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On the rigidity of lagrangian products

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Motivated by work of the first author, this paper studies symplectic embedding problems of lagrangian products that are sufficiently symmetric. In general, lagrangian products arise naturally in the study of billiards. The main result of the paper is the rigidity of a large class of symplectic embedding problems of lagrangian products in any dimension. This is achieved by showing that the lagrangian products under consideration are symplectomorphic to toric domains, and by using the Gromov width and the cube capacity introduced by Gutt and Hutchings to obtain rigidity.

1. Introduction

The study of symplectic embeddings lies at the heart of symplectic topology and was kickstarted by Gromov's celebrated non-squeezing theorem [16]. Since then, many surprising results have been discovered highlighting the boundary between flexibility and rigidity in symplectic topology (cf. [8, 22, 32] for thorough overviews). For the purposes of this paper, it is important to remark the role that symplectic capacities play in solving symplectic embedding problems, especially in the case of four-dimensional toric domains (cf. [7, 10, 28]).

Recently, in [30], the first author studied symplectic embedding problems involving the 4-dimensional lagrangian bidisk, an example of a class of symplectic manifolds that are known as *lagrangian products* and arise naturally in the study of billiards (cf. [2, 29]). The main result in [30] is the computation of the optimal symplectic embeddings of the lagrangian bidisk into a ball and an ellipsoid. The novelty of [30] is to identify the lagrangian bidisk with a concave toric domain using the standard billiard in the disk, thus allowing one to use the machinery of embedded contact homology (ECH) capacities to solve the problem.

Inspired by [30], this paper studies symplectic embedding problems for a large class of lagrangian products in any dimension. The main result of the paper is that, for all lagrangian products under consideration, the corresponding symplectic embedding problems are rigid, meaning that one cannot do better than inclusion (see Theorem 4 for a precise statement). The strategy for the proof is similar to that employed in [30]. Firstly, the relevant lagrangian products are shown to be symplectomorphic to some toric domains (see Theorem 7). It is worthwhile observing that these symplectomorphisms are constructed by understanding the symplectic geometry of the billiard in the interval (see Section 3). Secondly, the above identification allows us to use two symplectic capacities, the well-known Gromov width and the cube capacity recently introduced by Gutt and Hutchings in [20], to solve the problem (see Theorem 11). To the best of our knowledge, the results of this paper are the first in the study of symplectic embeddings of lagrangian products in any dimension.

The results of the present paper, as well as those of [30], corroborate the connection between integrable billiards and lagrangian products admitting an integrable Hamiltonian torus action. We plan on investigating this relation further in future papers.

1.1. Lagrangian products

We start by defining the main object of study of this paper.

Definition 1. Given $A, B \subset \mathbb{R}^n$, the *lagrangian product* of A and B, denoted by $A \times_L B$, is the following subset of $\mathbb{R}^{2n}$

$$A \times_L B = \left\{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} \mid (x_1, \ldots, x_n) \in A \text{ and } (y_1, \ldots, y_n) \in B\right\},$$

endowed with the restriction of the symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$.

This article studies symplectic embedding problems of lagrangian products and the main result is that many of these embeddings problems are rigid (see Theorem 4). Inspired by [30], one of the key ingredients in the proof of the main result is to endow lagrangian products that are 'sufficiently symmetric' with an integrable Hamiltonian toric action (see Theorem 7). To make the above notion precise, we introduce the following terminology.

Definition 2. An open and bounded subset $A \subset \mathbb{R}^n$ is said to be

- *a balanced region* if $(x_1, \ldots, x_n) \in A \Rightarrow [-|x_1|, |x_1|] \times \cdots \times [-|x_n|, |x_n|] \subset A$;
- *a symmetric region* if it is balanced and invariant under permutation of any two coordinates.

A balanced or symmetric region A is *convex* if $A \subset \mathbb{R}^n$ is a convex subset, while it is *concave* if $\mathbb{R}^n_{\geq 0} \setminus A$ is a convex subset of $\mathbb{R}^n$ (see Figure 1 (a) and (b)).

Example 3. For any $n \geq 1$ and any $p \in [1, \infty]$, the open unit ball in the L^p-norm in $\mathbb{R}^n$, denoted by B^n_p, is a symmetric region.

Given $X_1, X_2 \subset \mathbb{R}^{2n}$, we say that X_1 *symplectically embeds into* X_2 if there exists a smooth embedding from X_1 into X_2 preserving ω_0. If X_1 symplectically embeds in X_2, we write $X_1 \hookrightarrow X_2$. The following theorem is the main result of this paper.

Theorem 4. *Let A and A' be subsets of $\mathbb{R}^n$ satisfying one of the conditions below.*

(i) $A \in \{B^n_1, B^n_\infty\}$ and A' is a convex or concave balanced region,

(ii) A is a convex symmetric region, and $A' \in \{B^n_1, B^n_\infty\}$,

(iii) A is a convex symmetric region, and A' is a concave symmetric region,

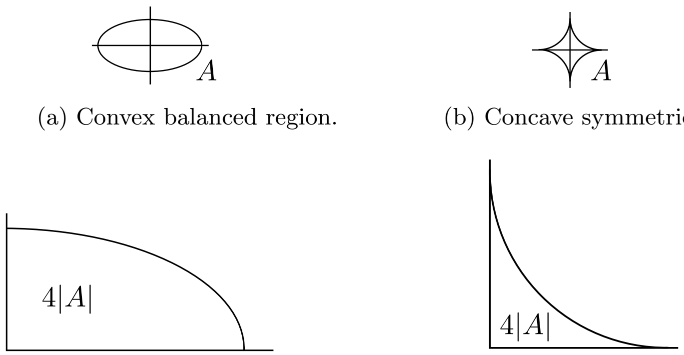
(iv) $A = B^n_p$ and $A' = r \cdot B^n_q$ for some $p, q \in [1, \infty]$ and $r \in]0, \infty[$.

Then

$$B^n_\infty \times_L A \hookrightarrow B^n_\infty \times_L A' \iff A \subset A'.$$

Remark 5. To the best of our knowledge, Theorem 4 is one of very few symplectic embedding results in dimensions greater than four, particularly for (families of) bounded sets. Some results in higher dimensions can be found in [6, 11, 19, 21].

The proof of Theorem 4 goes in two steps. First, we prove the existence of a symplectomorphism between any lagrangian product of the form



(a) Convex balanced region.
(b) Concave symmetric region.
(c) Convex toric domain.
(d) Concave symmetric toric domain.

Figure 1. Balanced regions and their corresponding toric domains

$B^n_\infty \times_L A$, where A is balanced, and an appropriate toric domain (see Definition 6 and Theorem 7). This allows to reformulate Theorem 4 in terms of symplectic embeddings between certain toric domains and their moment map images (see Theorem 11). To solve the latter problem, we use two symplectic capacities to show that we cannot do better than inclusion in the corresponding cases: the Gromov width and the cube capacity. The latter was recently introduced in [20].

1.2. Toric domains

Consider the standard integrable toric action $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \curvearrowright \left(\mathbb{R}^{2n} = \mathbb{C}^n, \omega_0\right)$ defined by

$$(\theta_1, \dots, \theta_n) \cdot (z_1, \dots, z_n) = \left(e^{2\pi i\theta_1} z_1, \dots, e^{2\pi i\theta_n} z_n\right).$$

Identifying the dual of the Lie algebra of $\mathbb{T}^n$ with $\mathbb{R}^n$, one of the moment maps for the above action is $\boldsymbol{\mu}(z_1, \dots, z_n) = \left(\pi|z_1|^2, \dots, \pi|z_n|^2\right)$.

Definition 6. Given an open subset $\Omega \subset \mathbb{R}^n_{\geq 0}$, the *toric domain associated to* Ω is the symplectic manifold $(X_\Omega, \omega_\Omega)$, where $X_\Omega := \boldsymbol{\mu}^{-1}(\Omega)$ and $\omega_\Omega = \omega_0|_{X_\Omega}$.

Henceforth, to simplify notation, we denote the toric domain associated to Ω by X_Ω.

Given a balanced region $A \subset \mathbb{R}^n$, we set $|A| := A \cap \mathbb{R}^n_{\geq 0}$. We note that, since A is balanced, A is determined by $|A|$; moreover, $|A| \subset \mathbb{R}^n_{\geq 0}$ is open. For any subset $U \subset \mathbb{R}^n$, we set

$$4U := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \left(\frac{1}{4} x_1, \ldots, \frac{1}{4} x_n \right) \in U \right\}.$$

The following result is the first step towards proving Theorem 4.

Theorem 7. *Let $A \subset \mathbb{R}^n$ be a balanced region. Then there is a symplectomorphism*

$$B^n_\infty \times_L A \cong X_{4|A|}.$$

Figures 1 (c) and (d) show the moment map images of the toric domains obtained from the regions in Figures 1 (a) and (b) respectively. The proof of Theorem 7, carried out in Section 3, uses the integrability of a system which models billiards on an interval and the fact that we decompose $B^n_\infty \times_L \mathbb{R}^n$ as a product of n symplectic factors (see Section 3). This is the main novelty of this paper and might be of independent interest. In spirit, it is a similar result to the existence of a symplectomorphism between the lagrangian bidisk and a concave toric domain proved in [30], although the integrable system in [30] is different from the one in the current paper.

Remark 8. Some particular cases of Theorem 7 are known or could be easily deduced from existing results in the literature. For instance, the ideas in [31, Section 2] allow to prove Theorem 7 in the case in which A is a parallelepiped. Besides this family, little seems to be known in general, although it is worth mentioning that, if $n = 2$, [25, Corollary 4.2] proves the case $A = B^2_1$ by using a non-trivial result due to McDuff (cf. [27, Theorem 1.1]), which is intrinsically different from our constructive techniques and only applicable in this specific case.

Assuming Theorem 7, Theorem 4 can be restated in terms of toric domains. To this end, we introduce the following notion.

Definition 9. Given a convex (respectively concave) balanced region $A \subset \mathbb{R}^n$, $X_{|A|}$ is said to be a *convex* (respectively *concave*) toric domain. If, in addition, A is symmetric, $X_{|A|}$ is said to be *symmetric*.

Remark 10. If $\Omega \subset \mathbb{R}^n_{\geq 0}$ is open and bounded, the existence of a balanced region A such that $\Omega = |A|$ is equivalent to the following condition:

$$(x_1, \ldots, x_n) \in \Omega \Rightarrow [0, x_1] \times \cdots \times [0, x_n] \subset \Omega. \tag{1}$$

In particular, the notions of convex and concave toric domains from [7, 23] coincide with those of Definition 9, except that we consider open domains instead of compact domains. We note that the definition of convex toric domain in [10] is slightly different and allows for toric domains that do not satisfy (1).

For any $n \geq 1$ and any $p \in [1, \infty]$, we set $\Omega^n_p := |B^n_p|$. Assuming Theorem 7, Theorem 4 is a straightforward consequence of the following result.

Theorem 11. *Let Ω and Ω' be open subsets of $\mathbb{R}^n_{\geq 0}$ satisfying one of the conditions below.*

(i) $\Omega \in \{\Omega^n_1, \Omega^n_\infty\}$ *and* $X_{\Omega'}$ *is a convex or concave toric domain,*

(ii) X_Ω *is a convex symmetric toric domain and* $\Omega' \in \{\Omega^n_1, \Omega^n_\infty\}$,

(iii) X_Ω *is a convex symmetric toric domain and* $X_{\Omega'}$ *is a concave symmetric toric domain,*

(iv) $\Omega = \Omega^n_p$ *and* $\Omega' = r \cdot \Omega^n_q$ *for some* $p, q \in [1, \infty]$ *and* $r \in]0, \infty[$.

Then

$$X_\Omega \hookrightarrow X_{\Omega'} \iff \Omega \subset \Omega',$$

Remark 12. The domains $X_{\Omega^n_1}$ and $X_{\Omega^n_\infty}$ are usually known as the ball $B(1) = E(1, \ldots, 1)$ and the polydisk $P(1, \ldots, 1)$, respectively.

1.3. Symplectic capacities

The proof of Theorem 11 provided below uses symplectic capacities. A symplectic capacity is a map c from a certain class of symplectic manifolds to $[0, \infty]$ with the following properties:

(a) $c(X, r \cdot \omega) = r \cdot c(X, \omega)$, for $r \in]0, \infty[$;

(b) $(X_1, \omega_1) \hookrightarrow (X_2, \omega_2) \Rightarrow c(X_1, \omega_1) \leq c(X_2, \omega_2)$.

For star-shaped domains $X \subset \mathbb{R}^{2n}$ equipped with the standard symplectic form[1], the following quantities are symplectic capacities:

$$c_1(X) = \sup\left\{r \in \mathbb{R} \mid X_{r\cdot\Omega_1^n} \hookrightarrow X\right\},$$
$$c_\infty(X) = \sup\left\{r \in \mathbb{R} \mid X_{r\cdot\Omega_\infty^n} \hookrightarrow X\right\}.$$

Remark 13. The capacity $c_1(X)$ was first introduced by Gromov in [16] and is known in the literature as the Gromov width of X, while $c_\infty(X)$ is shown to be a capacity by Gutt and Hutchings in [20] and is the analog of $c_1(X)$ for a cube.

The following result is a simple consequence of some results in [20].

Theorem 14. *Let X_Ω be a convex or concave toric domain. Then*

$$c_1(X_\Omega) = \max\{r \in \mathbb{R} \mid r \cdot \Omega_1^n \subset \Omega\}, \tag{2}$$
$$c_\infty(X_\Omega) = \max\{r \in \mathbb{R} \mid r \cdot \Omega_\infty^n \subset \Omega\}. \tag{3}$$

Proof. In [20] Gutt and Hutchings define a normalized symplectic capacity c_1^{SH} for star-shaped domains in $\mathbb{R}^{2n}$, *i.e.* on balls and cylinders, c_1^{SH} agrees with c_1. In particular, for any star-shaped domain $X \subset \mathbb{R}^{2n}$,

$$c_1(X) \leq c_1^{SH}(X). \tag{4}$$

Moreover, they show in [20, Theorem 1.6] that for a convex toric domain X_Ω,

$$c_1^{SH}(X_\Omega) = \min_{i=1,\dots,n} \sup\{r \in \mathbb{R} \mid r \cdot e_i \in \Omega\}.$$

From the convexity of Ω and the definition of c_1, we obtain

$$c_1^{SH}(X_\Omega) = \max\{r \in \mathbb{R} \mid r \cdot \Omega_1^n \subset \Omega\} \leq c_1(X_\Omega). \tag{5}$$

Combining (4) and (5), we obtain (2) for a convex toric domain. For a concave toric domain, (2) is a direct consequence of [20, Corollary 1.16].

To complete the proof, observe that [20, Theorem 1.18] gives (3) for convex and concave toric domains. □

The capacities c_1 and c_∞ can be used to prove Theorem 11, which, in turn, provides a proof of the main result of this paper assuming Theorem 7.

[1]Since we only use capacities of subsets of $\mathbb{R}^{2n}$ equipped with the standard symplectic form, we drop the symplectic form from the notation.

Proof of Theorem 11. Given open subsets $\Omega, \Omega' \subset \mathbb{R}^n_{\geq 0}$, the inclusion $\Omega \subset \Omega'$ implies the existence of a symplectic embedding $X_\Omega \hookrightarrow X_{\Omega'}$ (without imposing any restrictions). Therefore, it remains to prove the other implication. To this end, suppose that $X_\Omega \hookrightarrow X_{\Omega'}$ where Ω and Ω' satisfy one of the conditions (i) – (iv). The aim is to show that $\Omega \subset \Omega'$. We proceed case by case.

(i) This is a direct consequence of Theorem 14. First consider the case $\Omega = \Omega_1^n$. It follows from (2) that $c_1(X_{\Omega_1^n}) = 1$. So $1 = c_1(X_{\Omega_1^n}) \leq c_1(X_{\Omega'})$. Again from (2) we obtain $\Omega_1^n \subset \Omega'$. The case $\Omega = \Omega_\infty^n$ is dealt with analogously using c_∞ and (3).

(ii) Suppose first that $\Omega' = \Omega_1^n$. It follows from (3) that $1/n = c_\infty(X_{\Omega_1^n}) \geq c_\infty(X_\Omega)$. Since Ω is symmetric and convex, it follows from (3) that Ω lies below the hyperplane normal to $(1, \ldots, 1)$ passing through the point $(c_\infty(X_\Omega), \ldots, c_\infty(X_\Omega))$. So $x_1 + \cdots + x_n < n \cdot c_\infty(X_\Omega) \leq 1$ for all $(x_1, \ldots, x_n) \in \Omega$. Therefore $\Omega \subset \Omega_1^n$.

On the other hand, suppose that $\Omega' = \Omega_\infty^n$. If $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, Ω being symmetric implies that for all $i, j = 1, \ldots, n$,

$$\sup\{r > 0 \mid r \cdot e_i \in \Omega\} = \sup\{r > 0 \mid r \cdot e_j \in \Omega\}.$$

In particular, since Ω is convex, it follows from (2) that, for any $i = 1, \ldots, n$, $c_1(X_\Omega) = \sup\{r > 0 \mid r \cdot e_i \in \Omega\}$. Therefore, $\Omega \subset [0, c_1(X_\Omega)]^n$; since $c_1(X_\Omega) \leq c_1(X_{\Omega_\infty^n}) = 1$, it follows that $\Omega \subset [0, 1]^n$. As Ω is open, $\Omega \subset [0, 1[^n = \Omega_\infty^n$ as desired.

(iii) Arguing as in the first part of (ii), it follows that if $X_{\Omega'}$ is concave and symmetric, then $c_\infty(X_{\Omega'}) \cdot \Omega_1^n \subset \Omega'$. Since $X_\Omega \hookrightarrow X_{\Omega'}$, it follows from (ii) that

$$\Omega \subset c_\infty(X_\Omega) \cdot \Omega_1^n \subset c_\infty(X_{\Omega'}) \cdot \Omega_1^n \subset \Omega'.$$

(iv) Suppose first that $p \leq q$. It follows from (2) that $c_1(\Omega_p^n) = c_1(\Omega_q^n) = 1$. So $\Omega_p^n \hookrightarrow r \cdot \Omega_q^n$ implies that $1 \leq r$. Since $p \leq q$, we conclude that $\Omega_p^n \subset \Omega_q^n \subset r \cdot \Omega_q^n$.

Suppose that $q \leq p$. From (3) we obtain $c_\infty(X_{\Omega_s^n}) = \frac{1}{n^{1/s}}$. So

$$\frac{1}{n^{1/p}} \leq \frac{r}{n^{1/q}}. \tag{6}$$

Let $(x_1,\dots,x_n)\in\Omega_p^n$. It follows from Hölder's inequality and (6) that

$$\sum_{i=1}^n\left(\frac{x_i}{r}\right)^q\le\frac{1}{r^q}\left(\sum_{i=1}^n x_i^p\right)^{\frac{q}{p}}n^{\frac{p-q}{p}}\le\left(\frac{n^{\frac{p-q}{pq}}}{r}\right)^q\le 1,$$

which implies that $(x_1,\dots,x_n)\in r\cdot\Omega_q^n$. □

1.4. Outline of the paper

The rest of this paper is structured as follows. In Section 2, we give another application of Theorems 4 and 7 to symplectic embeddings in dimension 4. Section 3 contains the proof of Theorem 7, which relies on understanding a family of integrable systems modeling the billiard in the interval.

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2. Symplectic embeddings

In this section, we discuss the rigidity of some symplectic embeddings involving the lagrangian bidisk studied in [30], proving that both rigidity and flexibility occur (see Theorem 17 and Corollary 18). First we introduce an equivalence relation for four-dimensional lagrangian products.

Definition 15. Let A_1, B_1, A_2 and B_2 be open sets of $\mathbb{R}^2$ containing the origin. The lagrangian products $A_1\times_L B_1$ and $A_2\times_L B_2$ are *equivalent* if there exist $a>0$ and $U\in SO(2)$ such that $A_1=aU\cdot A_2$ and $B_1=a^{-1}U\cdot B_2$, or $B_1=aU\cdot A_2$ and $A_1=a^{-1}U\cdot B_2$. In this case, we write $A_1\times_L B_1\sim A_2\times_L B_2$

Observe that two equivalent lagrangian products are symplectomorphic.

Definition 16. Let A, B, C and D be connected, open sets of $\mathbb{R}^2$ containing the origin. The symplectic embedding problem $A\times_L B\overset{?}{\hookrightarrow}C\times_L D$ is *rigid* if

$$A \times_L B \hookrightarrow (aC) \times_L D \text{ for some } a > 0$$
$$\Rightarrow A \times_L B \sim A' \times_L B' \subset (aC') \times_L D' \sim C \times_L D,$$

for some open subsets A', B', C', D' of $\mathbb{R}^2$ containing the origin.

The following theorem is the main result of this section.

Theorem 17. *For any $p \in [2, +\infty]$ the symplectic embedding problems*

$$B_2^2 \times_L B_2^2 \overset{?}{\hookrightarrow} B_\infty^2 \times_L B_p^2 \quad \textit{and} \quad B_\infty^2 \times_L B_p^2 \overset{?}{\hookrightarrow} B_2^2 \times_L B_2^2$$

are rigid.

Combining Theorems 4 and 17, we obtain the following result.

Corollary 18. *For any $p, q, r, s \in \{1, 2, \infty\}$ with*

$$(7) \qquad (p, q, r, s) \notin \{(1, \infty, 2, 2), (\infty, 1, 2, 2), (2, 2, 1, \infty), (2, 2, \infty, 1)\},$$

the symplectic embedding problem $B_p^2 \times_L B_q^2 \overset{?}{\hookrightarrow} B_r^2 \times_L B_s^2$ is rigid.

In fact, Corollary 18 is optimal, in the sense that if (7) does not hold, then $B_p^2 \times_L B_q^2 \overset{?}{\hookrightarrow} B_r^2 \times_L B_s^2$ is not rigid (see Section 2.2).

2.1. The lagrangian bidisk

The lagrangian bidisk $B_2^2 \times_L B_2^2$ is the only lagrangian product that appears in Theorem 17 and not in Theorem 4. While the techniques of the present paper do not allow to identify the lagrangian bidisk with a toric domain, the first author proved in [30] that $B_2^2 \times_L B_2^2$ can be endowed with an effective Hamiltonian $\mathbb{T}^2$-action. This is the content of the following result, stated below without proof.

Theorem 19 ([30, Theorem 3]). *Let $\Omega_0 \subset \mathbb{R}^n_{\geq 0}$ be the open subset of bounded by the coordinate axes and the curve parametrized by*

$$\gamma(\alpha) = 2\left(\sin\alpha - \alpha\cos\alpha, \sin\alpha + (\pi - \alpha)\cos\alpha\right), \quad \alpha \in [0, \pi].$$

Then $B_2^2 \times_L B_2^2$ is symplectomorphic to the toric domain X_{Ω_0}.

Proof of Theorem 17. Fix $p \in [2, \infty[$. It follows from Theorem 19 and formulae (2) and (3) that

$$c_1\left(B_2^2 \times_L B_2^2\right) = c_1(X_{\Omega_0}) = 4, \tag{8}$$
$$c_\infty\left(B_2^2 \times_L B_2^2\right) = c_\infty(X_{\Omega_0}) = 2. \tag{9}$$

Suppose first that $B_2^2 \times_L B_2^2 \hookrightarrow B_\infty^2 \times_L aB_p^2$ for some $a > 0$. It follows from Theorems 7 and 19 that $X_{\Omega_0} \hookrightarrow X_{4a\cdot\Omega_p^2}$. From (2) and (8) we obtain

$$4 = c_1(X_{\Omega_0}) \leq 4a \cdot c_1(X_{\Omega_p^2}) = 4a.$$

So $a \geq 1$. Since $B_2^2 \subset B_p^2 \subset B_\infty^2$, it follows that $B_2^2 \times_L B_2^2 \subset B_\infty^2 \times_L B_p^2$. Therefore $B_2^2 \times_L B_2^2 \overset{?}{\hookrightarrow} B_\infty^2 \times_L B_p^2$ is rigid.

Suppose that $B_\infty^2 \times_L B_p^2 \hookrightarrow aB_2^2 \times_L B_2^2$ for some $a > 0$, so that $X_{4\cdot\Omega_p^2} \hookrightarrow X_{a\cdot\Omega_0}$. From (3) and (9) we obtain

$$\frac{4}{2^{1/p}} = c_\infty(X_{\Omega_p^2}) \leq a \cdot c_\infty(X_{\Omega_0}) = 2a.$$

So $a \geq \frac{2}{2^{1/p}}$. It follows from a simple calculation that

$$B_p^2 \subset \frac{2^{1/2}}{2^{1/p}} B_2^2 \quad \text{and} \quad B_\infty^2 \subset 2^{1/2} B_2^2.$$

Hence

$$B_\infty^2 \times_L B_p^2 \subset \left(\frac{2^{1/2}}{2^{1/p}} B_2^2\right) \times_L \left(2^{1/2} B_2^2\right) \sim \frac{2}{2^{1/p}} B_2^2 \times_L \times B_2^2.$$

Therefore $B_p^2 \times_L B_\infty^2 \overset{?}{\hookrightarrow} B_2^2 \times_L B_2^2$ is rigid.

The case $p = \infty$ can be dealt with analogously by substituting Ω_p^2 by Ω_∞^2 and $1/p$ by 0 in the above calculations. $\square$

2.2. Diamonds, disks and squares

The aim of this section is to prove Corollary 18 and explain why it is optimal.

Proof of Corollary 18. Begin by observing that there exist $a > 0$ and $U \in SO(2)$ such that $B_1^2 = aU \cdot B_\infty^2$, and that B_2^2 is $SO(2)$-invariant. So

$$B_2^2 \times_L B_1^2 \sim B_\infty^2 \times_L aB_2^2 \sim aB_\infty^2 \times_L B_2^2 \sim B_1^2 \times_L B_2^2, \tag{10}$$

Moreover

$$B_1^2 \times_L B_1^2 \sim a^2 B_\infty^2 \times_L B_\infty^2 \quad \text{and} \quad B_\infty^2 \times_L B_1^2 \sim B_\infty^2 \times_L B_1^2. \tag{11}$$

Fix $p, q, r, s \in \{1, 2, \infty\}$ satisfying (7). It follows from (10) and (11) that the $B_p^2 \times_L B_q^2 \overset{?}{\hookrightarrow} B_r^2 \times_L B_s^2$ is equivalent to one of the embedding problems considered either in Theorem 4 or in Theorem 17. $\square$

It remains to show that the symplectic embedding problems

$$B_\infty^2 \times_L B_1^2 \overset{?}{\hookrightarrow} B_2^2 \times_L B_2^2 \quad \text{and} \quad B_2^2 \times_L B_2^2 \overset{?}{\hookrightarrow} B_\infty^2 \times_L B_1^2$$

are not rigid. It follows from a simple calculation that $4\Omega_1^2 \subset \Omega_0$. So $B_\infty^2 \times_L B_1^2 \hookrightarrow B_2^2 \times_L B_2^2$. However, if $B_1^2 \subset aB_2^2$ and $B_\infty^2 \subset bB_2^2$, then $a \geq 1$ and $b \geq \sqrt{2}$. So $ab \geq \sqrt{2} > 1$. Therefore the embedding problem $B_\infty^2 \times_L B_1^2 \overset{?}{\hookrightarrow} B_2^2 \times_L B_2^2$ is not rigid.

On the other hand, it is shown in [30] that $X_{\Omega_0} \hookrightarrow X_{3\sqrt{3}\cdot\Omega_1^2}$. So $B_2^2 \times_L B_2^2 \hookrightarrow B_\infty^2 \times_L \frac{3\sqrt{3}}{4} B_1^2$. However, if $B_2^2 \subset aB_1^2$ and $B_2^2 \subset bB_\infty^2$, then $a \geq \sqrt{2}$ and $b \geq 1$ and hence

$$ab \geq \sqrt{2} > \frac{3\sqrt{3}}{4}.$$

Therefore the embedding problem $B_2^2 \times_L B_2^2 \hookrightarrow B_1^2 \times_L B_\infty^2$ is not rigid.

Remark 20. We could say that an embedding problem of toric domains $X_\Omega \overset{?}{\hookrightarrow} X_{\Omega'}$ is rigid if

$$X_\Omega \hookrightarrow X_{a\cdot\Omega'} \Rightarrow \Omega \subset a \cdot \Omega'.$$

Based on the calculations above, $B_\infty^2 \times_L B_1^2 \overset{?}{\hookrightarrow} B_2^2 \times_L B_2^2$ is not rigid as an embedding problem of lagrangian products, but it is rigid as an embedding problem of toric domains. However, $B_2^2 \times_L B_2^2 \overset{?}{\hookrightarrow} B_\infty^2 \times_L B_1^2$ is not rigid in either sense.

3. From balanced regions to toric domains

The aim of this section is to prove Theorem 7, thus completing the proof of the main result of the paper, Theorem 4. Our strategy to prove Theorem 7 is inspired by some of the arguments in [30, Section 2] and can be broken down in the following three steps:

(i) For any $n \geq 1$ and any $\varepsilon > 0$, we define an integrable system $\Phi_\varepsilon : B^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^n$ related to n uncoupled billiards in the interval. We prove that, for any $\varepsilon > 0$, $\Phi_\varepsilon : B^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^n$ is isomorphic to $\boldsymbol{\mu} : \mathbb{R}^{2n} \to \mathbb{R}^n$, the integrable system obtained by considering (one of) the moment map(s) of the standard Hamiltonian $\mathbb{T}^n$-action on $\mathbb{R}^{2n}$ (see Section 1.2). If $(\boldsymbol{\Psi}_\varepsilon, \mathbf{I}_\varepsilon)$ denotes the above isomorphism for a fixed ε, we also show that, in some sense, the maps $\mathbf{I}_\varepsilon$ possess a limit as $\varepsilon \to 0$, which we denote by $\mathbf{I}_0$.

(ii) Fix $n \geq 1$ and a balanced region $A \subset \mathbb{R}^n$. Using the family of isomorphisms of integrable systems $(\boldsymbol{\Psi}_\varepsilon, \mathbf{I}_\varepsilon)$ and the map $\mathbf{I}_0$ of (i), we construct a family of nested symplectic submanifolds of $B^n_\infty \times_L A$ parametrized by ε whose images under $\boldsymbol{\Psi}_\varepsilon$ are a nested family of symplectic submanifolds exhausting $X_{4|A|}$ (see Lemma 44 for a precise statement).

(iii) Using the symplectic isotopy extension theorem (cf. [3, Proposition 4] and [4]) and the compact exhaustions of (ii), we construct the desired symplectomorphism between $B^n_\infty \times_L A$ and $X_{4|A|}$.

The structure of this section is as follows. Section 3.1 constructs the desired family of integrable systems on $B^n_\infty \times_L \mathbb{R}^n$, while Section 3.2 deals with steps (ii) and (iii), thus completing the proof of Theorem 7 and, hence, of the main result, Theorem 4.

3.1. A family of integrable systems on $B^n_\infty \times_L \mathbb{R}^n$

3.1.1. The category of integrable systems. Before constructing the desired family of integrable systems on $B^n \times_L \mathbb{R}^n$, we recall some basic notions pertaining to integrable systems that are used throughout the paper.

Definition 21. An *integrable system* on a $2n$-dimensional symplectic manifold (M, ω) is a smooth map

$$\Phi := (h_1, \dots, h_n) : (M, \omega) \to \mathbb{R}^n$$

satisfying the following conditions

- for all $i, j = 1, \dots, n$, $\{h_i, h_j\} = 0$, where $\{\cdot, \cdot\}$ is the Poisson bracket on $C^\infty(M)$ induced by ω, and
- the map Φ is a submersion on a dense subset of M.

Example 22. For the purposes of this paper, the following are important examples of integrable systems:

(a) If $n = 1$, an integrable system on a surface (M, ω) is a function $H \in C^\infty(M)$ whose differential does not vanish on a dense subset.

(b) For $i = 1, 2$, let $\Phi_i : (M_i, \omega_i) \to \mathbb{R}^{n_i}$ be an integrable system. Then the map $\Phi := (\Phi_1, \Phi_2) : (M_1 \times M_2, \omega_1 \oplus \omega_2) \to \mathbb{R}^{n_1+n_2}$ is an integrable system, where $\omega_1 \oplus \omega_2 = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2$ and, for $i = 1, 2$, $\mathrm{pr} : M_1 \times M_2 \to M_i$ denotes the canonical projection.

(c) A *symplectic toric manifold* is a triple (M, ω, μ), where (M, ω) is a $2n$-dimensional symplectic manifold and μ is the moment map of an effective Hamiltonian $\mathbb{T}^n$-action on (M, ω). Given a symplectic toric manifold (M, ω) and identifying the dual of the Lie algebra of $\mathbb{T}^n$ with $\mathbb{R}^n$, the map $\mu : (M, \omega) \to \mathbb{R}^n$ defines an integrable system. (The fact that μ is a submersion on a dense set follows from the Marle-Guillemin-Sternberg local normal form for effective Hamiltonian actions, cf. [17, 26].) In particular, the following maps define integrable systems:
   - the moment map $\boldsymbol{\mu} : \mathbb{R}^{2n} \to \mathbb{R}^n$ of the standard Hamiltonian $\mathbb{T}^n$-action on $\mathbb{R}^{2n}$, and
   - the moment map of the cotangent lift of the $\mathbb{T}^n$-action on $\mathbb{T}^n$ by left (or right) multiplication. Using the canonical trivialization $T^*\mathbb{T}^n \cong \mathbb{R}^n \times \mathbb{T}^n$ so that the canonical symplectic form becomes $\sum_{i=1}^n da_i \wedge d\theta_i$, this moment map becomes the projection

$$\mathrm{pr}_1 : \left(\mathbb{R}^n \times \mathbb{T}^n, \sum_{i=1}^n da_i \wedge d\theta_i\right) \to \mathbb{R}^n$$

   onto the first component.

An important role in this paper is played by the following notion of equivalence of integrable systems.

Definition 23. Two integrable systems $\Phi_1 : (M_1, \omega_1) \to \mathbb{R}^{n_1}$ and $\Phi_2 : (M_2, \omega_2) \to \mathbb{R}^{n_2}$ are *isomorphic* if there exists a pair (Ψ, g) consisting of a symplectomorphism $\Psi : (M_1, \omega_1) \to (M_2, \omega_2)$ and a diffeomorphism[2] $g : \Phi_1(M_1) \to \Phi_2(M_2)$ such that $\Phi_2 \circ \Psi = g \circ \Phi_1$.

[2]A map $g : C \subset \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ between is said to be *smooth* if there exists an open set V containing C and a smooth map $\tilde{g} : U \to \mathbb{R}^{n_2}$ that extends g.

Remark 24. The above notion of isomorphism of integrable systems behaves well with respect to the product construction (b) of Example 22. More precisely, if, for $i = 1, 2$, (Ψ_i, g_i) is an isomorphism between $\Phi_i : (M_i, \omega_i) \to \mathbb{R}^{n_i}$ and $\Phi'_i : (M'_i, \omega'_i) \to \mathbb{R}^{n'_i}$, then the pair $(\Psi_1 \times \Psi_2, g_1 \times g_2)$ is an isomorphism between $\Phi = (\Phi_1, \Phi_2) : (M_1 \times M_2, \omega_1 \oplus \omega_2) \to \mathbb{R}^{n_1+n_2}$ and $\Phi' = (\Phi'_1, \Phi'_2) : (M'_1 \times M'_2, \omega'_1 \oplus \omega'_2) \to \mathbb{R}^{n'_1+n'_2}$.

Another construction that is relevant for our purposes is that of restricting integrable systems to suitable subsets.

Definition 25. Given an integrable system $\Phi : (M, \omega) \to \mathbb{R}^n$ and an open subset $U \subset M$, the *subsystem relative to* U is the integrable system $\Phi|_U : (U, \omega|_U) \to \mathbb{R}^n$.

For any $n \geq 1$, the family of integrable systems $\Phi_\varepsilon : B^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^n$ that we are interested in is going to be constructed using the product construction (b) of Example 22, since $B^n_\infty \times_L \mathbb{R}^n$ is symplectomorphic to the (symplectic) product of n copies of $B^1_\infty \times_L \mathbb{R}$. Thus, firstly we define the relevant family of integrable systems and prove all desired properties in the case $n = 1$ (see Section 3.1.2), and, secondly, we consider the general case (see Section 3.1.3).

3.1.2. The one dimensional case. For any $\varepsilon > 0$, consider the integrable system $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$, where $H_\varepsilon(x, y) = \frac{1}{2}\left(y^2 + \varepsilon \frac{1}{1-x^2}\right)$ and let x, y denote canonical coordinates on $\mathbb{R}^2$.

Remark 26. The family of integrable systems $\left\{H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}\right\}_{\varepsilon>0}$ is related to the dynamics of the billiard in the interval $[-1, +1]$ as follows. Firstly, observe that $B^1_\infty \times_L \mathbb{R}$ is symplectomorphic to $\left(T^* B^1_\infty, \omega_{\mathrm{can}}\right)$. Secondly, the potential $V(x) = \frac{1}{2(1-x^2)}$ satisfies the properties to fit in the approximation scheme first introduced in [5] that allow us to study the billiard in the interval $[-1, +1]$ as a limit of Hamiltonian systems on the cotangent bundle of B^1_∞.

The following result, stated below without proof, establishes some basic properties of $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ for any $\varepsilon > 0$.

Proposition 27. *For any* $\varepsilon > 0$,

(1) the image of H_ε *equals* $\left[\frac{\varepsilon}{2}, +\infty\right[$;

(2) the only singular point of H_ε *is* $(0, 0)$, *which equals the fiber* $H_\varepsilon^{-1}\left(\frac{\varepsilon}{2}\right)$;

(3) the Hessian of H_ε at $(0,0)$ is positive definite;

(4) the map $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ is proper.

Remark 28. Property (3) is equivalent to stating that, for any $\varepsilon > 0$, the point $(0,0)$ is a *non-degenerate* singular point of *elliptic type* for the integrable system $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$, cf. [15, Introduction] and [13, Section I.3] for more details.

An immediate consequence of Proposition 27 is the following simple, yet useful, result.

Corollary 29. *For any $\varepsilon > 0$, the fibers of H_ε are compact and connected.*

Proof. Fix $\varepsilon > 0$. Property (4) implies that the fibers of H_ε are compact. Using property (2), it remains to check that the fiber $H_\varepsilon^{-1}(c)$, for $c \in \left]\frac{\varepsilon}{2}, +\infty\right[$, is connected. To this end, consider the restriction of H_ε to $\left(B^1_\infty \times_L \mathbb{R}\right) \smallsetminus \{(0,0)\}$ as a map onto $\left]\frac{\varepsilon}{2}, +\infty\right[$. This is a proper surjective submersion by properties (2) and (4); thus it is a locally trivial fiber bundle. Since its codomain is simply connected and its domain is connected, the long exact sequence in homotopy for the above restriction implies that, for all $c \in \left]\frac{\varepsilon}{2}, +\infty\right[$, $H_\varepsilon^{-1}(c)$ is connected, as desired. $\square$

Proposition 27 and Corollary 29 provide a complete topological description of the map H_ε for any $\varepsilon > 0$: for any $c \in \left]\frac{\varepsilon}{2}, +\infty\right[$, the fiber $H_\varepsilon^{-1}(c)$ is regular and diffeomorphic to S^1, while $H_\varepsilon^{-1}\left(\frac{\varepsilon}{2}\right)$ is a point.

Next we study the symplectic geometry of $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ for a fixed $\varepsilon > 0$. Since the regular fibers of H_ε are compact and connected, the Liouville-Arnol'd theorem ensures the existence of *local action-angle variables* (cf. [1, Section 50], [14], [18, Section 44] for details in general). For the case at hand, this can be phrased as follows. By Properties (1) and (2) in Proposition 27, the intersection of the set of regular values of H_ε with $H_\varepsilon\left(B^1_\infty \times \mathbb{R}\right)$ equals $\left]\frac{\varepsilon}{2}, +\infty\right[$. Then the Liouville-Arnol'd theorem applied to $H_\varepsilon : B^1_\infty \times \mathbb{R} \to \mathbb{R}$ yields the following result, stated below without proof.

Lemma 30. *For any $\varepsilon > 0$ and for any $c_0 \in \left]\frac{\varepsilon}{2}, +\infty\right[$, there exist an open neighborhood $U \subset \left]\frac{\varepsilon}{2}, +\infty\right[$ of c_0, a local diffeomorphism $I^U_\varepsilon : U \to \mathbb{R}$, and a symplectomorphism $\Psi^U_\varepsilon : \left(H_\varepsilon^{-1}(U), dx \wedge dy\right) \to \left(I^U_\varepsilon(U) \times S^1, da \wedge d\theta\right)$ such that $\left(\Psi^U_\varepsilon, I^U_\varepsilon\right)$ is an isomorphism between the subsystems of $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ and $\mathrm{pr}_1 : \left(\mathbb{R} \times S^1, da \wedge d\theta\right) \to \mathbb{R}$ relative to $H_\varepsilon^{-1}(U)$ and $\mathrm{pr}^{-1}\left(I_\varepsilon(U)\right)$ respectively.*

Remark 31. The smooth map I_ε^U of Lemma 30 is referred to as a *local action* near c_0; an explicit, well-known formula for I_ε^U is given by

$$I_\varepsilon^U(c) = \oint_{H_\varepsilon^{-1}(c)} y(x,c)dx, \tag{12}$$

where $y(x,c)$ is the smooth function defined implicitly by the equation $H_\varepsilon(x,y) = c$ (cf. [1, Section 50])[3]. Moreover, the map

$$I_\varepsilon^U \circ H_\varepsilon : \big(H_\varepsilon^{-1}(U), dx \wedge dy\big) \to \mathbb{R}$$

is the moment map of an effective Hamiltonian S^1-action. This is because $\big(\psi_\varepsilon^U, I_\varepsilon^U\big)$ is an isomorphism of integrable systems and $\big(I_\varepsilon^U(U) \times S^1, da \wedge d\theta, \mathrm{pr}_1\big)$ is a symplectic toric manifold (see Example 22(c)).

A priori, Lemma 30 only holds locally, *i.e.* in a neighborhood of any given regular value. In general, there are well-known topological obstructions to gluing these local isomorphisms (cf. [14]). However, in the case at hand the situation is particularly simple.

Corollary 32. *For any $\varepsilon > 0$, there exist a smooth map $I_\varepsilon : \left]\frac{\varepsilon}{2}, +\infty\right[\to \mathbb{R}$ which is a diffeomorphism onto its image, and a symplectomorphism $\Psi_\varepsilon : \big(B^1_\infty \times_L \mathbb{R}\big) \smallsetminus \{(0,0)\} \to \big(I_\varepsilon\big(\left]\frac{\varepsilon}{2}, +\infty\right[\big) \times S^1, da \wedge d\theta\big)$ such that $(\Psi_\varepsilon, I_\varepsilon)$ is an isomorphism between the subsystems of $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ and $\mathrm{pr}_1 : \big(\mathbb{R} \times S^1, da \wedge d\theta\big) \to \mathbb{R}$ relative to $H_\varepsilon^{-1}\big(\left]\frac{\varepsilon}{2}, +\infty\right[\big)$ and $\mathrm{pr}_1^{-1}\big(I_\varepsilon\big(\left]\frac{\varepsilon}{2}, +\infty\right[\big)\big)$ respectively. In particular, the map $I_\varepsilon \circ H_\varepsilon : \big(B^1_\infty \times_L \mathbb{R}\big) \smallsetminus \{(0,0)\} \to \mathbb{R}$ is the moment map of an effective Hamiltonian S^1-action.*

Proof. Fix $\varepsilon > 0$. The topological obstructions to gluing the local isomorphisms of Lemma 30 depend on the topology of the intersection of the set of regular values of H_ε with the image of H_ε (cf. [14]). In particular, they vanish if this intersection is contractible. Therefore, since the intersection under consideration equals $\left]\frac{\varepsilon}{2}, +\infty\right[$, the local isomorphisms of Lemma 30 can be glued together to obtain an isomorphism $(\Psi_\varepsilon, I_\varepsilon)$ between the subsystems of $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ and $\mathrm{pr}_1 : \big(\mathbb{R} \times S^1, da \wedge d\theta\big) \to \mathbb{R}$ relative to $H_\varepsilon^{-1}\big(\left]\frac{\varepsilon}{2}, +\infty\right[\big)$ and $I_\varepsilon\big(\left]\frac{\varepsilon}{2}, +\infty\right[\big)$ respectively. It remains to show that I_ε is a diffeomorphism onto its image. Since, for any open subset U as in Lemma 30, $I_\varepsilon|_U = I_\varepsilon^U$, for any $c \in \left]\frac{\varepsilon}{2}, +\infty\right[$, $I'_\varepsilon(c) \neq 0$. Connectedness of $\left]\frac{\varepsilon}{2}, +\infty\right[$

[3]Equation (12) differs by the standard formula for local actions by a factor of 2π (cf. [1, Section 50]). This is due to the fact that, in this paper, we identify S^1 with $\mathbb{R}/\mathbb{Z}$ while it is customary in the literature to use the identification $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$.

gives that I_ε is strictly monotone and, therefore, a diffeomorphism onto its image. $\square$

Remark 33. As a consequence of the proof of Corollary 32, the right hand side of (12) equals the function $I_\varepsilon(c)$ for any $c \in \left]\frac{\varepsilon}{2}, +\infty\right[$. Substituting the function $y(x,c)$ obtained by solving explicitly $H_\varepsilon(x,y) = c$ in equation (12), we obtain

$$
\begin{aligned}
I_\varepsilon(c) &= 4 \int_0^{\sqrt{1-\frac{\varepsilon}{2c}}} \sqrt{2c - \frac{\varepsilon}{1-x^2}}\, dx \\
&= 4\sqrt{2c-\varepsilon} \int_0^{\sqrt{1-\frac{\varepsilon}{2c}}} \sqrt{1 - \frac{\varepsilon x^2}{(2c-\varepsilon)(1-x^2)}}\, dx.
\end{aligned}
\tag{13}
$$

Formula (13) gives that the action I_ε varies continuously with ε.

For each $\varepsilon > 0$, Corollary 32 describes the symplectic geometry of the restriction of the integrable system $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ to its regular points. In fact, it is possible to strengthen Corollary 32 to provide a description of the integrable system that includes its singular point; this can be achieved by exploiting the *linearization* results for non-degenerate singular elliptic points (cf. [13, 15] for further details in general). For the purposes at hand, it suffices to state the linearization result in the simplest case, which is a consequence of the main theorem in [9].

Theorem 34 (Colin de Verdière and Vey, [9]). *Let $H : \left(\mathbb{R}^2, dx \wedge dy\right) \to \mathbb{R}$ be an integrable system such that $(0,0)$ is a singular point of H and the Hessian of H at $(0,0)$ is positive definite. Then there exist open neighborhoods $U \subset H(\mathbb{R}^2)$, $V \subset [0,+\infty[$ of $H(0,0)$ and of 0 respectively, a local diffeomorphism $I : U \to V$ with $I(H(0,0)) = 0$, and a symplectomorphism $\Psi : \left(H^{-1}(U), dx \wedge dy\right) \to \left(\mu^{-1}(V), du \wedge dv\right)$ such that (Ψ, I) is an isomorphism between the subsystems of $H : \left(\mathbb{R}^2, dx \wedge dy\right) \to \mathbb{R}$ and of $\mu : \left(\mathbb{R}^2, du \wedge dv\right) \to \mathbb{R}$ relative to $H^{-1}(U)$ and $\mu^{-1}(V)$ respectively, where $\mu(u,v) = \pi\left(u^2+v^2\right)$. In particular, $I \circ H : \left(H^{-1}(U), dx \wedge dy\right) \to \mathbb{R}$ is the moment map of an effective Hamiltonian S^1-action.*

Applying Theorem 34 to the family of integrable systems $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$, we obtain the following result.

Lemma 35. *For each $\varepsilon > 0$, the action function $I_\varepsilon : \left]\frac{\varepsilon}{2}, +\infty\right[\to \mathbb{R}$ of Corollary 32 extends to a smooth function defined on $\left[\frac{\varepsilon}{2}, +\infty\right[$.*

Proof. Fix $\varepsilon > 0$. The integrable system $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses of Theorem 34. Therefore it is possible to find open neighborhoods $U \subset \left[\frac{\varepsilon}{2}, +\infty\right[$ and $V \subset [0, +\infty[$ of $\frac{\varepsilon}{2}$ and 0 respectively, and an isomorphism (Ψ, I) between the subsystems of $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ and of $\mu : \left(\mathbb{R}^2, du \wedge dv\right) \to \mathbb{R}$ relative to $H_\varepsilon^{-1}(U)$ and $\mu^{-1}(V)$ respectively. Shrinking U if needed, it may be assumed that U is connected and that I is a diffeomorphism onto V. By abuse of notation denote the restrictions of I and Ψ to $U \cap \left]\frac{\varepsilon}{2}, +\infty\right[$ and $H_\varepsilon^{-1}\left(U \cap \left]\frac{\varepsilon}{2}, +\infty\right[\right)$ respectively by I and Ψ.

Since $\mathrm{pr}_1 \circ \Psi_\varepsilon \circ \Psi^{-1}$ is the moment map of an effective Hamiltonian S^1-action, so is $I_\varepsilon \circ I^{-1} \circ \mu$. Moreover, if $\mathcal{X}_\mu$ and $\mathcal{X}_{I_\varepsilon \circ I^{-1} \circ \mu}$ denote the Hamiltonian vector fields of the functions μ and $\mathcal{X}_{I_\varepsilon \circ I^{-1} \circ \mu}$ respectively, then

$$\mathcal{X}_{I_\varepsilon \circ I^{-1} \circ \mu} = \left(\frac{d\left(I_\varepsilon \circ I^{-1}\right)}{dc} \circ \mu\right) \mathcal{X}_\mu.$$

Since μ and $\frac{d(I_\varepsilon \circ I^{-1})}{dc} \circ \mu$ Poisson commute and are moment maps of effective Hamiltonian S^1-actions, it follows that the function $\frac{d(I_\varepsilon \circ I^{-1})}{dc} \circ \mu$ takes values in $\{\pm 1\}$. Since $\mu^{-1}(V) \smallsetminus \{(0,0)\}$ is connected, then $\frac{d(I_\varepsilon \circ I^{-1})}{dc} \circ \mu$ is constant. Moreover, since μ is a submersion restricted to $\mu^{-1}(V) \smallsetminus \{(0,0)\}$, it follows that $\frac{d(I_\varepsilon \circ I^{-1})}{dc}$ is constant and equal to ± 1. Thus the function $I_\varepsilon \circ I^{-1}$ is the restriction of an element h of $\mathrm{AGL}(1;\mathbb{Z}) := \mathrm{GL}(1;\mathbb{Z}) \ltimes \mathbb{R}$ to $V \smallsetminus \{0\}$. In particular, since I can be extended smoothly at $\frac{\varepsilon}{2}$, so can I_ε, which proves the desired result. $\square$

By abuse of notation, denote the extension given by Lemma 35 also by $I_\varepsilon : \left[\frac{\varepsilon}{2}, +\infty\right[\to \mathbb{R}$.

Lemma 36. *For a fixed $\varepsilon > 0$, the map $I_\varepsilon : \left[\frac{\varepsilon}{2}, +\infty\right[\to [0, +\infty[$ is a diffeomorphism.*

Proof. Fix $\varepsilon > 0$. The proofs of Theorem 34 and of Lemma 35 imply that the derivative of I_ε at $\frac{\varepsilon}{2}$ does not vanish, which, together with Corollary 32, gives that I'_ε does not vanish on $\left[\frac{\varepsilon}{2}, +\infty\right[$. To prove the desired result, it suffices to show that $I_\varepsilon\left(\frac{\varepsilon}{2}\right) = 0$, that I_ε is strictly increasing, and that the image of I_ε is not bounded. To prove the first result, we have to show that $\lim\limits_{c \to \frac{\varepsilon}{2}^+} I_\varepsilon(c) = 0$. Using equation (13), it suffices to prove that the integral

$\int\limits_0^{\sqrt{1-\frac{\varepsilon}{2c}}} \sqrt{1-\frac{\varepsilon x^2}{(2c-\varepsilon)(1-x^2)}}dx$ is bounded. Since the integrand is non-negative, this integral is certainly non-negative; on the other hand, the integrand is less than 1, which implies that the integral is, in fact, bounded as required. This shows that $I_\varepsilon\left(\frac{\varepsilon}{2}\right)=0$.

Let $c_1 > c_2 \geq \frac{\varepsilon}{2}$. Then

$$\begin{aligned} I_\varepsilon(c_1) &= 4\int\limits_0^{\sqrt{1-\frac{\varepsilon}{2c_1}}} \sqrt{2c_1-\frac{\varepsilon}{1-x^2}}dx > 4\int\limits_0^{\sqrt{1-\frac{\varepsilon}{2c_2}}} \sqrt{2c_1-\frac{\varepsilon}{1-x^2}}dx \\ &> 4\int\limits_0^{\sqrt{1-\frac{\varepsilon}{2c_2}}} \sqrt{2c_2-\frac{\varepsilon}{1-x^2}}dx = I_\varepsilon(c_2), \end{aligned}$$

where the first inequality follows from the fact that the function $\sqrt{2c_1-\frac{\varepsilon}{1-x^2}}$ is positive on $\left[\sqrt{1-\frac{\varepsilon}{2c_2}}, \sqrt{1-\frac{\varepsilon}{2c_1}}\right]$, while the second is a consequence of the fact that $c_1 > c_2$ implies that, for all $x \in \left[0, \sqrt{1-\frac{\varepsilon}{2c_2}}\right]$, $\sqrt{2c_1-\frac{\varepsilon}{1-x^2}} > \sqrt{2c_2-\frac{\varepsilon}{1-x^2}}$. Therefore, I_ε is strictly increasing as desired.

Finally, to see that I_ε is unbounded, observe that, by equation (13), it suffices to show that, for all c sufficiently large, the integral

$$\int\limits_0^{\sqrt{1-\frac{\varepsilon}{2c}}} \sqrt{1-\frac{\varepsilon x^2}{(2c-\varepsilon)(1-x^2)}}dx$$

is bounded away from 0. To this end, observe that this integral depends continuously on c so that, as $c \to +\infty$, the above integral tends to 1, thus implying the desired property. □

In order to strengthen Corollary 32 to include the singular point of $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$, we need the following result, which is a consequence of [24, Theorem 1.3] (and a generalization of the well-known classification of *compact* symplectic toric manifolds due to Delzant, cf. [12]), and is stated below without proof.

Theorem 37. *For $i = 1, 2$, let (M_i, ω_i, μ_i) be a symplectic toric manifold with connected fibers with $\mu_i(M_i)$ contractible. Then there exists a symplectomorphism $\Psi : (M_1, \omega_1) \to (M_2, \omega_2)$ with $\mu_2 \circ \Psi = \mu_1$ if and only if $\mu_1(M_1) = \mu_2(M_2)$.*

Remark 38. While not explicitly stated in [24], it follows from ideas therein that if $\{(M, \omega, \mu_\varepsilon)\}_{\varepsilon>0}$ is a family of symplectic toric manifolds depending continuously on a parameter ε such that

- for all $\varepsilon > 0$, the fibers of μ_ε are connected, and
- there exists a symplectic toric manifold (M', ω', μ') with $\mu'(M') = \mu_\varepsilon(M)$ for all $\varepsilon > 0$,

then the family of symplectomorphisms $\Psi_\varepsilon : (M, \omega) \to (M', \omega')$ can be chosen to depend continuously on ε.

In analogy with Corollary 32, we have the following result describing the symplectic geometry of the integrable system $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$.

Corollary 39. *For any $\varepsilon > 0$, there exists a symplectomorphism $\Psi_\varepsilon : B^1_\infty \times_L \mathbb{R} \to (\mathbb{R}^2, du \wedge dv)$ such that $(\Psi_\varepsilon, I_\varepsilon)$ is an isomorphism between $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ and $\mu : (\mathbb{R}^2, du \wedge dv) \to \mathbb{R}$, where $\mu(u, v) = \pi(u^2 + v^2)$. Moreover, the family $\{\Psi_\varepsilon\}_{\varepsilon>0}$ may be chosen to depend continuously on ε.*

Proof. Fix $\varepsilon > 0$. By construction, the composite $I_\varepsilon \circ H_\varepsilon$ is the moment map of an effective Hamiltonian S^1-action with connected fibers whose image equals $[0, +\infty[$ by Lemma 36. Thus $(B^1_\infty \times_L \mathbb{R}, dx \wedge dy, I_\varepsilon \circ H_\varepsilon)$ and $(\mathbb{R}^2, du \wedge dv, \mu)$ are symplectic toric manifolds satisfying the hypotheses of Theorem 37. Seeing as they have equal moment map images, Theorem 37 ensures the existence of the desired symplectomorphism Ψ_ε. The fact that Ψ_ε may be chosen to depend continuously on ε follows from Remark 38. $\square$

An important consequence of Corollary 39, which plays a key role in the proof of Theorem 7, is the following result.

Proposition 40. *For all $\varepsilon_1 > \varepsilon_2$ and for all $c \geq \frac{\varepsilon_1}{2}$,*

$$\Psi_{\varepsilon_1}\left(H^{-1}_{\varepsilon_1}\left(\left[\frac{\varepsilon_1}{2}, c\right[\right)\right) \subset \Psi_{\varepsilon_2}\left(H^{-1}_{\varepsilon_2}\left(\left[\frac{\varepsilon_2}{2}, c\right[\right)\right). \tag{14}$$

Proof. Fix $\varepsilon_1 > \varepsilon_2$ and $c \geq \frac{\varepsilon_1}{2}$. Firstly, observe that (14) is equivalent to

$$\mu^{-1}\left([0, I_{\varepsilon_1}(c)[\right) \subset \mu^{-1}\left([0, I_{\varepsilon_2}(c)[\right). \tag{15}$$

This can be seen as follows: for $i = 1, 2$, we have that

$$\mu \circ \Psi_{\varepsilon_i}\left(H_{\varepsilon_i}^{-1}\left(\left[\frac{\varepsilon_i}{2}, c\right[\right)\right) = I_{\varepsilon_i}\left(\left[\frac{\varepsilon_i}{2}, c\right[\right) = [0, I_{\varepsilon_i}(c)[\,; \tag{16}$$

the first equality follows from Corollary 39, while the second from Lemma 36. Corollary 39 also implies that, for $i = 1, 2$, the subset $\Psi_{\varepsilon_i}\left(H_{\varepsilon_i}^{-1}\left(\left[\frac{\varepsilon_i}{2}, c\right[\right)\right)$ is saturated with respect to μ. This fact, together with equation (16) implies the inclusion of equation (14) holds if and only if that of equation (15) does. To show that equation (15) is true, it suffices to prove that $[0, I_{\varepsilon_1}(c)[\, \subset [0, I_{\varepsilon_2}(c)[$ or, equivalently, that $I_{\varepsilon_1}(c) < I_{\varepsilon_2}(c)$. The proof of this last statement is analogous to an argument used in Lemma 36. Using equation (13), it can be seen that, for fixed c, $I_\varepsilon(c)$ is a continuous, decreasing function of ε. This yields the desired result. □

To conclude this section, we observe that equation (13) implies that, in some sense, the family of diffeomorphisms $\{I_\varepsilon\}_{\varepsilon>0}$ converges uniformly as ε goes to 0.

Lemma 41. *For all $c > 0$, $\lim_{\varepsilon\to 0^+} I_\varepsilon(c) = 4\sqrt{2c} =: I_0(c)$. Moreover, for any $\varepsilon_0 > 0$, any decreasing sequence ε_k converging to 0 with the property that $\varepsilon_0 > \varepsilon_1$, and any compact subset $K \subset \mathbb{R}_{\geq 0}$, $I_{\varepsilon_k} \to I_0$ uniformly in the set $K \cap \left[\frac{\varepsilon_0}{2}, +\infty\right[$.*

Proof. Fix $c > 0$. Then c is in the domain of I_ε for all ε sufficiently small; therefore, it makes sense to consider $\lim_{\varepsilon\to 0^+} I_\varepsilon(c)$. The result follows from observing that I_ε depends continuously on ε; thus equation (13) yields that

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(c) = 4\int_0^1 \sqrt{2c}\, dx = 4\sqrt{2c}.$$

This proves the first assertion. To prove the second, fix ε_0, a decreasing sequence ε_k converging to 0 and a compact set K as in the statement. Then $K' := K \cap \left[\frac{\varepsilon_0}{2}, +\infty\right[$ is compact, the family of functions $\{I_{\varepsilon_k}|_{K'}\}_n$ is monotone (see the proof of Proposition 40), and the function $I_0|_{K'}$ is continuous. The result then follows by Dini's theorem. □

3.1.3. The general case. For any $n \geq 1$, consider the family of smooth maps $\{\Phi_\varepsilon : B^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^n\}_{\varepsilon>0}$, where

$$\Phi_\varepsilon(\mathbf{x}, \mathbf{y}) = (H_\varepsilon(x_1, y_1), \ldots, H_\varepsilon(x_n, y_n)), \tag{17}$$

and $H_\varepsilon : B^1_\infty \times_L \mathbb{R} \to \mathbb{R}$ is the smooth function introduced in Section 3.1.2. Viewing $B^n_\infty \times_L \mathbb{R}^n$ as the symplectic product of n copies of $B^1_\infty \times_L \mathbb{R}$, it follows from the construction (b) in Example 22 that, for each $\varepsilon > 0$, $\Phi_\varepsilon : B^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^n$ is an integrable system. In fact, much more is true.

Corollary 42. *For any $\varepsilon > 0$, there exist a diffeomorphism $\mathbf{I}_\varepsilon : \left[\frac{\varepsilon}{2}, +\infty\right[^n \to [0, +\infty[^n$ and a symplectomorphism $\mathbf{\Psi}_\varepsilon : B^n_\infty \times_L \mathbb{R}^n \to \left(\mathbb{R}^{2n}, \omega_0\right)$ such that $(\mathbf{\Psi}_\varepsilon, \mathbf{I}_\varepsilon)$ is an isomorphism between*

$$\Phi_\varepsilon : B^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^n \quad \textit{and} \quad \boldsymbol{\mu} : \left(\mathbb{R}^{2n}, \sum_{i=1}^n du_i \wedge dv_i\right) \to \mathbb{R}^n,$$

where $\boldsymbol{\mu}(\mathbf{u}, \mathbf{v}) = \pi\left(u_1^2 + v_1^2, \ldots, u_n^2 + v_n^2\right)$. In particular, $\mathbf{I}_\varepsilon \circ \Phi_\varepsilon$ is the moment map of an effective Hamiltonian $\mathbb{T}^n$-action on $B^n \times_L \mathbb{R}^n$. Moreover, the family $\{\mathbf{\Psi}_\varepsilon\}_{\varepsilon>0}$ may be chosen to depend continuously on ε.

Proof. Setting $\mathbf{I}_\varepsilon(\mathbf{c}) := (I_\varepsilon(c_1), \ldots, I_\varepsilon(c_n))$ and $\mathbf{\Psi}_\varepsilon(\mathbf{x}, \mathbf{y}) := (\Psi_\varepsilon(x_1, y_1), \ldots, \Psi_\varepsilon(x_n, y_n))$, where I_ε and Ψ_ε are as in equation (13) and Corollary 39 respectively, the desired result follows from Remark 24 and Corollary 39. $\square$

Lemma 41 and Corollary 42 imply that the family of diffeomorphisms $\{\mathbf{I}_\varepsilon\}_{\varepsilon>0}$ converges uniformly as ε goes to 0.

Corollary 43. *For any $\mathbf{c} \in]0, +\infty[^n$, $\lim_{\varepsilon\to 0^+} \mathbf{I}_\varepsilon(\mathbf{c}) = \mathbf{I}_0(\mathbf{c})$, where $\mathbf{I}_0(\mathbf{c}) := (I_0(c_1), \ldots, I_0(c_n))$ and $I_0(c) = 4\sqrt{2c}$. Moreover, for any $\varepsilon_0 > 0$, any decreasing sequence ε_k converging to 0 with $\varepsilon_0 > \varepsilon_1$, and any compact subset $K \subset \mathbb{R}^n_{\geq 0}$, $\mathbf{I}_{\varepsilon_k} \to \mathbf{I}_0$ uniformly in $K \cap \left[\frac{\varepsilon_0}{2}, +\infty\right[^n$.*

Proof. The first statement is an immediate consequence of Lemma 41 and Corollary 42. The second statement follows similarly upon observing that, without loss of generality, it may be assumed that K is of the form $K_1 \times \cdots \times K_n \subset \mathbb{R}_{\geq 0} \times \cdots \times \mathbb{R}_{\geq 0} = \mathbb{R}^n_{\geq 0}$, where, for each $i = 1, \ldots, n$, $K_i \subset \mathbb{R}_{\geq 0}$ is compact. $\square$

3.2. Constructing the symplectomorphism

The aim of this section is to prove Theorem 7, which endows any lagrangian product of the form $B^n_\infty \times_L A$, where $A \subset \mathbb{R}^n$ is a balanced region (see Definition 2), with an effective Hamiltonian $\mathbb{T}^n$-action. As a first step, we construct a suitable compact exhaustion of any lagrangian product of the above form (see Step (ii)). Henceforth, given $B \subset \mathbb{R}^l$, we denote its closure by $\mathrm{cl}\,(B)$.

Lemma 44. *For any balanced region $A \subset \mathbb{R}^n$, there exists a family of symplectic submanifolds $\{P_\varepsilon\}_{\varepsilon>0}$ of $B^n_\infty \times_L A$, with compact closure in $B^n_\infty \times_L A$, satisfying the following properties:*

(a) $\bigcup_{\varepsilon>0} \mathrm{cl}\,(P_\varepsilon) = B^n_\infty \times_L A$ *and* $\bigcup_{\varepsilon>0} \mathbf{\Psi}_\varepsilon\,(\mathrm{cl}\,(P_\varepsilon)) = X_{4|A|}$*;*

(b) if $\varepsilon_1 > \varepsilon_2$*, then* $\mathrm{cl}\,(P_{\varepsilon_1}) \subset \mathrm{cl}\,(P_{\varepsilon_2})$ *and* $\mathbf{\Psi}_{\varepsilon_1}\,(\mathrm{cl}\,(P_{\varepsilon_1})) \subset \mathbf{\Psi}_{\varepsilon_2}\,(\mathrm{cl}\,(P_{\varepsilon_2}))$*,*

where $4|A| \subset \mathbb{R}^n_{\geq 0}$ *is as in Section 1.2, and* $\left\{\mathbf{\Psi}_\varepsilon : B^n_\infty \times_L \Sigma \to \mathbb{R}^{2n}\right\}_{\varepsilon>0}$ *is the family of symplectomorphisms of Corollary 42 depending continuously on* ε*.*

Proof. Fix a balanced region $A \subset \mathbb{R}^n$. For any $\varepsilon > 0$, let $\Phi_\varepsilon : B^n_\infty \times_L \mathbb{R}^n \to \mathbb{R}^n$ be the integrable system defined by equation (17). For $\varepsilon > 0$, set

$$P_\varepsilon := \Phi_\varepsilon^{-1}\left(\mathbf{I}_0^{-1}\,(4|A|)\right) \subset B^n_\infty \times_L \mathbb{R}^n,$$

where $\mathbf{I}_0 : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is the map of Corollary 43. The claim is that $\{P_\varepsilon\}_{\varepsilon>0}$ is the required family. Begin by observing that, since A is open, so is $4|A| \subset \mathbb{R}^n_{\geq 0}$. Continuity of Φ_ε for any $\varepsilon > 0$ and of $\mathbf{I}_0$ implies that, for each $\varepsilon > 0$, P_ε is an open subset of $B^n_\infty \times_L \mathbb{R}^n$ and, thus, a symplectic submanifold of $B^n_\infty \times_L \mathbb{R}^n$. For each $\varepsilon > 0$, the closure of P_ε is mapped to the closure of $4|A|$ in $\mathbb{R}^n_{\geq 0}$ under $\mathbf{I}_0 \circ \Phi_\varepsilon$. Since A is bounded, so is $4|A|$ is bounded, which implies that $\mathrm{cl}\,(4|A|) \subset \mathbb{R}^n_{\geq 0}$ is compact. Moreover, the maps Φ_ε and $\mathbf{I}_0$ are proper, the former by Property (4) of Proposition 27 and by construction, while the latter by virtue of being a homeomorphism onto a closed subset of $\mathbb{R}^n$. Therefore, for each $\varepsilon > 0$, the closure of P_ε is contained in a compact subset and is, therefore, compact. To simplify the argument of the rest of the proof, we deal with each statement separately.

Claim 45. $\bigcup_{\varepsilon>0} \mathrm{cl}\,(P_\varepsilon) = B^n_\infty \times_L A.$

Proof of Claim 45. Fix $\varepsilon > 0$ and let $(\mathbf{x}, \mathbf{y}) \in \mathrm{cl}\,(P_\varepsilon)$. By definition, $\mathbf{x} \in B^n_\infty$ and $\mathbf{I}_0\left(\Phi_\varepsilon(\mathbf{x}, \mathbf{y})\right) \in \mathrm{cl}\,(4|A|)$. Using the definition of $\mathbf{I}_0$ and Φ_ε, the latter condition gives that

$$\left(4\sqrt{y_1^2 + \frac{\varepsilon}{1 - x_1^2}}, \ldots, 4\sqrt{y_n^2 + \frac{\varepsilon}{1 - x_n^2}}\right) \in \mathrm{cl}\,(4|A|)\,. \tag{18}$$

However, since $4|A|$ satisfies (1), equation (18) implies that

$$\left[0, 4\sqrt{y_1^2 + \frac{\varepsilon}{1 - x_1^2}}\right[\times \cdots \times \left[0, 4\sqrt{y_n^2 + \frac{\varepsilon}{1 - x_n^2}}\right[\subset 4|A|,$$

which, in particular, yields that $(4|y_1|, \ldots, 4|y_n|) \in 4|A|$. By definition of $4|A|$, this last condition gives that $\mathbf{y} \in A$. Thus $(\mathbf{x}, \mathbf{y}) \in B^n_\infty \times_L A$; since $(\mathbf{x}, \mathbf{y}) \in \mathrm{cl}\,(P_\varepsilon)$ and $\varepsilon > 0$ are arbitrary, for all $\varepsilon > 0$, $\mathrm{cl}\,(P_\varepsilon) \subset B^n_\infty \times_L A$. Hence, for all $\varepsilon > 0$, P_ε is a symplectic submanifold of $B^n_\infty \times_L A$ with compact closure in $B^n_\infty \times_L A$, and $\bigcup_{\varepsilon > 0} \mathrm{cl}\,(P_\varepsilon) \subset B^n_\infty \times_L A$.

It remains to prove the opposite inclusion. Suppose that $(\mathbf{x}, \mathbf{y}) \in B^n_\infty \times_L A$. Then, by definition, $(4|y_1|, \ldots, 4|y_n|) \in 4|A|$; since $4|A| \subset \mathbb{R}^n_{\geq 0}$ is open, for all sufficiently small $\varepsilon > 0$, $\left(4\sqrt{y_1^2 + \frac{\varepsilon}{1 - x_1^2}}, \ldots, 4\sqrt{y_n^2 + \frac{\varepsilon}{1 - x_n^2}}\right) \in 4|A|$, which is equivalent to $(\mathbf{x}, \mathbf{y}) \in P_\varepsilon$. Since $(\mathbf{x}, \mathbf{y}) \in B^n_\infty \times_L A$ is arbitrary, this gives that $B^n_\infty \times_L A \subset \bigcup_{\varepsilon > 0} P_\varepsilon \subset \bigcup_{\varepsilon > 0} \mathrm{cl}\,(P_\varepsilon)$. ☐

Claim 46. $\bigcup_{\varepsilon > 0} \mathbf{\Psi}_\varepsilon\left(\mathrm{cl}\,(P_\varepsilon)\right) = X_{4|A|}$.

Proof of Claim 46. Fix $\varepsilon > 0$; firstly we show that $\mathbf{\Psi}_\varepsilon\left(\mathrm{cl}\,(P_\varepsilon)\right) \subset X_{4|A|}$. Since $X_{4|A|}$ is saturated with respect to $\boldsymbol{\mu}$, it suffices to prove that $\boldsymbol{\mu}\left(\mathbf{\Psi}_\varepsilon\left(\mathrm{cl}\,(P_\varepsilon)\right)\right) \subset \boldsymbol{\mu}\left(X_{4|A|}\right) = 4|A|$. By Corollary 42, $\boldsymbol{\mu} \circ \mathbf{\Psi}_\varepsilon = \mathbf{I}_\varepsilon \circ \Phi_\varepsilon$ and, by definition, $P_\varepsilon = \Phi_\varepsilon^{-1}\left(\mathbf{I}_0^{-1}(4|A|)\right)$, so that $\Phi_\varepsilon\left(\mathrm{cl}\,(P_\varepsilon)\right) \subset \mathbf{I}_0^{-1}\left(\mathrm{cl}\,(4|A|)\right)$; therefore it suffices to prove that $\mathbf{I}_\varepsilon\left(\mathbf{I}_0^{-1}\left(\mathrm{cl}\,(4|A|)\right)\right) \subset 4|A|$. In fact, since the domain of $\mathbf{I}_\varepsilon$ is $\left[\frac{\varepsilon}{2}, +\infty\right[^n$, it suffices to show that

$$\mathbf{I}_\varepsilon\left(\mathbf{I}_0^{-1}\left(\mathrm{cl}\,(4|A|)\right) \cap \left[\frac{\varepsilon}{2}, +\infty\right[^n\right) \subset 4|A|.$$

Suppose that $\mathbf{a} \in \mathrm{cl}\,(4|A|)$ is such that $\mathbf{I}_0^{-1}(\mathbf{a}) \in \left[\frac{\varepsilon}{2}, +\infty\right[^n$; the aim is to show that $\mathbf{I}_\varepsilon\left(\mathbf{I}_0^{-1}(\mathbf{a})\right) \in 4|A|$. Observe that, by definition of $\mathbf{I}_0$ (see Corollary 43), $\mathbf{I}_0^{-1}(\mathbf{a}) = \left(I_0^{-1}(a_1), \ldots, I_0^{-1}(a_n)\right)$; moreover, by definition of $\mathbf{I}_\varepsilon$ (see the proof of Corollary 42),

$$\mathbf{I}_\varepsilon\left(\mathbf{I}_0^{-1}(\mathbf{a})\right) = \left(I_\varepsilon\left(I_0^{-1}(a_1)\right), \ldots, I_\varepsilon\left(I_0^{-1}(a_n)\right)\right).$$

By assumption, for each $i = 1, \ldots, n$, $I_0^{-1}(a_i) \geq \frac{\varepsilon}{2}$. The definitions of I_ε and of I_0 (see (13) and Lemma 41) imply that, for all $i = 1, \ldots, n$, $I_\varepsilon\left(I_0^{-1}(a_i)\right) < I_0\left(I_0^{-1}(a_i)\right) = a_i$. In particular,

$$\mathbf{I}_\varepsilon\left(\mathbf{I}_0^{-1}(\mathbf{a})\right) \in [0, a_1[\times \cdots \times [0, a_n[\,;$$

on the other hand, the right hand side of the above equation is a subset of $4|A|$ since $\mathbf{a} \in \mathrm{cl}\,(4|A|)$ and $4|A|$ satisfies (1). Thus $\mathbf{I}_\varepsilon\left(\mathbf{I}_0^{-1}(\mathbf{a})\right) \in 4|A|$; since $\mathbf{a} \in 4|A|$ is arbitrary, the above argument shows that $\mathbf{I}_\varepsilon\left(\mathbf{I}_0^{-1}(\mathrm{cl}\,(4|A|))\right) \subset 4|A|$ and, therefore, $\boldsymbol{\Psi}_\varepsilon(\mathrm{cl}\,(P_\varepsilon)) \subset X_{4|A|}$. Since $\varepsilon > 0$ is arbitrary, we have that $\bigcup_{\varepsilon>0} \boldsymbol{\Psi}_\varepsilon(\mathrm{cl}\,(P_\varepsilon)) \subset X_{4|A|}$.

To prove the opposite inclusion, suppose that $\mathbf{z} \in X_{4|A|}$; it suffices to show that there exists $\varepsilon > 0$ and $(\mathbf{x}, \mathbf{y}) \in P_\varepsilon \subset \mathrm{cl}\,(P_\varepsilon)$ such that $\boldsymbol{\Psi}_\varepsilon(\mathbf{x}, \mathbf{y}) = \mathbf{z}$. By Corollary 42, we know that, for any $\varepsilon > 0$, there exists a unique point $(\mathbf{x}_\varepsilon, \mathbf{y}_\varepsilon) \in B^n_\infty \times_L \mathbb{R}^n$ with $\boldsymbol{\Psi}_\varepsilon(\mathbf{x}_\varepsilon, \mathbf{y}_\varepsilon) = \mathbf{z}$. Hence, it suffices to show that, for some $\varepsilon > 0$, $(\mathbf{x}_\varepsilon, \mathbf{y}_\varepsilon) \in P_\varepsilon$, which is equivalent to $\mathbf{I}_0(\Phi_\varepsilon(\mathbf{x}_\varepsilon, \mathbf{y}_\varepsilon)) \in 4|A|$, since P_ε is saturated with respect to Φ_ε. To see that this holds, we argue as follows. Choose a decreasing sequence ε_k converging to 0 and, for any k, set $\mathbf{c}_k = (c_{k,1}, \ldots, c_{k,n}) := \Phi_{\varepsilon_k}(\mathbf{x}_{\varepsilon_k}, \mathbf{y}_{\varepsilon_k})$, and $\boldsymbol{\mu}(\mathbf{z}) =: (a_1, \ldots, a_n)$. By assumption, we have that, for all k and all $i = 1, \ldots, n$, $I_{\varepsilon_k}(c_{k,i}) = a_i$. Let $l > k$ and suppose that there exists $i = 1, \ldots, n$ such that $c_{l,i} > c_{k,i}$. Then

$$a_i = I_{\varepsilon_k}(c_{k,i}) < I_{\varepsilon_l}(c_{k,i}) < I_{\varepsilon_l}(c_{l,i}) < a_i, \tag{19}$$

where the first inequality follows from the fact that if $\varepsilon > \varepsilon'$ and $c \geq \frac{\varepsilon}{2}$, then $I_\varepsilon(c) < I_{\varepsilon'}(c)$ (see the proof of Proposition 40), while the second follows from the fact that I_{ε_l} is a strictly increasing function (see the proof of Lemma 36). The inequalities (19) yield a contradiction; thus, for all $l > k$ and all $i = 1, \ldots, n$, $c_{l,i} \leq c_{k,i}$. Together with the fact that, for all k, $\mathbf{c}_k \in \mathbb{R}^n_{\geq 0}$, this fact implies that the sequence $\{\mathbf{c}_k\}_k \subset \mathbb{R}^n_{\geq 0}$ is bounded. Therefore, without loss of generality, it may be assumed that $\mathbf{c}_k \to \mathbf{c}_\infty \in \mathbb{R}^n_{\geq 0}$. Hence,

$$\lim_{k\to+\infty}\left(\lim_{j\to+\infty} \mathbf{I}_{\varepsilon_j}(\mathbf{c}_k)\right) = \lim_{k\to+\infty} \mathbf{I}_0(\mathbf{c}_k) = \mathbf{I}_0(\mathbf{c}_\infty), \tag{20}$$

where the first equality follows from Corollary 43 and the second from continuity of $\mathbf{I}_0$. On the other hand,

$$\lim_{k\to+\infty} \mathbf{I}_{\varepsilon_k}(\mathbf{c}_k) = \boldsymbol{\mu}(\mathbf{z}), \tag{21}$$

since, by definition, for all k, $\mathbf{I}_{\varepsilon_k}(\mathbf{c}_k) = \boldsymbol{\mu}(\mathbf{z})$. Comparing equations (20) and (21), we obtain that $\mathbf{I}_0(\mathbf{c}_\infty) = \boldsymbol{\mu}(\mathbf{z})$. Since $4|A| \subset \mathbb{R}^n_{\geq 0}$ is open and $\boldsymbol{\mu}(\mathbf{z}) \in 4|A|$, there exists a $\delta > 0$ such that if $\mathbf{a}' \in \mathbb{R}^n_{\geq 0}$ and $\|\boldsymbol{\mu}(\mathbf{z}) - \mathbf{a}'\| < \delta$, then $\mathbf{a}' \in 4|A|$. Choose k sufficiently large so that $\|\mathbf{I}_0(\mathbf{c}_\infty) - \mathbf{I}_0(\mathbf{c}_k)\| < \frac{\delta}{2}$; this can be achieved since $\mathbf{I}_0$ is continuous and $\mathbf{c}_k \to \mathbf{c}_\infty$ as $k \to +\infty$. Hence,

$$\|\boldsymbol{\mu}(\mathbf{z}) - \mathbf{I}_0(\Phi_{\varepsilon_k}(\mathbf{x}_{\varepsilon_k}, \mathbf{y}_{\varepsilon_k}))\| = \|\mathbf{I}_0(\mathbf{c}_\infty) - \mathbf{I}_0(\mathbf{c}_k)\| < \frac{\delta}{2}; \tag{22}$$

moreover, by definition of $\mathbf{I}_0$, $\mathbf{I}_0(\Phi_{\varepsilon_k}(\mathbf{x}_{\varepsilon_k}, \mathbf{y}_{\varepsilon_k})) \in \mathbb{R}^n_{\geq 0}$. Thus

$$\mathbf{I}_0(\Phi_{\varepsilon_k}(\mathbf{x}_{\varepsilon_k}, \mathbf{y}_{\varepsilon_k})) \in 4|A|$$

as desired, which, unraveling the above argument, implies that $X_{4|A|} \subset \bigcup_{\varepsilon>0} \boldsymbol{\Psi}_\varepsilon(P_\varepsilon) \subset \bigcup_{\varepsilon>0} \boldsymbol{\Psi}_\varepsilon(\mathrm{cl}(P_\varepsilon))$ and completes the proof. ☐

Claims 45 and 46 yield that the family of symplectic submanifolds $\{P_\varepsilon\}_{\varepsilon>0}$ satisfies property (a).

Claim 47. If $\varepsilon_1 > \varepsilon_2$, then $\mathrm{cl}(P_{\varepsilon_1}) \subset \mathrm{cl}(P_{\varepsilon_2})$.

Proof of Claim 47. Fix $\varepsilon_1 > \varepsilon_2$. It suffices to show that $P_{\varepsilon_1} \subset P_{\varepsilon_2}$. Fix $(\mathbf{x}, \mathbf{y}) \in P_{\varepsilon_1}$. By definition, $\mathbf{I}_0(\Phi_{\varepsilon_1}(\mathbf{x}, \mathbf{y})) \in 4|A|$, *i.e.*

$$\left(4\sqrt{y_1^2 + \frac{\varepsilon_1}{1-x_1^2}}, \dots, 4\sqrt{y_n^2 + \frac{\varepsilon_1}{1-x_n^2}}\right) \in 4|A|.$$

On the other hand, observe that, since $\varepsilon_1 > \varepsilon_2$, for all $i = 1, \dots, n$,

$$\sqrt{y_1^2 + \frac{\varepsilon_1}{1-x_1^2}} > \sqrt{y_1^2 + \frac{\varepsilon_2}{1-x_1^2}}.$$

Since $4|A|$ satisfies property (1), arguing as in the proof of Claim 45, we obtain that

$$\left(4\sqrt{y_1^2 + \frac{\varepsilon_2}{1-x_1^2}}, \dots, 4\sqrt{y_n^2 + \frac{\varepsilon_2}{1-x_n^2}}\right) \in 4|A|,$$

which gives that $\mathbf{I}_0(\Phi_{\varepsilon_2}(\mathbf{x},\mathbf{y})) \in 4|A|$. By definition of P_{ε_2}, $(\mathbf{x},\mathbf{y}) \in P_{\varepsilon_2}$. Since $(\mathbf{x},\mathbf{y}) \in P_{\varepsilon_1}$ is arbitrary, this shows that $P_{\varepsilon_1} \subset P_{\varepsilon_2}$ as desired. □

Claim 48. If $\varepsilon_1 > \varepsilon_2$, then $\boldsymbol{\Psi}_{\varepsilon_1}(\mathrm{cl}(P_{\varepsilon_1})) \subset \boldsymbol{\Psi}_{\varepsilon_2}(\mathrm{cl}(P_{\varepsilon_2}))$.

Proof of Claim 48. Fix $\varepsilon_1 > \varepsilon_2$. Since, for $i = 1, 2$, $\boldsymbol{\Psi}_{\varepsilon_i}$ is a homeomorphism, it suffices to show that $\boldsymbol{\Psi}_{\varepsilon_1}(P_{\varepsilon_1}) \subset \boldsymbol{\Psi}_{\varepsilon_2}(P_{\varepsilon_2})$. As, for $i = 1, 2$, the subset $\boldsymbol{\Psi}_{\varepsilon_i}(P_{\varepsilon_i})$ is saturated with respect to $\boldsymbol{\mu}$, in order to prove the desired result it suffices to show that $\boldsymbol{\mu}(\boldsymbol{\Psi}_{\varepsilon_1}(P_{\varepsilon_1})) \subset \boldsymbol{\mu}(\boldsymbol{\Psi}_{\varepsilon_2}(P_{\varepsilon_2}))$, which is equivalent to $\mathbf{I}_{\varepsilon_1}(\Phi_{\varepsilon_1}(P_{\varepsilon_1})) \subset \mathbf{I}_{\varepsilon_2}(\Phi_{\varepsilon_2}(P_{\varepsilon_2}))$ in light of Corollary 42. Observe that, for $i = 1, 2$, $\Phi_{\varepsilon_i}(P_{\varepsilon_i}) = \mathbf{I}_0^{-1}(4|A|) \cap \left[\frac{\varepsilon_i}{2}, +\infty\right[^n$; thus, since $\varepsilon_1 > \varepsilon_2$, $\Phi_{\varepsilon_1}(P_{\varepsilon_1}) \subset \Phi_{\varepsilon_2}(P_{\varepsilon_2})$. Let $\mathbf{c} = (c_1, \ldots, c_n) \in \Phi_{\varepsilon_1}(P_{\varepsilon_1}) \subset \Phi_{\varepsilon_2}(P_{\varepsilon_2})$; the fact that $4|A|$ satisfies property (1) implies that

$$\left[\frac{\varepsilon_2}{2}, c_1\right] \times \cdots \times \left[\frac{\varepsilon_2}{2}, c_n\right] \subset \Phi_{\varepsilon_2}(P_{\varepsilon_2}). \tag{23}$$

For, the condition $\mathbf{c} \in \Phi_{\varepsilon_2}(P_{\varepsilon_2})$ implies that $\mathbf{I}_0(\mathbf{c}) = \left(4\sqrt{2c_1}, \ldots, 4\sqrt{2c_n}\right) \in 4|A|$. Since $4|A|$ satisfies property (1), then

$$\left[0, 4\sqrt{2c_1}\right] \times \cdots \times \left[0, 4\sqrt{2c_n}\right] \subset 4|A|.$$

Thus

$$\mathbf{I}_0^{-1}\left(\left[0, 4\sqrt{2c_1}\right] \times \cdots \times \left[0, 4\sqrt{2c_n}\right]\right) \subset \mathbf{I}_0^{-1}(4|A|); \tag{24}$$

however, by definition of $\mathbf{I}_0$ (see Corollary 43),

$$\begin{aligned} &\mathbf{I}_0^{-1}\left(\left[0, 4\sqrt{2c_1}\right] \times \cdots \times \left[0, 4\sqrt{2c_n}\right]\right) \\ &= \left(I_0^{-1}\left(\left[0, 4\sqrt{2c_1}\right]\right)\right) \times \cdots \times \left(I_0^{-1}\left(\left[0, 4\sqrt{2c_n}\right]\right)\right) \\ &= [0, c_1] \times \cdots \times [0, c_n]. \end{aligned} \tag{25}$$

Equation (23) follows by combining equations (24) and (25) with the equality $\Phi_{\varepsilon_2}(P_{\varepsilon_2}) = \mathbf{I}_0^{-1}(4|A|) \cap \left[\frac{\varepsilon_2}{2}, +\infty\right[^n$. Equation (23) implies that

$$\begin{aligned} [0, I_{\varepsilon_2}(c_1)] \times \cdots \times [0, I_{\varepsilon_2}(c_n)] &= \mathbf{I}_{\varepsilon_2}\left(\left[\frac{\varepsilon_2}{2}, c_1\right] \times \cdots \times \left[\frac{\varepsilon_2}{2}, c_n\right]\right) \\ &\subset \mathbf{I}_{\varepsilon_2}(\Phi_{\varepsilon_2}(P_{\varepsilon_2})), \end{aligned} \tag{26}$$

where the first equality follows from the definition of $\mathbf{I}_{\varepsilon_2}$ and properties of I_{ε_2} (see the proof of Lemma 36). Since $\varepsilon_1 > \varepsilon_2$, the proof of Proposition 40 gives that, for all $i = 1, \ldots, n$, $I_{\varepsilon_1}(c_i) < I_{\varepsilon_2}(c_i)$, which, together with equation (26) gives that $\mathbf{I}_{\varepsilon_1}(\mathbf{c}) = (I_{\varepsilon_1}(c_1), \dots, I_{\varepsilon_1}(c_n)) \in \mathbf{I}_{\varepsilon_2}(\Phi_{\varepsilon_2}(P_{\varepsilon_2}))$. Since $\mathbf{c} \in \Phi_{\varepsilon_1}(P_{\varepsilon_1})$ is arbitrary, the above argument shows that $\mathbf{I}_{\varepsilon_1}(\Phi_{\varepsilon_1}(P_{\varepsilon_1})) \subset \mathbf{I}_{\varepsilon_2}(\Phi_{\varepsilon_2}(P_{\varepsilon_2}))$ as desired. □

Claims 47 and 48 yield that the family of symplectic submanifolds $\{P_\varepsilon\}_{\varepsilon>0}$ satisfies property (b). This completes the proof. □

Lemma 44 allows to prove Theorem 7.

Proof of Theorem 7. Fix a balanced region $A \subset \mathbb{R}^n$. The aim is to construct a symplectomorphism between $B^n_\infty \times_L A$ and the toric domain $X_{4|A|}$. Let $\{P_\varepsilon\}_{\varepsilon>0}$ be the family of symplectic submanifolds with compact closure of $B^n_\infty \times_L A$ as in Lemma 44. Pick a decreasing sequence ε_k converging to 0. By property (b), for all $l > k$, $\mathrm{cl}(P_{\varepsilon_l}) \subset \mathrm{cl}(P_{\varepsilon_k})$; moreover, combining properties (a) and (b) in Claim 44, $\bigcup_{k\geq 1} \mathrm{cl}(P_{\varepsilon_k}) = B^n_\infty \times_L A$ and $\bigcup_{k\geq 1} \mathbf{\Psi}_{\varepsilon_k}(\mathrm{cl}(P_{\varepsilon_k})) = X_{4|A|}$.

To construct the desired symplectomorphism we use an argument of [27] which also appears in [30, Proof of Theorem 3]. Fix $k \geq 2$. Observe that, for any $t \in [\varepsilon_k, \varepsilon_{k-1}]$,

$$\mathbf{\Psi}_{\varepsilon_{k-1}}\left(\mathrm{cl}\left(P_{\varepsilon_{k-1}}\right)\right) \subset \mathbf{\Psi}_t(\mathrm{cl}(P_t)) \subset \mathbf{\Psi}_{\varepsilon_k}(\mathrm{cl}(P_{\varepsilon_k})),$$

where the inclusions follow from property (b) in Claim 44. Thus it is possible to consider an isotopy of symplectic embeddings $\mathbf{\Psi}_t^{-1} \circ \mathbf{\Psi}_{\varepsilon_{k-1}} : \mathrm{cl}\left(P_{\varepsilon_{k-1}}\right) \hookrightarrow \mathrm{cl}(P_{\varepsilon_k})$ for $t \in [\varepsilon_k, \varepsilon_{k-1}]$. Using the symplectic isotopy extension theorem (cf. [3, Proposition 4] and [4]), there exists an isotopy of symplectomorphisms $\chi_t : \mathrm{cl}(P_{\varepsilon_k}) \to \mathrm{cl}(P_{\varepsilon_k})$ for $t \in [\varepsilon_k, \varepsilon_{k-1}]$ such that

- $\chi_t|_{P_{\varepsilon_{k-1}}} = \mathbf{\Psi}_t^{-1} \circ \mathbf{\Psi}_{\varepsilon_{k-1}}$, and
- χ_t is the identity away from some neighborhood of $\mathrm{cl}\left(P_{\varepsilon_{k-1}}\right)$.

The map $\widetilde{\mathbf{\Psi}}_{\varepsilon_k} := \mathbf{\Psi}_{\varepsilon_k} \circ \chi_{\varepsilon_k} : \mathrm{cl}(P_{\varepsilon_k}) \hookrightarrow \mathbb{R}^{2n}$ is a symplectic embedding satisfying

- $\widetilde{\mathbf{\Psi}}_{\varepsilon_k}|_{\mathrm{cl}\left(P_{\varepsilon_{k-1}}\right)} = \mathbf{\Psi}_{\varepsilon_{k-1}}$, and
- $\widetilde{\mathbf{\Psi}}_{\varepsilon_k}$ equals $\mathbf{\Psi}_{\varepsilon_k}$ away from some neighborhood of $\mathrm{cl}\left(P_{\varepsilon_{k-1}}\right)$.

Setting

$$\mathbf{\Psi}(\mathbf{x}, \mathbf{y}) := \begin{cases} \mathbf{\Psi}_{\varepsilon_1}(\mathbf{x}, \mathbf{y}) & \text{if } (\mathbf{x}, \mathbf{y}) \in \mathrm{cl}(P_{\varepsilon_1}), \\ \widetilde{\mathbf{\Psi}}_{\varepsilon_k}(\mathbf{x}, \mathbf{y}) & \text{if } (\mathbf{x}, \mathbf{y}) \in \mathrm{cl}(P_{\varepsilon_k}) \setminus \mathrm{cl}\left(P_{\varepsilon_{k-1}}\right), \end{cases}$$

we obtain a well-defined map $\mathbf{\Psi} : \bigcup_{k\geq 1} \mathrm{cl}\,(P_{\varepsilon_k}) = B^n_\infty \times_L A \to \mathbb{R}^{2n}$. The above properties imply that Ψ is a symplectic embedding of $B^n_\infty \times_L A$ into $\mathbb{R}^{2n}$ whose image equals $\bigcup_{k\geq 1} \mathbf{\Psi}_{\varepsilon_k}\,(\mathrm{cl}\,(P_{\varepsilon_k})) = X_{4|A|}$ as desired. $\square$

References

[1] V. I. Arnol'd, Mathematical Methods of Classical Mechanics, Vol. 60 of Graduate Texts in Mathematics, Springer-Verlag, New York, second edition (1989), ISBN 0-387-96890-3. Translated from the Russian by K. Vogtmann and A. Weinstein.

[2] S. Artstein-Avidan and Y. Ostrover, *Bounds for Minkowski billiard trajectories in convex bodies*, Int. Math. Res. Not. IMRN (2014), no. 1, 165–193.

[3] D. Auroux, *Asymptotically holomorphic families of symplectic submanifolds*, Geom. Funct. Anal. **7** (1997), no. 6, 971–995.

[4] A. Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. **53** (1978), no. 2, 174–227.

[5] V. Benci and F. Giannoni, *Periodic bounce trajectories with a low number of bounce points*, Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1989), no. 1, 73–93.

[6] O. Buse and R. Hind, *Symplectic embeddings of ellipsoids in dimension greater than four*, Geom. Topol. **15** (2011), no. 4, 2091–2110.

[7] K. Choi, D. Cristofaro-Gardiner, D. Frenkel, M. Hutchings, and V. G. B. Ramos, *Symplectic embeddings into four-dimensional concave toric domains*, J. Topol. **7** (2014), no. 4, 1054–1076.

[8] K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk, *Quantitative symplectic geometry*, in: Dynamics, Ergodic Theory, and Geometry, Vol. 54 of Math. Sci. Res. Inst. Publ., 1–44, Cambridge Univ. Press, Cambridge (2007).

[9] Y. Colin de Verdière and J. Vey, *Le lemme de Morse isochore*, Topology **18** (1979), no. 4, 283–293.

[10] D. Cristofaro-Gardiner, *Symplectic embeddings from concave toric domains into convex ones*, arXiv:1409.4378.

[11] D. Cristofaro-Gardiner and R. Hind, *Symplectic embeddings of products*, to appear in Comm. Math. Helv.

[12] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, Bull. Soc. Math. France **116** (1988), no. 3, 315–339.

[13] J.-P. Dufour and P. Molino, *Compactification d'actions de $\mathbf{R}^n$ et variables action-angle avec singularités*, in: Symplectic Geometry, Groupoids, and Integrable Systems (Berkeley, CA, 1989), Vol. 20 of Math. Sci. Res. Inst. Publ., 151–167, Springer, New York (1991).

[14] J. J. Duistermaat, *On global action-angle coordinates*, Comm. Pure Appl. Math. **33** (1980), no. 6, 687–706.

[15] L. H. Eliasson, *Hamiltonian systems with Poisson commuting integrals*, PhD thesis, University of Stockholm (1984).

[16] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347.

[17] V. Guillemin and S. Sternberg, *A normal form for the moment map*, in: Differential Geometric Methods in Mathematical Physics (Jerusalem, 1982), Vol. 6 of Math. Phys. Stud., 161–175, Reidel, Dordrecht (1984).

[18] V. Guillemin and S. Sternberg, Symplectic Techniques in Physics, Cambridge University Press, Cambridge, second edition (1990), `ISBN 0-521-38990-9`.

[19] L. Guth, *Symplectic embeddings of polydisks*, Invent. Math. **172** (2008), no. 3, 477–489.

[20] J. Gutt and M. Hutchings, *Symplectic capacities from positive S^1-equivariant symplectic homology*, `arXiv:1707.06514`.

[21] R. Hind and E. Kerman, *New obstructions to symplectic embeddings*, Invent. Math. **196** (2014), no. 2, 383–452.

[22] M. Hutchings, *Recent progress on symplectic embedding problems in four dimensions*, Proc. Natl. Acad. Sci. USA **108** (2011), no. 20, 8093–8099.

[23] M. Hutchings, *Beyond ECH capacities*, Geom. Topol. **20** (2016), no. 2, 1085–1126.

[24] Y. Karshon and E. Lerman, *Non-compact symplectic toric manifolds*, SIGMA Symmetry Integrability Geom. Methods Appl. **11** (2015), Paper 055, 37.

[25] J. Latschev, D. McDuff, and F. Schlenk, *The Gromov width of 4-dimensional tori*, Geom. Topol. **17** (2013), no. 5, 2813–2853.

[26] C.-M. Marle, *Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique*, Rend. Sem. Mat. Univ. Politec. Torino **43** (1985), no. 2, 227–251 (1986).

[27] D. McDuff, *Blow ups and symplectic embeddings in dimension* 4, Topology **30** (1991), no. 3, 409–421.

[28] D. McDuff, *Symplectic embeddings of 4-dimensional ellipsoids*, J. Topol. **2** (2009), no. 1, 1–22.

[29] Y. Ostrover, *When symplectic topology meets Banach geometry*, in: S. Jang, Y. Kim, D.-W. Lee, and I. Yie, editors, Proceedings of the International Congress of Mathematicians (Seoul 2014), Vol. II, 959 – 981, Kyung Moon SA Co. Ltd., Seoul, South Korea (2014).

[30] V. G. B. Ramos, *Symplectic embeddings and the Lagrangian bidisk*, Duke Math. J. **166** (2017), no. 9, 1703–1738.

[31] F. Schlenk, *Symplectic embeddings of ellipsoids*, Israel J. Math. **138** (2003), 215–252.

[32] F. Schlenk, *Symplectic embedding problems, old and new*, in preparation (2017).

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