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Bohr-Sommerfeld Lagrangian submanifolds as minima of convex functions

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We prove more convexity properties for Lagrangian submanifolds in symplectic and Kähler manifolds. Namely, every closed Bohr-Sommerfeld Lagrangian submanifold Q of a symplectic/Kähler manifold X can be realised as a Morse-Bott minimum for some 'convex' exhausting function defined in the complement of a symplectic/complex hyperplane section Y. In the Kähler case, 'convex' means strictly plurisubharmonic while, in the symplectic case, it refers to the existence of a Liouville pseudogradient. In particular, $Q \subset X \setminus Y$ is a regular Lagrangian submanifold in the sense of Eliashberg-Ganatra-Lazarev.

1. Introduction

Rational convexity properties of Lagrangian submanifolds were first discovered in $\mathbf{C}^2$ by Duval and then investigated further by Duval-Sibony, Gayet and Guedj. In particular, generalising a result established by Duval-Sibony [DS95] in $\mathbf{C}^n$, Guedj [Gue99] obtained the following theorem: in a complex projective manifold X, every closed Lagrangian submanifold Q is rationally convex, which means that $X \setminus Q$ is filled up with smooth complex hypersurfaces. More precisely, these complex hypersurfaces Y are very ample divisors of arbitrarily large degrees, so their complements are affine manifolds and possess exhausting $\mathbf{C}$-convex functions $f : X \setminus Y \to \mathbf{R}$. In this work, which was motivated by the study of vanishing cycles in global Picard-Lefschetz theory, we give a necessary and sufficient condition for the existence of such a function f admitting Q as a Morse-Bott (*i.e.* transversally non-degenerate) minimum. This condition refers to a Kähler class and can be more generally stated as follows in the symplectic setting:

Definition 1. Let (X, ω) be an integral symplectic manifold, meaning that X is a closed manifold and ω a symplectic form with integral periods. We say that a Lagrangian submanifold Q *satisfies the Bohr-Sommerfeld condition*

— or simply *is Bohr-Sommerfeld* — if the homomorphism $\mathrm{H}_2(X, Q, \mathbf{Z}) \to \mathbf{R}$ defined by integration of ω takes its values in $\mathbf{Z}$.

In the Kähler setting, our main result is:

Theorem 2. *Let (X, ω) be a closed integral Kähler manifold and Q a closed Lagrangian submanifold satisfying the Bohr-Sommerfeld condition. Then, for every sufficiently large integer k, there exist a complex hyperplane section Y of degree k in X avoiding Q and an exhausting $\mathbf{C}$-convex function $f : X \setminus Y \to \mathbf{R}$ that has a Morse-Bott minimum at Q and is Morse away from Q with finitely many critical points.*

To be more explicit, there exists a holomorphic line bundle $L \to X$ with first Chern class ω such that the complex hypersurface Y is the zero-set of a holomorphic section of some large tensor power of L.

In [AGM01], Auroux-Gayet-Mohsen reproved Guedj's above theorem and extended it to the symplectic setting using the ideas and techniques developed by Donaldson in [Don96]. Theorem 2 also has a symplectic version, whose statement below appeals to the following terminology:

- A *symplectic hyperplane section of degree k* in a closed integral symplectic manifold (X, ω) is a symplectic submanifold Y of codimension 2 that is Poincaré dual to $k\omega$.
- A function $f : X \setminus Y \to \mathbf{R}$ is *ω-convex* if it admits a pseudogradient that is a Liouville (*i.e.* ω-dual to some primitive of ω) vector field.

With this wording, Donaldson's main theorem in [Don96] is that every closed integral symplectic manifold contains symplectic hyperplane sections of all sufficiently large degrees. Furthermore, according to Auroux-Gayet-Mohsen [AGM01], such symplectic hyperplane sections can be constructed away from any given closed Lagrangian submanifold. On the other hand, Giroux showed in [Gir18] that, for all sufficiently large degrees, the complements of Donaldson's symplectic hyperplane sections admit exhausting ω-convex functions (and hence are Weinstein manifolds). Mixing these ingredients, we obtain:

Theorem 3. *Let (X, ω) be a closed integral symplectic manifold and Q a closed Bohr-Sommerfeld Lagragian submanifold of X. Then, for every sufficiently large integer k, there exist a symplectic hyperplane section Y of degree k in X avoiding Q and an exhausting ω-convex function $f : X \setminus Y \to$*

$\mathbf{R}$ *that has a Morse-Bott minimum at Q and is Morse away from Q with finitely many critical points.*

In [EGL15], Eliashberg-Ganatra-Lazarev introduced the following definition: a Lagrangian submanifold Q in a Weinstein manifold (W, ω) is 'regular' if there exists a Liouville pseudogradient on W that is tangent to Q (or equivalently there exists a primitive of ω vanishing on Q). This property, which implies that Q is an exact Lagrangian submanifold, is known for quite a long time to be a strong constraint. For instance, it is elementary to see (without any holomorphic curve theory) that a closed Lagrangian submanifold in $\mathbf{C}^n$ cannot be regular. In the same time, though we do not have any example of a non-regular closed exact Lagrangian submanifold in a Weinstein manifold, we do not know any general method to prove that exact Lagrangian submanifolds should *a priori* be regular. Theorems 2 and 3 show that, in the complement of the complex and symplectic hyperplane sections constructed, the Bohr-Sommerfeld Lagrangian submanifold Q is included in the zero-set of a Liouville pseudogradient and is therefore regular.

In Section 2 we explain why the Bohr-Sommerfeld condition is necessary for our purposes and describe some of properties of Bohr-Sommerfeld Lagrangians. In Section 3 we prove Theorem 3, applying the main technical result from [Gir18]. In Section 4 we prove Theorem 2 and a complex-geometric analogue, using techniques that go back to [DS95].

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2. Bohr-Sommerfeld Lagrangian submanifolds

Let us first remark that Cieliebak-Mohnke proved, in [CM17, Thm. 8.3], a version of the main theorem of [AGM01] that is specific to Bohr-Sommerfeld Lagrangian submanifolds.

The Bohr-Sommerfeld condition in Theorems 2 and 3 is necessary, indeed:

Lemma 4. *Let (X,ω) be a closed symplectic manifold and Q a Lagrangian submanifold. Suppose that there exist a closed submanifold $Y \subset X$ Poincaré-dual to ω avoiding Q and λ a primitive of ω over $X \setminus Y$ such that $\lambda|_Q$ is exact. Then Q is a Bohr-Sommerfeld Lagrangian submanifold of (X,ω).*

Proof. It suffices to prove the following (well-known) claim: Let X be a closed connected oriented manifold, $Y \subset X$ a closed codimension 2 submanifold and ω a non-exact closed 2-form on X that is Poincaré-dual to Y. Then, for every compact surface $\Sigma \subset X$ with boundary disjoint from Y and primitive λ of ω on $X \setminus Y$ whose restriction to the submanifold Q is exact,

$$\int_\Sigma \omega = \Sigma.Y\,.$$

We first suppose that Y is connected. For any embedded 2-disc D intersecting Y transversely at one point, with sign $\epsilon(D) = \pm 1$, set $r := \epsilon(D)(\int_D \omega - \int_{\partial D} \lambda)$. The 'residue' r does not depend on the disc D. To see this we will prove that, for two such discs D and D',

$$\epsilon(D')\int_{\partial D'} \lambda - \epsilon(D)\int_{\partial D} \lambda = \int_C \omega = \epsilon(D')\int_{D'} \omega - \epsilon(D)\int_D \omega.$$

Connectedness of Y gives an oriented cylinder C in $X \setminus Y$ bounding $-\epsilon(D')\partial D'$ and $\epsilon(D)\partial D$. On the one hand, by Stokes theorem,

$$\int_C \omega = \epsilon(D')\int_{\partial D'} \lambda - \epsilon(D)\int_{\partial D} \lambda.$$

On the other hand, the capped cylinder $C + \epsilon(D)D - \epsilon(D')D'$ is a boundary in X and ω is closed so

$$\int_C \omega + \epsilon(D)\int_D \omega - \epsilon(D')\int_{D'} \omega = 0.$$

Finally, the 'residue' r is independent of D.

Let $\Sigma \subset X$ be a compact surface intersecting Y away from $\partial\Sigma$. By a general position argument we may suppose the intersection is transverse. For each point $p_i \in \Sigma \cap Y$, take a disc $D_i \subset \Sigma$ that intersects Y only at p_i. Stokes theorem gives $\int_{\Sigma \setminus \cup_i D_i} \omega = -\sum_i \int_{\partial D_i} \lambda$, then:

$$\int_\Sigma \omega = \Sigma.Y\ r\,. \tag{1}$$

Since ω is not exact, we can apply (1) to some closed surface Σ_0 with $\Sigma_0.Y = \int_{\Sigma_0} \omega \neq 0$. This gives $r = 1$; so (1) proves the claim.

Suppose Y is not connected. If $\dim X \geq 3$, the cycle $[Y]$ may be represented by a closed connected submanifold, namely an embedded (away from $\partial\Sigma_0$) connected sum of the connected components of Y. If $\dim X = 2$, we may represent $[Y]$ by some integral multiple of any point. Consequently, we reduce to the previous case. □

Meanwhile, the Bohr-Sommerfeld condition can be easily obtained after a modification of the symplectic form:

Lemma 5 (Approximation and rescaling). *Let (X, ω) be a closed symplectic manifold and Q a closed Lagrangian submanifold. Then there exists a small closed 2-form ϵ and an integer k such that Q is a Bohr-Sommerfeld Lagrangian submanifold of $(X, k(\omega + \epsilon))$.*

Proof. We argue as in [AGM01]: the 2-form ω vanishes on Q so, in view of the exact sequence $\cdots \to H^2(X, Q; \mathbf{R}) \to H^2(X; \mathbf{R}) \to H^2(Q; \mathbf{R}) \to \cdots$, it is the image of a relative class $c \in H^2(X, Q; \mathbf{R})$. We approximate c by some $r \in H^2(X, Q; \mathbf{Q})$ and take a small closed form ϵ vanishing on Q that represents $c - r$. Then the closed form $\omega - \epsilon$ is symplectic, vanishes on Q and its relative periods — given by evaluation of r — are rational. □

We now give the characterisation of Bohr-Sommerfeld Lagrangian submanifolds that we will use to prove Theorems 2 and 3.

Lemma 6 (Hermitian flat line bundles). *Let (X, ω) be an integral symplectic manifold and Q a submanifold. Then Q is a Bohr-Sommerfeld Lagrangian submanifold if and only if there exist a Hermitian line bundle $L \to X$ and a unitary connection ∇ of curvature $-2i\pi\omega$ such that $(L, \nabla)|_Q$ is a trivial flat bundle. If Q is a Bohr-Sommerfeld Lagrangian and, in addition, (X, ω) is Kähler, then one can take for (L, ∇) a holomorphic Hermitian line bundle with its Chern connection.*

Proof. Suppose that Q is a Bohr-Sommerfeld Lagrangian submanifold. Since ω has integral periods, we may fix a lift c of its cohomology class to $\mathrm{H}^2(X, \mathbf{Z})$. We take a Hermitian line bundle $L_0 \to X$ with first Chern class c and a unitary connection ∇_0 of curvature $-2i\pi\omega$. The submanifold Q is Lagrangian so the restriction $(L_0, \nabla_0)|_Q$ is a flat Hermitian bundle.

We will construct a flat Hermitian line bundle $(L_1, \nabla_1) \to X$ whose restriction to Q is isomorphic to $(L_0, \nabla_0)|_Q$. Then the desired line bundle will be $L_0 \otimes L_1^{-1}$.

Recall that flat Hermitian line bundles over a manifold Y are classified up to isomorphism by their holonomy representation $\mathrm{H}_1(Y,\mathbf{Z}) \to \mathrm{U}(1)$ (cf. proposition 3.6.15 in [Thu97]). To construct the flat bundle (L_1,∇_1) it suffices to extend the holonomy representation $\rho : \mathrm{H}_1(Q,\mathbf{Z}) \to \mathrm{U}(1)$ of the flat bundle $(L_0,\nabla_0)|_Q$ to a homomorphism $\mathrm{H}_1(X,\mathbf{Z}) \to \mathrm{U}(1)$.

We first show that ρ is trivial on the kernel of the group homomorphism $i : \mathrm{H}_1(Q,\mathbf{Z}) \to \mathrm{H}_1(X,\mathbf{Z})$ induced by inclusion. Consider the exact sequence of the pair (X,Q):

$$\cdots \to \mathrm{H}_2(X,Q;\mathbf{Z}) \xrightarrow{\partial} \mathrm{H}_1(Q,\mathbf{Z}) \xrightarrow{i} \mathrm{H}_1(X,\mathbf{Z}) \to \cdots$$

where ∂ is the homomorphism given by the boundary of chains. It suffices to show that $\rho \circ \partial = 0$. Every $a \in \mathrm{H}_2(X,Q;\mathbf{Z})$ can be represented by an embedded surface $\Sigma \subset X$ whose (possibly empty) boundary is included in Q. It then follows from (well-known) lemma 7 that:

$$\rho(\partial a) = \exp\left(2i\pi \int_a \omega\right). \tag{2}$$

Since the Lagrangian submanifold Q is Bohr-Sommerfeld, $\rho(\partial a) = 1$.

Thus ρ factors through a homomorphism $\tilde{\rho} : \mathrm{H}_1(Q,\mathbf{Z})/\ker i \to \mathrm{U}(1)$ where $\mathrm{H}_1(Q,\mathbf{Z})/\ker i$ injects into $\mathrm{H}_1(X,\mathbf{Z})$. Now $\mathrm{U}(1)$ is a divisible abelian group so it is an injective $\mathbf{Z}$-module (see for instance [Wei95, Corollary 2.3.2]). Hence $\tilde{\rho}$ extends to $H_1(X,\mathbf{Z})$.

In the case where (X,ω) is Kähler, the above Hermitian line bundle (L_0,∇_0) can be chosen holomorphic with its Chern connection (see, *e.g.*, [Dem12, Theorem 13.9.b]). On the other hand the flat line bundle (L_1,∇_1) is isomorphic to the quotient of the trivial flat bundle $\tilde{X} \times \mathbf{C}$ by the diagonal action of the fundamental group, acting on its universal cover $\tilde{X}$ by deck transformations and on $\mathbf{C}$ by the holonomy representation $\mathrm{H}_1(X,\mathbf{Z}) \to \mathrm{U}(1)$ (cf. proposition 3.6.15 in [Thu97]). Therefore the trivial holomorphic structure and the trivial connection on $\tilde{X} \times \mathbf{C}$ respectively induce a holomorphic structure and the Chern connection on L_1. Consequently, the bundle $L_0 \otimes L_1^{-1}$ has the desired properties.

Conversely, let (X,ω) be a symplectic manifold and a Hermitian line bundle $L \to X$ with a unitary connexion of curvature $-2i\pi\omega$ such that $(L,\nabla)|_Q$ is a trivial flat bundle. Then the (trivial) holonomy representation ρ of $(L,\nabla)|_Q$ satisfies (2); so Q is a Bohr-Sommerfeld Lagrangian. □

Lemma 7 (Gauss-Bonnet). *Let X be a manifold and $L \to X$ a Hermitian line bundle with a unitary connection ∇ whose curvature 2-form* *is written* $-2i\pi\omega$. *Let* Σ *be a connected oriented surface with non-empty boundary and* $f : \Sigma \to X$ *a map. The holonomy of* ∇ *along the loop* $f|_{\partial\Sigma}$ *is* $\exp(2i\pi \int_\Sigma f^\star\omega) \in \mathrm{U}(1)$.

Proof. We may assume $X = \Sigma$ and $f = \mathrm{Id}_\Sigma$ by pulling back the line bundle L by f. There is a unit section $s : \Sigma \to L$. In the trivialisation of L given by s there is a primitive α of ω such that the connection ∇ reads $d - 2i\pi\alpha$. By Stokes theorem

$$\int_\Sigma \omega = \int_{\partial\Sigma} \alpha \ .$$

We may assume that $\partial\Sigma$ is connected. Take $\beta : [0,1] \to \partial\Sigma$ a parametrisation of $\partial\Sigma$. For every unit parallel lift $\gamma : [0,1] \to L$ of β and for all $t \in [0,1]$, $\gamma'(t) = 2i\pi\gamma(t)\ (\beta^\star\alpha)_t(\partial_t)$ hence

$$2i\pi \int_{\partial\Sigma} \alpha = \int_{[0,1]} \frac{\gamma'(t)}{\gamma(t)}\ dt = \log\frac{\gamma(1)}{\gamma(0)}.$$

An exponentiation gives the result. □

3. As minima of ω-convex functions

In this section we prove Theorem 3 so Q is a closed Bohr-Sommerfeld Lagrangian submanifold in a closed integral symplectic manifold (X,ω). We will adopt the following notation:

- J: a fixed ω-compatible almost complex structure on X;
- $g = \omega(\cdot, J\cdot)$: the corresponding Riemannian metric;
- λ_0: the Liouville form on $T^\star Q$;
- $f_0 : p \in T^\star Q \mapsto \pi|p|^2 \in \mathbf{R}$ where $|\cdot|$ is the norm on each fibre of $T^\star Q \to Q$ induced by the restriction of the metric g to Q.

Using Weinstein's normal form theorem, we identify a neighbourhood N of $Q \subset (X,\omega)$ with a tube $\{f_0 < c\}$ around the zero section Q in $(T^\star Q, \mathrm{d}\lambda_0)$ in such a way that, for all $q \in Q$, the subspaces $T_qQ, T_q^\star Q \subset T_q(T^\star Q)$ are g-orthogonal.

Using Lemma 6, we fix a Hermitian line bundle $L \to X$ with a unitary connection ∇ of curvature $-2\pi i\omega$ and a unit parallel section s_0 of the flat bundle $(L,\nabla)|_Q$.

The main characters of the next lemmata are the two following sections of $L|_N$:

- $s : N \to L|_N$: the extension of the section s_0 by parallel transport by ∇ along the rays in the fibres of $T^\star Q$;
- $s_0 := e^{-f_0} s : N \to L|_N$ (which is well-defined since $f_0|_Q = 0$).

For any positive integer k, we denote by L^k the k-th tensor power of the line bundle L, whose induced connection has curvature $-2k\pi i\omega$, and we set $g_k := kg$ the rescaled metric. For any integer $r \geq 0$, we endow the vector bundle $\bigotimes^r T^\star X \otimes L^k$ with the connection induced by the Levi-Civita connection for the metric g_k and our connection on L^k; we still write this connection ∇. We define the $\mathcal{C}^r$ norm of a section $u : X \to L^k$ by $\|u\|_{\mathcal{C}^r, g_k} := \sup|u| + \sup|\nabla u|_{g_k} + \cdots + \sup|\nabla^r u|_{g_k}$. The J-linear and $-J$-linear parts of the connection ∇ are written ∇' and ∇''.

For any 1-form λ on X, we will denote by $\overrightarrow{\lambda}$ the vector field that is ω-dual to λ.

Lemma 8. *There exists a constant $C > 0$ such that, for every integer $k \geq 1$, the function f_0 and the section s_0^k satisfy the following bounds on N:*

$$\overrightarrow{\lambda_0}.(kf_0) \geq C^{-1}(|\overrightarrow{\lambda_0}|^2_{g_k} + |\mathrm{d}(kf_0)|^2_{g_k}),\ C^{-1}(kf_0)^{1/2} \leq |\mathrm{d}(kf_0)|_{g_k} \leq C(kf_0)^{1/2},$$
$$|\nabla s_0^k|_{g_k} \leq C(kf_0)^{1/2} e^{-kf_0},\ \|\nabla^2 s_0^k\|_{\mathcal{C}^0, g_k} \leq C \text{ and } \|\nabla'' s_0^k\|_{\mathcal{C}^1, g_k} \leq Ck^{-1/2}.$$

Proof. By rescaling, it suffices to establish the first two bounds of the statement for $k = 1$. The function f_0 is Lyapounov for the vector field $\overrightarrow{\lambda_0}$. This implies the first bound. The submanifold Q is a Morse-Bott minimum for f_0, hence the second bound.

Since $s_0 = e^{-f_0} s$ with s parallel,

$$\nabla s_0 = -\mathrm{d}f_0 e^{-f_0} s + e^{-f_0} \nabla s = -(\mathrm{d}f_0 + 2\pi i \lambda_0) s_0.$$

Therefore, ∇s_0 vanishes identically on the zero section. Hence, there exists a constant $C > 0$ such that $|\nabla s_0|_g \leq C f_0^{1/2}$. Moreover, the 1-jet of $\nabla'' s_0$ vanishes at each point of Q. Indeed, by the identity $\lambda_0 = -\omega(\cdot, \overrightarrow{\lambda_0})$ (here $k = 1$) and by J-linearity of the 1-form $g(\cdot, \overrightarrow{\lambda_0}) - i\omega(\cdot, \overrightarrow{\lambda_0})$,

$$\nabla'' s_0 = -2\pi \left(\tfrac{\mathrm{d}f_0}{2\pi} + i\lambda_0 \right)'' s_0 = -2\pi (\tfrac{\mathrm{d}f_0}{2\pi} - g(\cdot, \overrightarrow{\lambda_0}))'' s_0,$$

so it suffices to show that the 1-jet of the 1-form $\frac{\mathrm{d}f_0}{2\pi} - g(\cdot, \overrightarrow{\lambda_0})$ vanishes identically along Q. Its 0-jet clearly vanishes, and, for each vector $v = (v_1, v_2)$

in the g-orthogonal sum $T(T^\star Q)|_Q = TQ \oplus T^\star Q$,

$$\mathrm{d}(g(\cdot, \overrightarrow{\lambda_0}))(v,v) = g(v, v.\overrightarrow{\lambda_0}) = g(v, v_2) = g(v_2, v_2) = (\mathrm{d}^2 f_0)(v,v)/(2\pi),$$

hence its 1-jet vanishes too. Consequently, there exists a constant $C > 0$ such that $|\nabla\nabla'' s_0|_g \leq C f_0^{1/2}$ and $|\nabla'' s_0|_g \leq C f_0$. Therefore, by the Leibniz rule, we obtain the desired bounds on ∇s_0^k and $\nabla^2 s_0^k$, and the two bounds $|\nabla'' s_0^k|_{g_k} \leq C k^{1/2} f_0 e^{-kf_0}$, $|\nabla\nabla'' s_0^k|_{g_k} \leq (k f_0^{3/2} + f_0^{1/2}) C e^{-kf_0}$. The two latter real-valued Gaussian functions of f_0 both reach their global maximum at $Constant \times k^{-1}$ so we obtain the last bound of the statement. □

In particular, our sections s_0^k are asymptotically holomorphic in the following sense:

Definition 9 (Donaldson, Auroux). Sections $s_k : X \to L^k$ are called *asymptotically holomorphic* if there exists a constant $C > 0$ such that for every positive integer k, $\|\nabla'' s_k\|_{\mathcal{C}^1, g_k} \leq C k^{-1/2}$ and $\|s_k\|_{\mathcal{C}^2, g_k} \leq C$.

The following result was already observed in Auroux-Gayet-Mohsen [AGM01, Remark p.746]. Recall that our neighbourhood N of Q is identified with the cotangent tube $\{f_0 < c\}$.

Lemma 10. *Let $\beta : N \to [0,1]$ be a compactly supported function (independent of k) with $\beta = 1$ on a tube $\{f_0 < b\}$. Then, the sections $s_{0,k} := \beta s_0^k : X \to L^k$ are asymptotically holomorphic.*

Proof of lemma 10. The sections s_0^k satisfy the estimates of lemma 8 on N. Then, there exists a constant $C > 0$ such that:

$$\begin{aligned} \|\nabla'' s_{0,k}\|_{\mathcal{C}^0, g_k} &\leq \|d\beta\|_{\mathcal{C}^0, g_k} \sup_{\{f_0 > b\}} |s_0^k| + \|\nabla'' s_0^k\|_{\mathcal{C}^0, g_k} \\ &\leq C k^{-1/2} e^{-bk} + C k^{-1/2}. \end{aligned}$$

Similarly:

$$\begin{aligned} \|\nabla\nabla'' s_{0,k}\|_{\mathcal{C}^0, g_k} \leq{}& \|d^2\beta\|_{\mathcal{C}^0, g_k} \sup_{\{f_0 > b\}} |s_0^k| \\ &+ 2\|d\beta\|_{\mathcal{C}^0, g_k} \sup_{\{f_0 > b\}} |\nabla s_0^k|_{g_k} + \|\nabla\nabla'' s_0^k\|_{\mathcal{C}^0, g_k} \\ \leq{}& C k^{-1} e^{-bk} + 2 C k^{-1/2} (C k^{1/2} c^{1/2} e^{-bk}) + C k^{-1/2}. \end{aligned}$$

Hence, there exists a constant $C > 0$ such that, for all k, $\|\nabla'' s_{0,k}\|_{\mathcal{C}^1, g_k} \leq C k^{-1/2}$. In the same way, we obtain the bound $\|s_{0,k}\|_{\mathcal{C}^2, g_k} \leq C$. □

Giroux's theorem below provides transverse perturbations of our sections $s_{0,k}$ with the following property.

Definition 11 (Giroux). Let $\kappa \in (0,1)$. A section $s : X \to L^k$ is called κ-quasiholomorphic if $|\nabla'' s| \leq \kappa |\nabla' s|$ at each point of X.

Theorem 12 ([Gir18, Proposition 13]). *Let $\epsilon > 0$, $\kappa \in (0,1)$ and $s_{0,k} : X \to L^k$ asymptotically holomorphic sections. Then, for any sufficiently large integer k, there exists a section $s_{1,k} : X \to L^k$ with the following properties:*

- *$s_{1,k}$ vanishes transversally;*
- *$s_{1,k}$ is κ-quasiholomorphic;*
- *$\|s_{1,k} - s_{0,k}\|_{\mathcal{C}^1, g_k} < \epsilon$;*
- *$-\log |s_{1,k}| : \{p \in X,\ s_{1,k}(p) \neq 0\} \to \mathbf{R}$ is a Morse function.*

Let us now bring the previous facts together to prove Theorem 3.

Proof of Theorem 3. Using lemma 10, we fix sections $s_{0,k} : X \to L^k$ with $s_{0,k} = s_0^k$ on a tube $\{f_0 < b\}$. We then fix $\epsilon \in (0,1)$ and take sections $s_{1,k} : X \to L^k$ provided by Theorem 12. The subset $Y := \{s_{1,k} = 0\} \subset (X, \omega)$ is a symplectic hyperplane section of degree k (because of the first two properties of Theorem 12, see for instance proposition 3 in [Don96]) avoiding the submanifold Q (because $|s_0| = 1$ on Q and by the third property of Theorem 12).

It remains to construct an ω-convex exhaustion $f : X \setminus Y \to \mathbf{R}$ that has a Morse-Bott minimum at Q and is Morse away from Q with finitely many critical points. In order to do so, we will glue the function $f_{0,k} := k f_0 : N \to \mathbf{R}$, which clearly has a Morse-Bott minimum at Q, with the exhaustive function $f_{1,k} := -\log |s_{1,k}|$, which is Morse (by the last property of Theorem 12) and has finitely many critical points (because $s_{1,k}$ vanishes transversally).

Before gluing, let us note that, by lemma 8, $f_{0,k}$ is Lyapounov for the Liouville vector field $\overrightarrow{\lambda_0}$ with Lyapounov constant in the metric g_k that is independent of k. On the other hand, a Liouville pseudogradient for $f_{1,k}$ is provided by Giroux's following lemma. In order to state it, we set $\lambda_{1,k}$ the real 1-form such that, in the unitary trivialisation of $L^k|_{X \setminus Y}$ given by $s_{1,k}/|s_{1,k}|$, the connection ∇ reads $\mathrm{d} - 2k\pi i \lambda_{1,k}$. We also recall that the notation $\overrightarrow{\lambda}$ stands for the ω-dual vector field to a given 1-form λ.

Lemma 13 ([Gir18, Lemma 12]). *Let $\kappa \in (0,1)$ and $s_{1,k} : X \to L^k$ a κ-quasiholomorphic section. Then*

$$\overrightarrow{\lambda_{1,k}}.f_{1,k} \geq \tfrac{1}{2}\frac{1-\kappa^2}{1+\kappa^2}\left(|\mathrm{d}f_{1,k}|^2_{g_k} + |\overrightarrow{\lambda_{1,k}}|^2_{g_k}\right).$$

Hence the function $f_{1,k}$ is Lyapounov for the Liouville vector field $\overrightarrow{\lambda_{1,k}}$, with a uniform Lyapounov constant in the metric g_k.

Finally, the desired function f is constructed in the following lemma, by gluing, on an annular region $\{a < f_{0,k} < b\}$ about Q, the standard (Morse-Bott) Weinstein structure $(\overrightarrow{\lambda_0}, f_{0,k})$ on $T^\star Q$ with the Weinstein structure $(\overrightarrow{\lambda_{1,k}}, f_{1,k})$ given by Giroux's above theorem and lemma. □

In the following lemma, the number c still refers to the size of our cotangent tube $\{f_0 < c\}$ about Q.

Lemma 14. *Let $\kappa \in (0,1)$ and $a, b \in (0,c)$ with $a < b$. Then, for every sufficiently small $\epsilon \in (0,1)$ and for every $k \geq k_0(\epsilon)$ sufficiently large, there exist a Liouville vector field $\overrightarrow{\lambda}$ on $X \setminus Y$ and a Lyapounov function $f : X \setminus Y \to \mathbf{R}$ for $\overrightarrow{\lambda}$ such that $(\overrightarrow{\lambda}, f) = (\overrightarrow{\lambda_0}, f_{0,k})$ on $\{f_{0,k} \leq a\}$, $(\overrightarrow{\lambda}, f) = (\overrightarrow{\lambda_{1,k}}, f_{1,k})$ away from $\{f_{0,k} < b\}$ and f has no critical point on $\{a \leq f_{0,k} \leq b\}$.*

Proof. We will omit the indices k in the proof.

For now, we admit the following two facts: there exists a constant $C > 0$ (independent of k, ϵ) such that

$$\|f_0 - f_1\|_{\mathcal{C}^1(N), g_k} \leq C\epsilon \tag{3}$$

and, for sufficiently small $\epsilon > 0$, the form $\lambda_1 - \lambda_0$ is exact on N.

We will glue the Weinstein structures in two steps. Let us fix two numbers $a < a_+ < b_- < b$. For $\epsilon < \min(\frac{a_+ - a}{2C}, \frac{b - b_-}{2C})$, the annular region $\{a < f_0 < b\}$ contains the level sets $\{f_1 = a_-\}$ and $\{f_1 = b_+\}$ (by the bound (3)).

First, let us glue the functions in the inner collar $\{f_0 \geq a\} \cap \{f_1 \leq a_-\}$; more precisely, let us construct a Lyapounov function $f : X \to \mathbf{R}$ for the vector field $\overrightarrow{\lambda_0}$ with $f = f_0$ on $\{f_0 \leq a\}$ and $f = f_1$ away from $\{f_1 < a_+\}$. It suffices to show that $\overrightarrow{\lambda_0}$ is transverse to the level sets of f_0 and f_1 in this inner collar; indeed, increasing from a to b along each trajectory of $\overrightarrow{\lambda_0}$ gives a function f transverse to the level sets of f_0 and f_1. There exists a constant $C' > 0$ such that $\|\mathrm{d}f_0\|_{g_k} \geq C'$ (by lemma 8). By the latter bound and (3), $\|\mathrm{d}f_1\|_{g_k} \geq C' - C\epsilon$. So, by the Lyapounov conditions, there exists a constant $C' > 0$ such that $\overrightarrow{\lambda_0}.f_0 \geq C'$ (in particular $\overrightarrow{\lambda_0}$ is transverse to the level sets of f_0) and $\overrightarrow{\lambda_1}.f_1 \geq C'$. By the latter bound and again (3), $\overrightarrow{\lambda_0}$ is transverse to the level sets of f_1.

Second, we glue the Liouville vector fields in the outer collar $\{a_- < f_1 \leq b_+\}$ (where $f = f_1$); more precisely we construct a Liouville vector field $\overrightarrow{\lambda}$ which is transverse to the level sets of the function f in the outer collar and coincides with $\overrightarrow{\lambda_0}$ on $\{f_1 < a_-\}$ and with $\overrightarrow{\lambda_1}$ outside $\{f_1 \leq b_+\}$. The 1-form $\lambda_1 - \lambda_0$ is exact (by our initial claim) so we have a function H such that $\overrightarrow{\lambda_1} - \overrightarrow{\lambda_0} = \overrightarrow{\mathrm{d}H}$. Let us fix a cutoff function $\beta : \mathbf{R} \to [0,1]$ such that $\beta = 0$ near $\mathbf{R}_{\leq (b-a)/2}$ and $\beta = 1$ near $\mathbf{R}_{\geq b}$ and set $\beta_1 := \beta \circ f_1$. Then the vector field $\overrightarrow{\lambda} := \overrightarrow{\lambda_0} + \overrightarrow{\mathrm{d}(\beta_1 H)}$ is Liouville and satisfies the desired boundary conditions. Moreover, $\overrightarrow{\beta_1}$ is tangent to the level sets of f_1 and, by the above paragraph, $\overrightarrow{\lambda_0}$ and $\overrightarrow{\lambda_1}$ are positively transverse to these, so $\overrightarrow{\lambda} = (1-\beta_1)\overrightarrow{\lambda_0} + \beta_1\overrightarrow{\lambda_1} + \overrightarrow{\beta_1}H$ is transverse to these too.

It remains to prove the two initial claims. On the one hand, $f_0 - f_1 = \mathrm{Re}\log(s_1 s_0^{-1})$ and, since $u_j := s_j/|s_j|$ satisfies $\nabla u_j = -2k\pi i \lambda_j u_j$,

$$\lambda_1 - \lambda_0 = \tfrac{1}{2k\pi i}\left(u_0^{-1}\nabla u_0 - u_1^{-1}\nabla u_1\right) = \tfrac{1}{2k\pi i}\mathrm{d}\log(u_0 u_1^{-1}) = \tfrac{1}{2k\pi}\mathrm{d}\arg(s_1 s_0^{-1}).$$

On the other hand, $\|\log(s_1 s_0^{-1})\|_{\mathcal{C}^1, g_k} \leq C\epsilon$; this is a consequence of the three bounds $\|s_1 - s_0\|_{\mathcal{C}^1, g_k} < \epsilon$, $\inf|s_0| > e^{-c}$, and $\|\nabla s_0\|_{\mathcal{C}^0, g_k} \leq Constant$ (from lemma 8). In particular we obtain the bound (3) and for ϵ sufficiently small, $\|\arg(s_1 s_0^{-1})\|_{\mathcal{C}^0} < \pi/3$ so $\lambda_1 - \lambda_0$ is exact. ☐

Remark 15 (An alternative proof of the regularity of $Q \subset X \setminus Y$). For sufficiently large k, it is possible to choose our κ-quasiholomorphic perturbation $s_{1,k} : X \to L^k$ (vanishing transversally and away from Q) of $s_{0,k}$ in such a way that the quotient function $(s_{1,k}/s_{0,k})|_Q$ is real-valued. The latter property, which can be achieved by implementing techniques from Auroux-Munoz-Presas' [AMP05] in the proof of [Gir18, Proposition 13], implies that the Liouville pseudogradient $\overrightarrow{\lambda_{1,k}}$ of the function $-\log|s_{1,k}|$ is tangent to Q.

4. As minima of C-convex functions

This section deals with the proof of Theorem 2, so Q is a closed Bohr-Sommerfeld Lagrangian submanifold in a closed integral Kähler manifold (X,ω). Using Lemma 6, we fix a holomorphic Hermitian line bundle $L \to X$ with Chern curvature $-2\pi i\omega$ and a parallel unit section $s_0 : Q \to L|_Q$. We denote by ∇ the Chern connection. Let k be a positive integer. We will use the following notation:

- $g_k = k\omega(\cdot, i\cdot)$: the rescaled metric ;
- d_k: the distance function to Q in the metric g_k ;
- $B_k(Q, c) = \{d_k < c\}$.

We recall that we endow the vector bundle $\bigotimes^r T^\star X \otimes L^k$ with the connection induced by the Levi-Civita connection for the metric g_k and the connection on L^k — we still write this connection ∇. We define the $\mathcal{C}^r$ norm of a section $u : X \to L^k$ by $\|u\|_{\mathcal{C}^r, g_k} := \sup |u| + \sum_{j=1}^r \sup |\nabla^j u|_{g_k}$.

Since Q is a totally real submanifold, there are many **C**-convex functions on a neighbourhood of Q having a Morse-Bott minimum at Q (namely the squared distance function d_1^2). In the next two lemmas construct such a function under the form $f_0 = -\log |s_0|$, where s_0 is a section of L over a neighbourhood of Q whose powers are asymptotically holomorphic (in a sense made precise in lemma 18) and will be later modified into genuine global holomorphic sections of L^k (see proposition 19).

Lemma 16. *There exists a number $c > 0$ such that the restriction of the line bundle L to $N := B_1(Q, c)$ admits a non-vanishing holomorphic section $s : N \to L|_N$. Furthermore, given any integer $r \geq 1$, the complex-valued function $s_0/(s|_Q)$ extends to a smooth function $F : N \to \mathbf{C}$ such that the form $\mathrm{d}''F$ vanishes identically along Q together with its r-jet.*

We will eventually choose $r = n$, the complex dimension of the manifold X.

Proof. Since Q is a totally real submanifold of X, it has a neighbourhood on which the squared distance function d_1^2 is **C**-convex (see for instance Proposition 2.15 in [CE12]), for sufficiently small $c > 0$, the neighbourhood N is a Stein manifold. So the first assertion follows from results of Oka [Oka39] and Grauert [Gra58].

For any positive integer r, [CE12, Proposition 5.55] shows that the complex-valued function $s_0/(s|_Q)$ extends to a smooth function $F : N \to \mathbf{C}$ such that, at each point of Q, $\mathrm{d}''F$ vanishes together with its r-jet. □

The desired local section and local function are respectively:

- $s_0 := Fs : N \to L|_N$, extending the section $s_0 : Q \to L|_Q$;
- $f_0 := -\log |s_0| : N \to \mathbf{R}$.

Remark 17 (The real-analytic case). If the submanifold Q is real-analytic, then one can take for $s_0 : N \to L|_N$ a holomorphic section. Indeed, one may ensure that the connection ∇ on the bundle L provided by Lemma 6 is real-analytic. In that case, the section $s_0 : Q \to L|_Q$ is real-analytic and can be complexified.

Lemma 18. *Suppose that $r \geq 1$. Then, the submanifold Q is a Morse-Bott minimum for the function f_0. Besides, there exists a constant $C > 0$ such that, for every integer $k \geq 1$, the section $s_0^k : N \to L^k|_N$ satisfies the following bounds on N:*

$$|2\pi k\omega - \mathrm{dd}^c \log |s_0^k||_{g_k} \leq Ck^{-1/2}d_k,$$
$$e^{-Cd_k^2} \leq |s_0^k| \leq e^{-d_k^2/C},$$
$$|\nabla s_0^k|_{g_k} \leq Cd_k e^{-d_k^2/C}, \|\nabla'' s_0^k\|_{\mathcal{C}^1, g_k} \leq Ck^{-r/2}.$$

Proof. We first observe that ∇s_0 vanishes at every point $p \in Q$. Indeed, $T_pX = T_pQ \oplus i\, T_pQ$ (because Q is totally real of middle dimension), $\nabla s_0(p) = \nabla' s_0(p)$ (because $\nabla'' s_0(p)$ vanishes) and $\nabla s_0(p)$ vanishes on T_pQ (because $s_0|_Q$ is parallel). Thus, there exists a constant $C > 0$ such that $|\nabla s_0| \leq Cd_1$. Similarly, since the r-jet of $\nabla'' s_0$ vanishes identically on Q, there exists a constant $C > 0$ such that $|\nabla'' s_0|_{g_1} \leq Cd_1^{r+1}$ and $|\nabla\nabla'' s_0|_{g_1} \leq Cd_1^r$.

The function $f_0 = -\log|s_0|$ vanishes together with its 1-jet at p; indeed, $f_0(p) = 0$ and

$$\mathrm{d}f_0(p) = \tfrac{1}{2}\mathrm{d}\log(|s_0|^2) = \tfrac{1}{2}|s_0|^{-2}\,\mathrm{d}(|s_0|^2) = |s_0|^{-2}\,\mathrm{Re}\langle\nabla s_0, s_0\rangle = 0.$$

Moreover,

$$2\pi\omega_p + (\mathrm{dd}^c f_0)_p = \mathrm{dd}^c \log\left|\frac{s}{s_0}\right| = -i\mathrm{d}'\mathrm{d}'' \log\left|\frac{s}{s_0}\right|^2 = -(i\mathrm{d}'\mathrm{d}''\log|F|^2)_p = 0$$

because the 1-jet of the form $\mathrm{d}''F$ vanishes at p. Therefore, there exists a constant $C > 0$ such that $|2\pi\omega + \mathrm{dd}^c f_0|_g \leq Cd_1$. Multiplicating this by k gives the first bound of the statement. On the other hand, the Hessian quadratic form $(\mathrm{d}^2 f_0)_p : T_pX \to \mathbf{R}$ vanishes on T_pQ and satisfies, for every vector $v \in T_pX$,

$$(\mathrm{d}^2 f_0)(v,v) + (\mathrm{d}^2 f_0)(iv,iv) = -(\mathrm{dd}^c f_0)(v,iv) = 2\pi\omega(v,iv) = 2\pi g(v,v).$$

Hence, $(\mathrm{d}^2 f_0)_p$ is positive definite on $i\,T_pQ$ and Q is a Morse-Bott minimum for f_0. Since Q is compact, one can find a constant $C > 0$ such that, on some neighbourhood of Q for the metric g_1:

$$C^{-1}d_1^2 \leq f_0 \leq Cd_1^2.$$

In other words, $e^{-Cd_1^2} \leq |s| \leq e^{-d_1^2/C}$. We obtain the second bound of the statement by taking the k-th power. The third bound and the bounds

$$|\nabla'' s_0^k|_g \leq Ckd_1^{r+1}e^{-kd_1^2/C}, \qquad |\nabla\nabla'' s_0^k|_g \leq Ckd_1^r(1+kd_1^2)e^{-kd_1^2/C}$$

follow from this bound and the bounds on ∇s_0, $\nabla'' s_0$ and $\nabla\nabla'' s_0$ by the Leibniz rule applied to s_0^k. The two latter real-valued Gaussian functions of d_1 both reach their global maximum at $Constant \times k^{-1/2}$. By expressing these bounds in the rescaled metric g_k, we obtain the last bound of the statement. □

The following is the main result of this section. Recall that the number c is the size of the tube $N = B_1(Q, C)$.

Proposition 19. *Let $\rho \in (0, c)$. There exist holomorphic sections $s_k : X \to L^k$ such that, for every $\epsilon > 0$ and for $k \geq k_0(\epsilon)$ sufficiently large, s_k vanishes transversally and $\|s_k - s_0^k\|_{\mathcal{C}^1, g_k} < \epsilon$ on $B_1(Q, \rho)$, the ρ-neighbourhood of Q in the metric g.*

We postpone the proof of proposition 19 and first explain how it implies Theorem 2.

Proof of Theorem 2. We fix a radius $\rho \in (0, c)$ and, by proposition 19, holomorphic sections $s_k : X \to L^k$: for every $\epsilon > 0$ and for $k \geq k_1(\epsilon)$ sufficiently large, the zero-set $Y := s_k^{-1}(0)$ is a (smooth) complex hyperplane section and $\|s_k - s_0^k\|_{\mathcal{C}^1, g_k} < \epsilon$ on $B_1(Q, \rho)$. By the second and third inequalities in lemma 18, there exists a constant $C > 0$ (independent of k and ϵ) such that, for $\epsilon > 0$ sufficiently small, on $B_k(Q, \rho)$, the functions $f_1 := -\log |s_k|$ and $f_0 = -\log |s_0^k|$ satisfy

$$\|f_1 - f_0\|_{\mathcal{C}^1, g_k} < C\epsilon.$$

Take a cutoff function $\beta_k : X \to [0, 1]$ supported in $B_k(Q, \rho)$, with $\beta_k = 1$ on $B_k(Q, \rho/2)$ and $\|\beta_k\|_{\mathcal{C}^2, g_k} \leq C'$ for some constant $C' > 0$ (independent of k). The function $f := \beta_k f_0 + (1 - \beta_k) f_1 : X \setminus Y \to \mathbf{R}$ is exhausting, reaches a Morse-Bott minimum at Q and its critical points remain in a compact subset. (We remark that, for sufficiently small ϵ, this minimum is global. Indeed, on $\{\beta_k = 1\}$, $f = f_0$, and on $\{\beta_k < 1\}$, $f_1 \geq -\log(|s_0| + \epsilon) \geq -\log(e^{-\rho^2/C} + \epsilon) > 0$.)

Let us show that f is $\mathbf{C}$-convex. First, since s_k is holomorphic, $-\mathrm{dd}^c f_1 = 2k\pi\omega$. Then, by the first bound of lemma 18, there exists a constant $C'' > 0$ such that $\|\mathrm{dd}^c(f_0 - f_1)\|_{\mathcal{C}^0, g_k} \leq C'' k^{-1/2}$. Hence,

$$
\begin{aligned}
(4) \qquad \|2k\pi\omega + \mathrm{dd}^c f\|_{\mathcal{C}^0, g_k} &= \|\mathrm{dd}^c(\beta_k(f_0 - f_1))\|_{\mathcal{C}^0, g_k} \\
&\leq \|\beta_k\| C'' k^{-1/2} + \|(f_0 - f_1)\mathrm{dd}^c \beta_k\| \\
&\quad + \|\mathrm{d}(f_0 - f_1) \wedge \mathrm{d}^c \beta_k\| + \|\mathrm{d}^c(f_1 - f_0) \wedge \mathrm{d}\beta_k\| \\
&\leq C'' k^{-1/2} + 3(C\epsilon)C'.
\end{aligned}
$$

Consequently, for every $\epsilon > 0$ sufficiently small and for every $k \geq k_0(\epsilon)$ sufficiently large, $\|2k\pi\omega + \mathrm{dd}^c f\|_{\mathcal{C}^0, g_k} < 2\pi$. This inequality ensures that the function f is $\mathbf{C}$-convex. A $\mathcal{C}^2$-small perturbation of the function f with support in a compact subset of $Y \setminus Q$ is Morse away from Q and satisfies the properties of Theorem 2. □

Our next aim is to prove proposition 19. The following lemma defines global smooth sections of L^k which will be later modified into genuine holomorphic sections. The L^2-norm of a section $s : X \to \bigotimes^r T^\star X \otimes L^k$ for the rescaled metric g_k is defined by

$$\|s\|_{\mathrm{L}^2, g_k} := \left(\int_X |s|^2_{g_k} \frac{(k\omega)^n}{n!} \right)^{1/2}.$$

Lemma 20. *Let $\beta : X \to [0,1]$ a function supported in N with $\beta = 1$ on a tube $B(Q, \rho)$. There exists a constant $C > 0$ such that the sections $s_{0,k} := \beta s_0^k : X \to L^k$ satisfy the following bounds:*

$$\|\nabla'' s_{0,k}\|_{\mathcal{C}^1, g_k} \leq C k^{-r/2}, \|\nabla'' s_{0,k}\|_{\mathrm{L}^2, g_k} \leq C k^{(n-r)/2}$$

Proof. The sections s_0^k satisfy the bounds of Lemma 18 on N. Then, there exists a constant $C > 0$ such that:

$$
\begin{aligned}
\|\nabla'' s_{0,k}\|_{\mathcal{C}^0, g} &\leq \|\mathrm{d}\beta\|_{\mathcal{C}^0, g} \sup_{\{d_1 > \rho\}} |s_0^k| + \sup_{B(Q, 2\rho)} |\nabla'' s_0^k|_g \\
&\leq C(e^{-k/C} + k^{-(r-1)/2}).
\end{aligned}
$$

In the same way:

$$\begin{aligned}\|\nabla\nabla'' s_{0,k}\|_{\mathcal{C}^0,g} &\leq \|\mathrm{d}^2\beta\|_{\mathcal{C}^0,g} \sup_{\{d_1>\rho\}} |s_0^k| \\ &\quad + 2\|\mathrm{d}\beta\|_{\mathcal{C}^0,g} \sup_{\{d_1>\rho\}} |\nabla s_0^k|_g + \sup_{B(Q,2\rho)} |\nabla\nabla'' s_0^k|_g \\ &\leq Ce^{-k/C} + Ce^{-k/C} + Ck^{-(r-2)/2}.\end{aligned}$$

Since

$$\|\nabla'' s_{0,k}\|_{\mathrm{L}^2,g_k} \leq Ck^{n/2}\|\nabla'' s_{0,k}\|_{\mathcal{C}^0,g_k},$$

the $\mathcal{C}^1$ and the L^2 norms, in the metric g_k, satisfy the bounds of the statement. □

We now use the following version of Hörmander's L^2-estimates:

Theorem 21 (cf. [Dem12, Theorem VIII.6.5] and the discussion thereafter). *Let (X,ω) be a closed integral Kähler manifold and $L \to X$ a holomorphic Hermitian line bundle with Chern curvature $-2\pi i\omega$. Set $C := \sup|\frac{\mathrm{Ricci}(\omega)}{2\pi}|_g$. Then, for every $k > C$ and for every smooth section $u : X \to \bigwedge^{1,0} T^\star X \otimes L^k$ such that $\nabla'' u = 0$, there exists a smooth section $t : X \to L^k$ satisfying:*

$$\nabla'' t = u \ \textit{and} \ \ \|t\|^2_{\mathrm{L}^2} \leq \frac{1}{n(k-C)}\|u\|^2_{\mathrm{L}^2}.$$

Applying this theorem to the sections $s_{0,k}$ of lemma 20, we obtain smooth sections $t_k : X \to L^k$ satisfying $\|t_k\|_{\mathrm{L}^2,g_k} \leq Ck^{(n-r-1)/2}$, and, for k sufficiently large, $\nabla''(s_{0,k} - t_k) = 0$. The following lemma converts our L^2-estimates to $\mathcal{C}^1$-estimates.

Lemma 22. *Let (X,ω) be a closed integral Kähler manifold, $L \to X$ a holomorphic Hermitian line bundle with Chern curvature $-2\pi i\omega$. There exists a constant $C > 0$ such that for every integer k and for every section $t : X \to L^k$:*

$$\|t\|_{\mathcal{C}^1,g_k} \leq C(\|\nabla'' t\|_{\mathcal{C}^1,g_k} + \|t\|_{\mathrm{L}^2,g_k}).$$

Proof. The desired bound is local. At a given point $p \in X$, we will obtain it on a g_k-ball of uniform radius about p — where, for sufficiently large k, the geometry of L^k compares with the trivial line bundle over the unit ball of euclidean space $(\mathbf{C}^n, g_0)$. There exist constants $R, C > 0$ and a family (indexed by $p \in X, k \geq 1$) of holomorphic charts $\underline{z}_p^k : B_k(p,R) \to \mathbf{C}^n$ centered at p such that,

$$\|(\underline{z}_p^k)_\star g_k - g_0\|_{\mathcal{C}^1, g_0} \leq C k^{-1/2} \text{ over } (\underline{z}_p^k)(B_k(p,R)). \tag{5}$$

We first explain this when $k = 1$. There exist constants $R, C_0 > 0$ and a family of holomorphic charts $\underline{z}_p : B_1(p,R) \to \mathbf{C}^n$ centered at p with $||\nabla(\underline{z}_p^\star g)||_{\mathcal{C}^0} < C_0$, where the covariant derivative and the norm are taken for the flat metric. Furthermore, after post-composing each chart by and element of $\mathrm{GL}(n, \mathbf{C})$, we may assume that $(\underline{z}_p^\star g)(p) = g_0$. Then, the family $\underline{z}_p$ satisfies the bound (5) with $C = C_0(1+R)$. In the general case $k \geq 1$, to get the desired charts $\underline{z}_p^k$, it suffices to post-compose each chart $\underline{z}_p$ by the centered dilation $\mathbf{C}^n \to \mathbf{C}^n$ of ratio $k^{1/2}$.

Let us take a Hörmander holomorphic peak section at p (see for instance [Don96, Proposition 34]): for sufficiently large k, there exists a holomorphic section $s_p : X \to L^k$ satisfying the bounds:

$$|s_p(p)| = 1, \inf_{B_k(p,R)} |s_p| \geq C^{-1} \text{ and } \|s_p\|_{\mathcal{C}^1, g_k} \leq C,$$

for some constant $C > 0$ independent of p and k.

Let t be a section of L^k and $p \in X$. We set $f := \frac{t}{s_p}$. In view of the identities $\nabla t = \mathrm{d}f \;\; s_p + f \nabla s_p$, $\nabla\nabla t = \mathrm{d}^2 f \;\; s_p + 2\mathrm{d}f \otimes \nabla s_p + f \nabla\nabla s_p$, and the bounds on the peak sections, it suffices to show that for sufficiently large k,

$$\|f\|_{\mathcal{C}^1(B_k(p,R/6)), g_k} \leq C \|\mathrm{d}'' f\|_{\mathcal{C}^1(B_k(p,R)), g_k} + C \|f\|_{\mathrm{L}^2(B_k(p,R)), g_k}.$$

In the following, we will identify the domain of the chart $\underline{z}_p^k$ with its image in $\mathbf{C}^n$. We denote by $B_0(q,R)$ the ball of radius R at a point q in $\mathbf{C}^n$ and by μ the Euclidean volume form on $\mathbf{C}^n$. Let us prove the (standard) following bound:

$$\|f\|_{\mathcal{C}^1(B_0(0,R/5)), g_0} \leq C \|\mathrm{d}'' f\|_{\mathcal{C}^1(B_0(0,R/2)), g_0} + C \|f\|_{\mathrm{L}^2(B_0(0,R/2)), g_0}.$$

This will end the proof because, in view of the comparison (5) of the rescaled metric g_k with the flat metric g_0, for sufficiently large k, we have the inclusions $B_k(p,R/6) \subset B_0(0,R/5)$ and $B_0(0,R/2) \subset B_k(p,R)$, and there exists a constant $C > 0$ (independent on k and p) such that, over $B_0(0, R/2)$,

$\mu \leq (1 + Ck^{-n/2})\frac{(k\omega)^n}{n!}$ and $(1 - Ck^{-1/2})|\cdot|_{g_0} \leq |\cdot|_{g_k} \leq (1 + Ck^{-1/2})|\cdot|_{g_0}$.

On the one hand, [HW68, Lemma 4.4] gives:

$$\|f\|_{\mathcal{C}^0(B_0(0,R/4))} \leq C\|\mathrm{d}''f\|_{\mathcal{C}^0(B_0(0,R/2))} + C\|f\|_{\mathrm{L}^2(B_0(0,R/2)),g_0}.$$

On the other hand, we have the following standard bound (cf. [CE12, Lemma 8.37] for instance):

$$\|f\|_{\mathcal{C}^1(B_0(0,R/5)),g_0} \leq C\|\mathrm{d}''f\|_{\mathcal{C}^1(B_0(0,R/4)),g_0} + C\|f\|_{\mathcal{C}^0(B_0(0,R/4))}.$$

In the two above estimates the constants depend only on R and n. Therefore we obtain the desired bound. □

By lemmas 22 and 20, and using $r = n$, we obtain the following estimate: for every $\epsilon > 0$, for $k \geq k_1(\epsilon)$ sufficiently large,

$$\|t_k\|_{\mathcal{C}^1,g_k} \leq C(\|\nabla'' s_{0,k}\|_{\mathcal{C}^1,g_k} + k^{-1/2}\|s_{0,k}\|_{\mathrm{L}^2,g_k}) \leq Ck^{(n-r-1)/2} < \epsilon/2.$$

On the other hand, by Bertini theorem, for sufficiently large k there exists a holomorphic section $s_k : X \to L^k$ vanishing transversally with

$$\|s_k - (s_{0,k} - t_k)\|_{\mathcal{C}^1,g_k} < \epsilon/2.$$

Therefore the sections s_k satisfy the conclusions of proposition 19. This ends the proof of Theorem 2.

Let us finish with a complex-geometric variant of Theorem 2:

Theorem 23. *Let X be a closed complex manifold, a a Kähler class and Q a closed submanifold. Suppose that Q is a Bohr-Sommerfeld Lagrangian submanifold for some Kähler form in a. Then, there exists a holomorphic line bundle $L \to X$ with first Chern class a, and, for every sufficiently large k, there exist a Hermitian metric h_k on L^k with positive Chern curvature and a holomorphic section $s_k : X \to L^k$ vanishing transversally such that the function $-\log|s_k|_{h_k} : X \setminus s_k^{-1}(0) \to \mathbf{R}$ has a Morse-Bott minimum at Q and is Morse elsewhere.*

Proof of Theorem 23. We fix a Kähler form $\omega \in a$ with $\omega|_Q = 0$ as well as a Hermitian holomorphic line bundle $L \to X$ with Chern curvature $-2i\pi\omega$ whose restriction to Q is a trivial flat bundle (by lemma 6). We fix $\epsilon, \rho > 0$

and repeat the construction of section 4 to obtain sections $s_0^k, s_k : X \to L^k$ with the properties stated in lemma 18 and proposition 19. We keep the notation $f_0 = -\log |s_0^k|$ and $f_1 = -\log |s_k|$.

To construct the desired Hermitian metric on L^k, we will proceed as in the final step of the proof of Theorem 2 but we will modify the initial Hermitian metric h^k of L^k instead of the function f_1. Take a cutoff function $\beta_k : X \to [0,1]$ with support in $B_k(Q,\rho)$ with $\beta_k = 1$ on $B_k(Q,\rho/2)$ and such that $\|\mathrm{d}\beta_k\|_{\mathcal{C}^1,g_k} < C'$, for some constant $C' > 0$ independent of k. We define a new Hermitian metric on L^k by:

$$h'_k = e^{2\beta_k(f_1 - f_0)} h^k.$$

The exhaustion function $-\log |s_k|_{h'_k} : \{s_k \neq 0\} \to \mathbf{R}$ equals f_0 on $B_k(Q,\rho/2)$ hence has a Morse-Bott local minimum at Q. Furthermore,

$$2k\pi\omega - \mathrm{dd}^c \log |s_k|_{h'_k} = -\mathrm{dd}^c(\beta_k(f_1 - f_0)).$$

Therefore, by repeating the estimation (4), for every $\epsilon < \epsilon_0$ sufficiently small and for $k \geq k_0(\epsilon)$ sufficiently large, $\|2k\pi\omega - \mathrm{dd}^c \log |s_k|_{h'_k}\|_{\mathcal{C}^0,g_k} < 2\pi$. This inequality ensures that the function $-\log |s_k|_{h'_k}$ is $\mathbf{C}$-convex. Finally, there exists a $\mathcal{C}^2$-small function $\eta_k : X \setminus Y \to \mathbf{R}$ with compact support away from Q such that, setting the Hermitian metric $h''_k := e^{-2\eta_k} h'_k$, the function $-\log |s_k|_{h''_k} = -\log |s_k|_{h'} + \eta_k$ is Morse away from Q.

In conclusion, the Hermitian metric h''_k and the sections $s_k : X \to L^k$ have the desired properties. □

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