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On the lower bounds of the L^2-norm of the Hermitian scalar curvature

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On a pre-quantized symplectic manifold, we show that the symplectic Futaki invariant, which is an obstruction to the existence of constant Hermitian scalar curvature almost-Kähler metrics, is actually an asymptotic invariant. This allows us to deduce a lower bound for the L^2-norm of the Hermitian scalar curvature as obtained by S. Donaldson [15] in the Kähler case.

1. Introduction

Let (M, ω) be a symplectic manifold of (real) dimension $2n$. An almost-complex structure J is ω-compatible if the tensor $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ defines a Riemannian metric. The metric g is called then an almost-Kähler metric. When J is integrable, g is a Kähler metric. Given an almost-Kähler metric g, one can define the *canonical Hermitian connection* (see [24, Section 2], [32])

$$\nabla_X Y = D^g_X Y - \frac{1}{2} J(D^g_X J)Y,$$

where D^g is the Levi-Civita connection of g and X, Y any vector fields on M. The curvature of the induced Hermitian connection on the anti-canonical bundle $\Lambda^n(T^{1,0}_J M)$ is of the form $\sqrt{-1}\rho^\nabla$. The closed (real) 2-form ρ^∇ is called the *Hermitian Ricci form* and it is a de Rham representative of $2\pi c_1(M,\omega)$ the first Chern class of the tangent bundle TM. The *Hermitian scalar curvature* s^∇ of the almost-Kähler structure (ω, J) is then the normalized trace of ρ^∇, i.e.

$$s^\nabla \omega^n = 2n\rho^\nabla \wedge \omega^{n-1}.$$

When the metric is Kähler, s^∇ coincides with the (usual) Riemannian scalar curvature.

We fix now a $2n$-dimensional compact (connected) symplectic manifold (M,ω). We denote by AK_ω the (infinite dimensional) Fréchet space of all ω-compatible almost-complex structures and C_ω the subspace of ω-compatible complex structures. It turns out that the natural action of the Hamiltonian symplectomorphism group $Ham(M,\omega)$ on AK_ω is Hamiltonian [16, 21] with moment map $\mu : AK_\omega \longrightarrow (Lie(Ham(M,\omega)))^*$ given by $\mu(J)(f) = \int_M s^\nabla f \frac{\omega^n}{n!}$, where s^∇ is the Hermitian scalar curvature of (ω, J). The induced metrics by the critical points of the functional (defined on AK_ω)

$$\|\mu\|^2 : J \longmapsto \int_M (s^\nabla)^2 \frac{\omega^n}{n!}$$

are called *extremal almost-Kähler metrics* [4, 30]. These metrics appear then as a natural extension of Calabi's extremal Kähler metrics [8, 9] to the symplectic setting. The symplectic gradient of the Hermitian scalar curvature of an extremal almost-Kähler metric turns out to be an infinitesimal isometry of the metric. In particular, constant Hermitian scalar curvature almost-Kähler (*cHscaK* in short) metrics are extremal.

Furthermore, one can define a (geometric) *symplectic Futaki invariant* (in the Kähler case, see [22]). Explicitly, we fix a compact group G in the Hamiltonian symplectomorphism group $Ham(M,\omega)$. Let $\mathfrak{g}_\omega$ be the space of smooth functions (with zero integral) which are Hamiltonians with respect to ω of elements of $\mathfrak{g} = Lie(G)$. Denote by AK^G_ω (resp. C^G_ω) the space of all G-invariant ω-compatible almost-complex structures (resp G-invariant ω-compatible complex structures). Then, we define the map

$$\begin{aligned} \mathcal{F}^G_\omega : \mathfrak{g} &\longrightarrow \mathbb{R} \\ \mathcal{F}^G_\omega(X) &= \int_M s^\nabla \mathsf{h} \frac{\omega^n}{n!}, \end{aligned}$$

where $\mathsf{h} \in \mathfrak{g}_\omega$ is the Hamiltonian induced by X and s^∇ is the Hermitian scalar curvature induced by any $J \in AK^G_\omega$. It turns out that $\mathcal{F}^G_\omega$ is independent of the choice of $J \in AK^G_\omega$ [23, Proposition 9.7.1] [30, Lemma 3.4]. The map $\mathcal{F}^G_\omega$

is called the symplectic Futaki invariant relative to AK^G_ω. It readily follows that if AK^G_ω contains a cHscaK metric, then $\mathcal{F}^G_\omega \equiv 0$.

In the Kähler setting, the *Donaldson–Futaki invariant* defined in [18] gives (non-trivial) lower bounds on the Calabi functional [8, 9] as proved by S. Donaldson in [15, Theorem 1]. The existence of constant scalar curvature Kähler (*cscK* in short) metrics is then related to an algebro-geometric stability condition, called K-stability, introduced by G. Tian [44] for Fano manifolds (see also [14]). The Donaldson–Futaki invariant [15, 18] is an algebraic invariant which can be defined for singular manifolds and coincide with the geometric Futaki invariant [22] when the central fiber of the degeneration is smooth. Furthermore, the Donaldson-Futaki invariant has been also defined recently for Sasakian manifolds in [11].

In this paper, we point out that the Donaldson–Futaki invariant may be extended to the symplectic case. Our motivation is that, in the toric case, the existence of an extremal Kähler metric is conjecturally equivalent to the existence of non-integrable extremal almost-Kähler metrics [18] (see also [2, Conjecture 2]). Moreover, the examples of toric manifolds studied in [18] which are not K-stable do not admit even a cHscaK metric. A related question and also part of the motivation of this work is the *almost-Kähler Calabi-Yau equation* on 4-manifolds which has a unique solution if a conjecture of S. Donaldson [20] holds (see also [31, Question 6.9] and [45]).

More explicitly, let us consider (M,ω) a compact symplectic manifold *pre-quantized* by a Hermitian line bundle (L,h). We fix a compact group G in $Ham(M,\omega)$. We consider a G-invariant ω-compatible almost-complex structure J. For an integer k, we define the *renormalized* Bochner–Laplacian operator Δ_k acting on the smooth sections of L^k. For a sufficiently large $k > 0$, the space $\mathcal{H}_k$ of the eigensections of Δ_k, with eigenvalues in some interval depending only on L, is finite dimensional. An orthonormal basis of $\mathcal{H}_k$ gives a 'nearly' symplectic and 'nearly' holomorphic embedding $\Phi_k : M \longrightarrow \mathbb{P}\mathcal{H}^*_k$ [36, 37], where the space $\mathbb{P}\mathcal{H}^*_k$ can be identified with a $N_k + 1$ complex projective space. Moreover, the line bundles L^k and $\Phi^*_k(\mathcal{O}(1))$ over M are canonically isomorphic. The Hermitian metrics h^k on L^k and $h^{\Phi^*_k(\mathcal{O}(1))}$ (induced by the Hermitian metric on $\mathcal{O}(1)$) on $\Phi^*_k(\mathcal{O}(1))$ are then related by

$$h^{\Phi^*_k(\mathcal{O}(1))} = \frac{h^k}{B_k},$$

where B_k is the *generalized Bergman function* defined in (3) (see [37, Theorem 8.3.11]).

Furthermore, the dimension of the space $\mathcal{H}_k$ has an asymptotic expansion of the following type (as consequence of Theorem 2.2),

$$\begin{aligned}\dim \mathcal{H}_k &= a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),\\ &= k^n \int_M \frac{\omega^n}{n!} + \frac{k^{n-1}}{4\pi} \int_M s^{\nabla} \frac{\omega^n}{n!} + O(k^{n-2}).\end{aligned}$$

where s^{∇} is the Hermitian scalar curvature of (ω, J). Observe that the integral $\int_M s^{\nabla} \frac{\omega^n}{n!} = \frac{4\pi}{(n-1)!} \int_M c_1(M,\omega) \wedge [\omega]^{n-1}$ is independent of the choice of J.

We choose a S^1-action Γ on (M,ω) generated by a Hamiltonian vector field in $Lie(G)$. The S^1-action on M can be lifted to L^k and induces a linear action A_k on the smooth sections of L^k. Furthermore, this linear action fixes the space $\mathcal{H}_k$ since the S^1-action Γ preserves the almost-Kähler metric induced by J. The trace of this linear action admits an asymptotic expansion (as a consequence of Theorem 2.5)

$$\begin{aligned}\mathrm{Tr}(A_k) &= b_0 k^{n+1} + b_1 k^n + O(k^{n-1}),\\ &= -k^{n+1} \int_M \mathsf{h} \frac{\omega^n}{n!} - \frac{k^n}{4\pi} \int_M \mathsf{h} s^{\nabla} \frac{\omega^n}{n!} + O(k^{n-1}),\end{aligned}$$

where the function h is a Hamiltonian of the S^1-action with respect to ω. We remark that the integral $\int_M \mathsf{h} s^{\nabla} \frac{\omega^n}{n!}$ is independent of the choice of $J \in AK_\omega^G$ the space of all G-invariant ω-compatible almost-complex structures [30, Lemma 3.1].

Definition 1.1. The *symplectic Donaldson–Futaki invariant* $\mathcal{F}^G(\Gamma)$ of the S^1-action Γ on (M,L) generated by a Hamiltonian vector field in $Lie(G)$ is defined by

$$\mathcal{F}^G(\Gamma) = \frac{a_1}{a_0} b_0 - b_1.$$

Let $\chi_\Gamma : \mathbf{C}^* \hookrightarrow GL(N_k+1)$ be a one-parameter subgroup, such that $\chi_\Gamma(S^1) \subset U(N_k+1)$ corresponds to the linear action induced by the S^1-action Γ on $\mathcal{H}_k$, i.e. $\chi_\Gamma(t) = t^{A_k}$, for $t \in S^1$. Now, we consider the *degeneration* induced by the family $\chi_\Gamma(t) \circ \Phi_k(M)$ in $\mathbb{P}\mathcal{H}_k^*$. We do the following assumptions:

Assumption (A1). The limit

$$M_0 = \lim_{t\to 0} \chi_\Gamma(t) \circ \Phi_k(M)$$

exists as a compact connected *symplectic stratified space* in the sense of Sjamaar-Lerman [42, Section 1].

Assumption (A2). In $\mathbb{P}\mathcal{H}_k^*$, the current of integration over $\Phi_k(M)$ is equal to the current of integration over S_0, connected open dense stratum in M_0.

Remark 1.2. If there is only one stratum then M_0 is smooth. The existence of an open dense stratum is an essential result of [42], and this stratum S_0 is a manifold of full measure. If we were considering the stronger notion of Kähler stratified space for M_0, S_0 would be a complex manifold and its closure a complex-analytic subvariety of the Kähler space M_0. In the Kähler case, a *normality* assumption was historically introduced in [14, Section 1] to ensure the convergence of the integrals. In the Fano case, for the anticanonical polarization, it turns out the Yau-Tian-Donaldson conjecture is valid if one considers actually only *normal* limit central fibers.

By definition, M_0 is preserved under the action of χ_Γ. Then, our main result is that the L^2-norm of the zero mean value of the Hermitian scalar curvature of any G-invariant almost-Kähler structure whose symplectic form is ω is bounded below by the symplectic Donaldson–Futaki invariant.

Theorem 1. *Let AK_ω^G be the space of all G-invariant ω-compatible almost-complex structures. Assume that for all k large and for any S^1-subgroup $\Gamma \subset G$, the limit M_0 exists in the above sense, i.e (A1) and (A2) hold. Then,*

$$\inf_{J\in AK_\omega^G} \|s^\nabla - S^\nabla\|_{L^2} \geqslant \sup_{\Gamma\subset G} \left(-4\pi \frac{\mathcal{F}^G(\Gamma)}{\|\chi_\Gamma\|}\right),$$

where we denoted s^∇ the Hermitian scalar curvature of (ω, J) with normalized average $S^\nabla = \frac{\int_M s^\nabla \omega^n}{\int_M \omega^n}$ and $\|\chi_\Gamma\|$ is the leading term of the asymptotic expansion of the norm of the trace-free part $\underline{A_k}$ of A_k i.e.

$$\mathrm{Tr}(\underline{A_k^2}) = \|\chi_\Gamma\|^2 k^{n+2} + O(k^{n+1}). \tag{1}$$

The L^2-norm $\|\cdot\|_{L^2}$ is with respect to the volume form $\frac{\omega^n}{n!}$.

The asymptotic expansion of $\mathrm{Tr}(A_k^2)$ is computed in Lemma 3.4 while the expression of $\|\chi_\Gamma\|$ is given by Corollary 4. Our proof of (1) is direct and differs in part from [15, Theorem 2] (see also the reference [43]). In the toric almost-Kähler case, a lower bound of the norm of the Hermitian scalar curvature is given by C. LeBrun in [28, Theorem A] and [29, Proposition 2].

Theorem 1 indicates that one can *possibly* define a notion of *stability* for the existence of almost-Kähler metrics with constant Hermitian scalar curvature and study the uniqueness of such metrics as done by S. Donaldson in [17] in the complex projective case. In order to do so, we suggest the following definition for a *symplectic test configuration* based on [43, Section 6.3] and [18].

Definition 1.3. A symplectic test configuration of exponent k, for an almost-Kähler manifold (M, ω, J) pre-quantized by a Hermitian complex line bundle (L, h), is given by:

(i) An embedding $\Phi_k : M \longrightarrow \mathbb{P}\mathcal{H}_k^*$ using the vector space $\mathcal{H}_k$ built using the sections of L^k. We can then identify $\mathbb{P}\mathcal{H}_k^*$ with $\mathbb{CP}^{N_k}$.

(ii) A one-parameter subgroup $\chi_\Gamma : \mathbf{C}^* \hookrightarrow GL(N_k + 1)$.

(iii) The existence of a limit at $t = 0$ of $\chi_\Gamma(t) \circ \Phi_k(M)$ as a compact connected symplectic stratified space $M_0 \subset \mathbb{P}\mathcal{H}_k^*$ with connected open dense stratum S_0.

(iv) The existence for all $t \neq 0$ of a surjective continuous map

$$\varphi_t : \chi_\Gamma(t) \circ \Phi_k(M) \to M_0$$

such that there exists an open connected dense submanifold $U_t \subset \chi_\Gamma(t) \circ \Phi_k(M)$ for which the restriction $\varphi_{t\,|U_t}$ is a symplectomorphism on S_0.

Remark 1.4. Note that both conditions (A1) and (A2) are satisfied under this definition. In the complex projective case (J is integrable), a test configuration implies the existence of the map φ_t satisfying (iv), as proved by M. Harada and K. Kaveh in [26, Theorem A, Remark (i)], while an algebraic variety can always be seen as a topologically stratified space. In particular, this definition encompasses the case of (complex normal) symplectic varieties in the sense of Beauville for which there exists a canonical stratification, see [27, Theorem 2.3].

Let us discuss some applications of Theorem 1. A direct corollary is the following result.

Corollary 2. *Under the assumptions of Theorem 1, if an almost-Kähler structure* (ω, J) *has a constant Hermitian scalar curvature, for any* $J \in AK_\omega^G$*, then* $\mathcal{F}^G(\Gamma) \geqslant 0$ *for any* S^1*-subgroup* $\Gamma \subset G$*.*

A consequence of Corollary 2 is that if for a S^1-action $\Gamma \subset G$ on a Kähler manifold (M, ω, J), $\mathcal{F}^G(\Gamma) < 0$, then there is no cscK metrics in the Kähler class $[\omega]$ on the complex manifold (M, J) since the symplectic Donaldson–Futaki invariant coincides with the Donaldson–Futaki invariant. Furthermore, we want to stress the fact that there is *no cHscaK metric in* AK_ω^G. In other words, a destabilizing test configuration in the Kähler setting would imply non existence even of cHscaK metrics. We observe that the Kähler metrics in the Kähler class $[\omega]$ can be seen as a subspace of AK_ω^G via Moser's Lemma (see for example [40, Section 3.2]). If we consider the K-unstable toric examples studied in [18, Section 7.2] for which the destabilizing test configurations satisfy our assumptions, we recover this way the fact that they don't carry cHscaK structures. We have extra examples of such phenomena for projective bundles.

Corollary 3. *Consider* E *a holomorphic vector bundle over a complex curve of genus* $g \geq 2$ *of rank* $\mathrm{rk}(E)$*. Let* $\mathbb{P}(E)$ *be the complex manifold underlying the total space of the projectivization of* E*.*

- *If* $\mathrm{rk}(E) = 2$*, then the ruled surface* $\mathbb{P}(E)$ *admits a cHscaK metric if and only if* E *is polystable.*
- *If* $\mathrm{rk}(E) > 2$*, then the ruled manifold* $\mathbb{P}(E)$ *admits a cHscaK metric* ω *with* $C_\omega^{S^1} \neq \emptyset$ *if and only if* E *is polystable.*

We briefly present the organization of the paper. In Section 2, we introduce the necessary material recalling the key results of W. Lu– X. Ma– G. Marinescu. In Section 3, we prove Theorem 1 and Corollary 3.

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2. Generalized Bergman kernel

In order to generalize the lower bounds on the Calabi functional as done by S. Donaldson [15] to the symplectic case, we use the eigensections of the *renormalized Laplacian* operator [6, 25], defined on smooth sections of a Hermitian line bundle over a compact symplectic manifold, as natural substitutes for the holomorphic sections. Note that we are not working with another natural operator, the spin^c Dirac operator for which other results about Bergman kernel exist, see [12].

More precisely, let (M,ω) be a compact symplectic manifold of dimension $2n$. Suppose that (M,ω) is *pre-quantized* by a Hermitian complex line bundle (L,h) which means that the curvature R^{∇^L} of some Hermitian connection ∇^L of L satisfies

$$\frac{\sqrt{-1}}{2\pi}R^{\nabla^L}=\omega.$$

This means that the de Rham class $[\omega]$ is integral.

Fix an almost-complex structure J compatible with ω and denote by $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$ the induced almost-Kähler metric. This defines a Laplacian operator Δ^{L^k} on L^k acting on smooth sections of L^k, for $k>0$. Explicitly,

$$\Delta^{L^k}=-\sum_{i=1}^{2n}\left(\nabla^{L^k}_{e_i}\right)^2-\nabla^{L^k}_{(D^g_{e_i}e_i)},$$

where D^g is the Levi-Civita connection with respect to g and $\{e_i\}$ is a local g-orthonormal basis of TM. The Hermitian metric h^k and connection ∇^{L^k} on L^k are induced by h and ∇^L. The renormalized Laplacian is given then by

$$\Delta_k=\Delta^{L^k}-2\pi nk.$$

From [38, Corollary 1.2], there exists two constants $C_1,C_2>0$ independent of k such that the spectrum of Δ_k is contained in $(-C_1,C_1)\cup(k\,C_2,+\infty)$ (see also [6, 25]). Let $\mathcal{H}_k\subset C^\infty(M,L^k)$ be the span of the eigensections of Δ_k with eigenvalues in $(-C_1,C_1)$. The space $\mathcal{H}_k$ is then finite dimensional and for large k (see [6, 25, 38])

$$\dim\mathcal{H}_k=\int_M e^{k[\omega]}\,Td(T^{1,0}_JM),$$

where $Td(T^{1,0}_JM)$ is the Todd class of the (complex) vector bundle $T^{1,0}_JM$.

Remark 2.1. When g is Kähler and L is a holomorphic Hermitian line bundle, the operator Δ_k coincides with the $\overline{\partial}$-Laplacian, by the Bochner–Kodaira formula (e.g [5, Proposition 3.71]). Then, for large k, the space $\mathcal{H}_k$ is exactly the space of holomorphic sections of L^k.

On sections of L^k, we define the inner product

$$\langle s_1, s_2 \rangle_{L^2} = \int_M (s_1, s_2)_{h^k} \frac{(k\omega)^n}{n!}. \tag{2}$$

Let $\{s_0, \cdots, s_{N_k}\}$ be an orthonormal basis of $\mathcal{H}_k$. At $x \in M$, the *generalized Bergman function* is defined as the restriction to the diagonal of the Bergman kernel, i.e by the formula

$$B_k(x) = \sum_{i=0}^{N_k} |s_i(x)|^2_{h^k}. \tag{3}$$

X. Ma– G. Marinescu proved the following asymptotic expansion.

Theorem 2.2 ([36, Theorem 0.2] [37, Theorem 8.3.4]). *We have the following expansion when $k \to \infty$,*

$$B_k = 1 + \frac{s^\nabla}{4\pi} k^{-1} + O(k^{-2}), \tag{4}$$

valid in C^l for any $l \geq 0$. Here, s^∇ denotes the Hermitian scalar curvature of (ω, J).

Let $\mathbb{P}\mathcal{H}_k^*$ be the projective space associated to the dual of $\mathcal{H}_k$. Moreover, once we fix a basis of $\mathcal{H}_k$, we have an identification $\mathbb{P}\mathcal{H}_k^* \cong \mathbb{CP}^{N_k}$. We have then the following

Theorem 2.3 ([36, Theorem 3.6],[37, Theorem 8.3.11]). *For large k, the Kodaira maps $\Phi_k : M \longrightarrow \mathbb{P}\mathcal{H}_k^*$, given by*

$$\Phi_k(x) = \{s \in \mathcal{H}_k \mid s(x) = 0\}$$

are well-defined.

Observe that there is a well-defined Fubini-Study form ω_{FS} on $\mathbb{P}\mathcal{H}_k^*$ with a compatible metric g_{FS}. We have then

Theorem 2.4 ([36, Theorem 3.6],[37, Theorem 8.3.11]). *For large k, we have in C^∞-norm*

$$\frac{1}{k}\Phi_k^*(\omega_{FS}) - \omega = O(k^{-1}),$$
$$\frac{1}{k}\Phi_k^*(g_{FS}) - g = O(k^{-1}).$$

Moreover, the maps Φ_k are embeddings and 'nearly holomorphic' i.e.

$$\frac{1}{k}\|\bar\partial\Phi_k\| = O(k^{-1}), \quad \frac{1}{k}\|\partial\Phi_k\| \geqslant C, \quad \text{for some } C > 0.$$

Very recently, W. Lu– X. Ma– G. Marinescu improved the speed rate of the approximation of the symplectic form. This improvement is actually crucial to obtain the main result of the paper.

Theorem 2.5 ([34, Theorem 0.1]). *For large k, we have in C^∞-norm*

$$\frac{1}{k}\Phi_k^*(\omega_{FS}) - \omega = O(k^{-2}).$$

3. Lower bounds on the L^2-norm of the Hermitian scalar curvature

Let (M,ω) be a compact symplectic manifold pre-quantized by a Hermitian complex line bundle (L,h). We fix an ω-compatible almost-complex structure J.

Given an embedding $\Phi_k : M \longrightarrow \mathbb{P}\mathcal{H}_k^*$, for a sufficiently large $k>0$ as in Theorem 2.4, we define a matrix $M(\Phi_k)$ with entries

$$M(\Phi_k)_{ij} = \int_M \Phi_k^*\left(\frac{Z^i\overline{Z}^j}{|Z|^2}\right)\frac{(\Phi_k^*\omega_{FS})^n}{n!},$$

where Z^j are homogeneous coordinates on $\mathbb{P}\mathcal{H}_k^*$. Let $\underline{M}(\Phi_k)$ denote the trace-free part of $M(\Phi_k)$.

Lemma 3.1. *Consider (M,ω,J) a compact almost-Kähler manifold pre-quantized by a Hermitian complex line bundle (L,h). Then, there is a sequence of embeddings $\Phi_k : M \longrightarrow \mathbb{P}\mathcal{H}_k^*$ such that*

$$\|\underline{M}(\Phi_k)\| \leqslant \frac{k^{n/2-1}}{4\pi}\|s^\nabla - S^\nabla\|_{L^2} + O(k^{n/2-2}).$$

Here $\|\underline{M}(\Phi_k)\| = \left(\mathrm{Tr}\,(\underline{M}(\Phi_k))^2\right)^{1/2}$, s^∇ *is the Hermitian scalar curvature of* (ω, J) *and* $S^\nabla = \frac{\int_M s^\nabla \omega^n}{\int_M \omega^n}$ *is the normalized average of* s^∇.

Proof. This is done as in the Kähler case. For the reader's convenience, we reproduce here the proof. We use the sequence of embeddings Φ_k defined by the orthonormal bases $\{s_0, \cdots, s_{N_k}\}$ of $\mathcal{H}_k$. Using Theorem 2.5, we have that

$$M(\Phi_k)_{ij} = \int_M \Phi_k^* \left(\frac{Z^i \overline{Z}^j}{|Z|^2}\right) \frac{(\Phi_k^* \omega_{FS})^n}{n!},$$
$$= \int_M \frac{(s_i, s_j)_{h^k}}{B_k} \frac{(k\omega)^n \left(1 + O(k^{-2})\right)}{n!}.$$

We can assume that M is diagonal. Then, using Theorem 2.2, we obtain

$$\begin{aligned}
(5) \qquad M(\Phi_k)_{ii} &= k^n \int_M \frac{|s_i|^2_{h^k}}{B_k} \frac{\omega^n \left(1 + O(k^{-2})\right)}{n!}, \\
&= k^n \int_M |s_i|^2_{h^k} \left(1 - \frac{s^\nabla}{4\pi} k^{-1}\right) \frac{\omega^n}{n!} + O(k^{n-2}), \\
&= 1 - \frac{k^{-1}}{4\pi} \int_M |s_i|^2_{h^k}\, s^\nabla \frac{(k\omega)^n}{n!} + O(k^{n-2}).
\end{aligned}$$

From Theorem 2.2, the dimension of $\mathcal{H}_k$ is given by

$$(6) \qquad N_k + 1 = k^n \int_M \frac{\omega^n}{n!} + \frac{k^{n-1}}{4\pi} \int_M s^\nabla \frac{\omega^n}{n!} + O(k^{n-2}).$$

It follows that

$$\sum_{i=0}^{N_k} M(\Phi_k)_{ii} = N_k + 1 - \frac{k^{-1}}{4\pi} \int_M B_k\, s^\nabla \frac{(k\omega)^n}{n!} + O(k^{n-2}),$$
$$= N_k + 1 - \frac{k^{-1}}{4\pi} \int_M s^\nabla \frac{(k\omega)^n}{n!} + O(k^{n-2}),$$

Hence

$$\frac{\mathrm{Tr}(M(\Phi_k))}{N_k + 1} = 1 - \frac{k^{-1}}{4\pi} S^\nabla + O(k^{-2}).$$

Combined with (5), the trace free part $\underline{M}(\Phi_k)$ of $M(\Phi_k)$ is

$$\underline{M}(\Phi_k)_{ii} = -\frac{k^{-1}}{4\pi} \int_M |s_i|^2_{h^k} (s^\nabla - S^\nabla) \frac{(k\omega)^n}{n!} + O(k^{-2}).$$

By the Cauchy–Schwarz inequality, we have

$$|\underline{M}(\Phi_k)_{ii}|^2 \leqslant \frac{k^{-2}}{16\pi^2}\int_M |s_i|^2_{h^k}\frac{(k\omega)^n}{n!}\int_M |s_i|^2_{h^k}(s^\nabla - S^\nabla)^2\frac{(k\omega)^n}{n!} + O(k^{-3}),$$
$$= \frac{k^{-2}}{16\pi^2}\int_M |s_i|^2_{h^k}(s^\nabla - S^\nabla)^2\frac{(k\omega)^n}{n!} + O(k^{-3}).$$

Taking the sum, we obtain that

$$\|\underline{M}(\Phi_k)\|^2 \leqslant \frac{k^{-2}}{16\pi^2}\int_M B_k(s^\nabla - S^\nabla)^2\frac{(k\omega)^n}{n!} + O(k^{n-3}),$$
$$= \frac{k^{n-2}}{16\pi^2}\int_M (s^\nabla - S^\nabla)^2\frac{\omega^n}{n!} + O(k^{n-3}).$$

The Lemma follows. □

Our aim now is to find a lower bound for $\|\underline{M}(\Phi_k)\|$. First, we fix a compact group G in $Ham(M,\omega)$. We consider a G-invariant ω-compatible almost-complex structure J. We choose a S^1-action Γ on (M,ω) generated by a Hamiltonian vector field in $Lie(G)$. The S^1-action can be lifted to an action on L^k (preserving h^k and ∇^{L^k}) (for any $k \geqslant 1$). This induces a linear action of S^1 on smooth sections of L^k. Furthermore, since the S^1-action preserves the induced metric by J, the induced action maps $\mathcal{H}_k$ to itself. We denote by $-\sqrt{-1}A_k$ the infinitesimal generator of the linearized S^1-action Γ on $\mathcal{H}_k$ with A_k having integral entries.

For large $k > 0$, let $\Phi_k : M \longrightarrow \mathbb{P}\mathcal{H}_k^*$ be an embedding of M using an orthonormal bases $\{s_0, \cdots, s_{N_k}\}$ of $\mathcal{H}_k$. Let $\chi_\Gamma : \mathbf{C}^* \hookrightarrow GL(N_k+1)$ be a one-parameter subgroup, such that $\chi_\Gamma(S^1) \subset U(N_k+1)$ satisfying $\chi_\Gamma(t) = t^{A_k}$ (normalized so that $\chi_\Gamma(1)$ is the identity map). By definition, $\chi_\Gamma(S^1)$ preserves both the Fubini-Study form ω_{FS} and g_{FS} on $\mathbb{P}\mathcal{H}_k^*$. A Hamiltonian function (with respect to ω_{FS}) for the corresponding S^1-action is given by

$$\mathsf{h}_{A_k} = \frac{-\sum_{i,j}(A_k)_{ij}\, Z^i\overline{Z}^j}{|Z|^2}.$$

so that

$$\Phi_k^*(\mathsf{h}_{A_k}) = \frac{-\sum_{i,j}(A_k)_{ij}\,(s_i,s_j)_{h^k}}{B_k}. \tag{7}$$

Now, let $\Phi_k^t = \chi_\Gamma(t)\circ\Phi_k$ and define the function

$$f(t) = -\mathrm{Tr}(\underline{A_k M}(\Phi_k^t)) = -\mathrm{Tr}\left(\underline{A_k}M(\Phi_k^t)\right),$$

where $\underline{A_k}$ is the trace-free part of A_k. Then

$$f(t) = \int_M \Phi_k^{t*}(\mathsf{h}_{A_k}) \frac{(\Phi_k^{t*}\omega_{FS})^n}{n!} + \frac{\mathrm{Tr}(A_k)}{N_k+1} \int_M \frac{(\Phi_k^{t*}\omega_{FS})^n}{n!}.$$

A calculation shows that for real numbers $t > 0$ we have $f'(t) \geqslant 0$.

Lemma 3.2. *With the above definition, one has* $\forall t > 0$,

$$f'(t) \geqslant 0.$$

Proof. We consider the one-parameter group of diffeomorphisms generated by the vector field $-\mathrm{grad}\,\mathsf{h}_{A_k}$ so we are approaching 0 along the positive real axis in $\mathbb{C}^*$. Then, we have the following derivative at $s = 0$

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} & \int_M \Phi_k^{s*}(\mathsf{h}_{A_k}) \frac{(\Phi_k^{s*}\omega_{FS})^n}{n!} \\ &= -\int_{\Phi_k(M)} |\mathrm{grad}\,\mathsf{h}_{A_k}|^2 \frac{\omega_{FS}^n}{n!} \\ &\quad + \int_{\Phi_k(M)} \mathsf{h}_{A_k} \frac{\mathfrak{L}_{-\mathrm{grad}\,\mathsf{h}_{A_k}}\omega_{FS} \wedge \omega_{FS}^{n-1}}{(n-1)!}. \end{aligned} \tag{8}$$

The second term in the r.h.s of (8) can be written as

$$\begin{aligned} & \int_{\Phi_k(M)} \mathsf{h}_{A_k}\, \mathfrak{L}_{-\mathrm{grad}\,\mathsf{h}_{A_k}}\omega_{FS} \wedge \omega_{FS}^{n-1} \\ &= -\int_M d\left(\Phi_k^*\mathsf{h}_{A_k}\right) \wedge \Phi_k^*(d^c\mathsf{h}_{A_k}) \wedge \Phi_k^*\left(\omega_{FS}^{n-1}\right), \\ &= -\int_{\Phi_k(M)} (d\mathsf{h}_{A_k})_M \wedge d^c\mathsf{h}_{A_k} \wedge \omega_{FS}^{n-1}, \\ &= \frac{1}{n}\int_{\Phi_k(M)} |d\mathsf{h}_{A_k}|_M^2\, \omega_{FS}^n, \end{aligned} \tag{9}$$

where $|d\mathsf{h}_{A_k}|_M^2 = |\mathrm{grad}\,\mathsf{h}_{A_k}|_M^2$ is the norm of the tangential part to $\Phi_k(M)$. We deduce

$$\left.\frac{d}{ds}\right|_{s=0} \int_M \Phi_k^{s*}(\mathsf{h}_{A_k}) \frac{(\Phi_k^{s*}\omega_{FS})^n}{n!} = -\int_{\Phi_k(M)} |\mathrm{grad}\,\mathsf{h}_{A_k}|_N^2 \frac{\omega_{FS}^n}{n!},$$

where $|\mathrm{grad}\,\mathsf{h}_{A_k}|_N^2$ is the norm of the normal component. On the other hand

$$\left.\frac{d}{ds}\right|_{s=0} \int_M \frac{(\Phi_k^{s*}\omega_{FS})^n}{n!} = 0$$

Increasing t corresponds to flowing along $\operatorname{grad} \mathsf{h}_{A_k}$. We deduce that $f'(t) \geqslant 0$ for real numbers $t > 0$. □

Now it follows, since $\lim_{t\to 0} f(t)$ exists (cf. (11)), that

$$-\mathrm{Tr}(\underline{A_k\, M}(\Phi_k)) = f(1) \geqslant \lim_{t\to 0} f(t),$$

and so by the Cauchy–Schwarz inequality

$$\|\underline{A_k}\|\, \|\underline{M}(\Phi_k)\| \geqslant \lim_{t\to 0} f(t). \tag{10}$$

In particular if $\lim_{t\to 0} f(t) > 0$, then we get a positive lower bound on $\|\underline{M}(\Phi_k)\|$.

Under our assumption (A2), we can actually consider

$$\lim_{t\to 0} f(t) = \int_{S_0} \mathsf{h}_{A_k} \frac{\omega_{FS}^n}{n!} + \frac{\mathrm{Tr}(A_k)}{N_k + 1} \int_{S_0} \frac{\omega_{FS}^n}{n!}. \tag{11}$$

It follows from Theorem 2.5 that one can choose a Hamiltonian h with respect to ω such that

$$\frac{1}{k}\Phi_k^*\,(\mathsf{h}_{A_k}) - \mathsf{h} = O(k^{-2}). \tag{12}$$

Then

$$\begin{aligned}\int_M \mathsf{h}\, B_k \frac{\omega^n}{n!} &= \frac{1}{k}\int_M \Phi_k^*(\mathsf{h}_{A_k}) B_k \frac{\omega^n}{n!} + O(k^{-2}),\\ &= -\frac{1}{k^{n+1}} \sum_{i,j} (A_k)_{ij} \int_M (s_i, s_j)_{h^k} \frac{(k\omega)^n}{n!} + O(k^{-2}),\\ &= -\frac{1}{k^{n+1}} \mathrm{Tr}(A_k) + O(k^{-2}).\end{aligned}$$

It follows from Theorem 2.2 that

$$\begin{aligned}\mathrm{Tr}(A_k) &= b_0 k^{n+1} + b_1 k^n + O(k^{n-1}),\\ &= -k^{n+1} \int_M \mathsf{h} \frac{\omega^n}{n!} - \frac{k^n}{4\pi} \int_M \mathsf{h} s^{\nabla} \frac{\omega^n}{n!} + O(k^{n-1}).\end{aligned} \tag{13}$$

Using (A2), Equation (12) and Theorem 2.5, we get

$$\begin{aligned}(14)\qquad \int_{S_0} \mathsf{h}_{A_k} \frac{\omega_{FS}^n}{n!} &= \int_{\Phi_k(M)} \mathsf{h}_{A_k} \frac{\omega_{FS}^n}{n!}, \\ &= \int_M (k\,\mathsf{h} + O(k^{-1}))\left(k^n \frac{\omega^n}{n!} + O(k^{n-2})\right), \\ &= -b_0 k^{n+1} + O(k^{n-1}).\end{aligned}$$

Then, from (10), (11), (6), (13) and (14), we deduce

$$\begin{aligned}\|\underline{A_k}\|\,\|\underline{M}(\Phi_k)\| &\geqslant -b_0k^{n+1} + \frac{b_0k^{n+1} + b_1k^n + O(k^{n-1})}{a_0k^n + a_1k^{n-1} + O(k^{n-2})} a_0k^n + O(k^{n-1}), \\ &= -b_0k^{n+1} \\ &\quad + \left(b_0k^{n+1} + b_1k^n + O(k^{n-1})\right)\left(1 - \frac{a_1}{a_0}k^{-1} + O(k^{-2})\right) \\ &\quad + O(k^{n-1}), \\ &= k^n\left(b_1 - \frac{a_1}{a_0}b_0\right) + O(k^{n-1}).\end{aligned}$$

It follows then from Lemma 3.1 that

$$\begin{aligned}(15)\qquad &\|\underline{A_k}\|\left(\frac{k^{n/2-1}}{4\pi}\|s^\nabla - S^\nabla\|_{L^2} + O(k^{n/2-2})\right) \\ &\geqslant k^n\left(b_1 - \frac{a_1}{a_0}b_0\right) + O(k^{n-1}).\end{aligned}$$

Now, we need to compute the asymptotic expansion for $\|A_k\|^2 = \mathrm{Tr}(A_k^2)$. Let us denote $\nu = \omega^n/n!$ and consider $P_{\nu,k}$ the smooth kernel of the L^2-orthogonal projection from $C^\infty(M, L^k)$ to $\mathcal{H}_k$. Set

$$K_k(x,y) = |P_{\nu,k}(x,y)|^2_{h^k\otimes(h^k)^*},$$

where $x, y \in M$. We can write

$$K_k(x,y) = k^n \sum_{i,j=1} (s_i(x), s_j(x))_{h^k}(s_j(y), s_i(y))_{h^k},$$

for $\{s_i\}$ an L^2-orthonormal basis with respect to the inner product (2). We consider the integral operator associated to K_k which is defined for any

$f \in C^\infty(M)$ as

$$Q_{K_k}(f)(x) = \int_M K_k(x,y) f(y) \frac{\omega_y^n}{n!}.$$

The Q-operator has been studied by S. Donaldson [19], K. Liu– X. Ma [33] and X. Ma– G. Marinescu [39] in the context of Kähler compact manifolds. They provided an asymptotic result for this operator. We quote a generalization of this result obtained by W. Lu– X. Ma– G. Marinescu to the context of pre-quantized symplectic compact manifolds.

Theorem 3.3 ([35, Theorem 1.1]). *For any integer $m \geq 0$, there exists a constant $c > 0$ such that for any $f \in C^\infty(M)$,*

$$\left\| Q_{K_k}(f) - f \right\|_{C^m} \leq \frac{c}{k} \|f\|_{C^{m+2}}. \tag{16}$$

Moreover, (16) *is uniform in the sense that there is an integer s_0 such that if the hermitian metric h on L varies in a bounded set in C^{s_0} topology then the constant c is independent of h.*

Lemma 3.4. *With notations as above,*

$$\mathrm{Tr}(A_k^2) = k^{n+2} \int_M \mathsf{h}^2 \frac{\omega^n}{n!} + O(k^{n+1}),$$

where h is a hamiltonian defined by ω.

Proof. Let us write

$$\tilde{A}_{ij} = k^n \int_M (s_i, \Phi_k^*(\mathsf{h}_{A_k}) s_j)_{h^k} \frac{\omega^n}{n!}, \tag{17}$$

where $\Phi_k^*(\mathsf{h}_{A_k})$ is given by (7) and $\{s_i\}$ is a fixed L^2-orthonormal basis of eigensections with respect to the inner product (2). Now, set

$$\begin{aligned} Q(A_k)_{ij} &= k^n \int_M \left(s_i, \sum_{p,q} (A_k)_{pq} (s_p, s_q)_{h^k} s_j \right)_{h^k} \frac{\omega^n}{n!}, \\ &= k^n \int_M \sum_{p,q} (A_k)_{pq} (s_p, s_q)_{h^k} (s_i, s_j)_{h^k} \frac{\omega^n}{n!}. \end{aligned}$$

With the map $\iota : Met(\mathcal{H}_k) \to C^\infty(M)$ given by

$$\iota(A_{ij}) = \sum_{i,j} A_{ij} (s_i, s_j)_{h^k},$$

one can write $\iota \circ Q(A_k) = Q_{K_k} \circ \iota(A_k)$. The map ι is linear and invertible on its image. From Theorem 3.3, we have

$$Q(A_k) = A_k(Id + O(k^{-1})). \tag{18}$$

The Bergman function has a uniform asymptotic expansion as stated in Theorem 2.2. From the higher order term of this expansion, we can deduce using (17), (7) and (18) that

$$\tilde{A}_{ij} = -Q(A_k)(Id + O(k^{-1})) = -A_k(Id + O(k^{-1})).$$

Consequently,

$$\mathrm{Tr}(A_k^2) = \mathrm{Tr}(\tilde{A}^2)(1 + O(k^{-1})).$$

Now, let us compute $\mathrm{Tr}(\tilde{A}^2)$. By a direct computation, we have

$$\begin{aligned}
&\mathrm{Tr}(\tilde{A}^2)\\
&\quad = k^{2n} \int_{M\times M} \sum_{i,j} \left(s_i(x), \Phi_k^*(\mathsf{h}_{A_k})(x) s_j(x)\right) \left(s_j(y), \Phi_k^*(\mathsf{h}_{A_k})(y) s_i(y)\right) \omega_x^n \omega_y^n,\\
&\quad = k^n \int_M \mathrm{Tr}(Q_{K_k}(\Phi_k^*(\mathsf{h}_{A_k}))\Phi_k^*(\mathsf{h}_{A_k}))\omega^n,\\
&\quad = k^{n+2} \int_M \mathrm{Tr}\left(Q_{K_k}\left(\frac{1}{k}\Phi_k^*(\mathsf{h}_{A_k})\right) \frac{1}{k}\Phi_k^*(\mathsf{h}_{A_k})\right) \omega^n.
\end{aligned}$$

We have $Q(\frac{1}{k}\Phi_k^*(\mathsf{h}_{A_k})) = \frac{1}{k}\Phi_k^*(\mathsf{h}_{A_k})(1 + O(\frac{1}{k}))$ from Theorem 3.3 and also $\frac{1}{k}\Phi_k^*(\mathsf{h}_{A_k}) = \mathsf{h}(1 + O(\frac{1}{k}))$ from (12). Combining all previous results, we obtain the asymptotic of $\mathrm{Tr}(A_k^2)$. ☐

Let us write $\mathrm{Tr}(\underline{A_k^2})$ as

$$\mathrm{Tr}(\underline{A_k^2}) = \|\chi_\Gamma\|^2 k^{n+2} + O(k^{n+1}).$$

Then, the expression of $\|\chi_\Gamma\|$ is given by the following result.

Corollary 4. *With notations as above*

$$\|\chi_\Gamma\|^2 = \int_M (\mathsf{h} - \widehat{\mathsf{h}})^2 \frac{\omega^n}{n!},$$

with $\widehat{\mathsf{h}}$ *the normalized average of* h.

Proof of Theorem 1. The proof is now obtained by combining Lemma 3.4 and (15) and letting $k \to \infty$. □

Proof of Corollary 3. We know from Narasimhan and Seshadri that if E is polystable then $\mathbb{P}(E)$ admits a cscK metric (in any Kähler class) and thus a cHscaK metric, see [2, Theorem 1] for details. Now, assume that we have a symplectic form such that $C_\omega^{S^1} \neq \emptyset$ i.e there is an S^1-invariant integrable compatible almost-complex structure J. If $E = E_1 \oplus \cdots \oplus E_s$ is not polystable and F is a destabilizing subbundle of one component of E, say E_1, one can consider the test configuration associated to the deformation to the normal cone of $\mathbb{P}(F \oplus E_2 \cdots \oplus E_s)$ whose central fibre is $\mathbb{P}(F \oplus E_1/F \oplus E_2 \oplus \cdots \oplus E_s)$ and in particular is smooth. This test configuration admits a $\mathbb{C}^*$ action that covers the usual action on the base $\mathbb{C}$ and whose restriction to $F \oplus E_1/F \oplus E_2 \oplus \cdots \oplus E_s$ scales the fibers of F with weight 1 and acts trivially on the other components. Seeing $(\mathbb{P}(E), \omega, J)$ as a Kähler manifold, the computations of [41, Section 5] (see also [13]) show that the Futaki invariant of this test configuration is negative. Actually, the Futaki invariant is a positive multiple of the difference of the slopes $\mu(E_1) - \mu(F) < 0$. Then, we apply Corollary 2 to deduce the non existence of cHscaK structure in $AK_\omega^{S^1}$. In the case of $\mathrm{rk}(E) = 2$, any symplectic rational ruled surface admits a compatible integrable complex structure, see [1, Section 3] and references therein. Note that for the general case, it is unclear whether we can drop the assumption on C_ω as there exist projective manifolds with symplectic forms ω such that $C_\omega = \emptyset$, see for instance [10]. □

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