

Scalar curvature as moment map in generalized Kähler geometry

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It is known that the scalar curvature arises as the moment map in Kähler geometry. In pursuit of the analogy, we develop the moment map framework in generalized Kähler geometry of symplectic type. Then we establish the definition of the scalar curvature on a generalized Kähler manifold of symplectic type from the moment map view point. We also obtain the generalized Ricci form which is a representative of the first Chern class of the anticanonical line bundle. We show that infinitesimal deformations of generalized Kähler structures with constant generalized scalar curvature are finite dimensional on a compact manifold. Explicit descriptions of the generalized Ricci form and the generalized scalar curvature are given on a generalized Kähler manifold of type $(0, 0)$. Poisson structures constructed from a Kähler action of T^m on a Kähler-Einstein manifold give rise to intriguing deformations of generalized Kähler-Einstein structures. In particular, the anticanonical divisor of three lines on $\mathbb{C}P^2$ in a general position yields nontrivial examples of generalized Kähler-Einstein structures

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1. Introduction

Let (X, ω) be a compact symplectic manifold with a symplectic structure ω . An almost complex structure J is compatible with ω if a pair (J, ω) gives an almost Kähler structure on M . We denote by $\tilde{\mathcal{C}}_\omega$ the set of almost complex structures which are compatible with ω . Then $\tilde{\mathcal{C}}_\omega$ is an infinite dimensional Kähler manifold on which Hamiltonian diffeomorphisms of (M, ω) act $\tilde{\mathcal{C}}_\omega$ preserving the Kähler structure. Each $J \in \tilde{\mathcal{C}}_\omega$ gives a Riemannian metric $g(J)$ and we denote by $s(J)$ the scalar curvature of $g(J)$ which is regarded as a function on $\tilde{\mathcal{C}}_\omega$. Then the following theorem was established in Kähler geometry by Fujiki and Donaldson.

Theorem 1.1. [Fu], [Do] *The scalar curvature is the moment map on $\tilde{\mathcal{C}}_\omega$ for the action of Hamiltonian diffeomorphisms.*

The moment map framework in Kähler geometry suggests that the existence of constant scalar curvature Kähler metrics is inevitably linked with the certain stability in algebraic geometry which leads to well-known Donaldson-Tian-Yau conjecture in Kähler geometry.

Generalized Kähler geometry is a successful generalization of ordinary Kähler geometry which is equivalent to bihermitian geometry satisfying the certain torsion conditions.

Many interesting examples of generalized Kähler manifolds were already constructed by holomorphic Poisson structures [Go1], [Go2], [Go3], [Go4], [Gu1], [Hi1], [Hi2], [Lin1].

Main theme of this paper is to pursue an analogue of moment map framework in generalized Kähler geometry and to establish the notion of the scalar curvature on a generalized Kähler manifold. In this paper we assume that a generalized Kähler structure consists of commuting two generalized complex structures $(\mathcal{J}, \mathcal{J}_\psi)$, where \mathcal{J} is an arbitrary almost generalized complex structure and \mathcal{J}_ψ is induced from a d -closed nondegenerate, pure spinor ψ of symplectic type^{*1}. We construct an invariant function from \mathcal{J} and ψ which is referred to the generalized scalar curvature GR. Then it turns out that a moment map in generalized Kähler geometry is given by the generalized scalar curvature GR. From the view point of moment map, the notion of generalized Ricci curvature is introduced and the definition of generalized Kähler-Einstein structure is provided.

In order to obtain moment map framework, one may try to follow the same way as in Kähler geometry by using the Levi-civita connection and the curvature. However, we need to pave a way without the use of the Levi-civita connection and the curvature in this paper because the notion of Levi-civita connection and the curvature in generalized Kähler geometry are very different from the ones in Kähler geometry and are not suitable for our purpose. Nondegenerate, pure spinors play a central role rather than generalized complex structures in this paper. A nondegenerate, pure spinor is a differential form on a manifold M which induces an almost generalized complex structure \mathcal{J} by $\ker \phi = E_{\mathcal{J}}$ and $\overline{\ker \phi} = \overline{E}_{\mathcal{J}}$. Conversely, the canonical line bundle $K_{\mathcal{J}}$ of an almost generalized complex structure \mathcal{J} gives a nondegenerate, pure spinor unique up to multiplication by nonzero complex functions. Let $\{\phi_\alpha\}$ be trivializations of $K_{\mathcal{J}}$, where ϕ_α is a nondegenerate, pure spinor which induces an almost generalized complex structure \mathcal{J} . Then the exterior derivative of the differential form ϕ_α is given by

$$d\phi_\alpha = \eta_\alpha \cdot \phi_\alpha + N_\alpha \cdot \phi_\alpha$$

where η_α is a real section $T_M \oplus T_M^*$ and N_α is also a real section of $\wedge^3 E_{\mathcal{J}} \oplus \wedge^3 \overline{E}_{\mathcal{J}}$. A real function ρ_α is defined by

$$\langle \phi_\alpha, \overline{\psi} \rangle_s = \rho_\alpha \langle \psi, \overline{\psi} \rangle_s,$$

^{*1}Thus ψ is given by $\psi = e^{b+\sqrt{-1}\omega}$, where ω is a real symplectic form. A pair $(\mathcal{J}, \mathcal{J}_\psi)$ is called a *generalized Kähler structure of symplectic type*. We can obtain further generalization of moment map framework for any d -closed nondegenerate, pure spinor ψ .

where $\langle \cdot, \cdot \rangle_s$ denotes the inner metric of Spin representation which is a $2n$ -form on M . Then \mathcal{J} acts on η_α by $\mathcal{J}\eta_\alpha \in T_M \oplus T_M^*$. By Spin representation, $\mathcal{J}\eta_\alpha$ acts on the differential form ψ by $\mathcal{J}\eta_\alpha \cdot \psi$. Taking the exterior derivative d , we have a differential form $d(\mathcal{J}\eta_\alpha \cdot \psi)$ which is locally defined. We also obtain a differential form $d(\mathcal{J}d \log \rho_\alpha \cdot \psi)$. Then it turns out that $-2d(\mathcal{J}\eta_\alpha \cdot \psi) + d(\mathcal{J}d \log \rho_\alpha \cdot \psi)$ does not depend on the choice of trivializations of $K_{\mathcal{J}}$ which defines a differential form on M (see Proposition 5.1). Since $\psi = e^{b+\sqrt{-1}\omega}$, it follows that $d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d \log \rho_\alpha) \cdot \bar{\psi}$ is given by

$$(1.1) \quad d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d \log \rho_\alpha) \cdot \bar{\psi} = (P - \sqrt{-1}Q) \cdot \bar{\psi},$$

where P, Q are real d -closed 2-forms. Thus we define a generalized Ricci form and a generalized scalar curvature GR by

$$\begin{aligned} \text{GRic} &:= -P \quad \text{generalized Ricci form ,} \\ \text{GR} &:= n \frac{P \wedge \omega^{n-1}}{\omega^n} \quad \text{generalized scalar curvature} \end{aligned}$$

where ω is a symplectic form and GR is a real function (see Definition 5.3). An almost generalized complex structure \mathcal{J} is compatible with ψ if a pair $(\mathcal{J}, \mathcal{J}_\psi)$ is an almost generalized Kähler structure. We denote by $\tilde{\mathcal{A}}_\psi(M)$ the set of almost generalized complex structures which are compatible with ψ . Then it turns out that $\tilde{\mathcal{A}}_\psi(M)$ admits a Kähler structure on which the generalized Hamiltonian group acts preserving its Kähler structure. The Lie algebra of generalized Hamiltonian group is given by real smooth functions.

Then our main theorem is the following:

THEOREM 6.4 There exists a moment map $\mu : \tilde{\mathcal{A}}_\psi(M) \rightarrow C_0^\infty(M)^*$ for the generalized Hamiltonian action which is given by the generalized scalar curvature GR ,

$$\langle \mu(\mathcal{J}), f \rangle = (\sqrt{-1})^{-n} \int_M f(GR_{\mathcal{J}}) \langle \psi, \bar{\psi} \rangle_s$$

In Section 2, we shall give a brief review of almost generalized complex structures focusing on nondegenerate, pure spinors and in Section 3, we define an almost generalized Kähler structure. In Section 4, we recall the stability theorem of generalized Kähler structures which is crucial to construct non-trivial examples of generalized Kähler manifolds. In Section 5, we define a generalized Ricci form GRic and we show that GRic is a representative of the first Chern class of the anticanonical line bundle $K_{\mathcal{J}}$. The generalized scalar curvature is obtained from the generalized Ricci form. The generalized

scalar curvature is an invariant function under the action of the extension of volume-preserving diffeomorphisms by d -closed b fields. In Section 6, we formulate the moment map framework of generalized Kähler geometry. After preliminary results are shown in Section 7, our main theorem is proved in Section 8. In Section 9, we show that infinitesimal deformations of generalized Kähler structures with constant generalized scalar curvature are given by an elliptic complex. In particular, the infinitesimal deformations are finite dimensional on a compact manifold. In Section 10, we give simple expressions of the generalized Ricci form and the generalized scalar curvature of a generalized Kähler structure of type $(0, 0)$.[†] A generalized Kähler structure $(\mathcal{J}_\phi, \mathcal{J}_\psi)$ of type $(0, 0)$ is, by definition, induced from a pair

$$(\phi = e^{B+\sqrt{-1}\omega_1}, \psi = e^{\sqrt{-1}\omega_2})$$

of d -closed, nondegenerate, pure spinors of symplectic types, where B is a real 2-form and ω_1 and ω_2 are symplectic forms respectively. Then the 2-form GRic and the function GR are given by

$$\begin{aligned} \text{GRic} &= -dB\omega_1^{-1} \left(d \log \frac{\omega_1^n}{\omega_2^n} \right) \\ (\text{GR}) \omega_2^n &= \omega_2^{n-1} \wedge dB\omega_1^{-1} \left(d \log \frac{\omega_1^n}{\omega_2^n} \right), \end{aligned}$$

where $B : T_M \rightarrow T_M^*$ and $\omega_i^{-1} : T_M^* \rightarrow T_M$ ($i = 1, 2$). Then it turns out that the generalized Kähler structures coming from hyperKähler structures have vanishing GRic form. In Section 11, we define a generalized Kähler-Einstein structure. In Section 12, we provide nontrivial examples of generalized Kähler-Einstein structures which arise as Poisson deformations from Kähler-Einstein manifolds on which T^m acts preserving its Kähler structure. In particular, the anticanonical divisor of three lines in a general position on $\mathbb{C}P^2$ gives a nontrivial example of a generalized Kähler-Einstein structure.

Boulanger obtained remarkable results on the moment map in the cases of toric generalized Kähler manifolds from the view point of toric Kähler manifolds [Bou]. A generalized Kähler structure is equivalent to a bihermitian structure with the certain torsion condition. From the viewpoint of bihermitian geometry, generalized Kähler Ricci flow was introduced [St1]. Apostolov and Streets discuss Calabi-Yau problem in generalized Kähler geometry [AS]. It is interesting to find out an expression of our moment map

[†]A generalized Kähler structure of type $(0, 0)$ corresponds to a degenerate bihermitian structure, i.e, $[J_+, J_-]_x \neq 0$ for all $x \in M$.

in terms of bihermitian geometry. There is a remarkable link between generalized geometry and noncommutative algebraic geometry. It is quite natural to ask whether the existence of generalized Kähler structure with constant generalized scalar curvature is related with a stability on a noncommutative algebraic manifold.

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2. Generalized complex structures

Let M be a differentiable manifold of real dimension $2n$. The bilinear form $\langle \cdot, \cdot \rangle_{T \oplus T^*}$ on the direct sum $T_M \oplus T_M^*$ over a differentiable manifold M of $\dim = 2n$ is defined by

$$\langle v + \xi, u + \eta \rangle_{T \oplus T^*} = \frac{1}{2} (\xi(u) + \eta(v)), \quad u, v \in T_M, \xi, \eta \in T_M^*.$$

Let $\text{SO}(T_M \oplus T_M^*)$ be the fibre bundle over M with fibre $\text{SO}(2n, 2n)$ which is a subbundle of $\text{End}(T_M \oplus T_M^*)$ preserving the bilinear form $\langle \cdot, \cdot \rangle_s$. An almost generalized complex structure \mathcal{J} is a section of $\text{SO}(T_M \oplus T_M^*)$ satisfying $\mathcal{J}^2 = -\text{id}$. Then as in the case of almost complex structures, an almost generalized complex structure \mathcal{J} yields the eigenspace decomposition: $(T_M \oplus T_M^*)^{\mathbb{C}} = E_{\mathcal{J}} \oplus \overline{E}_{\mathcal{J}}$, where $E_{\mathcal{J}}$ is $-\sqrt{-1}$ -eigenspaces and $\overline{E}_{\mathcal{J}}$ is the complex conjugate of $E_{\mathcal{J}}$. The Courant bracket of $T_M \oplus T_M^*$ is defined by

$$[u + \xi, v + \eta]_{\text{cou}} = [u, v] + \mathcal{L}_u \eta - \mathcal{L}_v \xi - \frac{1}{2} (di_u \eta - di_v \xi),$$

where $u, v \in TM$ and ξ, η is T^*M . If $E_{\mathcal{J}}$ is involutive with respect to the Courant bracket, then \mathcal{J} is a generalized complex structure, that is, $[e_1, e_2]_{\text{cou}} \in \Gamma(E_{\mathcal{J}})$ for any two elements $e_1 = u + \xi$, $e_2 = v + \eta \in \Gamma(E_{\mathcal{J}})$.

Let $\text{CL}(T_M \oplus T_M^*)$ be the Clifford algebra bundle which is a fibre bundle with fibre the Clifford algebra $\text{CL}(2n, 2n)$ with respect to $\langle \cdot, \cdot \rangle_{T \oplus T^*}$ on M . Then a vector v acts on the space of differential forms $\bigoplus_{p=0}^{2n} \wedge^p T^*M$ by the interior product i_v and a 1-form acts on $\bigoplus_{p=0}^{2n} \wedge^p T^*M$ by the exterior product $\theta \wedge$, respectively. Then the space of differential forms gives a

representation of the Clifford algebra $\text{CL}(T_M \oplus T_M^*)$ which is the spin representation of $\text{CL}(T_M \oplus T_M^*)$. Thus the spin representation of the Clifford algebra arises as the space of differential forms

$$\wedge^\bullet T_M^* = \oplus_p \wedge^p T_M^* = \wedge^{\text{even}} T_M^* \oplus \wedge^{\text{odd}} T_M^*.$$

The inner product $\langle \cdot, \cdot \rangle_s$ of the spin representation is given by

$$\langle \alpha, \beta \rangle_s := (\alpha \wedge \sigma \beta)_{[2n]},$$

where $(\alpha \wedge \sigma \beta)_{[2n]}$ is the component of degree $2n$ of $\alpha \wedge \sigma \beta \in \oplus_p \wedge^p T^* M$ and σ denotes the Clifford involution which is given by

$$\sigma \beta = \begin{cases} +\beta & \deg \beta \equiv 0, 1 \pmod{4} \\ -\beta & \deg \beta \equiv 2, 3 \pmod{4} \end{cases}$$

We define $\ker \Phi := \{e \in (T_M \oplus T_M^*)^{\mathbb{C}} \mid e \cdot \Phi = 0\}$ for a differential form $\Phi \in \wedge^{\text{even/odd}} T_M^*$. If $\ker \Phi$ is maximal isotropic, i.e., $\dim_{\mathbb{C}} \ker \Phi = 2n$, then Φ is called a *pure spinor* of even/odd type.

A pure spinor Φ is *nondegenerate* if $\ker \Phi \cap \overline{\ker \Phi} = \{0\}$, i.e., $(T_M \oplus T_M^*)^{\mathbb{C}} = \ker \Phi \oplus \overline{\ker \Phi}$. Then a nondegenerate, pure spinor $\Phi \in \wedge^\bullet T_M^*$ gives an almost generalized complex structure \mathcal{J}_Φ which satisfies

$$\mathcal{J}_\Phi e = \begin{cases} -\sqrt{-1}e, & e \in \ker \Phi \\ +\sqrt{-1}e, & e \in \overline{\ker \Phi} \end{cases}$$

Conversely, an almost generalized complex structure \mathcal{J} locally arises as \mathcal{J}_Φ for a nondegenerate, pure spinor Φ which is unique up to multiplication by non-zero functions. Thus an almost generalized complex structure yields the canonical line bundle $K_{\mathcal{J}} := \mathbb{C}\langle \Phi \rangle$ which is a complex line bundle locally generated by a nondegenerate, pure spinor Φ satisfying $\mathcal{J} = \mathcal{J}_\Phi$. An generalized complex structure \mathcal{J}_Φ is integrable if and only if $d\Phi = \eta \cdot \Phi$ for a section $\eta \in T_M \oplus T_M^*$. The *type number* of $\mathcal{J} = \mathcal{J}_\Phi$ is defined as the minimal degree of the differential form Φ . Note that type number $\text{Type } \mathcal{J}$ is a function on a manifold which is not a constant in general.

Example 2.1. Let J be a complex structure on a manifold M and \mathcal{J}^* the complex structure on the dual bundle T^*M which is given by $J^*\xi(v) = \xi(Jv)$ for $v \in TM$ and $\xi \in T^*M$. Then a generalized complex structure \mathcal{J}_J is given

by the following matrix

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix},$$

Then the canonical line bundle is the ordinary one which is generated by complex forms of type $(n, 0)$. Thus we have $\text{Type } \mathcal{J}_J = n$.

Example 2.2. Let ω be a symplectic structure on M and $\hat{\omega}$ the isomorphism from TM to T^*M given by $\hat{\omega}(v) := i_v\omega$. We denote by $\hat{\omega}^{-1}$ the inverse map from T^*M to TM . Then a generalized complex structure \mathcal{J}_ψ is given by the following

$$\mathcal{J}_\psi = \begin{pmatrix} 0 & -\hat{\omega}^{-1} \\ \hat{\omega} & 0 \end{pmatrix}, \quad \text{Type } \mathcal{J}_\psi = 0$$

Then the canonical line bundle is given by the differential form $\psi = e^{\sqrt{-1}\omega}$. Thus $\text{Type } \mathcal{J}_\psi = 0$.

Example 2.3 (b-field action). A d -closed 2-form b acts on a generalized complex structure by the adjoint action of Spin group e^b which provides a generalized complex structure $\text{Ad}_{e^b}\mathcal{J} = e^b \circ \mathcal{J} \circ e^{-b}$.

Example 2.4 (Poisson deformations). Let β be a holomorphic Poisson structure on a complex manifold. Then the adjoint action of Spin group e^β gives deformations of new generalized complex structures by $\mathcal{J}_{\beta t} := \text{Ad}_{e^{\beta t}}\mathcal{J}$. Then $\text{Type } \mathcal{J}_{\beta t_x} = n - 2$ rank of β_x at $x \in M$, which is called the Jumping phenomena of type number.

Let (M, \mathcal{J}) be a generalized complex manifold and $\overline{E}_\mathcal{J}$ the eigenspace of eigenvalue $\sqrt{-1}$. Then we have the Lie algebroid complex $\wedge^\bullet \overline{E}_\mathcal{J}$:

$$0 \longrightarrow \wedge^0 \overline{E}_\mathcal{J} \xrightarrow{\overline{\partial}_\mathcal{J}} \wedge^1 \overline{E}_\mathcal{J} \xrightarrow{\overline{\partial}_\mathcal{J}} \wedge^2 \overline{E}_\mathcal{J} \xrightarrow{\overline{\partial}_\mathcal{J}} \wedge^3 \overline{E}_\mathcal{J} \longrightarrow \dots$$

The Lie algebroid complex is the deformation complex of generalized complex structures. In fact, $\varepsilon \in \wedge^2 \overline{E}_\mathcal{J}$ gives deformed isotropic subbundle $E_\varepsilon := \{e + [\varepsilon, e] \mid e \in E_\mathcal{J}\}$. Then E_ε yields deformations of generalized complex structures if and only if ε satisfies Generalized Maurer-Cartan equation

$$\overline{\partial}_\mathcal{J}\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon]_{\text{Sch}} = 0,$$

where $[\varepsilon, \varepsilon]_{\text{Sch}}$ denotes the Schouten bracket. The Kuranishi space of generalized complex structures is constructed.

Then the second cohomology group $H^2(\wedge^\bullet \bar{E}_{\mathcal{J}})$ of the Lie algebraic complex gives the infinitesimal deformations of generalized complex structures and the third one $H^3(\wedge^\bullet \bar{E}_{\mathcal{J}})$ is the obstruction space to deformations of generalized complex structures.

Let $\{e_i\}_{i=1}^n$ be a local basis of $E_{\mathcal{J}}$ for an almost generalized complex structure \mathcal{J} , where $\langle e_i, \bar{e}_j \rangle_{T \oplus T^*} = \delta_{i,j}$. The almost generalized complex structure \mathcal{J} is written as an element of Clifford algebra,

$$\mathcal{J} = \frac{\sqrt{-1}}{2} \sum_i e_i \cdot \bar{e}_i,$$

where \mathcal{J} acts on $T_M \oplus T_M^*$ by the adjoint action $[\mathcal{J}, \cdot]$. Thus we have $[\mathcal{J}, e_i] = -\sqrt{-1}e_i$ and $[\mathcal{J}, \bar{e}_i] = \sqrt{-1}e_i$. An almost generalized complex structure \mathcal{J} acts on differential forms by the Spin representation which gives the decomposition:

$$(2.1) \quad \wedge^\bullet T_M^* = U^{-n} \oplus U^{-n+1} \oplus \dots \oplus U^n$$

3. Almost generalized Kähler structures

Definition 3.1. An almost generalized Kähler structure is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ consisting of two commuting almost generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ such that $\hat{G} := -\mathcal{J}_1 \circ \mathcal{J}_2 = -\mathcal{J}_2 \circ \mathcal{J}_1$ gives a positive definite symmetric form $G := \langle \hat{G}, \cdot \rangle$ on $T_M \oplus T_M^*$. We call G a generalized metric. A generalized Kähler structure is an almost generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ such that both \mathcal{J}_1 and \mathcal{J}_2 are generalized complex structures.

\mathcal{J}_i gives the decomposition $(T_M \oplus T_M^*)^{\mathbb{C}} = E_{\mathcal{J}_i} \oplus \bar{E}_{\mathcal{J}_i}$ for $i = 1, 2$. Since \mathcal{J}_1 and \mathcal{J}_2 are commutative, we have the simultaneous eigenspace decomposition

$$(T_M \oplus T_M^*)^{\mathbb{C}} = (E_{\mathcal{J}_1} \cap E_{\mathcal{J}_2}) \oplus (\bar{E}_{\mathcal{J}_1} \cap \bar{E}_{\mathcal{J}_2}) \oplus (E_{\mathcal{J}_1} \cap \bar{E}_{\mathcal{J}_2}) \oplus (\bar{E}_{\mathcal{J}_1} \cap E_{\mathcal{J}_2}).$$

Since $\hat{G}^2 = +\text{id}$, The generalized metric \hat{G} also gives the eigenspace decomposition: $T_M \oplus T_M^* = C_+ \oplus C_-$, where C_{\pm} denote the eigenspaces of \hat{G} of

eigenvalues ± 1 . We denote by $E_{\mathcal{J}_1}^\pm$ the intersection $E_{\mathcal{J}_1} \cap C_{\pm}^{\mathbb{C}}$. Then it follows

$$\begin{aligned} E_{\mathcal{J}_1} \cap E_{\mathcal{J}_2} &= E_{\mathcal{J}_1}^+, & \overline{E}_{\mathcal{J}_1} \cap \overline{E}_{\mathcal{J}_2} &= \overline{E}_{\mathcal{J}_1}^+ \\ E_{\mathcal{J}_1} \cap \overline{E}_{\mathcal{J}_2} &= E_{\mathcal{J}_1}^-, & \overline{E}_{\mathcal{J}_1} \cap E_{\mathcal{J}_2} &= \overline{E}_{\mathcal{J}_1}^- \end{aligned}$$

Example 3.2. Let $X = (M, J, \omega)$ be a Kähler manifold. Then the pair $(\mathcal{J}_J, \mathcal{J}_\psi)$ is a generalized Kähler where $\psi = \exp(\sqrt{-1}\omega)$.

4. The stability theorem of generalized Kähler manifolds

It is known that the stability theorem of ordinary Kähler manifolds holds

Theorem 4.1 (Kodaira-Spencer). *Let $X = (M, J)$ be a compact Kähler manifold and X_t small deformations of $X = X_0$ as complex manifolds. Then X_t inherits a Kähler structure.*

The following stability theorem of generalized Kähler structures shows that there are many intriguing examples of generalized Kähler manifolds of symplectic type.

Theorem 4.2. [Go2] *Let $X = (M, J, \omega)$ be a compact Kähler manifold and $(\mathcal{J}_J, \mathcal{J}_\psi)$ the induced generalized Kähler structure, where $\psi = e^{\sqrt{-1}\omega}$. If there are analytic deformations $\{\mathcal{J}_t\}$ of $\mathcal{J}_0 = \mathcal{J}_J$ as generalized complex structures, then there are deformations of d -closed nondegenerate, pure spinors $\{\psi_t\}$ such that pairs $(\mathcal{J}_t, \mathcal{J}_{\psi_t})$ are generalized Kähler structures, where $\psi_0 = \psi$*

5. Generalized Ricci curvature and generalized scalar curvature

We use the same notation as before. Let \mathcal{J} be an almost complex structure on M with trivializations $\{\phi_\alpha\}$ of the canonical line bundle $K_{\mathcal{J}}$. Then recall that η_α is given by

$$(5.1) \quad d\phi_\alpha = \eta_\alpha \cdot \phi_\alpha + N_\alpha \cdot \phi_\alpha,$$

where $\eta_\alpha \in T_M \oplus T_M^*$ and $N_\alpha \in \wedge^3 E_{\mathcal{J}} \oplus \wedge^3 \overline{E}_{\mathcal{J}}$ are real sections, i.e., $\eta_\alpha = \overline{\eta}_\alpha$, $N_\alpha = \overline{N}_\alpha$. Because of the reality condition, η_α and N_α are uniquely determined. Let (\mathcal{J}, ψ) be an almost generalized Kähler structure of symplectic

type. Then recall that a real function ρ_α on U_α is given by

$$(5.2) \quad \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s = \rho_\alpha \langle \psi, \bar{\psi} \rangle_s$$

Proposition 5.1. *A differential form $d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d\log\rho_\alpha) \cdot \bar{\psi}$ does not depend on the choice of trivializations $\{\phi_\alpha\}$ of $K_{\mathcal{J}}$.*

Proof. Let $e^{\kappa_{\alpha,\beta}}$ be the transition function on the intersection $U_\alpha \cap U_\beta$. Then we have $\phi_\alpha = e^{\kappa_{\alpha,\beta}} \phi_\beta$. Since $d\phi_\beta = (\eta_\beta + N_\beta) \cdot \phi_\beta$, we have

$$\begin{aligned} d\phi_\alpha &= d(e^{\kappa_{\alpha,\beta}} \phi_\beta) = d\kappa_{\alpha,\beta} \cdot e^{\kappa_{\alpha,\beta}} \phi_\beta + e^{\kappa_{\alpha,\beta}} (\eta_\beta + N_\beta) \phi_\beta \\ &= d\kappa_{\alpha,\beta} \cdot \phi_\alpha + (\eta_\beta + N_\beta) \cdot \phi_\alpha \end{aligned}$$

Thus we have

$$(\eta_\alpha + N_\alpha) \cdot \phi_\alpha = (\eta_\beta + d\kappa_{\alpha,\beta} + N_\beta) \cdot \phi_\alpha.$$

Since $N_\alpha, N_\beta \in \wedge^3 E_{\mathcal{J}} \oplus \wedge^3 \bar{E}_{\mathcal{J}}$ and $\eta_\alpha, \eta_\beta, d\kappa_{\alpha,\beta} \in T_M \oplus T_M^*$, we have $N_\alpha = N_\beta$ and $\eta_\alpha \cdot \phi_\alpha = (\eta_\beta + d\kappa_{\alpha,\beta}) \cdot \phi_\alpha$. Since η_α, η_β are real, it follows that we have

$$(5.3) \quad \eta_\alpha = \eta_\beta + \bar{\partial}_{\mathcal{J}} \kappa_{\alpha,\beta} + \partial_{\mathcal{J}} \bar{\kappa}_{\alpha,\beta}$$

We also have

$$(5.4) \quad \rho_\alpha = \rho_\beta e^{\kappa_{\alpha,\beta} + \bar{\kappa}_{\alpha,\beta}}$$

Since $d((d\kappa_{\alpha,\beta}) \cdot \psi) = 0$, it follows from $d\kappa_{\alpha,\beta} = \partial_{\mathcal{J}} \kappa_{\alpha,\beta} + \bar{\partial}_{\mathcal{J}} \kappa_{\alpha,\beta}$ that we have

$$(5.5) \quad d(\partial_{\mathcal{J}} \kappa_{\alpha,\beta}) \cdot \psi + d(\bar{\partial}_{\mathcal{J}} \kappa_{\alpha,\beta}) \cdot \psi = 0$$

Since $d((d\bar{\kappa}_{\alpha,\beta}) \cdot \psi) = 0$, we also have

$$(5.6) \quad d(\bar{\partial}_{\mathcal{J}} \bar{\kappa}_{\alpha,\beta}) \cdot \psi + d(\partial_{\mathcal{J}} \bar{\kappa}_{\alpha,\beta}) \cdot \psi = 0$$

Applying (5.3), (5.5) and (5.6), we have

$$\begin{aligned} & d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d\log\rho_\alpha) \cdot \psi - d(-2\mathcal{J}\eta_\beta + \mathcal{J}d\log\rho_\beta) \cdot \psi \\ &= -2d\mathcal{J}(\bar{\partial}_{\mathcal{J}} \kappa_{\alpha,\beta} + \partial_{\mathcal{J}} \bar{\kappa}_{\alpha,\beta}) \cdot \psi + d\mathcal{J}d(\kappa_{\alpha,\beta} + \bar{\kappa}_{\alpha,\beta}) \cdot \psi \\ &= -2\sqrt{-1}d(\bar{\partial}_{\mathcal{J}} \kappa_{\alpha,\beta} - \partial_{\mathcal{J}} \bar{\kappa}_{\alpha,\beta}) \cdot \psi \\ &\quad + \sqrt{-1}d(\bar{\partial}_{\mathcal{J}} \kappa_{\alpha,\beta} - \partial_{\mathcal{J}} \kappa_{\alpha,\beta} + \bar{\partial}_{\mathcal{J}} \bar{\kappa}_{\alpha,\beta} - \partial_{\mathcal{J}} \bar{\kappa}_{\alpha,\beta}) \cdot \psi \\ &= -2\sqrt{-1}d\bar{\partial}_{\mathcal{J}}(\kappa_{\alpha,\beta} + \bar{\kappa}_{\alpha,\beta}) \cdot \psi + 2\sqrt{-1}d\bar{\partial}_{\mathcal{J}}(\kappa_{\alpha,\beta} + \bar{\kappa}_{\alpha,\beta}) \cdot \psi = 0 \end{aligned}$$

Thus we have the result^{*2}. □

Hence $d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d\log\rho_\alpha) \cdot \bar{\psi}$ yields a globally defined differential form on M . Since $\psi = e^{b+\sqrt{-1}\omega}$, it follows that $d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d\log\rho_\alpha) \cdot \bar{\psi}$ is given by

$$(5.7) \quad d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d\log\rho_\alpha) \cdot \bar{\psi} = (P - \sqrt{-1}Q) \cdot \bar{\psi},$$

where P, Q are real d -closed 2-forms. In fact, $-2\mathcal{J}\eta_\alpha + \mathcal{J}d\log\rho_\alpha$ is written as $v + \theta \in T_M \oplus T_M^*$ for a vector v and a 1-form θ and then $-2\mathcal{J}\eta_\alpha + \mathcal{J}d\log\rho_\alpha \cdot \bar{\psi}$ is given by $(i_v b - \sqrt{-1}i_v \omega + \theta) \wedge \bar{\psi}$. Thus P and Q are given by $P = di_v b + d\theta$ and $Q = di_v \omega$.

Remark 5.2. Since $N_\alpha = N_\beta$, we have a globally defined section $N \in \wedge^3 E_{\mathcal{J}} \oplus \wedge^3 \bar{E}_{\mathcal{J}}$ which is the Nijenhuis type tensor, that is, \mathcal{J} is integrable if and only if $N = 0$.

Definition 5.3. [Generalized Ricci form and generalized scalar curvature] We define a generalized Ricci form GRic to be a d -closed 2-form P in (5.7) and we define a generalized scalar curvature GR to be a real function on M which is given by the following,

$$\begin{aligned} \text{GRic} &:= -P \quad \text{generalized Ricci form ,} \\ \text{GR} &:= \frac{nP \wedge \omega^{n-1}}{\omega^n} : \text{generalized scalar curvature} \end{aligned}$$

where ω is a symplectic form.

A diffeomorphism F of M acts on (\mathcal{J}, ψ) to give an almost generalized Kähler structure (\mathcal{J}', ψ') . We denote by GR' generalized scalar curvature of (\mathcal{J}', ψ') . Then we have

Proposition 5.4.

$$\text{GR}' = F^*(\text{GR}),$$

that is, GR is equivalent under the action of diffeomorphisms. Further GR is invariant under the action of d -closed b -fields.

Proof. A diffeomorphism F of M induces the bundle map $F_\#$ of $T_M \oplus T_M^*$ by $F_\#(v + \theta) = F_*^{-1}(v) + F^*\theta$ for $v \in T_M$ and $\theta \in T_M^*$. Then we see that

^{*2}In this proof, note that we do not use the integrability of \mathcal{J} .

$F_{\#}(v + \theta) \cdot F^*(\alpha) = F^*((v + \theta) \cdot \alpha)$ for a differential form α . Let b be a real d -closed 2-form. Then e^b is regarded as an element of Spin group of the Clifford algebra of $T_M \oplus T_M^*$ which acts on differential forms by the wedge produce of e^b . Then we have the adjoint action Ad_{e^b} on $T_M \oplus T_M^*$ by $\text{Ad}_{e^b}(v + \theta) := e^b(v + \theta)e^{-b} = v - i_v b + \theta$. Then we see that

$$(\mathcal{J}', \psi') = (F_{\#} \circ \mathcal{J} \circ F_{\#}^{-1}, F^* \psi).$$

Then it follows that $\phi'_\alpha = F^* \phi_\alpha$ is the nondegenerate pure spinor which induces \mathcal{J}' . We define η'_α by $d\phi'_\alpha = \eta'_\alpha \cdot \phi'_\alpha$. Thus we have

$$d\phi'_\alpha = F^* d\phi_\alpha = F^*(\eta_\alpha \cdot \phi_\alpha) = F_{\#}(\eta_\alpha) \cdot \phi'_\alpha$$

Thus we see that $\eta'_\alpha = F_{\#}(\eta_\alpha)$. The function ρ'_α is given by

$$\langle \phi'_\alpha, \bar{\phi}'_\alpha \rangle_s = \rho'_\alpha \langle \psi', \bar{\psi}' \rangle_s$$

Thus we have $\rho'_\alpha = F^* \rho_\alpha$. Then we see

$$(5.8) \quad (\mathcal{J}' \eta'_\alpha) \cdot \psi' = F_{\#} \circ \mathcal{J} \circ F_{\#}^{-1}(F_{\#}(\eta_\alpha)) \cdot F^* \psi$$

$$(5.9) \quad = F_{\#}(\mathcal{J} \eta_\alpha) \circ F^* \psi$$

$$(5.10) \quad = F^*(\mathcal{J} \eta_\alpha \cdot \psi)$$

We also have

$$(5.11) \quad \mathcal{J}'(d \log \rho'_\alpha) \cdot \psi' = F_{\#} \circ \mathcal{J} \circ F_{\#}^{-1}(F^*(d \log \rho_\alpha) F^* \psi)$$

$$(5.12) \quad = F_{\#}(\mathcal{J}(d \log \rho_\alpha)) \cdot F^* \psi$$

$$(5.13) \quad = F^*(\mathcal{J}(d \log \rho_\alpha) \cdot \psi)$$

Thus we obtain

$$(5.14) \quad d(-2\mathcal{J}' \eta'_\alpha + \mathcal{J}' d \log \rho'_\alpha) \cdot \bar{\psi}' = F^*(d(-2\mathcal{J} \eta_\alpha + \mathcal{J} d \log \rho_\alpha) \cdot \bar{\psi})$$

Since GR is given by the real part of the following:

$$GR := \text{Re} \frac{\sqrt{-1} \langle \psi, d(-2\mathcal{J} \eta_\alpha + \mathcal{J} d \log \rho_\alpha) \cdot \bar{\psi} \rangle_s}{2 \langle \psi, \bar{\psi} \rangle_s}$$

From (5.14), we have $\text{GR}' = F^*(\text{GR})$.

We denote by (\mathcal{J}_b, ψ_b) the pair given by the action of e^b on (\mathcal{J}, ψ) . Then \mathcal{J}_b is induced from $e^b \cdot \phi_\alpha$ and $\psi_b = e^b \cdot \psi$. Thus we have $\eta_\alpha^b = \text{Ad}_{e^b}(\eta_\alpha)$ and $\rho_\alpha^b = \rho_\alpha$. Then we see that

$$d(-2\mathcal{J}_b\eta_\alpha^b + \mathcal{J}_bd \log \rho_\alpha^b) \cdot \bar{\psi}_b = e^b (d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d \log \rho_\alpha) \cdot \bar{\psi})$$

Since $\langle \cdot, \cdot \rangle_s$ is invariant under the action of e^b , we see that GR is invariant under the action of e^b . \square

We denote by [GRic] the cohomology class of a real d -closed 2-form GRic. Then we have

Proposition 5.5. *The cohomology class [GRic] is given by the 1-st Chern class,*

$$[\text{GRic}] = 4\pi c_1(K_{\mathcal{J}}^{-1}) \in H^2(M)$$

Proof. We calculate the spectral sequence from de Rham to Čeck cohomology to determine a representative of Čeck cohomology group given by d -closed form GRic. $d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d \log \rho_\alpha) \cdot \psi$ is d -exact on U_α . On $U_\alpha \cap U_\beta$, it follows from (5.3) and (5.4) that we have

$$\begin{aligned} & (-2\mathcal{J}\eta_\alpha + \mathcal{J}d \log \rho_\alpha) \cdot \psi - (-2\mathcal{J}\eta_\beta + \mathcal{J}d \log \rho_\beta) \cdot \psi \\ &= (-2\mathcal{J}(\eta_\alpha - \eta_\beta) + \mathcal{J}d(\kappa_{\alpha,\beta} + \bar{\kappa}_{\alpha,\beta})) \cdot \psi \\ &= -2\mathcal{J}(\bar{\partial}_{\mathcal{J}}\kappa_{\alpha,\beta} + \partial\bar{\kappa}_{\alpha,\beta}) + \mathcal{J}(\bar{\partial}_{\mathcal{J}}\kappa_{\alpha,\beta} + \partial_{\mathcal{J}}\kappa_{\alpha,\beta} + \bar{\partial}_{\mathcal{J}}\bar{\kappa}_{\alpha,\beta} + \partial_{\mathcal{J}}\bar{\kappa}_{\alpha,\beta}) \cdot \psi \\ &= -\sqrt{-1}d(\kappa_{\alpha,\beta} - \bar{\kappa}_{\alpha,\beta}) \cdot \psi \\ &= 2dk_{\alpha,\beta}^{\text{Im}} \cdot \psi, \end{aligned}$$

where $\kappa_{\alpha,\beta}^{\text{Im}}$ denotes the imaginary part of $\kappa_{\alpha,\beta}$. Thus we have a Čeck representative,

$$2(\kappa_{\alpha,\beta}^{\text{Im}} + \kappa_{\beta,\gamma}^{\text{Im}} + \kappa_{\gamma,\alpha}^{\text{Im}}) \cdot \psi$$

Thus the representative of the class $[P]$ is given by $2(\kappa_{\alpha,\beta}^{\text{Im}} + \kappa_{\beta,\gamma}^{\text{Im}} + \kappa_{\gamma,\alpha}^{\text{Im}})$.

The 1-st Chern class $c_1(K_{\mathcal{J}})$ has a Čeck representative

$$c_{\alpha,\beta,\gamma} = \frac{1}{2\pi\sqrt{-1}}(\kappa_{\alpha,\beta} + \kappa_{\beta,\gamma} + \kappa_{\gamma,\alpha}) \equiv \frac{1}{2\pi}(\kappa_{\alpha,\beta}^{\text{Im}} + \kappa_{\beta,\gamma}^{\text{Im}} + \kappa_{\gamma,\alpha}^{\text{Im}})$$

Thus we have $[P] = 4\pi c_1(K_{\mathcal{J}})$. Since $\text{GRic} = -P$, we obtain the result. \square

Example 5.6. A GK structure $(\mathcal{J}_J, \psi = e^{\sqrt{-1}\omega})$ is induced from the genuine Kähler structure. Then GRic and GR are the ordinary Ricci curvature

and scalar curvature, respectively. In fact, we have ϕ_α to be a holomorphic n form $\phi_\alpha = dz_1 \wedge \cdots \wedge dz_n$ and $\psi = e^{\sqrt{-1}\omega}$ and $\langle \Omega_\alpha, \bar{\Omega}_\alpha \rangle_S = \rho_\alpha \langle \psi, \bar{\psi} \rangle_S$. Thus $d\mathcal{J}d \log \rho_\alpha = -2\sqrt{-1}\partial\bar{\partial} \log \det g_{i,\bar{j}}$ is the ordinary Ricci form.

Remark 5.7. We can generalize our construction of GR to the cases where ψ is an arbitrary d -closed, nondegenerate, pure spinor. In fact, $d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d \log \rho_\alpha) \cdot \bar{\psi}$ is still a representative of the first Chern class of $K_{\mathcal{J}}$ together with the class $[\bar{\psi}]$ and

$$GR^{\mathbb{C}} := \frac{\sqrt{-1}}{2} \frac{\langle \psi, \cdot, d(-2\mathcal{J}\eta_\alpha + \mathcal{J}d \log \rho_\alpha) \cdot \bar{\psi} \rangle_s}{\langle \psi, \bar{\psi} \rangle_s}$$

is an equivalent complex function under the action of diffeomorphisms which is invariant under the action of d -closed b -fields. In this general case, we define GR to be the real part of $GR^{\mathbb{C}}$. Then we have

$$(\sqrt{-1})^{-n}(GR)\langle \psi, \bar{\psi} \rangle_s = \text{Re}(\sqrt{-1})^{-n+1} \langle \psi, d(-\mathcal{J}\eta_\alpha + \frac{1}{2}\mathcal{J}d \log \rho_\alpha) \cdot \bar{\psi} \rangle_s,$$

where Re stands for the real part. The real part is also written as

$$(5.15) \quad \begin{aligned} (\sqrt{-1})^{-n}(GR)\langle \psi, \bar{\psi} \rangle_s &= c_n \left\langle \psi, d \left(-\mathcal{J}\eta_\alpha + \frac{1}{2}\mathcal{J}d \log \rho_\alpha \right) \cdot \bar{\psi} \right\rangle_s \\ &\quad - c_n \left\langle d \left(-\mathcal{J}\eta_\alpha + \frac{1}{2}\mathcal{J}d \log \rho_\alpha \right) \cdot \psi, \bar{\psi} \right\rangle_s, \end{aligned}$$

where $c_n = \frac{1}{2}(\sqrt{-1})^{-n+1}$.

Example 5.8 (generalized Calabi-Yau metrical structure). If a generalized Kähler structure is induced from a pair (ϕ, ψ) which consists of d -closed, nondegenerate, pure spinors such that $\langle \phi, \bar{\phi} \rangle_S = \langle \psi, \bar{\psi} \rangle_S$, then it is called a *generalized Calabi-Yau metrical structure*. Since $\rho_\alpha = 1$ and $\eta_\alpha = 0$, it follows that we have $GR^{\mathbb{C}} = 0$.

6. Generalized scalar curvature as moment map

Let $\mathcal{GC}(M)$ be the set of generalized complex structures on a differentiable compact manifold M of dimension $2n$, that is,

$$\mathcal{GC}(M) := \{ \mathcal{J} : \text{generalized complex structure on } M \}.$$

We denote by $\mathcal{G}K(M)$ the set of generalized Kähler structures on M , that is,

$$\mathcal{G}K(M) := \{(\mathcal{J}_0, \mathcal{J}_1) : \text{generalized Kähler structure on } M\}.$$

We also define $\widetilde{\mathcal{G}C}(M)$ as the set of almost generalized complex structures on M ,

$$\widetilde{\mathcal{G}C}(M) := \{\mathcal{J} : \text{almost generalized complex structure on } M\}.$$

We denote by $\widetilde{\mathcal{G}K}(M)$ the set of almost generalized Kähler structures,

$$\widetilde{\mathcal{G}K}(M) := \{(\mathcal{J}_0, \mathcal{J}_1) : \text{almost generalized Kähler structure on } M\}.$$

Let ψ be a d -closed, non-degenerate, pure spinor which induces \mathcal{J}_ψ . The spinor inner product of ψ is given by $\langle \psi, \bar{\psi} \rangle_S = (\phi \wedge \sigma\psi)_{[2n]}$. In particular, if $\psi := e^{b + \frac{\sqrt{-1}}{2}\omega}$, then we have the volume form

$$\langle \psi, \bar{\psi} \rangle_S = \frac{(\sqrt{-1})^n}{n!} \omega^n.$$

An almost generalized complex structure \mathcal{J} is ψ -compatible if and only if the pair $(\mathcal{J}, \mathcal{J}_\psi)$ is an almost generalized Kähler structure. Let $\mathcal{A}_\psi(M)$ be the set of ψ -compatible generalized complex structure, that is

$$\mathcal{A}_\psi(M) := \{\mathcal{J} \in \mathcal{G}C : (\mathcal{J}, \mathcal{J}_\psi) \in \mathcal{G}K\}.$$

We also define $\widetilde{\mathcal{A}}_\psi(M)$ to be the set of ψ -compatible almost generalized complex structures,

$$\widetilde{\mathcal{A}}_\psi(M) := \{\mathcal{J} \in \widetilde{\mathcal{G}C} : (\mathcal{J}, \mathcal{J}_\psi) \in \widetilde{\mathcal{G}K}\}.$$

For each point $x \in M$, we define $\widetilde{\mathcal{A}}_\psi(M)_x$ to be the set of ψ_x -compatible almost generalized complex structures, that is,

$$\widetilde{\mathcal{A}}_\psi(M)_x := \{\mathcal{J}_x | (\mathcal{J}_x, \mathcal{J}_{\psi,x}) : \text{almost generalized Kähler structure at } x\}.$$

Then we see that $\widetilde{\mathcal{A}}_\psi(M)_x$ is given by the Riemannian Symmetric space of type AIII[†]

$$U(n, n)/U(n) \times U(n)$$

[†]In Kähler geometry, the set of almost complex structures compatible with a symplectic structure ω is given by the Riemannian symmetric space $\mathrm{Sp}(2n)/U(n)$ which is biholomorphic to the Siegel upper half plane $\{h \in \mathrm{GL}_n(\mathbb{C}) \mid 1_n - h^*h > 0, h^t = h\}$.

which is biholomorphic to the complex bounded domain $\{h \in M_n(\mathbb{C}) \mid 1_n - h^*h > 0\}$, where $M_n(\mathbb{C})$ denotes the set of complex matrices of $n \times n$. Let P_ψ be the fibre bundle over M with fibre $\tilde{\mathcal{A}}_\psi(M)_x$, that is,

$$P_\psi := \bigcup_{x \in M} \tilde{\mathcal{A}}_\psi(M)_x \rightarrow M,$$

Then $\tilde{\mathcal{A}}_\psi(M)$ is given by sections $\Gamma(M, P_\psi)$ which contains $\mathcal{A}_\psi(M)$. We can introduce a Sobolev norm on $\tilde{\mathcal{A}}_\psi(M)$ such that $\tilde{\mathcal{A}}_\psi(M)$ becomes a Banach manifold in the usual way. The tangent bundle of $\tilde{\mathcal{A}}_\psi(M)$ at \mathcal{J} is given by

$$T_{\mathcal{J}}\tilde{\mathcal{A}}_\psi(M) = \{ \dot{\mathcal{J}} \in \text{so}(T_M \oplus T_M^*) : \dot{\mathcal{J}}\mathcal{J} + \mathcal{J}\dot{\mathcal{J}} = 0, \dot{\mathcal{J}}\mathcal{J}_\psi = \mathcal{J}_\psi\dot{\mathcal{J}} \},$$

where $\text{so}(T_M \oplus T_M^*)$ denotes the set of sections of Lie algebra bundle of $\text{SO}(T_M \oplus T_M^*)$. Then it follows that there exists an almost complex structure $J_{\tilde{\mathcal{A}}_\psi}$ on $\tilde{\mathcal{A}}_\psi(M)$ which is given by

$$J_{\tilde{\mathcal{A}}_\psi}(\dot{\mathcal{J}}) := \mathcal{J}\dot{\mathcal{J}}, \quad (\dot{\mathcal{J}} \in T_{\mathcal{J}}\tilde{\mathcal{A}}_\psi(M))$$

We also have a Riemannian metric $g_{\tilde{\mathcal{A}}_\psi}$ and a 2-form $\omega_{\tilde{\mathcal{A}}_\psi}$ on $\tilde{\mathcal{A}}_\psi(M)$ by

$$(6.1) \quad g_{\tilde{\mathcal{A}}_\psi}(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) := \frac{1}{(\sqrt{-1})^n} \int_M \text{tr}(\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \langle \psi, \bar{\psi} \rangle_S$$

$$(6.2) \quad \omega_{\tilde{\mathcal{A}}_\psi}(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) := \frac{-1}{(\sqrt{-1})^n} \int_M \text{tr}(\mathcal{J} \dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \langle \psi, \bar{\psi} \rangle_S$$

for $\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2 \in T_{\mathcal{J}}\tilde{\mathcal{A}}_\psi(M)$.

Proposition 6.1. $J_{\tilde{\mathcal{A}}_\psi}$ is integrable almost complex structure on $\tilde{\mathcal{A}}_\psi(M)$ and $\omega_{\tilde{\mathcal{A}}_\psi}$ is a Kähler form on $\tilde{\mathcal{A}}_\psi(M)$.

Proof. Let \mathcal{J}_V be an almost generalized complex structure on a real vector space V of dimension $2n$. We denote by X_n the Riemannian symmetric space $U(n, n)/U(n) \times U(n)$ which is identified with the set of almost generalized complex structures compatible with \mathcal{J}_V . We already see that $\tilde{\mathcal{A}}_\psi(M)$ is the set of global sections of the fibre bundle P_ψ over a manifold M with fibre X_n which is biholomorphic to the bounded domain $\{h \in M_n(\mathbb{C}) \mid 1_n - h^*h > 0\}$. If $\tilde{\mathcal{A}}_\psi(M)$ is not empty, we have a global section \mathcal{J}_0 . Then the fibre bundle is identified with the space of maps from M to the complex bounded domain $\{h \in M_n(\mathbb{C}) \mid 1_n - h^*h > 0\}$ which is open set in the complex vector

space $M_n(\mathbb{C})$. Since the almost complex structure $J_{\tilde{\mathcal{A}}_\psi}$ is induced from the one of the complex bounded domain, we see that $J_{\tilde{\mathcal{A}}_\psi}$ is integrable. X_n admits a Riemannian metric g_{x_n} and a 2-form ω_{x_n} which are given by

$$\begin{aligned} g_{x_n}(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) &= \text{tr}(\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \\ \omega_{x_n}(\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2) &= -\text{tr}(\mathcal{J} \dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2), \end{aligned}$$

where $\dot{\mathcal{J}}_1, \dot{\mathcal{J}}_2 \in T_{\mathcal{J}}X_n$. The complex bounded domain $\{h \in \text{GL}_n(\mathbb{C}) \mid 1_n - h^*h > 0\}$ admits a Kähler structure which is given by

$$4\sqrt{-1}\partial\bar{\partial} \log \det(1_n - h^*h).$$

Then under the identification $X_n \cong \{h \in M_n(\mathbb{C}) \mid 1_n - h^*h > 0\}$ by using \mathcal{J}_V , we have $\omega = 4\sqrt{-1}\partial\bar{\partial} \log \det(1_n - h^*h)$. Then the space of maps $\tilde{\mathcal{A}}_\psi(M)$ inherits a Riemannian metric and a Kähler structure which are given by

$$\begin{aligned} \omega_{\tilde{\mathcal{A}}_\psi} &= \frac{1}{(\sqrt{-1})^n} \int_M \text{tr}(\dot{\mathcal{J}}_1 \dot{\mathcal{J}}_2) \langle \psi, \bar{\psi} \rangle_S \\ &\quad + \frac{4}{(\sqrt{-1})^{n-1}} \partial\bar{\partial} \int_M \log \det(1_n - h^*h) \langle \psi, \bar{\psi} \rangle_S \end{aligned}$$

Hence $\omega_{\tilde{\mathcal{A}}_\psi}$ is closed. Thus $(\tilde{\mathcal{A}}_\psi(M), J_{\tilde{\mathcal{A}}_\psi}, \omega_{\tilde{\mathcal{A}}_\psi})$ is a Kähler manifold. □

Let $\widetilde{\text{Diff}}(M)$ be an extension of diffeomorphisms of M by 2-forms which is defined as

$$\widetilde{\text{Diff}}(M) := \{e^b F : F \in \text{Diff}(M), b : \text{2-form}\}.$$

Note that the product of $\widetilde{\text{Diff}}(M)$ is given by

$$(e^{b_1} F_1)(e^{b_2} F_2) := e^{b_1 + F_1^*(b_2)} F_1 \circ F_2,$$

where $F_1, F_2 \in \text{Diff}(M)$ and b_1, b_2 are real 2-forms. The action of $\widetilde{\text{Diff}}(M)$ on $\mathcal{GC}(M)$ by

$$e^b F_{\#} \circ \mathcal{J} \circ F_{\#}^{-1} e^{-b},$$

where $F \in \text{Diff}(M)$ acts on \mathcal{J} by $F_{\#} \circ \mathcal{J} \circ F_{\#}^{-1}$ and e^b is regarded as an element of $\text{SO}(T_M \oplus T_M^*)$ and $F_{\#}$ denotes the bundle map of $T_M \oplus T_M^*$ which is the lift of F . We define $\widetilde{\text{Diff}}(M)_{\psi}$ to be a subgroup consists of elements of $\widetilde{\text{Diff}}(M)$ which preserves ψ ,

$$\widetilde{\text{Diff}}_{\psi}(M) = \{e^b F \in \widetilde{\text{Diff}}(M) : e^b F^* \psi = \psi\}.$$

Then from (6.1), we have the following,

Proposition 6.2. *The symplectic structure $\omega_{\widetilde{\mathcal{A}}_\psi}$ is invariant under the action of ψ -preserving group $\widetilde{\text{Diff}}_\psi(M)$.*

We assume that type number of \mathcal{J}_ψ is 0, i.e., ψ is given by $\psi = e^{b+\sqrt{-1}\omega}$, where b is a real 2-form and ω denotes a symplectic form. We denote by $\text{Ham}_\omega(M)$ the Hamiltonian diffeomorphisms of (M, ω) .

Definition 6.3. By using the 2-form b , we define *generalized Hamiltonian diffeomorphisms* $\text{Ham}_\omega^b(M)$ by

$$\text{Ham}_\omega^b(M) := \{ e^b F e^{-b} \mid F \in \text{Ham}_\omega(M) \}.$$

Since $e^b F e^{-b} \psi = \psi$, we see that $\text{Ham}_\omega^b(M)$ is a subgroup of $\widetilde{\text{Diff}}_\psi(M)$. Thus $\text{Ham}_\omega^b(M)$ acts on $\widetilde{\mathcal{A}}_\psi(M)$ preserving the symplectic structure $\omega_{\widetilde{\mathcal{A}}_\psi}$. The Lie algebra of $\text{Ham}_\omega^b(M)$ is also given by $C_0^\infty(M)$, where $C_0^\infty(M) = \{ f \in C^\infty(M) \mid \int_M f \langle \psi, \bar{\psi} \rangle_s = 0 \}$. A Hamiltonian vector field v is given by $i_v \omega = df$ for $f \in C_0^\infty(M)$. Then $e := v - i_v b = \mathcal{J}_\psi(df) \in T_M \oplus T_M^*$ is called a *generalized Hamiltonian element*. Note that we have $e \cdot \psi = \sqrt{-1} df \cdot \psi$.

We denote by $\text{GR}_\mathcal{J}$ the generalized scalar curvature of $(\mathcal{J}, \mathcal{J}_\psi)$ for $\mathcal{J} \in \widetilde{\mathcal{A}}_\psi(M)$, where $\text{GR}_\mathcal{J}$ is a real function on M . The following is our main theorem:

Theorem 6.4. *There exists a moment map $\mu : \widetilde{\mathcal{A}}_\psi(M) \rightarrow C_0^\infty(M)^*$ for the generalized Hamiltonian action which is given by the generalized scalar curvature GR ,*

$$\langle \mu(\mathcal{J}), f \rangle = (\sqrt{-1})^{-n} \int_M (\text{GR}_\mathcal{J}) f \langle \psi, \bar{\psi} \rangle_s,$$

where $f \in C_0^\infty(M)$ and $\langle \mu(\mathcal{J}), f \rangle$ denotes the coupling between $\mu(\mathcal{J})$ and f .

Our proof of Theorem 6.4 will be given in Section 8.

7. Preliminary results for proof of the main theorem

In order to show our main theorem, we shall rewrite the symplectic form $\omega_{\widetilde{\mathcal{A}}_\psi}$ by using the Clifford algebra and the pure spinors ϕ_α and ψ . Such descriptions in terms of the Clifford algebra and pure spinors are suitable to obtain our main theorem by using Stokes' theorem. Let \mathcal{J} be an almost generalized complex structure which is compatible with ψ . We denote by $\{ \phi_\alpha \}$ trivializations of the canonical line bundle $K_\mathcal{J}$, where each ϕ_α is a nondegenerate, pure spinor on U_α which induces the generalized complex

structure \mathcal{J} . Arbitrary small deformations of almost generalized complex structures of \mathcal{J} are given by the adjoint action,

$$e^{h(t)} \circ \mathcal{J} \circ e^{-h(t)},$$

where $h(t) = h^{2,0}(t) + h^{0,2}(t)$ denotes a real section depending smoothly on a parameter t which satisfies $h^{2,0}(t) \in \wedge^2 E_{\mathcal{J}}$ and $h^{0,2}(t) = \overline{h^{2,0}(t)} \in \wedge^2 \overline{E}_{\mathcal{J}}$. Then the infinitesimal deformation $\dot{\mathcal{J}}_h$ is given by

$$\dot{\mathcal{J}}_h := \frac{d}{dt} e^{h(t)} \circ \mathcal{J} \circ e^{-h(t)}|_{t=0} = [h, \mathcal{J}],$$

where h and \mathcal{J} are regarded as elements of the Clifford algebra $\text{CL}(T_M \oplus T_M^*)$ and $[h, \mathcal{J}]$ denotes the commutator of h and \mathcal{J} which is identified with the bracket of Lie algebra $\text{so}(T_M \oplus T_M^*)$. The real element $h \in \wedge^2 E_{\mathcal{J}} \oplus \wedge^2 \overline{E}_{\mathcal{J}} \subset \text{CL}(T_M \oplus T_M^*)$ acts on nondegenerate pure spinors ϕ_{α} on U_{α} by $\dot{\phi}_{\alpha} := h \cdot \phi_{\alpha}$. Let \mathcal{J}_1 and \mathcal{J}_2 are two almost generalized complex structures which are locally induced from $\{\phi_{\alpha,1}\}$ and $\{\phi_{\alpha,2}\}$ respectively. Two real elements h_1 and h_2 give rise to infinitesimal deformations $\dot{\mathcal{J}}_{h_1}$ of \mathcal{J}_1 and $\dot{\mathcal{J}}_{h_2}$ of \mathcal{J}_2 , respectively. We also denote by $\dot{\phi}_{\alpha, h_i}$ an element $h_i \cdot \phi_{\alpha, i}$ for $i = 1, 2$.

Then the symplectic form $\omega_{\tilde{\mathcal{A}}_{\psi}}$ as in (6.1) is given by

$$\omega_{\tilde{\mathcal{A}}_{\psi}}(\dot{\mathcal{J}}_{h_1}, \dot{\mathcal{J}}_{h_2}) = \frac{-1}{(\sqrt{-1})^n} \int_M \text{tr} \mathcal{J} \dot{\mathcal{J}}_{h_1} \dot{\mathcal{J}}_{h_2} \langle \psi, \bar{\psi} \rangle_s,$$

where h_1, h_2 are real elements of $\wedge^2 E_{\mathcal{J}} \oplus \wedge^2 \overline{E}_{\mathcal{J}}$. We shall begin to write the symplectic form $\omega_{\tilde{\mathcal{A}}_{\psi}}$ in terms of pure spinors.

Lemma 7.1.

$$\text{tr} \mathcal{J} \dot{\mathcal{J}}_{h_1} \dot{\mathcal{J}}_{h_2} \langle \psi, \bar{\psi} \rangle_s = -\frac{\sqrt{-1}}{2} \{ \rho_{\alpha}^{-1} \langle \dot{\phi}_{\alpha, h_1}, \bar{\phi}_{\alpha, h_2} \rangle_s - \rho_{\alpha}^{-1} \langle \dot{\phi}_{\alpha, h_2}, \bar{\phi}_{\alpha, h_1} \rangle_s \}$$

Proof. The formula is shown by a local calculation. Let $\{e_i\}_{i=1}^{2n}$ be a local basis of $E_{\mathcal{J}}$ such that $\langle e_i, \bar{e}_j \rangle_{T \oplus T^*} = \delta_{i,j}$. Then the basis of $\{\bar{e}_i\}$ of $\overline{E}_{\mathcal{J}}$ is regarded as the dual basis of $E_{\mathcal{J}}^*$. A real element $h \in \wedge^2 E_{\mathcal{J}} \oplus \wedge^2 \overline{E}_{\mathcal{J}}$ is written as $h = \sum_{i,j} h_{i,j} e_i \wedge e_j + \bar{h}_{i,j} \bar{e}_i \wedge \bar{e}_j$ and $\dot{\mathcal{J}}_h = [h, \mathcal{J}]$ is given by

$$\dot{\mathcal{J}}_h = h\mathcal{J} - \mathcal{J}h = \sqrt{-1} \sum_{i,j} h_{i,j} e_i \wedge e_j - \bar{h}_{i,j} \bar{e}_i \wedge \bar{e}_j$$

Thus we have

$$\mathrm{tr} \mathcal{J} \dot{\mathcal{J}}_{h_1} \dot{\mathcal{J}}_{h_2} = 4\sqrt{-1} \sum_{i,j} (\bar{h}_{1,ij} h_{2,ji} - h_{1,ij} \bar{h}_{2,ji}),$$

where $\dot{\mathcal{J}}_h$ acts on $T_M \oplus T_M^*$ by the adjoint $[\dot{\mathcal{J}}_h, \cdot]$. By using the formula $\langle e \cdot \phi_\alpha, \bar{\phi}_\alpha \rangle_s = -\langle \phi_\alpha, e \cdot \bar{\phi}_\alpha \rangle_s$, we have

$$\langle \bar{e}_i \cdot \bar{e}_j \cdot \phi_\alpha, e_k \cdot e_l \cdot \bar{\phi}_\alpha \rangle_s = \langle e_l \cdot e_k \cdot \bar{e}_i \cdot \bar{e}_j \cdot \phi_\alpha, \bar{\phi}_\alpha \rangle_s.$$

Applying $e_k \cdot \bar{e}_i + \bar{e}_i \cdot e_k = -2\langle e_k, \bar{e}_i \rangle_{T \oplus T^*}$, we have

$$\langle \bar{e}_i \cdot \bar{e}_j \cdot \phi_\alpha, e_k \cdot e_l \cdot \bar{\phi}_\alpha \rangle_s = -4(\delta_{kj} \delta_{li} - \delta_{ki} \delta_{lj}) \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s.$$

Thus we obtain

$$\begin{aligned} \langle \dot{\phi}_{\alpha, h_1}, \bar{\phi}_{\alpha, h_2} \rangle_s &= \sum_{i,j,k,l} \bar{h}_{1,ij} h_{2,kl} \langle \bar{e}_i \cdot \bar{e}_j \cdot \phi_\alpha, e_k \cdot e_l \cdot \bar{\phi}_\alpha \rangle_s \\ &= -8 \sum_{i,j} \bar{h}_{1,ij} h_{2,ji} \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s \end{aligned}$$

We also have

$$\langle \dot{\phi}_{\alpha, h_2}, \bar{\phi}_{\alpha, h_1} \rangle_s = -8 \sum_{i,j} h_{1,ij} \bar{h}_{2,ji} \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s$$

Applying $\langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s = \rho_\alpha \langle \psi, \bar{\psi} \rangle_s$, we have

$$\mathrm{tr} \mathcal{J} \dot{\mathcal{J}}_{h_1} \dot{\mathcal{J}}_{h_2} \langle \psi, \bar{\psi} \rangle_s = -\frac{\sqrt{-1}}{2} \{ \rho_\alpha^{-1} \langle \dot{\phi}_{\alpha, h_1}, \bar{\phi}_{\alpha, h_2} \rangle_s - \rho_\alpha^{-1} \langle \dot{\phi}_{\alpha, h_2}, \bar{\phi}_{\alpha, h_1} \rangle_s \}$$

□

Proposition 7.2. *The symplectic form $\omega_{\tilde{\mathcal{A}}_\psi}$ is given by*

$$c_n^{-1} \omega_{\tilde{\mathcal{A}}_\psi} (\dot{\mathcal{J}}_{h_1}, \dot{\mathcal{J}}_{h_2}) = \int_M \rho_\alpha^{-1} \langle \dot{\phi}_{\alpha, h_1}, \bar{\phi}_{\alpha, h_2} \rangle_s - \int_M \rho_\alpha^{-1} \langle \dot{\phi}_{\alpha, h_2}, \bar{\phi}_{\alpha, h_1} \rangle_s$$

where $c_n = \frac{1}{2(\sqrt{-1})^{n-1}}$ and ρ_α is the function as in (5.2) and $\dot{\phi}_{\alpha, h_i} = h \cdot \phi_{\alpha, i}$ for $i = 1, 2$.

Proof. [**Proposition 7.2**] The result directly follows from Lemma 7.1 . □

Remark 7.3. Since $\rho_\alpha^{-1}\langle\dot{\phi}_{\alpha,h_1}, \overline{\dot{\phi}_{\alpha,h_2}}\rangle_s = \rho_\beta^{-1}\langle\dot{\phi}_{\beta,h_1}, \overline{\dot{\phi}_{\beta,h_2}}\rangle_s$ for α , and β , the $2n$ -form $\rho_\alpha^{-1}\langle\dot{\phi}_{\alpha,h_1}, \overline{\dot{\phi}_{\alpha,h_2}}\rangle_s$ gives a globally defined $2n$ -form on M .

Note that $\omega_{\mathcal{A}}$ is also written as

$$(7.1) \quad c_n^{-1}\omega_{\mathcal{A}}(\mathcal{J}_{h_1}, \mathcal{J}_{h_2}) = \int_M h_1 \cdot \phi_\alpha \wedge \sigma(\overline{h_2} \cdot \overline{\phi_\alpha}) \rho_\alpha^{-1} \\ - \int_M h_2 \cdot \phi_\alpha \wedge \sigma(\overline{h_1} \cdot \overline{\phi_\alpha}) \rho_\alpha^{-1}$$

Lemma 7.4. *We have the following identity with respect to σ and d for a differential form ω*

$$d\sigma\omega = \begin{cases} +\sigma d\omega & (\deg \omega = \text{even}) \\ -\sigma d\omega & (\deg \omega = \text{odd}) \end{cases}$$

Proof. $\sigma\omega$ is given by

$$\sigma\omega = \begin{cases} +\omega, & (\deg \omega \equiv 0, 1 \pmod{4}) \\ -\omega, & (\deg \omega \equiv 2, 3 \pmod{4}) \end{cases}$$

Then the result follows. □

Lemma 7.5. *For $e_1, e_2 \in T_M \oplus T_M^*$ and differential forms $\omega_1, \omega_2 \in \wedge^\bullet T_M^*$, we have*

$$\langle e_1 \cdot \omega_1, e_2 \cdot \omega_2 \rangle_s + \langle e_2 \cdot \omega_1, e_1 \cdot \omega_2 \rangle_s = 2\langle e_1, e_2 \rangle_{T \oplus T^*} \langle \omega_1, \omega_2 \rangle_s$$

Proof. For $e \in T_M \oplus T_M^*$ and $\omega_1, \omega_2 \in \wedge^\bullet T_M^*$, we have

$$\langle e \cdot \omega_1, \omega_2 \rangle_s + \langle \omega_1, e \cdot \omega_2 \rangle_s = 0.$$

Then we have

$$(7.2) \quad \langle e_1 \cdot \omega_1, e_2 \cdot \omega_2 \rangle_s + \langle e_2 \cdot \omega_1, e_1 \cdot \omega_2 \rangle_s \\ = -\langle e_2 \cdot e_1 \cdot \omega_1, \omega_2 \rangle_s - \langle e_1 \cdot e_2 \cdot \omega_1, \omega_2 \rangle_s$$

$$(7.3) \quad = -\langle (e_2 \cdot e_1 + e_1 \cdot e_2) \cdot \omega_1, \omega_2 \rangle_s$$

$$(7.4) \quad = 2\langle e_1, e_2 \rangle_{T \oplus T^*} \langle \omega_1, \omega_2 \rangle_s$$

□

Lemma 7.6. *Let $\theta = \theta^{1,0} + \theta^{0,1}$ be a 1-form, where $\theta^{1,0} \in E_{\mathcal{J}}$ and $\theta^{0,1} \in \overline{E_{\mathcal{J}}}$. For $h = h^{2,0} + h^{0,2} \in \wedge^2 E_{\mathcal{J}} \oplus \wedge^2 \overline{E_{\mathcal{J}}}$, we have the following:*

$$[[h, \mathcal{J}], \theta] \cdot \bar{\psi} = 2\sqrt{-1} ([h^{2,0}, \theta^{0,1}] - [h^{0,2}, \theta^{1,0}]) \cdot \bar{\psi}$$

Proof. Let ω be a differential form satisfying $\mathcal{J}\omega = \sqrt{-1}k\omega$ for $-n \leq k \leq n$. Then we have

$$\begin{aligned} [h, \mathcal{J}]\omega &= h\mathcal{J}\omega - \mathcal{J}h\omega \\ &= \sqrt{-1}k(h^{2,0} + h^{0,2})\omega - \sqrt{-1}(k-2)h^{2,0}\omega - \sqrt{-1}(k+2)h^{0,2}\omega \\ &= 2\sqrt{-1}(h^{2,0} - h^{0,2})\omega \end{aligned}$$

Since $[h^{2,0}, \theta^{1,0}] = 0$, we have

$$\begin{aligned} [[h, \mathcal{J}], \theta] \cdot \bar{\psi} &= 2\sqrt{-1}[(h^{2,0} - h^{0,2}), \theta] \cdot \bar{\psi} \\ &= 2\sqrt{-1}([h^{2,0}, \theta^{0,1}] - [h^{0,2}, \theta^{1,0}]) \cdot \bar{\psi} \end{aligned}$$

□

Remark 7.7. If an infinitesimal deformation $[h, \mathcal{J}]$ preserves ψ , then it follows $[h, \mathcal{J}] \cdot \psi = 0$. We shall consider an infinitesimal deformation $[h, \mathcal{J}]$ preserving ψ . Since $\mathcal{J} \cdot \psi = 0$ and $[h, \mathcal{J}] \cdot \psi = 0$, then it follows that $[[h, \mathcal{J}], \theta] \cdot \psi = [h, \mathcal{J}]\theta \cdot \psi$.

Lemma 7.8. *For real elements $e, \theta \in T_M \oplus T_M^*$ and $h = h^{2,0} + h^{0,2} \in \wedge^2 E_{\mathcal{J}} \oplus \wedge^2 \overline{E_{\mathcal{J}}}$, we have*

$$(7.5) \quad \begin{aligned} &2\langle e \cdot \phi_{\alpha}, \theta \cdot h \cdot \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} - 2\langle \theta \cdot h \cdot \phi_{\alpha}, e \cdot \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} \\ &= \sqrt{-1}\langle e \cdot \psi, [[h, \mathcal{J}], \theta] \cdot \bar{\psi} \rangle_s + \sqrt{-1}\langle [[h, \mathcal{J}], \theta] \cdot \psi, e \cdot \bar{\psi} \rangle_s \end{aligned}$$

Proof. Since $h^{2,0} \cdot \phi_{\alpha} = 0$ and $[h^{2,0}, \theta^{1,0}] = 0$ and $\theta^{0,1} \cdot \bar{\phi}_{\alpha} = 0$, the left hand side of (7.5) is given by

$$\begin{aligned} (\text{L.H.S}) &= 2\langle e \cdot \phi_{\alpha}, \theta \cdot h^{2,0} \cdot \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} - 2\langle \theta \cdot h^{0,2} \cdot \phi_{\alpha}, e \cdot \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} \\ &= 2\langle e \cdot \phi_{\alpha}, \theta^{0,1} \cdot h^{2,0} \cdot \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} - 2\langle \theta^{1,0} \cdot h^{0,2} \cdot \phi_{\alpha}, e \cdot \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} \\ &= -2\langle e \cdot \phi_{\alpha}, [h^{2,0}, \theta] \cdot \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} + 2\langle [h^{0,2}, \theta] \cdot \phi_{\alpha}, e \cdot \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} \end{aligned}$$

By applying Lemma 7.5 and $[h^{2,0}, \theta] \cdot \phi_{\alpha} = 0$, we have

$$\begin{aligned} (\text{L.H.S}) &= -4\langle e, [h^{2,0}, \theta] \rangle_{T \oplus T^*} \langle \phi_{\alpha}, \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} \\ &\quad + 4\langle e, [h^{0,2}, \theta] \rangle_{T \oplus T^*} \langle \phi_{\alpha}, \bar{\phi}_{\alpha} \rangle_s \rho_{\alpha}^{-1} \end{aligned}$$

It follows from $\langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s = \rho_\alpha \langle \psi, \bar{\psi} \rangle_s$ and Lemma 7.5 that we have

$$\begin{aligned} (\text{L.H.S}) &= -2\langle e \cdot \psi, [h^{2,0}, \theta] \cdot \bar{\psi} \rangle_s - 2\langle [h^{2,0}, \theta] \cdot \psi, e \cdot \bar{\psi} \rangle_s \\ &\quad + 2\langle [h^{0,2}, \theta] \cdot \psi, e \cdot \bar{\psi} \rangle_s + 2\langle e \cdot \psi, [h^{0,2}, \theta] \cdot \bar{\psi} \rangle_s \end{aligned}$$

From $[h^{2,0}, \mathcal{J}] = 2\sqrt{-1}h^{2,0}$, $[h^{0,2}, \mathcal{J}] = -2\sqrt{-1}h^{0,2}$, we have

$$\begin{aligned} (\text{L.H.S}) &= \sqrt{-1}\langle e \cdot \psi, [[h^{2,0}, \mathcal{J}], \theta] \cdot \bar{\psi} \rangle_s + \sqrt{-1}\langle [[h^{2,0}, \mathcal{J}], \theta] \cdot \psi, e \cdot \bar{\psi} \rangle_s \\ &\quad + \sqrt{-1}\langle [[h^{0,2}, \mathcal{J}], \theta] \cdot \psi, e \cdot \bar{\psi} \rangle_s + \sqrt{-1}\langle e \cdot \psi, [h^{0,2}, \mathcal{J}], \theta] \cdot \bar{\psi} \rangle_s \\ &= \sqrt{-1}\langle e \cdot \psi, [[h, \mathcal{J}], \theta] \cdot \bar{\psi} \rangle_s + \sqrt{-1}\langle [[h, \mathcal{J}], \theta] \cdot \psi, e \cdot \bar{\psi} \rangle_s \end{aligned}$$

Thus we have the result. \square

Lemma 7.9. *Let $N = N^{3,0} + N^{0,3}$ be a real section of $\wedge^3 E_{\mathcal{J}} \oplus \wedge^3 \bar{E}_{\mathcal{J}}$, where $N^{3,0} \in \wedge^3 E_{\mathcal{J}}$ and $N^{0,3} = \overline{N^{3,0}} \in \wedge^3 \bar{E}_{\mathcal{J}}$. Then we have*

$$\langle e \cdot \phi_\alpha, N \cdot h \cdot \bar{\phi}_\alpha \rangle_s = \langle N \cdot h \cdot \phi_\alpha, e \cdot \bar{\phi}_\alpha \rangle_s = 0$$

Proof. Let $U_{\mathcal{J}}^k$ be the eigenspace of an eigenvalue $\sqrt{-1}k$ with respect to the action of \mathcal{J} . Then $e \cdot \phi_\alpha \in U_{\mathcal{J}}^{-n+1}$. Since $h^{0,2} \cdot \bar{\phi}_\alpha = 0$, we have

$$\begin{aligned} N \cdot h \cdot \bar{\phi}_\alpha &= (N^{3,0} \cdot h^{2,0} + N^{3,0} \cdot h^{0,2} + N^{0,3} \cdot h^{2,0} + N^{0,3} \cdot h^{0,2}) \cdot \bar{\phi}_\alpha \\ &= N^{3,0} \cdot h^{2,0} \cdot \bar{\phi}_\alpha + N^{0,3} \cdot h^{2,0} \cdot \bar{\phi}_\alpha \end{aligned}$$

Then we have $N^{3,0} \cdot h^{2,0} \cdot \bar{\phi}_\alpha \in U_{\mathcal{J}}^{n-5}$. Since $U_{\mathcal{J}}^{n+1} = \{0\}$, $N^{0,3} \cdot h^{2,0} \cdot \bar{\phi}_\alpha = 0 \in U_{\mathcal{J}}^{n+1}$. Thus we have $\langle e \cdot \phi_\alpha, N \cdot h \cdot \bar{\phi}_\alpha \rangle_s = 0$. Then it follows $\langle N \cdot h \cdot \phi_\alpha, e \cdot \bar{\phi}_\alpha \rangle_s = 0$. \square

Lemma 7.10. *Let N be as in before. Then we have*

$$N \cdot \psi = 0$$

Proof. Let $\{e_i\}$ be a local basis of $E_{\mathcal{J}}$. Since $d\phi_\alpha = \eta_\alpha \cdot \phi_\alpha + N \cdot \phi_\alpha$, then we have

$$\begin{aligned} \langle N \cdot \phi_\alpha, e_i \cdot e_j \cdot e_k \cdot \bar{\phi}_\alpha \rangle_s &= \langle d\phi_\alpha, e_i \cdot e_j \cdot e_k \cdot \bar{\phi}_\alpha \rangle_s \\ &= -\langle e_i \cdot e_j \cdot d\phi_\alpha, e_k \cdot \bar{\phi}_\alpha \rangle_s \\ &= \langle [e_i, e_j]_{\text{cou}} \cdot \phi_\alpha, e_k \cdot \bar{\phi}_\alpha \rangle_s \end{aligned}$$

Thus the component $N_{i,j,k}^{3,0} := N(e_i, e_j, e_k)$ is given by

$$N_{i,j,k}^{3,0} = \langle [e_i, e_j]_{\text{cou}}, e_k \rangle_{T \oplus T^*}.$$

Each \bar{e}_i is decomposed into $\bar{e}_i = \bar{e}_i^+ + \bar{e}_i^-$, where $\bar{e}_i^\pm \in \bar{E}_{\mathcal{J}}^\pm$. From $\bar{e}_i^- \cdot \psi = 0$ and $e_i^+ \cdot \psi = 0$, it suffices to show that $N(\bar{e}_i^+, \bar{e}_j^+, \bar{e}_k^+) \bar{e}_i^+ \cdot \bar{e}_j^+ \cdot \bar{e}_k^+ \cdot \psi = 0$ and $N(\bar{e}_i^-, \bar{e}_j^-, \bar{e}_k^-) \bar{e}_i^- \cdot \bar{e}_j^- \cdot \bar{e}_k^- \cdot \psi = 0$. Since \mathcal{J}_ψ is integrable, it follows from $[\bar{e}_i^+, \bar{e}_j^+]_{\text{cou}} \in \bar{L}_\psi$. From $\bar{e}_k^+ \in \bar{L}_\psi$, we have

$$N(\bar{e}_i^+, \bar{e}_j^+, \bar{e}_k^+) = \langle [\bar{e}_i^+, \bar{e}_j^+]_{\text{cou}}, \bar{e}_k^+ \rangle_{T \oplus T^*} = 0$$

Thus $N(\bar{e}_i^+, \bar{e}_j^+, \bar{e}_k^+) \cdot \psi = 0$. We also have $N(e_i^-, e_j^-, e_k^-) = 0$. Hence $N \cdot \psi = 0$. \square

Lemma 7.11. *If $[h, \mathcal{J}] \cdot \psi = 0$, then we have*

$$\langle e \cdot \psi, [[h, \mathcal{J}], N] \cdot \bar{\psi} \rangle_s = \langle [[h, \mathcal{J}], N] \cdot \psi, e \cdot \bar{\psi} \rangle_s = 0$$

Proof. Since $[h, \mathcal{J}] \cdot \psi = 0$, it follows from Lemma 7.10 that we have $[[h, \mathcal{J}], N] \cdot \psi = 0$. Thus we have the result. \square

Lemma 7.12. *If $[h, \mathcal{J}] \cdot \psi = 0$, then we have*

$$\langle e \cdot N \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s = 0$$

Proof. Since $h = h^{2,0} + h^{0,2} \in \wedge^2 E_{\mathcal{J}} \oplus \wedge^2 \bar{E}_{\mathcal{J}}$ and $N = N^{3,0} + N^{0,3} \in \wedge^3 E_{\mathcal{J}} \oplus \wedge^3 \bar{E}_{\mathcal{J}}$, we have

$$\begin{aligned} \langle e \cdot N \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s &= -\langle N \cdot \phi_\alpha, e \cdot h \cdot \bar{\phi}_\alpha \rangle_s \\ &= -\langle N^{0,3} \cdot \phi_\alpha, e^{1,0} \cdot h^{2,0} \cdot \bar{\phi}_\alpha \rangle_s \end{aligned}$$

Since $e^{1,0} \cdot h^{2,0} = h^{2,0} \cdot e^{1,0}$, we have

$$\begin{aligned} \langle e \cdot N \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s &= -\langle N^{0,3} \cdot \phi_\alpha, h^{2,0} \cdot e^{1,0} \cdot \bar{\phi}_\alpha \rangle_s \\ &= \langle h^{2,0} \cdot N^{0,3} \cdot \phi_\alpha, e^{1,0} \cdot \bar{\phi}_\alpha \rangle_s \end{aligned}$$

We denote by $[h^{2,0}, N^{0,3}]^{0,1} \in \overline{E}_{\mathcal{J}}$ the component of $[h^{2,0}, N^{0,3}]$. Then applying Lemma 7.5, we obtain

$$\begin{aligned} \langle h^{2,0} \cdot N^{0,3} \cdot \phi_\alpha, e^{1,0} \cdot \overline{\phi}_\alpha \rangle_s &= \langle [h^{2,0} \cdot N^{0,3}]^{0,1} \cdot \phi_\alpha, e^{1,0} \cdot \overline{\phi}_\alpha \rangle_s \\ &= 2 \langle [h^{2,0}, N^{0,3}]^{0,1}, e^{1,0} \rangle_{T \oplus T^*} \langle \phi_\alpha, \overline{\phi}_\alpha \rangle_s \\ &= 2 \langle [h^{2,0}, N^{0,3}]^{0,1}, e^{1,0} \rangle_{T \oplus T^*} \langle \psi, \overline{\psi} \rangle_s \rho_\alpha \\ &= \langle [h^{2,0}, N^{0,3}]^{0,1} \cdot \psi, e^{1,0} \cdot \overline{\psi} \rangle_s \rho_\alpha \\ &\quad - \langle e^{1,0} \psi, [h^{2,0}, N^{0,3}]^{0,1} \cdot \overline{\psi} \rangle_s \rho_\alpha \end{aligned}$$

From Lemma 7.10 and $h \cdot \psi = 0$, we have $[h, N] \cdot \psi = 0$. Thus we have $[h^{2,0}, N^{0,3}]^{0,1} \cdot \overline{\psi} = 0$ and $[h^{2,0}, N^{0,3}]^{0,1} \cdot \psi = 0$. Hence we obtain $\langle e \cdot N \cdot \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s = 0$. \square

8. Proof of main theorem

This section is devoted to show our main theorem: Theorem 6.4. In order to show the main theorem, it suffices to show that

$$\frac{d}{dt} \langle \mu(\mathcal{J}_t), f \rangle|_{t=0} = \omega_{\tilde{\mathcal{A}}_\psi} (L_e \mathcal{J}, \dot{\mathcal{J}}_h),$$

where f is a generalized Hamiltonian and e is a generalized Hamiltonian element satisfying $e \cdot \psi = \sqrt{-1} df \cdot \psi$ and \mathcal{J}_t denotes deformations of \mathcal{J} which satisfies $\dot{\mathcal{J}}_h = [h, \mathcal{J}]$. A generalized Hamiltonian f gives a generalized Hamiltonian element $e \in T_M \oplus T_M^*$ by $e = \mathcal{J}_\psi df$.

Let $\{(\phi_\alpha, U_\alpha)\}$ be trivialisations of the canonical line bundle $K_{\mathcal{J}}$, where $\{U_\alpha\}$ is a finite open cover of a compact manifold M of dimension $2n$. We denote by $\{\chi_\alpha\}$ a partition of unity such that the support of χ_α is contained in U_α . From Proposition 7.2, it suffices to show the following:

$$e_n^{-1} \frac{d}{dt} \langle \mu(\mathcal{J}_t), f \rangle|_{t=0} = \int_M \langle L_e \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1} - \int_M \langle h \cdot \phi_\alpha, L_e \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1}.$$

By using the partition of unity, f is given by $f = \sum_\alpha f_\alpha$, where $f_\alpha = \chi_\alpha f$ and a generalized Hamiltonian element $e \in T_M \oplus T_M^*$ is also written as $e = \sum_\alpha e_\alpha$, where $e_\alpha = \mathcal{J}_\psi df_\alpha$.

Lemma 8.1. *If $U_\alpha \cap U_\beta \neq \emptyset$, we have*

$$\langle L_e \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1} = \langle L_e \phi_\beta, h \cdot \overline{\phi}_\beta \rangle_s \rho_\beta^{-1}$$

Proof. Since $\phi_\alpha = e^{\kappa_{\alpha,\beta}} \phi_\beta$, the Lie derivative $L_e \phi_\alpha := de \cdot \phi_\alpha + e \cdot d\phi_\alpha$ is given by

$$\begin{aligned} L_e \phi_\alpha &= e^{\kappa_{\alpha,\beta}} L_e \phi_\beta + (e \cdot de^{\kappa_{\alpha,\beta}} + de^{\kappa_{\alpha,\beta}} \cdot e) \cdot \phi_\beta \\ &= e^{\kappa_{\alpha,\beta}} L_e \phi_\beta - 2\langle e, de^{\kappa_{\alpha,\beta}} \rangle_{T \oplus T^*} \phi_\beta \end{aligned}$$

Since $h \in \wedge^2 E_{\mathcal{J}} \oplus \wedge^2 \overline{E}_{\mathcal{J}}$, we have $\langle e, de^{\kappa_{\alpha,\beta}} \rangle_{T \oplus T^*} \langle \phi_\beta, h \cdot \overline{\phi}_\alpha \rangle_s = 0$. Since $\overline{\phi}_\alpha = e^{\overline{\kappa}_{\alpha,\beta}} \overline{\phi}_\beta$ and $\rho_\alpha = e^{\kappa_{\alpha,\beta} + \overline{\kappa}_{\alpha,\beta}} \rho_\beta$, we have the result. \square

Thus there is a $2n$ -form $F_1(e)$ such that $F_1(e)|_{U_\alpha} = \langle L_e \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1}$. Since $e = \sum_\alpha e_\alpha$, it follows that $F_1(e) = \sum_\alpha F_1(e_\alpha)$. Since the support e_α is contained in U_α , we have

$$F_1(e_\alpha) = \langle L_{e_\alpha} \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1}.$$

Applying Stokes' theorem and Lemma 7.4, we have

$$\int_M \langle de_\alpha \cdot \phi_\alpha, \rho_\alpha^{-1} h \cdot \overline{\phi}_\alpha \rangle_s = \int_M \langle e_\alpha \cdot \phi_\alpha, d(\rho_\alpha^{-1} h \cdot \overline{\phi}_\alpha) \rangle_s$$

Thus we have

$$\begin{aligned} \int_M F_1(e_\alpha) &= \int_M \langle L_{e_\alpha} \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ &= \int_M \langle de_\alpha \cdot \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1} + \int_M \langle e_\alpha \cdot d\phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ &= \int_M \langle e_\alpha \cdot \phi_\alpha, dh \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1} + \int_M \langle e_\alpha \cdot \phi_\alpha, (d\rho_\alpha^{-1}) \cdot h \cdot \overline{\phi}_\alpha \rangle_s \\ &\quad + \int_M \langle e_\alpha \cdot (\eta_\alpha + N_\alpha) \cdot \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \end{aligned}$$

We define F_{1-1}, F_{1-2} and F_{1-3} by

$$\begin{aligned} F_{1-1} &= \langle e_\alpha \cdot \phi_\alpha, dh \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ F_{1-2} &= \langle e_\alpha \cdot \phi_\alpha, (d\rho_\alpha^{-1}) \cdot h \cdot \overline{\phi}_\alpha \rangle_s \\ F_{1-3} &= \langle e_\alpha \cdot (\eta_\alpha + N_\alpha) \cdot \phi_\alpha, h \cdot \overline{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \end{aligned}$$

We denote by $F_2(e_\alpha)$ the $2n$ -form $\langle h \cdot \phi_\alpha, L_{e_\alpha} \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1}$. Applying Stokes' theorem again, we have

$$\begin{aligned} \int_M F_2(e_\alpha) &= \int_M h \cdot \phi_\alpha \wedge \sigma(L_{e_\alpha} \bar{\phi}_\alpha) \rho_\alpha^{-1} \\ &= \int_M \langle h \cdot \phi_\alpha, de_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} + \int_M \langle h \cdot \phi_\alpha, e_\alpha \cdot d\bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ &= \int_M \langle dh \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} + \int_M \langle (d\rho_\alpha^{-1}) \cdot h \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \\ &\quad + \int_M \langle h \cdot \phi_\alpha, e_\alpha \cdot (\eta_\alpha + N) \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \end{aligned}$$

We also define F_{2-1}, F_{2-2} and F_{2-3} by

$$\begin{aligned} F_{2-1} &= \langle dh \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ F_{2-2} &= \langle (d\rho_\alpha^{-1}) \cdot h \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \\ F_{2-3} &= \langle h \cdot \phi_\alpha, e_\alpha \cdot (\eta_\alpha + N) \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \end{aligned}$$

$F_1(e_\alpha) - F_2(e_\alpha)$ is divided into the following three parts

$$\begin{aligned} F_{1-1} - F_{2-1} &= \langle e_\alpha \cdot \phi_\alpha, dh \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} - \langle dh \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ F_{1-2} - F_{2-2} &= \langle e_\alpha \cdot \phi_\alpha, (d\rho_\alpha^{-1}) \cdot h \cdot \bar{\phi}_\alpha \rangle_s - \langle (d\rho_\alpha^{-1}) \cdot h \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \\ F_{1-3} - F_{2-3} &= \langle e_\alpha \cdot (\eta_\alpha + N_\alpha) \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ &\quad - \langle h \cdot \phi_\alpha, e_\alpha \cdot (\eta_\alpha + N_\alpha) \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \end{aligned}$$

Deformations of almost generalized complex structures $\{\mathcal{J}_t\}$ are given by the action of one parameter family e^{ht} in Spin group which are induced from nondegenerate, pure spinors $e^{ht} \cdot \phi_\alpha$ and we have

$$(8.1) \quad de^{ht} \cdot \phi_\alpha = (\eta_\alpha(t) + N_\alpha(t)) \cdot e^{ht} \phi_\alpha,$$

where $\eta_\alpha(t) \in T_M \oplus T_M^*$ and $N_\alpha(t) \in \wedge^3(T_M \oplus T_M^*)$ are real sections satisfying $\eta_\alpha(0) = \eta_\alpha$ and $N_\alpha(0) = N_\alpha$. Taking the derivative of both sides of (8.1) with respect to t , we have

$$dh \cdot \phi_\alpha = (\dot{\eta}_\alpha + \dot{N}_\alpha) \cdot \phi_\alpha + (\eta_\alpha + N_\alpha) \cdot h \cdot \phi_\alpha,$$

where $\dot{\eta}_\alpha = \frac{d}{dt} \eta_\alpha(t)|_{t=0}$ and $\dot{N}_\alpha = \frac{d}{dt} N_\alpha(t)|_{t=0}$. Since the real section $\dot{\eta}_\alpha$ is decomposed into $\dot{\eta}_\alpha^{1,0} + \dot{\eta}_\alpha^{0,1}$, where $\dot{\eta}_\alpha^{1,0} \in E_{\mathcal{J}}$ and $\dot{\eta}_\alpha^{0,1} \in \bar{E}_{\mathcal{J}}$ and \dot{N} is also decomposed into $\sum_{p+q=3} \dot{N}^{p,q}$, where $\dot{N}^{p,q} \in \wedge^p E_{\mathcal{J}} \oplus \wedge^q \bar{E}_{\mathcal{J}}$. Note that \dot{N} is not contained in $\wedge^3 E_{\mathcal{J}} \oplus \wedge^3 \bar{E}_{\mathcal{J}}$ in general.

Then we have $\eta_\alpha \cdot \phi_\alpha = \eta_\alpha^{0,1} \cdot \phi_\alpha$ and $\mathcal{J}\phi_\alpha = -n\sqrt{-1}\phi_\alpha$. We also have $\mathcal{J}\eta_\alpha \cdot \phi_\alpha = \mathcal{J}\eta_\alpha^{0,1} \cdot \phi_\alpha = (-n+1)\sqrt{-1}\eta_\alpha \cdot \phi_\alpha$. Then we have

$$\begin{aligned} [\mathcal{J}, \eta_\alpha] \cdot \phi_\alpha &= \mathcal{J}\eta_\alpha \cdot \phi_\alpha - \eta_\alpha \mathcal{J} \cdot \phi_\alpha \\ &= (-n+1)\sqrt{-1}\eta_\alpha \cdot \phi_\alpha + n\sqrt{-1}\eta_\alpha \cdot \phi_\alpha = \sqrt{-1}\eta_\alpha \cdot \phi_\alpha \end{aligned}$$

Then we have

$$\begin{aligned} (8.2) \quad dh \cdot \phi_\alpha &= (\dot{\eta}_\alpha + \dot{N}_\alpha) \cdot \phi_\alpha + (\eta_\alpha + N) \cdot h \cdot \phi_\alpha \\ &= -\sqrt{-1}[\mathcal{J}, \eta_\alpha] \cdot \phi_\alpha + \dot{N} \cdot \phi_\alpha + (\eta_\alpha + N) \cdot h \cdot \phi_\alpha \end{aligned}$$

We also have

$$\dot{N} \cdot \phi_\alpha = -\sqrt{-1}[\mathcal{J}, (\dot{N}^{2,1} + \dot{N}^{1,2})] \cdot \phi_\alpha - \frac{1}{3}\sqrt{-1}[\mathcal{J}, (\dot{N}^{3,0} + \dot{N}^{0,3})] \cdot \phi_\alpha.$$

Since $\langle e_\alpha \cdot \phi_\alpha, \dot{N} \cdot \bar{\phi}_\alpha \rangle_s = \langle e_\alpha \cdot \phi_\alpha, (\dot{N}^{2,1} + \dot{N}^{1,2}) \cdot \bar{\phi}_\alpha \rangle_s$, we have

$$(8.3) \quad \langle e_\alpha \cdot \phi_\alpha, \dot{N} \cdot \bar{\phi}_\alpha \rangle_s = -\sqrt{-1} \langle e_\alpha \cdot \phi_\alpha, [\mathcal{J}, \dot{N}] \cdot \bar{\phi}_\alpha \rangle_s$$

Substituting (8.2) into F_{1-1} and using (8.3), we obtain

$$\begin{aligned} \langle e_\alpha \cdot \phi_\alpha, dh \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} &= \langle e_\alpha \cdot \phi_\alpha, \sqrt{-1}[\mathcal{J}, (\eta_\alpha + \dot{N})] \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ &\quad + \langle e_\alpha \cdot \phi_\alpha, (\eta_\alpha + N) \cdot h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \end{aligned}$$

Thus the term F_{1-1} is divided into two terms F_{1-1-1} and F_{1-1-2} ,

$$F_{1-1} = F_{1-1-1} + F_{1-1-2}$$

where it follows from Lemma 7.9 that we have

$$\begin{aligned} F_{1-1-1} &= \langle e_\alpha \cdot \phi_\alpha, \sqrt{-1}[\mathcal{J}, \eta_\alpha + \dot{N}] \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ F_{1-1-2} &= \langle e_\alpha \cdot \phi_\alpha, \eta_\alpha \cdot h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \end{aligned}$$

The term F_{2-1} is also divided into two terms

$$F_{2-1} = F_{2-1-1} + F_{2-1-2}$$

where

$$\begin{aligned} F_{2-1-1} &= \langle -\sqrt{-1}[\mathcal{J}, \eta_\alpha + \dot{N}] \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\ F_{2-1-2} &= \langle \eta_\alpha \cdot h \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \end{aligned}$$

By using Lemma 7.5 and $\langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s = \rho_\alpha \langle \psi, \bar{\psi} \rangle_s$, we obtain

$$\begin{aligned}
F_{1-1-1} - F_{2-1-1} &= \sqrt{-1} \langle e_\alpha \cdot \phi_\alpha, [\mathcal{J}, \dot{\eta}_\alpha + \dot{N}] \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\
&\quad + \sqrt{-1} \langle [\mathcal{J}, \dot{\eta}_\alpha + \dot{N}] \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\
&= 2\sqrt{-1} \langle e_\alpha, [\mathcal{J}, \dot{\eta}_\alpha + \dot{N}] \rangle_{T \oplus T^*} \langle \phi_\alpha, \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\
&= 2\sqrt{-1} \langle e_\alpha, [\mathcal{J}, \dot{\eta}_\alpha + \dot{N}] \rangle_{T \oplus T^*} \langle \psi, \bar{\psi} \rangle_s \\
&= \langle e_\alpha \cdot \psi, \sqrt{-1} [\mathcal{J}, \dot{\eta}_\alpha + \dot{N}] \cdot \bar{\psi} \rangle_s \\
&\quad + \langle \sqrt{-1} [\mathcal{J}, \dot{\eta}_\alpha + \dot{N}] \cdot \psi, e_\alpha \cdot \bar{\psi} \rangle_s
\end{aligned}$$

It follows from Lemma 7.10 and $\mathcal{J} \cdot \psi = 0$ that we have $N(t) \cdot \psi = 0$. Thus we have $\dot{N} \cdot \psi = 0$. It follows

$$F_{1-1-1} - F_{2-1-1} = \langle e_\alpha \cdot \psi, \sqrt{-1} [\mathcal{J}, \dot{\eta}_\alpha] \cdot \bar{\psi} \rangle_s + \langle \sqrt{-1} [\mathcal{J}, \dot{\eta}_\alpha] \cdot \psi, e_\alpha \cdot \bar{\psi} \rangle_s$$

From Lemma 7.12, the term F_{1-3} is given by

$$\begin{aligned}
F_{1-3} &= \langle e_\alpha \cdot (\eta_\alpha + N) \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} = \langle e_\alpha \cdot \eta_\alpha \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\
&= -\langle \eta_\alpha \cdot e_\alpha \cdot \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} = \langle e_\alpha \cdot \phi_\alpha, \eta_\alpha \cdot h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} = F_{1-1-2}
\end{aligned}$$

The term F_{2-3} is also given by

$$\begin{aligned}
F_{2-3} &= \langle h \cdot \phi_\alpha, e_\alpha \cdot (\eta_\alpha + N) \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} = -\langle h \cdot \phi_\alpha, \eta_\alpha \cdot e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\
&= \langle \eta_\alpha \cdot h \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} = F_{2-1-2}
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
F_{1-1-2} + F_{1-3} - F_{2-1-2} - F_{2-3} &= 2\langle e_\alpha \cdot \phi_\alpha, \eta_\alpha \cdot h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\
&\quad - 2\langle \eta_\alpha \cdot h \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1}
\end{aligned}$$

Applying Lemma 7.8 and substituting $\theta = \eta_\alpha$, we obtain

$$\begin{aligned}
F_{1-1-2} + F_{1-3} - F_{2-1-2} - F_{2-3} &= \sqrt{-1} \langle e_\alpha \cdot \psi, [[h, \mathcal{J}], \eta_\alpha] \cdot \bar{\psi} \rangle_s \\
&\quad + \sqrt{-1} \langle [[h, \mathcal{J}], \eta_\alpha] \cdot \psi, e_\alpha \cdot \bar{\psi} \rangle_s
\end{aligned}$$

We also have

$$\begin{aligned}
 F_{1-2} - F_{2-2} &= \langle e_\alpha \cdot \phi_\alpha, d\rho_\alpha^{-1} \cdot h \cdot \bar{\phi}_\alpha \rangle_s - \langle d\rho_\alpha^{-1} \cdot h \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \rangle_s \\
 &= \langle e_\alpha \cdot \phi_\alpha, -\frac{d\rho_\alpha}{\rho_\alpha} \cdot h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\
 &\quad - \left\langle -\frac{d\rho_\alpha}{\rho_\alpha} \cdot h \cdot \phi_\alpha, e_\alpha \cdot \bar{\phi}_\alpha \right\rangle_s \rho_\alpha^{-1}
 \end{aligned}$$

Applying Lemma 7.8 and substituting $\theta = \frac{d\rho_\alpha}{\rho_\alpha}$, we obtain

$$\begin{aligned}
 F_{1-2} - F_{2-2} &= -\frac{\sqrt{-1}}{2} \left\langle e_\alpha \cdot \psi, \left[[h, \mathcal{J}], \frac{d\rho_\alpha}{\rho_\alpha} \right] \cdot \bar{\psi} \right\rangle_s \\
 &\quad - \frac{\sqrt{-1}}{2} \left\langle \left[[h, \mathcal{J}], \frac{d\rho_\alpha}{\rho_\alpha} \right] \cdot \psi, e_\alpha \cdot \bar{\psi} \right\rangle_s
 \end{aligned}$$

Hence $F_1(e_\alpha) - F_2(e_\alpha)$ is given by the following,

$$\begin{aligned}
 F_1(e_\alpha) - F_2(e_\alpha) &= \langle L_{e_\alpha} \phi_\alpha, h \cdot \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} - \langle h \cdot \phi_\alpha, L_{e_\alpha} \bar{\phi}_\alpha \rangle_s \rho_\alpha^{-1} \\
 &= \sqrt{-1} \langle e_\alpha \cdot \psi, [\mathcal{J}, \eta_\alpha] \cdot \bar{\psi} \rangle_s + \sqrt{-1} \langle [\mathcal{J}, \eta_\alpha] \cdot \psi, e_\alpha \cdot \bar{\psi} \rangle_s \\
 &\quad + \sqrt{-1} \langle e_\alpha \cdot \psi, [[h, \mathcal{J}], \eta_\alpha] \cdot \bar{\psi} \rangle_s \\
 &\quad + \sqrt{-1} \langle [[h, \mathcal{J}], \eta_\alpha] \cdot \psi, e_\alpha \cdot \bar{\psi} \rangle_s \\
 &\quad - \frac{\sqrt{-1}}{2} \left\langle e_\alpha \cdot \psi, \left[[h, \mathcal{J}], \frac{d\rho_\alpha}{\rho_\alpha} \right] \cdot \bar{\psi} \right\rangle_s \\
 &\quad - \frac{\sqrt{-1}}{2} \left\langle \left[[h, \mathcal{J}], \frac{d\rho_\alpha}{\rho_\alpha} \right] \cdot \psi, e_\alpha \cdot \bar{\psi} \right\rangle_s
 \end{aligned}$$

Since e_α is a generalized Hamiltonian element satisfying $e_\alpha \cdot \psi = \sqrt{-1} df_\alpha \cdot \psi$, we have

$$\begin{aligned}
 F_1(e_\alpha) - F_2(e_\alpha) &= -\langle df_\alpha \cdot \psi, [\mathcal{J}, \eta_\alpha] \cdot \bar{\psi} \rangle_s + \langle [\mathcal{J}, \eta_\alpha] \cdot \psi, df_\alpha \cdot \bar{\psi} \rangle_s \\
 &\quad - \langle df_\alpha \cdot \psi, [[h, \mathcal{J}], \eta_\alpha] \cdot \bar{\psi} \rangle_s + \langle [[h, \mathcal{J}], \eta_\alpha] \cdot \psi, df_\alpha \cdot \bar{\psi} \rangle_s \\
 &\quad + \frac{1}{2} \left\langle df_\alpha \cdot \psi, \left[[h, \mathcal{J}], \frac{d\rho_\alpha}{\rho_\alpha} \right] \cdot \bar{\psi} \right\rangle_s \\
 &\quad - \frac{1}{2} \left\langle \left[[h, \mathcal{J}], \frac{d\rho_\alpha}{\rho_\alpha} \right] \cdot \psi, df_\alpha \cdot \bar{\psi} \right\rangle_s
 \end{aligned}$$

The action of Spin group preserves the form $\langle \cdot, \cdot \rangle_s$. Since deformations $J_t := e^{h(t)} \circ \mathcal{J} \circ e^{-h(t)}$ is given by the action of Spin group $e^{h(t)}$, thus ρ_α does

not depend on t . Recall $\dot{\mathcal{J}} = [h, \mathcal{J}]$. Then $F_1(e_\alpha) - F_2(e_\alpha)$ is given by the following derivative at $t = 0$,

$$\begin{aligned} F_1(e_\alpha) - F_2(e_\alpha) &= -\frac{d}{dt} \langle df_\alpha \cdot \psi, [\mathcal{J}_t, \eta_\alpha(t)] \cdot \bar{\psi} \rangle_s \\ &\quad + \frac{d}{dt} \langle [\mathcal{J}_t, \eta_\alpha(t)] \cdot \psi, df_\alpha \cdot \bar{\psi} \rangle_s \\ &\quad + \frac{1}{2} \frac{d}{dt} \left\langle df_\alpha \cdot \psi, \left[\mathcal{J}_t, \frac{d\rho_\alpha}{\rho_\alpha} \right] \cdot \bar{\psi} \right\rangle_s \\ &\quad - \frac{1}{2} \frac{d}{dt} \left\langle \left[\mathcal{J}_t, \frac{d\rho_\alpha}{\rho_\alpha} \right] \cdot \psi, df_\alpha \cdot \bar{\psi} \right\rangle_s \end{aligned}$$

Since we consider deformations preserving ψ , we have $\mathcal{J}_t \cdot \psi = 0$. Thus we have $[\mathcal{J}_t, \eta_\alpha(t)] \cdot \bar{\psi} = \mathcal{J}_t \eta_\alpha(t) \cdot \bar{\psi}$. The support of f_α is contained in U_α . Applying Stokes' theorem, we obtain

$$\begin{aligned} \int_M F_1(e_\alpha) - \int_M F_2(e_\alpha) &= \frac{d}{dt} \int_M \left\langle f_\alpha \psi, d \left(-\mathcal{J}_t \eta_\alpha(t) + \frac{1}{2} \mathcal{J}_t d \log \rho_\alpha \right) \cdot \bar{\psi} \right\rangle \\ &\quad + \frac{d}{dt} \int_M \left\langle d \left(\mathcal{J}_t \eta_\alpha(t) - \frac{1}{2} \mathcal{J}_t \log \rho_\alpha \right) \cdot \psi, f_\alpha \bar{\psi} \right\rangle_s \end{aligned}$$

Since $d(\mathcal{J} \eta_\alpha + \frac{1}{2} \mathcal{J} d \log \rho_\alpha) \cdot \bar{\psi}$ is a globally defined d -closed $2n$ -form, we have

$$\begin{aligned} \int_M F_1(e) - F_2(e) &= \sum_\alpha \int_M F_1(e_\alpha) - \sum_\alpha \int_M F_2(e_\alpha) \\ &= \sum_\alpha \frac{d}{dt} \int_M \left\langle f_\alpha \psi, \left(-d\mathcal{J}_t \eta_\alpha(t) + \frac{1}{2} d\mathcal{J}_t d \log \rho_\alpha \right) \cdot \bar{\psi} \right\rangle \\ &\quad + \sum_\alpha \frac{d}{dt} \int_M \left\langle \left(d\mathcal{J}_t \eta_\alpha(t) - \frac{1}{2} d\mathcal{J}_t \log \rho_\alpha \right) \cdot \psi, f_\alpha \bar{\psi} \right\rangle_s \\ &= \frac{d}{dt} \int_M \left\langle f \psi, \left(-d\mathcal{J}_t \eta_\alpha(t) + \frac{1}{2} d\mathcal{J}_t d \log \rho_\alpha \right) \cdot \bar{\psi} \right\rangle \\ &\quad + \frac{d}{dt} \int_M \left\langle \left(d\mathcal{J}_t \eta_\alpha(t) - \frac{1}{2} d\mathcal{J}_t \log \rho_\alpha \right) \cdot \psi, f \bar{\psi} \right\rangle_s \end{aligned}$$

Hence we obtain

$$\begin{aligned} c_n^{-1} \omega_{\tilde{\mathcal{A}}_\psi}(L_e \mathcal{J}, \dot{\mathcal{J}}_h) &= \frac{d}{dt} \int_M \left\langle f \psi, d \left(-\mathcal{J}_t \eta_\alpha(t) + \frac{1}{2} \mathcal{J}_t d \log \rho_\alpha \right) \cdot \bar{\psi} \right\rangle_s \\ &\quad + \frac{d}{dt} \int_M \left\langle d \left(\mathcal{J}_t \eta_\alpha(t) - \frac{1}{2} \mathcal{J}_t \log \rho_\alpha \right) \cdot \psi, f \bar{\psi} \right\rangle_s \end{aligned}$$

Thus it follows from (5.15) that the moment map μ is given by

$$\begin{aligned} \langle \mu(\mathcal{J}), f \rangle &= c_n \int_M \left\langle f\psi, d \left(-\mathcal{J}\eta_\alpha + \frac{1}{2} \mathcal{J}d \log \rho_\alpha \right) \cdot \bar{\psi} \right\rangle_s \\ &\quad + c_n \int_M \left\langle d \left(\mathcal{J}\eta_\alpha - \frac{1}{2} \mathcal{J} \log \rho_\alpha \right) \cdot \psi, f\bar{\psi} \right\rangle_s \\ &= (\sqrt{-1})^{-n} \int_M f(GR_{\mathcal{J}}) \langle \psi, \bar{\psi} \rangle_s \end{aligned}$$

Hence we obtain the result.

9. Deformations of generalized Kähler structures with constant generalized scalar curvature

Definition 9.1. If the generalized scalar curvature GR of a generalized Kähler structure $(\mathcal{J}, \mathcal{J}_\psi)$ is constant, then $(\mathcal{J}, \mathcal{J}_\psi)$ is called a generalized Kähler structure with constant generalized scalar curvature, that is,

$$GR = \lambda \text{ (constant),}$$

where $\lambda = n \frac{c_1(K_{\mathcal{J}}) \cup [\omega]^{n-1}}{[\omega^n]}$

Theorem 9.2. *Infinitesimal deformations of generalized Kähler structures with constant generalized scalar curvature are given by the cohomology group*

$$\ker \bar{\partial}_{\mathcal{J}} \cap (\bar{E}_{\mathcal{J}}^+ \wedge \bar{E}_{\mathcal{J}}^-) / \text{Im} \bar{\partial}_{\mathcal{J}}^+ \bar{\partial}_{\mathcal{J}}^- \cap (\bar{E}_{\mathcal{J}}^+ \wedge \bar{E}_{\mathcal{J}}^-)$$

of the following elliptic complex :

$$0 \rightarrow C_{\mathbb{C}}^\infty(M) \xrightarrow{\bar{\partial}_{\mathcal{J}}^+ \bar{\partial}_{\mathcal{J}}^-} \bar{E}_{\mathcal{J}}^+ \wedge \bar{E}_{\mathcal{J}}^- \xrightarrow{\bar{\partial}_{\mathcal{J}}} (\wedge^2 \bar{E}_{\mathcal{J}}^+ \wedge \bar{E}_{\mathcal{J}}^-) \oplus (\bar{E}_{\mathcal{J}}^+ \wedge \wedge^2 \bar{E}_{\mathcal{J}}^-) \xrightarrow{\bar{\partial}_{\mathcal{J}}} \dots$$

Since the cohomology group is finite dimensional for a compact manifold M , Infinitesimal deformations are also finite dimensional.

Proof. Let $\mathcal{A}_\psi(M)$ be the set of generalized complex structures which are compatible with ψ and $\text{Ham}_\psi(M)$ the generalized Hamiltonian group which acts on $\mathcal{A}_\psi(M)$. The orbit of $\text{Ham}_\psi(M)$ on $\mathcal{A}_\psi(M)$ is denoted by $\mathcal{O}_{\text{Ham}}(M)$. Let $\mathcal{J} \in \mathcal{A}_\psi(M)$ be a generalized complex structure such that $(\mathcal{J}, \mathcal{J}_\psi)$ admits constant generalized scalar curvature. The formal tangent space of

$\mathcal{A}_\psi(M)$ at \mathcal{J} is given by $\varepsilon \in \overline{E}_\mathcal{J}^+ \wedge \overline{E}_\mathcal{J}^-$ satisfying $\overline{\partial}_\mathcal{J}\varepsilon = 0$, since deformations of \mathcal{J} preserves ψ . Since generalized Kähler structures with constant generalized scalar curvature are given by the inverse image $\mu^{-1}(0)$ of the moment map μ for the action of $\text{Ham}_\psi(M)$, infinitesimal deformations are the orthogonal complement of the direct sum of $(T_\mathcal{J}\mathcal{O}_{Ham} \oplus \mathcal{J}T_\mathcal{J}\mathcal{O}_{Ham})$, where $T_\mathcal{J}\mathcal{O}_{Ham}$ denotes the tangent space of the orbit \mathcal{O}_{Ham} at \mathcal{J} . The $T_\mathcal{J}\mathcal{O}_{Ham}$ consists of $L_e\mathcal{J}$ for Hamiltonian element $e = e^{1,0} + e^{0,1} \in T_M \oplus T_M^*$, where $e^{1,0} \in E_\mathcal{J}$ and $e^{0,1} \in \overline{E}_\mathcal{J}$. Thus $T_\mathcal{J}\mathcal{O}_{Ham}$ is given by $\{\overline{\partial}_\mathcal{J}e^{0,1} \mid e : \text{Hamiltonian element}\}$. Since a Hamiltonian element e is given by $e = \mathcal{J}_\psi df$ for a hamiltonian f , we have

$$\begin{aligned} \overline{\partial}_\mathcal{J}e^{0,1} &= \overline{\partial}_\mathcal{J}(\mathcal{J}_\psi df)^{0,1} \\ &= \sqrt{-1}(\overline{\partial}_\mathcal{J}^+ + \overline{\partial}_\mathcal{J}^-)(\overline{\partial}_\mathcal{J}^+ - \overline{\partial}_\mathcal{J}^-)f \\ &= -2\sqrt{-1}\overline{\partial}_\mathcal{J}^+\overline{\partial}_\mathcal{J}^-f \end{aligned}$$

Since \mathcal{J} acts on $2\sqrt{-1}\overline{\partial}_\mathcal{J}^+\overline{\partial}_\mathcal{J}^-f \in \wedge^2\overline{E}_\mathcal{J}$ by the multiplication of $2\sqrt{-1}$. Thus we have the complexification,

$$T_\mathcal{J}\mathcal{O}_{Ham} \oplus \mathcal{J}T_\mathcal{J}\mathcal{O}_{Ham} = \{ -2\sqrt{-1}\overline{\partial}_\mathcal{J}^+\overline{\partial}_\mathcal{J}^-F \mid F \in C_c^\infty(M) \}$$

Hence infinitesimal deformations of generalized Kähler structures with constant generalized scalar curvature are given by the cohomology group

$$\ker \overline{\partial}_\mathcal{J} \cap (\overline{E}_\mathcal{J}^+ \wedge \overline{E}_\mathcal{J}^-) / \text{Im} \overline{\partial}_\mathcal{J}^+ \overline{\partial}_\mathcal{J}^- \cap (\overline{E}_\mathcal{J}^+ \wedge \overline{E}_\mathcal{J}^-)$$

The ellipticity of the complex follows from checking its symbol complex. Hence we obtain the result. \square

Example 9.3. Let S be a K3 surface and $(\mathcal{J}_J, \mathcal{J}_\psi)$ a generalized Kähler structure induced from a Ricci flat Kähler structure. We have the generalized Hodge decomposition $H^\bullet(S) = \oplus H^{p,q}$,

$$\begin{array}{ccccc}
 & & H^{0,2} & & \\
 & & & & \\
 & & & & \\
 & & H^{-1,1} & & H^{1,1} \\
 & & & & \\
 & & & & \\
 H^{-2,0} & & H^{0,0} & & H^{2,0} \\
 & & & & \\
 & & H^{-1,-1} & & H^{1,-1} \\
 & & & & \\
 & & & & \\
 & & H^{0,-2} & &
 \end{array}$$

Then infinitesimal deformations of generalized Kähler structures with vanishing generalized scalar curvature are given by $H^{0,0}(S)$, where $\dim H^{0,0} = 20$. In the cases of ordinary K3 surfaces, deformations of complex structures with vanishing Ricci tensor preserving a symplectic structure is 19 dimensional. Hence there is one more dimensional deformations which deform to generalized Kähler structures of type $(0, 0)$ which is discussed next section.

10. Generalized Kähler structures of type $(0, 0)$

Definition 10.1. A generalized Kähler structure of type $(0, 0)$ is a generalized Kähler structure $(\mathcal{J}_\phi, \mathcal{J}_\psi)$ which is induced from a pair

$$(\phi = e^{B+\sqrt{-1}\omega_1}, \psi = e^{\sqrt{-1}\omega_2})$$

which consists of d -closed, nondegenerate, pure spinors of symplectic types, where B is a real d -closed 2-form and both ω_1 and ω_2 are real symplectic forms, respectively.

Proposition 10.2. A pair $(\phi = e^{B+\sqrt{-1}\omega_1}, \psi = e^{\sqrt{-1}\omega_2})$ gives a generalized Kähler structure if and only if (ϕ, ψ) satisfies the followings :

- (1) $\omega_{\mathbb{C}}^{\pm} := B + \sqrt{-1}(\omega_1 \mp \omega_2)$ defines complex structures I_{\pm} such that $\omega_{\mathbb{C}}^{\pm}$ are d -closed holomorphic symplectic forms with respect to I_{\pm} respectively.
- (2) ω_2 is tame w.r.t both I_{\pm} .

Proof. Let E_ϕ be the eigenspace with eigenvalue $-\sqrt{-1}$ with respect to \mathcal{J}_ϕ and \overline{E}_ϕ the complex conjugate of E_ϕ . We denote by E_ψ the eigenspace with

eigenvalue $-\sqrt{-1}$ with respect to \mathcal{J}_ψ and \overline{E}_ψ is the complex conjugate of E_ψ . Then we have

$$\begin{aligned} E_\phi &= \{ v - i_v(B + \sqrt{-1}\omega_1) \mid v \in T_M^{\mathbb{C}} \}, & E_\psi &= \{ u - \sqrt{-1}i_u\omega_2 \mid u \in T_M^{\mathbb{C}} \} \\ \overline{E}_\phi &= \{ v - i_v(B - \sqrt{-1}\omega_1) \mid v \in T_M^{\mathbb{C}} \}, & \overline{E}_\psi &= \{ u + \sqrt{-1}i_u\omega_2 \mid u \in T_M^{\mathbb{C}} \} \end{aligned}$$

The condition $\mathcal{J}_\phi\mathcal{J}_\psi = \mathcal{J}_\psi\mathcal{J}_\phi$ is equivalent to the followings :

$$\dim_{\mathbb{C}} E_\phi \cap E_\psi = \dim_{\mathbb{C}} E_\phi \cap \overline{E}_\psi = n.$$

Thus $u - \sqrt{-1}i_u\omega_2 \in E_\phi \cap E_\psi$ if and only if

$$u - \sqrt{-1}i_u\omega_2 = u - i_u(B + \sqrt{-1}\omega_1).$$

Hence $u - \sqrt{-1}i_u\omega_2 \in E_\phi \cap E_\psi$ if and only if $i_u(B + \sqrt{-1}(\omega_1 - \omega_2)) = 0$. Thus $\ker(B + \sqrt{-1}(\omega_1 - \omega_2)) := \{ u \in T_M^{\mathbb{C}} \mid i_u(B + \sqrt{-1}(\omega_1 - \omega_2)) = 0 \}$ is $2n$ dimensional if and only if $\dim E_\phi \cap E_\psi = n$. If $u \in E_\phi$, then it follows from $E_\phi \cap \overline{E}_\phi = \{0\}$ that we have $u \neq \bar{u}$. Thus we see that

$$\ker(B + \sqrt{-1}(\omega_1 - \omega_2)) \cap \overline{\ker(B + \sqrt{-1}(\omega_1 - \omega_2))} = \{0\}.$$

Hence $\omega_{\mathbb{C}}^+ := B + \sqrt{-1}(\omega_1 - \omega_2)$ defines a complex structure I_+ such that $\omega_{\mathbb{C}}^+$ is a holomorphic symplectic form with respect to I_+ . We also see that $\ker(B + \sqrt{-1}(\omega_1 + \omega_2)) := \{ u \in T_M^{\mathbb{C}} \mid i_u(B + \sqrt{-1}(\omega_1 + \omega_2)) = 0 \}$ is $2n$ dimensional if and only if $\dim E_\phi \cap \overline{E}_\psi = n$. Thus $\omega_{\mathbb{C}}^- := B + \sqrt{-1}(\omega_1 + \omega_2)$ defines a complex structure I_- such that $\omega_{\mathbb{C}}^-$ is a holomorphic symplectic form with respect to I_- . Hence the condition $[\mathcal{J}_\phi, \mathcal{J}_\psi] = 0$ is equivalent to the condition (1). The eigenspace with eigenvalue ± 1 with respect to $G := \mathcal{J}_\phi\mathcal{J}_\psi$ are denoted by C_\pm , respectively. Then we have $C_+^{\mathbb{C}} = (E_\phi \cap E_\psi) \oplus (\overline{E}_\phi \cap \overline{E}_\psi)$ and $C_-^{\mathbb{C}} = (E_\phi \cap \overline{E}_\psi) \oplus (\overline{E}_\phi \cap E_\psi)$. For $u \in \ker \omega_{\mathbb{C}}^+ = T_{I_+}^{0,1}$, we have

$$\begin{aligned} G(u - \sqrt{-1}i_u\omega_2, \bar{u} + \sqrt{-1}i_{\bar{u}}\omega_2) &= \langle u - \sqrt{-1}i_u\omega_2, \bar{u} + \sqrt{-1}i_{\bar{u}}\omega_2 \rangle \\ &= -2\sqrt{-1}\omega_2(u, \bar{u}) \end{aligned}$$

For $u \in \ker \omega_{\mathbb{C}}^- = T_{I_-}^{0,1}$, we also have

$$\begin{aligned} G(u + \sqrt{-1}i_u\omega_2, \bar{u} - \sqrt{-1}i_{\bar{u}}\omega_2) &= -\langle u + \sqrt{-1}i_u\omega_2, \bar{u} - \sqrt{-1}i_{\bar{u}}\omega_2 \rangle \\ &= -2\sqrt{-1}\omega_2(u, \bar{u}) \end{aligned}$$

Thus $G = \mathcal{J}_\phi\mathcal{J}_\psi$ gives a generalized metric if and only if $-\sqrt{-1}\omega_2(u, \bar{u}) > 0$ for all $u \neq 0 \in T_{I_\pm}^{0,1}$. A symplectic structure is tame with respect to I_\pm if and

only if $\omega_2(x, I_{\pm}x) > 0$ for every real tangent $x \neq 0 \in T_M$. Since $-\sqrt{-1}\omega_2(x - \sqrt{-1}I_{\pm}x, x + \sqrt{-1}I_{\pm}x) = 2\omega_2(x, I_{\pm}x)$, Hence $G := \mathcal{J}_{\phi}\mathcal{J}_{\psi}$ gives a generalized metric if and only if ω_2 is tame with respect to I_{\pm} . Hence we obtain the result. \square

Remark 10.3. On a 4 dimensional manifold, the condition (1) is equivalent to the followings

$$B \wedge \omega_1 = B \wedge \omega_2 = \omega_1 \wedge \omega_2 = 0, \quad B \wedge B = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 \neq 0.$$

In the case of a generalized Kähler structure of type $(0,0)$, the GRic and GR are explicitly written.

Proposition 10.4. For a generalized Kähler structure of type $(0,0)$, GRic and GR are given by

$$\begin{aligned} GRic &= -dB\omega_1^{-1} \left(d \log \frac{\omega_1^n}{\omega_2^n} \right) \\ (GR)\omega_2^n &= n\omega_2^{n-1} \wedge dB\omega_1^{-1} \left(d \log \frac{\omega_1^n}{\omega_2^n} \right), \end{aligned}$$

where $B : T_M \rightarrow T_M^*$ and $\omega_i^{-1} : T_M^* \rightarrow T_M$ ($i = 1, 2$) and the composition $B\omega_1^{-1}$ is an endomorphism of T_M^* and then $B\omega_1^{-1}(d \log \frac{\omega_1^n}{\omega_2^n})$ is a 1-form and then the exterior derivative of $B\omega_1^{-1}(d \log \frac{\omega_1^n}{\omega_2^n})$ is a 2-form which is the GRic form.

Proof. Substituting $\rho_{\alpha} = \frac{\omega_1^n}{\omega_2^n}$ and $\eta_{\alpha} = 0$ into (5.7), we have the result. \square

Example 10.5 (HyperKähler str.). Let (g, I, J, K) be a hyperKähler structure with three Kähler forms $(\omega_I, \omega_J, \omega_K)$. We define B and two symplectic forms ω_1, ω_2 by

$$B = \omega_J, \quad \omega_1 = \frac{1}{2}(\omega_I + \omega_K), \quad \omega_2 = \frac{1}{2}(\omega_I - \omega_K).$$

Then $(\phi = e^{B+\sqrt{-1}\omega_1}, \psi = e^{\sqrt{-1}\omega_2})$ is a generalized Kähler structure which satisfies $GRic = 0$.

11. Generalized Kähler-Einstein structures

Definition 11.1. A generalized Kähler structure $(\mathcal{J}_\beta, \psi = e^{b+\sqrt{-1}\omega})$ is a generalized Kähler-Einstein if we have the following:

$$\text{GRic} = \lambda\omega$$

for constants λ .

In the case of generalized Kähler structure of type $(0, 0)$, the generalized Kähler-Einstein condition implies that $\omega_1^n = \omega_2^n$, where the Einstein constant is zero.

12. Generalized Kähler-Einstein structures constructed from holomorphic Poisson deformations

Let (M, J, ω) be a Kähler manifold with an ordinary complex structure J and a Kähler structure ω . We assume that the m -dimensional torus T acts on M preserving the Kähler structure (J, ω) on M and there exists a moment map $\mu_T : M \rightarrow (t^m)^*$ for the action of T , where we assume $m \geq 2$. Let $\{\xi_i\}_{i=1}^m$ be a basis of the Lie algebra t^m of the Torus T and $\{V_i\}_{i=1}^m$ the corresponding real vector fields which are generated by $\{\xi_i\}_{i=1}^m$. Each V_i is decomposed into $V_i^{1,0} + V_i^{0,1}$, where $V_i^{1,0} \in T_J^{1,0}$ and $V_i^{0,1} \in T_J^{0,1}$. Since $\{V_i\}_{i=1}^m$ are commuting vector fields, we have a real Poisson structure $\beta_{\mathbb{R}}$ by

$$\beta_{\mathbb{R}} = \sum_{i,j} \lambda_{i,j} V_i \wedge V_j,$$

where $\lambda_{i,j}$ are constants. Holomorphic vector fields $\{V_i^{1,0}\}_{i=1}^m$ also gives a holomorphic Poisson structure $\beta = \sum \lambda_{i,j} V_i^{1,0} \wedge V_j^{1,0}$. Let (\mathcal{J}_J, J_ψ) be the generalized Kähler structure coming from the ordinary Kähler structure (J, ω) , where $\psi = e^{\sqrt{-1}\omega}$. Let $\{\phi_\alpha\}$ be trivializations of $K_{\mathcal{J}_J}$, that is, each ϕ_α is a holomorphic n -form with respect to J . Then the action of $e^{\beta_{\mathbb{R}}}$ on each ϕ_α coincides with the action of e^β on ϕ_α , that is,

$$e^{\beta_{\mathbb{R}}} \cdot \phi_\alpha = e^\beta \cdot \phi_\alpha.$$

Thus the action of $e^{\beta_{\mathbb{R}}}$ on \mathcal{J}_J gives Poisson deformations of $\mathcal{J}_{\beta t}$. Then the action of $e^{\beta_{\mathbb{R}}}$ gives deformations of almost generalized Kähler structures

$$(\mathcal{J}_{\beta t}, \mathcal{J}_{\psi_t}) := (e^{\beta_{\mathbb{R}} t} \mathcal{J}_J e^{-\beta_{\mathbb{R}} t}, e^{\beta_{\mathbb{R}} t} \mathcal{J}_\psi e^{-\beta_{\mathbb{R}} t}),$$

where \mathcal{J}_{ψ_t} are almost generalized complex structures induced from $\psi_t = e^{\beta_{\mathbb{R}}t} \cdot \psi$.

Theorem 12.1. *Let $\mu_{T,i}$ be the function which is the coupling $\langle \mu_T, \xi_i \rangle$ of the moment map μ_T and $\xi_i \in t^m$. Then ψ_t is given by*

$$\psi_t = \exp \left(- \sum_{i,j} \lambda_{i,j} d\mu_{T,i} \wedge d\mu_{T,j} + \sqrt{-1} \omega \right)$$

Thus $d\psi_t = 0$ and $(\mathcal{J}_{\beta_t}, \mathcal{J}_{\psi_t})$ are deformations of generalized Kähler structures.

Proof. The exponential $e^{\beta_{\mathbb{R}}}$ is given by

$$e^{\beta_{\mathbb{R}}} = \prod_{i,j} e^{\lambda_{i,j} V_i \wedge V_j} = \prod_{i,j} (1 + \lambda_{i,j} V_i \wedge V_j).$$

Since $\omega(V_i, V_j) = 0$ and $i_{V_i} \omega = d\mu_{T,i}$, we have

$$V_i \wedge V_j \cdot \psi = -d\mu_{T,i} \wedge d\mu_{T,j} \wedge e^{\sqrt{-1}\omega}$$

Since $i_{V_i} d\mu_{T,j} = 0$, we have

$$\begin{aligned} e^{\beta_{\mathbb{R}}} \cdot \psi &= \prod_{i,j} (1 + \lambda_{i,j} V_i \wedge V_j) \cdot \psi = \prod_{i,j} (1 - \lambda_{i,j} d\mu_{T,i} \wedge d\mu_{T,j}) \cdot \psi \\ &= \prod_{i,j} e^{-\lambda_{i,j} d\mu_{T,i} \wedge d\mu_{T,j}} \cdot \psi \\ &= \exp \left(- \sum_{i,j} \lambda_{i,j} d\mu_{T,i} \wedge d\mu_{T,j} + \sqrt{-1} \omega \right). \end{aligned}$$

Thus ψ_t is d -closed and \mathcal{J}_{ψ_t} are generalized complex structures. Thus we have the result. □

Proposition 12.2. *Let (X, J, ω) be a Kähler-Einstein manifold which admits an action of real torus T^m ($m \geq 2$) preserving the Kähler structure (J, ω) . We assume that there exists a moment map for the action of T^m . We denote by $\{\xi_i\}_{i=1}^m$ a basis of the Lie algebra t^m which yields vector fields $\{V_i\}_{i=1}^m$. We assume that $\beta_{\mathbb{R}} := \sum_{i,j} \lambda_{i,j} V_i \wedge V_j$ is a nontrivial real Poisson structure for some constants $\lambda_{i,j}$. Then there exist nontrivial deformations*

of generalized Kähler-Einstein manifolds $(\mathcal{J}_{\beta t}, \psi_t)$, where $\{\mathcal{J}_{\beta t}\}$ are Poisson deformations of \mathcal{J}_J , where β is the holomorphic Poisson structure given by

$$\beta = \sum_{i,j} \lambda_{i,j} V_i^{1,0} \wedge V_j^{1,0}$$

and $V_i = V^{1,0} + V^{0,1}$ and $V_i^{1,0} \in T_J^{1,0}$, and $V^{0,1} \in T_J^{0,1}$.

Proof. It suffices to show Proposition 12.2 in the case of $\beta_{\mathbb{R}} = V_1 \wedge V_2$ which is a real Poisson structure given by the wedge of V_1 and V_2 . Let $\{\phi_\alpha\}$ be trivializations of the canonical line bundle K_J which are given by the ordinary holomorphic n -forms. The action of T^m preserves the complex structure J and the canonical line bundle. Thus the action of V_1 and V_2 are the representations of weights n_1 and n_2 , respectively, that is,

$$L_{V_1} \phi_\alpha = \sqrt{-1} n_1 \phi_\alpha \quad L_{V_2} \phi_\alpha = \sqrt{-1} n_2 \phi_\alpha$$

From $[V_1, V_2] = 0$ and $d\phi_\alpha = 0$, it follows that we have

$$\begin{aligned} d(\beta_{\mathbb{R}} \cdot \phi_\alpha) &= d(V_1 \wedge V_2) \cdot \phi_\alpha = L_{V_1} V_2 \cdot \phi_\alpha - V_1 dV_2 \cdot \phi_\alpha \\ &= V_2 \cdot L_{V_1} \phi_\alpha - V_1 \cdot L_{V_2} \phi_\alpha \\ &= \sqrt{-1} n_1 V_2 \cdot \phi_\alpha - \sqrt{-1} n_2 V_1 \cdot \phi_\alpha \end{aligned}$$

Since $V_i \cdot \beta_{\mathbb{R}} = V_i \cdot V_1 \cdot V_2 = 0$, we have

$$de^{\beta_{\mathbb{R}}} \phi_\alpha = (\sqrt{-1} n_1 V_2 - \sqrt{-1} n_2 V_1) \cdot e^{\beta_{\mathbb{R}}} \phi_\alpha$$

Since $(V_1 - \sqrt{-1} \mathcal{J}_\beta V_1) \cdot e^{\beta_{\mathbb{R}}} \phi_\alpha = (V_2 - \sqrt{-1} \mathcal{J}_\beta V_2) \cdot e^{\beta_{\mathbb{R}}} \phi_\alpha = 0$, we have

$$de^{\beta_{\mathbb{R}}} \phi_\alpha = (-n_1 \mathcal{J}_\beta V_2 + n_2 \mathcal{J}_\beta V_1) \cdot e^{\beta_{\mathbb{R}}} \phi_\alpha.$$

Since $\mathcal{J}_\beta V_i = e^\beta \mathcal{J}_J e^{-\beta} V_i = \mathcal{J}_J V_i = J V_i$, we also have

$$de^{\beta_{\mathbb{R}}} \phi_\alpha = (-n_1 J V_2 + n_2 J V_1) \cdot e^{\beta_{\mathbb{R}}} \phi_\alpha.$$

Since $(-n_1 J V_2 + n_2 J V_1)$ is a real section, it follows that $\eta_\alpha = (-n_1 J V_2 + n_2 J V_1)$ ^{*3} Let μ_T be the moment map for the action of T^m . Then $\mu_{T,i}$ is

^{*3}If $\beta_{\mathbb{R}} = \sum_{i,j} \lambda_{i,j} V_i \wedge V_j$, then $\eta_\alpha = \sum_{i,j} \lambda_{i,j} (-n_j J V_i + n_i J V_j)$

denoted by $\langle \mu_T, \xi_i \rangle$. Since $\omega(V_1, V_2) = 0$, $i_{V_i}\omega = d\mu_i$, we have

$$\begin{aligned}
 (12.1) \quad e^{\beta_{\mathbb{R}}} \cdot \psi &= e^{\beta_{\mathbb{R}}} \cdot e^{\sqrt{-1}\omega} = e^{\sqrt{-1}\omega} + V_1 \wedge V_2 \cdot e^{\sqrt{-1}\omega} \\
 &= e^{\sqrt{-1}\omega} - d\mu_{T,1} \wedge d\mu_{T,2} e^{\sqrt{-1}\omega} \\
 &= \exp(-d\mu_{T,1} \wedge d\mu_{T,2} + \sqrt{-1}\omega)
 \end{aligned}$$

Since $\psi_{\beta_{\mathbb{R}}} := e^{\beta_{\mathbb{R}}} \cdot \psi$ is d -closed, then \mathcal{J}_{ψ} is integrable. Hence $(\mathcal{J}_{\beta_{\mathbb{R}}}, \mathcal{J}_{\psi})$ is a generalized Kähler structure. Since $\eta_{\alpha} \in T_M$, it follows that $e^{\beta_{\mathbb{R}}}\eta_{\alpha}e^{-\beta_{\mathbb{R}}} = \eta_{\alpha}$. Thus we have

$$\begin{aligned}
 d\mathcal{J}_{\beta_{\mathbb{R}}}\eta_{\alpha} \cdot \psi_{\beta_{\mathbb{R}}} &= de^{\beta_{\mathbb{R}}}\mathcal{J}_J e^{-\beta_{\mathbb{R}}}\eta_{\alpha}e^{\beta_{\mathbb{R}}} \cdot \psi \\
 &= de^{\beta_{\mathbb{R}}}J\eta_{\alpha}\psi = de^{\beta_{\mathbb{R}}}(n_1V_2 - n_2V_1)e^{-\beta_{\mathbb{R}}}e^{\beta_{\mathbb{R}}}\psi \\
 &= d(n_1V_2 - n_2V_1)e^{\beta_{\mathbb{R}}} \cdot \psi
 \end{aligned}$$

Since $V_i \cdot d\mu_j = 0$ for $i, j = 1, 2$, we have $dV_i e^{\beta_{\mathbb{R}}} \cdot \psi = dV_i \psi$. Thus we have

$$\begin{aligned}
 (12.2) \quad d\mathcal{J}_{\beta_{\mathbb{R}}}\eta_{\alpha} \cdot \psi_{\beta_{\mathbb{R}}} &= d(n_1V_2 - n_2V_1) \cdot \psi \\
 &= \sqrt{-1}d(n_1i_{V_2}\omega - n_2i_{V_1}\omega) \cdot \psi = 0.
 \end{aligned}$$

We calculate the term $d\mathcal{J}_{\beta_{\mathbb{R}}}d\log\rho_{\alpha} \cdot \bar{\psi}_{\beta_{\mathbb{R}}}$. Since V_i preserves the function ρ_{α} , we have $L_{V_i}\rho_{\alpha} = 0$. Thus we have

$$e^{-\beta_{\mathbb{R}}}\frac{d\rho_{\alpha}}{\rho_{\alpha}}e^{\beta_{\mathbb{R}}} = \frac{d\rho_{\alpha}}{\rho_{\alpha}} - \left[V_1 \wedge V_2, \frac{d\rho_{\alpha}}{\rho_{\alpha}} \right] = \frac{d\rho_{\alpha}}{\rho_{\alpha}}.$$

Thus we have

$$\begin{aligned}
 d\mathcal{J}_{\beta_{\mathbb{R}}}d\log\rho_{\alpha} \cdot \bar{\psi}_{\beta_{\mathbb{R}}} &= de^{\beta_{\mathbb{R}}}\mathcal{J}_J e^{-\beta_{\mathbb{R}}}\frac{d\rho_{\alpha}}{\rho_{\alpha}}e^{\beta_{\mathbb{R}}} \cdot \bar{\psi} \\
 &= de^{\beta_{\mathbb{R}}}\mathcal{J}_J \frac{d\rho_{\alpha}}{\rho_{\alpha}} \cdot \bar{\psi} = de^{\beta_{\mathbb{R}}}\mathcal{J}_J \frac{d\rho_{\alpha}}{\rho_{\alpha}} e^{-\beta_{\mathbb{R}}}\bar{\psi}_{\beta_{\mathbb{R}}} \\
 &= d\mathcal{J}_J \frac{d\rho_{\alpha}}{\rho_{\alpha}} \cdot \bar{\psi}_{\beta_{\mathbb{R}}} + d \left[\beta_{\mathbb{R}}, \mathcal{J}_J \frac{d\rho_{\alpha}}{\rho_{\alpha}} \right] \cdot \bar{\psi}_{\beta_{\mathbb{R}}} \\
 &= dJ \left(\frac{d\rho_{\alpha}}{\rho_{\alpha}} \right) \wedge \psi_{\beta_{\mathbb{R}}} + d \left(V_1 \left\langle V_2, J \frac{d\rho_{\alpha}}{\rho_{\alpha}} \right\rangle - V_2 \left\langle V_1, J \frac{d\rho_{\alpha}}{\rho_{\alpha}} \right\rangle \right) \cdot \bar{\psi}_{\beta_{\mathbb{R}}} \\
 &= dJ \frac{d\rho_{\alpha}}{\rho_{\alpha}} \cdot \psi_{\beta_{\mathbb{R}}} + \sqrt{-1}d \left(\left\langle V_2, J \frac{d\rho_{\alpha}}{\rho_{\alpha}} \right\rangle i_{V_1}\omega - \left\langle V_1, J \frac{d\rho_{\alpha}}{\rho_{\alpha}} \right\rangle i_{V_2}\omega \right) \cdot \bar{\psi}_{\beta_{\mathbb{R}}}
 \end{aligned}$$

Thus we have

(12.3)

$$\begin{aligned} & -2d\mathcal{J}_{\beta_{\mathbb{R}}}\eta_{\alpha} \cdot \bar{\psi}_{\beta_{\mathbb{R}}} + d\mathcal{J}_{\beta_{\mathbb{R}}}d\log\rho_{\alpha} \cdot \bar{\psi}_{\beta_{\mathbb{R}}} \\ &= dJ\frac{d\rho_{\alpha}}{\rho_{\alpha}} \cdot \bar{\psi}_{\beta_{\mathbb{R}}} + \sqrt{-1}d\left(\left\langle V_2, J\frac{d\rho_{\alpha}}{\rho_{\alpha}} \right\rangle i_{V_1}\omega - \left\langle V_1, J\frac{d\rho_{\alpha}}{\rho_{\alpha}} \right\rangle i_{V_2}\omega\right) \cdot \bar{\psi}_{\beta_{\mathbb{R}}} \end{aligned}$$

As in Definition 5.3, $-2d\mathcal{J}_{\beta_{\mathbb{R}}}\eta_{\alpha} \cdot \bar{\psi}_{\beta_{\mathbb{R}}} + d\mathcal{J}_{\beta_{\mathbb{R}}}d\log\rho_{\alpha} \cdot \bar{\psi}_{\beta_{\mathbb{R}}}$ is written as

$$-2d\mathcal{J}_{\beta_{\mathbb{R}}}\eta_{\alpha} \cdot \bar{\psi}_{\beta_{\mathbb{R}}} + d\mathcal{J}_{\beta_{\mathbb{R}}}d\log\rho_{\alpha} \cdot \bar{\psi}_{\beta_{\mathbb{R}}} = (P - \sqrt{-1}Q) \cdot \bar{\psi}_{\beta_{\mathbb{R}}},$$

where $P = \text{GRic}$ and Q are real 2-forms.

Since $\sqrt{-1}d\left(\left\langle V_2, J\frac{d\rho_{\alpha}}{\rho_{\alpha}} \right\rangle i_{V_1}\omega - \left\langle V_1, J\frac{d\rho_{\alpha}}{\rho_{\alpha}} \right\rangle i_{V_2}\omega\right)$ is a pure imaginary 2-form and $dJ\frac{d\rho_{\alpha}}{\rho_{\alpha}}$ is a real 2-form, we obtain

$$\text{GRic} = -dJd\log\rho_{\alpha}$$

Since (X, J, ω) is a Kähler-Einstein manifold, we also have

$$-dJ\log\rho_{\alpha} = \lambda\omega.$$

Since $\psi_{\beta_{\mathbb{R}}} = \exp(-d\mu_{T,1} \wedge \mu_{T,2} + \sqrt{-1}\omega)$, we have $\text{GRic} = \lambda\omega$. \square

Let $X = (M, J)$ be a compact complex surface with effective anticanonical divisor. Let β be a nontrivial section of K^{-1} . Then β is a holomorphic Poisson structure. We denote by \mathcal{J}_{β} Poisson deformations of generalized complex structures. Then from the stability theorem of generalized Kähler structures, there is a generalized Kähler structure $(\mathcal{J}_{\beta}, \mathcal{J}_{\psi})$, where $\psi = e^{B+\sqrt{-1}\omega}$ is a d -closed, nondegenerate, pure spinor. We denote by $D = \{\beta = 0\}$ the divisor given by zero of β . Then we have

Proposition 12.3. *Let β be a Poisson structure on $X = \mathbb{C}P^2$ which is an anticanonical divisor D given by three lines in general position. Then there exists a generalized Kähler-Einstein structure $(\mathcal{J}_{\beta}, \mathcal{J}_{\psi})$ such that*

$$\text{GRic} = 3\omega,$$

where $\psi = e^{b+\sqrt{-1}\omega}$.

Proof. In our case, Poisson structure β is given by an action of 2-dimensional torus preserving the Kähler structure of $\mathbb{C}P^2$. Then the result follows from Proposition 12.2. \square

Proposition 12.4. *Let (M, J, ω) be a toric Kähler-Einstein manifold of dimension m . Then there exist deformations of nontrivial generalized Kähler-Einstein structures from the ordinary Kähler-Einstein structure, where $m \geq 2$.*

Proof. Since (M, J, ω) is a toric Kähler-Einstein manifold, there exists an action of T^m preserving the Kähler structure. Then the result follows from Proposition 12.2 \square

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