$ is the symplectic leaf $\mathcal{N}_{s\xi}$. The following lemma will allow us to apply the implicit function theorem at certain points in $\mathcal{N}_{s\xi}$.

Lemma 3.1. *For all $(\lambda_{\mathbf{R}}, \varphi_{\mathbf{S}}) \in \mathcal{C}^\delta \times \mathbb{T}^m$, the derivatives*

$$\begin{aligned} D_{\lambda_{\boldsymbol{R}^-}} F_{s\xi} &= I_r + O(e^{2s\delta}); \\ D_{\lambda_{\boldsymbol{R}^+}} F_{s\xi} &= O(e^{2s\delta}); \\ D_{\varphi_{\boldsymbol{S}}} F_{s\xi} &= O(e^{2s\delta}). \end{aligned} \tag{13}$$

(Here I_r is the $r \times r$ identity matrix and $O(e^{s\delta})$ denotes a matrix of the appropriate dimensions whose entries are $O(e^{2s\delta})$ as functions of s.)

Proof. By the formula for f_{-i}, Equations (10), and the comment directly following Equations (10),

$$e^{2sf_{-i}(\lambda_{\boldsymbol{R}},\varphi_{\boldsymbol{S}})} = e^{2s\lambda_{-i}} \left(1 + \sum_{j,k} c_{j,k} e^{2sL_{j,k}(\lambda_{\boldsymbol{R}},\varphi_{\boldsymbol{S}})} \right).$$

for $-i = -r, \dots, -1$, constants $c_{j,k}$, and some linear combinations $L_{j,k}(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}})$. Differentiating these equations gives

$$\begin{aligned} \frac{\partial f_{-i}}{\partial \lambda_k} &= e^{2s(\lambda_{-i} - f_{-i}(\lambda_{\boldsymbol{R}},\varphi_{\boldsymbol{S}}))} \left(\delta_{-i,k} + \sum_{j,k} \left(\frac{\partial L_{j,k}}{\partial \lambda_k} + \delta_{-i,k} \right) c_{j,k} e^{2sL_{j,k}(\lambda_{\boldsymbol{R}},\varphi_{\boldsymbol{S}})} \right); \\ \frac{\partial f_{-i}}{\partial \varphi_k} &= e^{2s(\lambda_{-i} - f_{-i}(\lambda_{\boldsymbol{R}},\varphi_{\boldsymbol{S}}))} \sum_{j,k} \frac{\partial L_{j,k}}{\partial \varphi_k} c_{j,k} e^{2sL_{j,k}(\lambda_{\boldsymbol{R}},\varphi_{\boldsymbol{S}})}. \end{aligned}$$

Here $\delta_{-i,k}$ is the Kronecker-delta function. By Proposition 2.5, for $(\lambda_{\mathbf{R}}, \varphi_{\mathbf{S}}) \in \mathcal{C}^{\delta} \times \mathbb{T}^m$,

$$e^{2s(\lambda_{-i} - f_{-i}(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}}))} = 1 + O(e^{2s\delta});$$
$$e^{2sL_{j,k}(\lambda_{\boldsymbol{R}}, \varphi_{\boldsymbol{S}})} = O(e^{2s\delta}),$$

which completes the proof. □

Fix an arbitrary element $p = (\lambda_{\mathbf{R}^-}, \lambda_{\mathbf{R}^+}, \varphi_{\mathbf{S}}) \in \mathcal{P}_\xi$ and consider the subspace

$$\mathcal{S}_p := \mathbb{R}^r \times \{\lambda_{\mathbf{R}^+}\} \times \{\varphi_{\mathbf{S}}\} \subseteq \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{T}^m.$$

By an intermediate value theorem argument, we can show that for $s \ll 0$, $\mathcal{N}_{s\xi}$ intersects $\mathcal{S}_p$ near p:

Lemma 3.2. *For all $\xi \in \mathfrak{t}^*_+$ and for all $\delta, \upsilon > 0$ sufficiently small, there exists $s_0 < 0$ such that for all $s \leqslant s_0$ and $p \in \mathcal{P}^\delta_\xi$, the intersection $\mathcal{S}_p \cap \mathcal{N}_{s\xi} \cap B_\upsilon(\mathcal{P}_\xi)$ is non-empty (see Figure 3).*

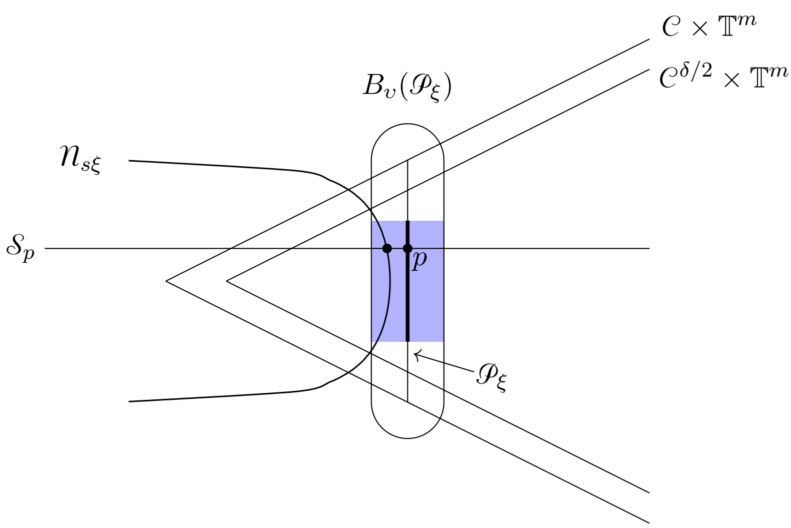


Figure 3. The intersection described in Lemma 3.2. The intersection of $\mathcal{N}_{s\xi}$ with the shaded region is locally the graph of a function defined on $\mathcal{P}^\delta_\xi$ (Proposition 3.3). In the figure, $\mathcal{P}^\delta_\xi$ is the thick part of $\mathcal{P}_\xi$.

Proof. Consider the equivalent problem of showing there is a s_0 such that for all $s \leqslant s_0$ and $p \in \mathscr{P}_\xi^\delta$, the submanifold $\mathfrak{L}_s(\mathcal{S}_p \cap B_\upsilon(\mathscr{P}_\xi))$ intersects the dressing orbit $\mathcal{D}_{s\xi}$. Since dressing orbits are joint level sets of the Casimir functions C_i, showing this intersection is non-empty is equivalent to showing that $\lambda_{\mathbf{R}^-}$ is contained in the image of $\mathcal{S}_p \cap B_\upsilon(\mathscr{P}_\xi)$ under the map $F_{s\xi}$ defined in Equations (11) and (12).

Fix $\delta > 0$ (small enough that $\mathscr{P}_\xi^\delta$ is nonempty). By Corollary 2.6, for $\varepsilon > 0$ sufficiently small,

$$\lim_{s \to -\infty} f_{-i}(\lambda_{-r}, \ldots, \lambda_{-i} \pm \varepsilon, \ldots, \lambda_{-1}, \lambda_{\mathbf{R}^+}, \varphi_{\mathbf{S}}) = \lambda_{-i} \pm \varepsilon.$$

Thus, for all $p \in \overline{\mathscr{P}}_\xi^\delta$, there is a s_p such that for $s \leqslant s_p$, the map $F_{s\xi}$ satisfies the assumptions of the Poincaré-Miranda Theorem on the box

$$[\lambda_{-r} - \varepsilon, \lambda_{-r} + \varepsilon] \times \cdots \times [\lambda_{-1} - \varepsilon, \lambda_{-1} + \varepsilon] \times \{\lambda_{\boldsymbol{R}^+}\} \times \{\varphi_{\boldsymbol{S}}\} \subset \mathcal{S}_p.$$

Take $\varepsilon > 0$ sufficiently small so that the box is contained in $\mathcal{S}_p \cap B_\upsilon(\mathscr{P}_\xi)$ and, without loss of generality (making υ smaller if necessary), assume that $\mathcal{S}_p \cap B_\upsilon(\mathscr{P}_\xi) \subset \mathcal{C}^{\delta/2} \times \mathbb{T}^m$ for all $p \in \overline{\mathscr{P}}_\xi^\delta$. It follows by the Poincaré-Miranda theorem that $\lambda_{\mathbf{R}^-}$ is contained in the image of the box under the map $F_{s\xi}$ for $s \leqslant s_p$.

By transversality of the intersection of $\mathcal{S}_p$ and $\mathcal{N}_{s\xi}$ at points in $\mathcal{C}^{\delta/2} \times \mathbb{T}^m$, for s less than some s' (Lemma 3.1), each $p \in \overline{\mathscr{P}}_\xi^\delta$ has a neighborhood U_p such that for $p' \in U_p$ and $s \leqslant s_p$, the intersection $\mathcal{S}_{p'} \cap \mathcal{N}_{s\xi} \cap B_\upsilon(\mathscr{P}_\xi)$ is non-empty. Passing to a finite subcover U_{p_k}, $k = 1, \ldots, n$ and letting $s_0 = \min\{s', s_{p_k}\}$ completes the proof. □

Define

$$\mathcal{U}_{\xi,\delta} := \bigcup_{p \in \mathscr{P}_\xi^\delta} \mathcal{S}_p.$$

From this point forward, take $\upsilon > 0$ sufficiently small so that $\mathcal{U}_{\xi,\delta} \cap B_\upsilon(\mathscr{P}_\xi) \subset \mathcal{C}^{\delta/2} \times \mathbb{T}^m$. The region $\mathcal{U}_{\xi,\delta} \cap B_\upsilon(\mathscr{P}_\xi)$ is shaded blue in Figure 3.

Proposition 3.3. *For all $\delta > 0$ and $s \leqslant s_0$ as in Lemma 3.2, the intersection $\mathcal{N}_{s\xi} \cap \mathcal{U}_{\xi,\delta} \cap B_\upsilon(\mathscr{P}_\xi)$ is locally the graph of a function*

$$g_s \colon \mathscr{P}_\xi^\delta \to \mathbb{R}^r.$$

Proof. Combine Lemmas 3.1, 3.2, and the implicit function theorem. □

3.2. Comparing symplectic volumes on the leaves of π_s

In this subsection, we compare the symplectic volumes of $(\mathcal{P}_\xi, \nu_\xi)$ and $(\mathcal{N}_{s\xi}, \eta_{s\xi})$. By Proposition 3.3, the intersection of $\mathcal{N}_{s\xi}$ with $\mathcal{U}_{\xi,\delta} \cap B_v(\mathcal{P}_\xi)$ is locally the graph of a function g_s. i.e. locally there is a diffeomorphism

$$G_s : \mathcal{P}_\xi^\delta \to \mathcal{N}_{s\xi},\ (\lambda_{\mathbf{R}}, \varphi_{\mathbf{S}}) \mapsto (g_s(\lambda_{\mathbf{R}^+}, \varphi_{\mathbf{S}}), \lambda_{\mathbf{R}^+}, \varphi_{\mathbf{S}})$$

Lemma 3.4. *For $s \leqslant s_0$ as in Lemma 3.2, at points in $\mathcal{N}_{s\xi} \cap \mathcal{U}_{\xi,\delta} \cap B_v(\mathcal{P}_\xi) \subset \mathcal{C}^{\delta/2} \times \mathbb{T}^m$,*

$$(G_s)_* \nu_\xi = \eta_{s\xi} + O(e^{s\delta})$$

(here $O(e^{s\delta})$ denotes a 2-form whose coefficients in coordinates $(\lambda_{\mathbf{R}}, \varphi_{\mathbf{S}})$ are $O(e^{s\delta})$ as functions of s).

Proof. Fix $p = (\lambda_{\mathbf{R}}, \varphi_{\mathbf{S}}) \in \mathcal{P}_\xi^\delta$. By the implicit function theorem, for all $(X, Y) \in T_p\mathcal{P}_\xi^\delta = \mathbb{R}^m \times \mathbb{R}^m$,

$$D_pG_s(X, Y) = \big(-(D_{\lambda_{\mathbf{R}^-}} F_{s\xi})^{-1}(D_{\lambda_{\mathbf{R}^+}} F_{s\xi} X + D_{\varphi_{\mathbf{S}}} F_{s\xi} Y), X, Y\big)$$

The constant bivector π_{PT} has the form

$$\pi_{PT} = \sum_k X_k \wedge Y_k$$

for some $X_k, Y_k \in T_p\mathcal{P}_\xi^\delta$. By Lemma 3.1 and the formula for D_pG_s above, we find $(G_s)_*\pi_{PT} = \pi_{PT} + O(e^{s\delta})$, where $O(e^{s\delta})$ denotes a bivector whose coefficients in coordinates $(\lambda_{\mathbf{R}}, \varphi_{\mathbf{S}})$ are $O(e^{s\delta})$ as functions of s. The 2-form

$$(G_s)_*\nu_\xi = ((G_s)_*\pi_{PT})^{-1} = \pi_{PT}^{-1} + O(e^{s\delta}).$$

On the other hand, by the proof of [2, Theorem 6.18], at $G_s(p) \in \mathcal{C}^{\delta/2} \times \mathbb{T}^m$,

$$\eta_{s\xi} = (\pi_s)^{-1} = \Big(\pi_{PT} + O(e^{s\delta})\Big)^{-1} = \pi_{PT}^{-1} + O(e^{s\delta}).$$

□

Finally, we show that for $s \ll 0$, the symplectic volume of $\mathcal{N}_{s\xi}$ is concentrated on the piece that lies in $\mathcal{C}^{\delta/2} \times \mathbb{T}^m$.

Proposition 3.5. *For ξ, δ ,υ, and $s \leqslant s_0$ as in Lemma 3.2, the symplectic volume of $\mathcal{N}_{s\xi} \cap \mathcal{U}_{\xi,\delta} \cap B_\upsilon(\mathcal{P}_\xi) \subset \mathcal{C}^{\delta/2} \times \mathbb{T}^m$ satisfies the inequalities*

$$\begin{aligned}\mathrm{Vol}(\mathcal{N}_{s\xi}, \eta_{s\xi}) &\geqslant \mathrm{Vol}(\mathcal{N}_{s\xi} \cap \mathcal{U}_{\xi,\delta} \cap B_\upsilon(\mathcal{P}_\xi), \eta_{s\xi}) \\ &\geqslant \mathrm{Vol}(\mathcal{N}_{s\xi}, \eta_{s\xi}) - \mathrm{Vol}(\mathcal{P}_\xi \setminus \mathcal{P}_\xi^\delta, \nu_\xi) + O(e^{\delta s}).\end{aligned}$$

Note that $\mathrm{Vol}(\mathcal{P}_\xi \setminus \mathcal{P}_\xi^\delta, \nu_\xi) \to 0$ as $\delta \to 0$.

Remark 3.6. In the proof of Theorem 1.3, we choose $\delta, \upsilon > 0$ sufficiently small and let D_ε be the closure of $\mathcal{U}_{\xi,\delta} \cap B_\upsilon(\mathcal{P}_\xi) \subseteq \mathcal{C}^{\delta/2} \times \mathbb{T}^m$.

Proof. The first inequality follows since volume is monotonic. By Proposition 3.3 and Lemma 3.4, $\mathcal{N}_{s\xi} \cap \mathcal{U}_{\xi,\delta} \cap B_\upsilon(\mathcal{P}_\xi)$ is locally the image of a diffeomorphism G_s with domain in $\mathcal{P}_\xi^\delta$ and $(G_s)_*\nu_\xi = \eta_{s\xi} + O(e^{s\delta})$, so

$$\mathrm{Vol}(\mathcal{N}_{s\xi} \cap \mathcal{U}_{\xi,\delta} \cap B_\upsilon(\mathcal{P}_\xi), \eta_{s\xi}) \geqslant \mathrm{Vol}(\mathcal{P}_\xi^\delta, \nu_\xi) + O(e^{s\delta}).$$

By definition of $\mathcal{P}_\xi^\delta = \mathcal{P}_\xi \cap (\mathcal{C}^\delta \times \mathbb{T}^m)$,

$$\mathrm{Vol}(\mathcal{P}_\xi^\delta, \nu_\xi) = \mathrm{Vol}(\mathcal{P}_\xi, \nu_\xi) - \mathrm{Vol}(\mathcal{P}_\xi \setminus \mathcal{P}_\xi^\delta, \nu_\xi).$$

Finally, by Theorem 2.2,

$$\begin{aligned}&\mathrm{Vol}(\mathcal{P}_\xi, \nu_\xi) - \mathrm{Vol}(\mathcal{P}_\xi \setminus \mathcal{P}_\xi^\delta, \nu_\xi) + O(e^{s\delta/2}) \\ &= \mathrm{Vol}(\mathcal{N}_{s\xi}, \eta_\xi) - \mathrm{Vol}(\mathcal{P}_\xi \setminus \mathcal{P}_\xi^\delta, \nu_\xi) + O(e^{s\delta}).\end{aligned}$$

□

4. Preimages of points in $PT(K^*)$

The goal of this section is to show that for a fixed value of $\xi \in \mathfrak{t}_+^*$ and $s \ll 0$, if $\mathfrak{C}_s(\mathrm{Ad}_k^* \xi) \in \mathfrak{L}_s(PT(K^*))$, then $\mathrm{Ad}_k^* \xi$ must be close to ξ in the coadjoint orbit $\mathcal{O}_\xi$.

Fix a faithful irreducible representation (ρ, V) of G. Let $n = \dim(V)$, and fix a Hermitian inner product on V which is preserved by $\rho(K)$. For the representation V, fix a unitary weight basis $v_1, \ldots, v_n$. Consider the wedge product $(\rho^l, \wedge^l V)$ of the representation (ρ, V). Note that $\wedge^l V$ has basis

$$\{v_{\mathbf{I}} := v_{i_1} \wedge \cdots \wedge v_{i_l} \mid \mathbf{I} = (i_1, \ldots, i_l) \text{ and } i_1 < \cdots < i_l\}.$$

We can reorder the unitary weight basis $\{v_i\}$ so that, for all $l \in [n]$, the vector $v_{[l]} = v_1 \wedge \cdots \wedge v_l$ is a minimal weight vector of $\wedge^l V$. For $\mathbf{I}, \mathbf{J} \subset [n]$

with $|\mathbf{I}| = |\mathbf{J}| = l$ denote by $\Delta_{\mathbf{I},\mathbf{J}}$ the $l \times l$ minor of elements of $\mathrm{GL}(V)$ in the basis v_i, with rows $\mathbf{I}$ and columns $\mathbf{J}$. Define the map

$$\mathrm{pr}_{\mathfrak{t}^*} : PT(K^*) \to \mathfrak{t}^*; \qquad x \in \mathscr{P}_\xi \mapsto \xi.$$

Lemma 4.1. *Let $l \in [n]$, and let $\mathbf{J} \subset [n]$ with $|\mathbf{J}| = l$ and $[l] \neq \mathbf{J}$. For all $\delta > 0$ and $s < 0$, define*

$$U_s = \{k \in K \mid \mathfrak{E}_s(\mathrm{Ad}_k^* \xi) = \mathfrak{L}_s(p) \text{ for some } p \in \mathcal{C}^\delta \times \mathbb{T}^m,\ \xi \in \mathrm{pr}_{\mathfrak{t}^*}(\mathcal{C}^\delta \times \mathbb{T}^m)\}.$$

Then there exists $a > 0$ such that for all $k \in U_s$,

$$|\Delta_{[l],\mathbf{J}}(\rho(k))| \leqslant a e^{s\delta},$$

in the unitary weight basis $\{v_i\}$.

Proof. Let $\mathrm{wt}(v_{[l]}) = w_0\zeta$, where $\zeta \in P_+$ is a dominant weight, and consider the irreducible subrepresentation $G \cdot v_{[l]}$ of $\wedge^l V$ which is generated by $v_{[l]}$. Then in this subrepresentation, $v_{[l]}$ will be of lowest weight. Let $\mathbf{L}$ denote the index of the highest weight vector of this subrepresentation. It follows that $\mathrm{wt}(v_{\mathbf{L}}) = \zeta$. Write the matrix entries of $\rho^l(g)$ in the basis $\{v_{\mathbf{I}}\}$ as $\rho^l_{\mathbf{I},\mathbf{J}}(g)$. Note that $\rho^l_{\mathbf{I},\mathbf{J}}(g) = \Delta_{\mathbf{I},\mathbf{J}}(\rho(g))$. Because $v_{[l]}$ is of lowest weight in the subrepresentation $G \cdot v_{[l]}$, we have

$$\rho^l(g) v_{[l]} = \sum_{\substack{w_0\zeta < \mathrm{wt}(v_{\mathbf{J}}) \\ \text{or } \mathbf{J}=[l]}} \rho^l_{\mathbf{J},[l]}(g) v_{\mathbf{J}}, \tag{14}$$

where the sum on the right hand side is over weight vectors $v_{\mathbf{J}}$ such that $w_0\zeta - \mathrm{wt}(v_{\mathbf{J}})$ is a negative weight or $\mathbf{J} = [l]$. In other words, $\rho^l_{\mathbf{J},[l]}(g) = 0$ unless $w_0\zeta < \mathrm{wt}(v_{\mathbf{J}})$ or $\mathbf{J} = [l]$.

Using the definition of the dressing action and the fact that the map $\mathfrak{E}_s$ is K-equivariant, we have

$$k \cdot (\mathfrak{E}_s(\xi))^2 \cdot k^* = \mathfrak{E}_s(\mathrm{Ad}_k^* \xi) \cdot \mathfrak{E}_s(\mathrm{Ad}_k^* \xi)^*. \tag{15}$$

Rewrite (15) as

$$k \cdot d_s^2 \cdot k^* = b_s \cdot b_s^* \tag{16}$$

where $d_s = \exp\left(s\sqrt{-1}\psi(\xi)\right)$ and $b_s = \mathfrak{L}_s(p)$.

Let us apply the representation ρ^l to both sides of (16), and consider the $([l],[l])$-entry of these matrices. Using the fact that $\{v_{\mathbf{I}}\}$ is a unitary basis for $\wedge^l V$, matrix multiplication and (14) gives us:

$$\sum_{\substack{w_0\zeta<\mathrm{wt}(v_{\mathbf{J}})\\ \text{or } \mathbf{J}=[l]}} \left|\rho^l_{\mathbf{J},[l]}(k^*)\right|^2 \cdot \left|\rho^l_{\mathbf{J},\mathbf{J}}(d_s)\right|^2 = \sum_{\substack{w_0\zeta<\mathrm{wt}(v_{\mathbf{J}})\\ \text{or } \mathbf{J}=[l]}} \left|\rho^l_{\mathbf{J},[l]}(b_s^*)\right|^2. \tag{17}$$

Since $\rho^l(k)\cdot\rho^l(k^*) = \rho^l(kk^*) = 1$, we have

$$\sum_{\substack{w_0\zeta<\mathrm{wt}(v_{\mathbf{J}})\\ \text{or } \mathbf{J}=[l]}} \left|\rho^l_{\mathbf{J},[l]}(k^*)\right|^2 = 1. \tag{18}$$

Rewrite (18) as

$$\left|\rho^l_{[l],[l]}(k^*)\right|^2 = 1 - \sum_{w_0\zeta<\mathrm{wt}(v_{\mathbf{J}})} \left|\rho^l_{\mathbf{J},[l]}(k^*)\right|^2$$

and plug it into (17). After rearranging, we get

$$\begin{aligned} \left|\rho^l_{[l],[l]}(d_s)\right|^2 = &\sum_{\substack{w_0\zeta<\mathrm{wt}(v_{\mathbf{J}})\\ \text{or } \mathbf{J}=[l]}} \left|\rho^l_{\mathbf{J},[l]}(b_s^*)\right|^2 \\ &+ \sum_{w_0\zeta<\mathrm{wt}(v_{\mathbf{J}})} \left|\rho^l_{\mathbf{J},[l]}(k^*)\right|^2 \cdot \left(\left|\rho^l_{[l],[l]}(d_s)\right|^2 - \left|\rho^l_{\mathbf{J},\mathbf{J}}(d_s)\right|^2\right). \end{aligned} \tag{19}$$

Since $w_0\zeta < \mathrm{wt}(v_{\mathbf{L}})$ and the terms $\left|\rho^l_{[l],[l]}(d_s)\right|^2 - \left|\rho^l_{\mathbf{J},\mathbf{J}}(d_s)\right|^2$ are positive, by discarding terms on the right hand side of (19), one has for any $\mathbf{J}$ with $w_0\zeta < \mathrm{wt}(v_{\mathbf{J}})$,

$$\left|\rho^l_{[l],[l]}(d_s)\right|^2 > \left|\rho^l_{\mathbf{L},[l]}(b_s^*)\right|^2 + \left|\rho^l_{\mathbf{J},[l]}(k^*)\right|^2 \cdot \left(\left|\rho^l_{[l],[l]}(d_s)\right|^2 - \left|\rho^l_{\mathbf{J},\mathbf{J}}(d_s)\right|^2\right).$$

Hence

$$\left|\rho^l_{\mathbf{J},[l]}(k^*)\right|^2 < \frac{\left|\rho^l_{[l],[l]}(d_s)\right|^2 - \left|\rho^l_{\mathbf{L},[l]}(b_s^*)\right|^2}{\left|\rho^l_{[l],[l]}(d_s)\right|^2 - \left|\rho^l_{\mathbf{J},\mathbf{J}}(d_s)\right|^2} = \frac{1 - \left|\rho^l_{[l],\mathbf{L}}(b_s)\right|^2/\left|\rho^l_{[l],[l]}(d_s)\right|^2}{1 - \left|\rho^l_{\mathbf{J},\mathbf{J}}(d_s)\right|^2/\left|\rho^l_{[l],[l]}(d_s)\right|^2}. \tag{20}$$

From Proposition 2.5, because $p \in \mathcal{C}^\delta \times \mathbb{T}^m$, we have

$$C_i(b_s)^2 = |\Delta_{w_0\omega_i,\omega_i}(b_s)|^2 \left(1 + O(e^{2s\delta})\right).$$

On the other hand, from (15), for $s < 0$,

$$C_i(b_s)^2 = \mathrm{Tr}(\rho^{\omega_i}(d_s^2)) = \sum_j c_j e^{2s(\gamma_j,\sqrt{-1}\xi)} = e^{2s(w_0\omega_i,\sqrt{-1}\xi)}\left(1 + O(e^{2s\delta})\right).$$

Here, the weights γ_j are those which appear in the representation ρ^{ω_i}, and $c_j = 1$ when γ_j is the extremal weight $w_0\omega_i$. The last equality holds because, by assumption, $\xi \in \mathrm{pr}_{\mathfrak{t}^*}(\mathcal{C}^\delta \times \mathbb{T}^m)$, which in turn guarantees that $(\alpha_i, \sqrt{-1}\xi) > \delta$ for all $i \in I$.

Combining the previous two equations, since

$$e^{s(w_0\omega_i,\sqrt{-1}\xi)} = \Delta_{w_0\omega_i,w_0\omega_i}(d_s),$$

we have

$$\left|\left|\frac{\Delta_{w_0\omega_i,\omega_i}(b_s)}{\Delta_{w_0\omega_i,w_0\omega_i}(d_s)}\right|^2 - 1\right| = O(e^{2s\delta}), \qquad \text{for all } i \in I.$$

For $\zeta \in P_+$, by using (7) we get

$$\left|\left|\frac{\Delta_{w_0\zeta,\zeta}(b_s)}{\Delta_{w_0\zeta,w_0\zeta}(d_s)}\right|^2 - 1\right| = O(e^{2s\delta}), \tag{21}$$

for $s \ll 0$. By the discussion at the end of Section 2, we know

$$\rho^l_{[l],[l]} = c\Delta_{w_0\zeta,w_0\zeta} \quad \text{and} \quad \rho^l_{[l],\mathbf{L}} = c\Delta_{w_0\zeta,\zeta}$$

for some $c \in \mathbb{C}^\times$. By (21) and (20), we find $|\Delta_{[l],\mathbf{J}}(\rho(k))| = |\Delta_{\mathbf{J},[l]}(\rho(k^*))| = O(e^{s\delta})$. □

Lemma 4.2. *Let $g\colon (-\infty, 0) \to \mathrm{U}(n)$ be an element of $\mathrm{U}(n)$ depending on a parameter s. Assume there exists $\delta > 0$ such that*

$|\Delta_{[l],\mathbf{J}}(g(s))| = O(e^{s\delta})$
for all $l \in [n]$ and all $\mathbf{J} \subset [n]$ with $|\mathbf{J}| = l$ and $[l] \neq \mathbf{J}$.

Then, the matrix entries satisfy $|g_{i,j}(s)| = O(e^{s\delta})$ for all $i \neq j$.

Proof. We proceed by induction on i. When $i = 1$, we have $|g_{1,j}| = O(e^{s\delta})$ for $j \neq 1$. Assume the statement is known for $1, \ldots, i-1$. By induction hypothesis and the fact that g is unitary, we have $1 - |g_{j,j}| = O(e^{s\delta})$ for $j < i$. By taking inner product of the i^{th} row with the previous rows and again using the fact that g is unitary, we have $|g_{i,j}| = O(e^{s\delta})$ for $j < i$. For $j > i$, consider the minor $\Delta_{[i],\mathbf{J}}(g)$, where $\mathbf{J} = \{1, \ldots, i-1, j\}$. By assumption, $|\Delta_{[i],\mathbf{J}}(g)| = O(e^{s\delta})$. Expanding this minor along the j^{th} column and applying the induction hypothesis, we have that $|g_{i,j}| = O(e^{s\delta})$. □

Recall that $\mathcal{N}_{s\xi}$ is the preimage $(\mathfrak{E}_s^{-1} \circ \mathfrak{L}_s)^{-1}(\mathcal{O}_\xi)$.

Proposition 4.3. *For all $\xi \in \mathfrak{t}_+^*$, if $U \subset \mathcal{O}_\xi$ is an open subset with $\xi \in U$, then for all sufficiently small $\delta > 0$, there exists $s_0 \in \mathbb{R}$ so that, for all $s \leqslant s_0$,*

$$\mathfrak{E}_s^{-1} \circ \mathfrak{L}_s \left(\mathcal{N}_{s\xi} \cap (\mathcal{C}^\delta \times \mathbb{T}^m) \right) \subseteq U.$$

Proof. Fix $\xi \in \mathfrak{t}_+^*$, $U \subseteq \mathcal{O}_\xi$ open with $\xi \in U$, and $\delta > 0$ sufficiently small so that $\xi \in \mathrm{pr}_{\mathfrak{t}^*}(\mathcal{C}^\delta \times \mathbb{T}^m)$. Observe that for all $s < 0$,

$$U_s' = \{k \in K \mid \mathfrak{E}_s(\mathrm{Ad}_k^* \xi) \in \mathfrak{L}_s(\mathcal{N}_{s\xi} \cap (\mathcal{C}^\delta \times \mathbb{T}^m))\} \subseteq U_s.$$

By Lemma 4.1, there exists $a > 0$ such that for all $k \in U_s'$,

$$|\Delta_{[l],\mathbf{J}}(\rho(k))| \leqslant a e^{s\delta}.$$

By Lemma 4.2 and since ρ faithful, there exists $s_0 < 0$ such that for all $s \leqslant s_0$,

$$\mathfrak{E}_s^{-1} \circ \mathfrak{L}_s \left(\mathcal{N}_{s\xi} \cap (\mathcal{C}^\delta \times \mathbb{T}^m) \right) \subseteq U.$$

□

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