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Moser–Greene–Shiohama stability for families

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Let M be a noncompact oriented connected manifold and let B be a compact manifold. We give conditions on two smooth families of volume forms $\{\omega_p\}_{p\in B}$, $\{\tau_p\}_{p\in B}$ which guarantee the existence of a smooth family of diffeomorphisms $\{\varphi_p\}_{p\in B}$ such that $\varphi_p^*\omega_p = \tau_p$ for all $p \in B$. If B is a point, our result recovers a theorem of Greene and Shiohama from 1979, which itself extended a theorem of Moser for compact manifolds.

1. Introduction

Throughout this paper *smooth* means C^∞ smooth, and manifolds are always assumed to be smooth manifolds without boundary except where explicitly stated otherwise.

1.1. Past works

It is a well known theorem due to Moser [9] that if two volume forms ω and τ on a compact manifold M satisfy $\int_M \omega = \int_M \tau$ then one can find a diffeomorphism φ of M such that $\varphi^*\omega = \tau$. Later Greene and Shiohama [7] realized that a version of Moser's theorem also holds even if M is not compact. The statement and the proof in [7] are more complicated than Moser's proof because the authors have to deal with the behavior at infinity of the forms. Their proof has three stages: first, they extend Moser's proof to forms which are compactly supported. Then they chop their noncompact manifold into pieces, and finally, a careful analysis of the behavior at the boundaries and interiors, allows them to construct a global diffeomorphism by pasting together the local diffeomorphisms, bypassing any analytic estimates.

1.2. Smooth families

If one mimics the Greene–Shiohama argument in the case of two smooth families of volume forms ω_p, τ_p, indexed by some compact manifold B which plays the role of the parameter space, this produces for each p a diffeomorphism φ_p such that $\varphi_p^*\omega_p = \tau_p$, but there is no information given about how φ_p changes when p changes in B. The goal of our paper is to give sufficient conditions for the variation of φ_p with respect to p to be smooth.

Definition 1.1. Let M be a manifold of dimension m and let B be a compact manifold. Let q be an integer with $0 \leqslant q \leqslant m$. A family of q-forms $\{\omega_p\}_{p\in B} \subset \Omega^q(M)$ is *smooth* if the map $B \times M \to \wedge^q T^*M, (p,x) \mapsto \omega_p(x)$ is smooth. A family $\{\varphi_p\}_{p\in B}$ of diffeomorphisms of M is *smooth* if the map $B \times M \to M, (p,x) \mapsto \varphi_p(x)$ is smooth. Two smooth families of volume forms $\{\omega_p\}_{p\in B}$ and $\{\tau_p\}_{p\in B}$ are *commensurable* on M if for any compact set $K \subset M$, for any connected component C of $M \setminus K$, one of the following holds:

- for any $p \in B$, $\int_C \omega_p = \int_C \tau_p = +\infty$;
- the integrals $\int_C \omega_p$ and $\int_C \tau_p$ are finite and continuous with respect to $p \in B$, and their difference is smooth with respect to $p \in B$.

1.3. Main theorem

Our main result is the following parametric version of the Moser and Greene–Shiohama result:

Theorem 1.2. *Let M be a noncompact oriented connected manifold. Let B be a compact manifold. Let $\{\omega_p\}_{p\in B}$ and $\{\tau_p\}_{p\in B}$ be commensurable smooth families of volume forms on M such that $\int_M \omega_p = \int_M \tau_p$ for any $p \in B$. Then there is a smooth family of diffeomorphisms $\{\varphi_p \colon M \to M\}_{p\in B}$ such that $\varphi_p^*\omega_p = \tau_p$ for each $p \in B$.*

The case when M is compact was proved by Moser [9]. If B is a point, Theorem 1.2 was proved by Greene–Shiohama [7]. If $\{\tau_p\}_{p\in B}$ is a constant family, we obtain:

Corollary 1.3. *Let M be a noncompact oriented connected manifold and B be a compact manifold. Let $p_0 \in B$. Let $\{\omega_p\}_{p\in B}$ be a smooth family of volume forms on M such that $\int_M \omega_p$ is independent of $p \in B$. Suppose moreover for any connected component C of the complement of a compact subset* *of M, either $\int_C \omega_p = +\infty$ for all $p \in B$, or $\int_C \omega_p$ is smooth with respect to $p \in B$. Then there is a smooth family of diffeomorphisms $\{\varphi_p \colon M \to M\}_{p\in B}$ such that $\varphi_p^* \omega_p = \omega_{p_0}$ for each $p \in B$.*

The remaining of the paper is devoted to proving Theorem 1.2. The proof is inductive and requires the introduction of certain topological-combinatorial constructions (Section 2), and geometric-analytic constructions (Section 3). This allows us to prove a filtration lemma for noncompact manifolds (Section 4), from which Theorem 1.2 easily follows (Section 5).

2. Topological-combinatorial constructions

In this section, we prepare the topological-combinatorial ingredients needed to prove our main theorem. We will first show a result about general topological spaces, which we will then use to give a slicing of a smooth manifold which satisfies certain properties (in terms of an exhaustion function for the manifold). Then we use this slicing to define a tree structure on the manifold itself, which will be an essential ingredient for the proof of the main theorem.

2.1. A topological statement about connected components

We start with a general topological statement which we shall need.

Lemma 2.1. *Let X be a locally connected locally compact Hausdorff space. Let $\mathcal{K}(X)$ be the collection of compact subsets of X. Let $K \in \mathcal{K}(X)$ and let $A, A' \subset X$ be connected and precompact. If A, A' lie in the same connected component C of X then there is $L \in \mathcal{K}(X)$ such that they lie in the same connected component of $L \cap C$.*

Proof. For any topological space Y and a nonempty connected subset E, denote by $\operatorname{conn}(Y, E)$ the unique connected component of Y containing E. For any $\mathcal{P} \subset \mathcal{K}(X)$ denote $\operatorname{conn}(\mathcal{P}, E) = \bigcup_{L\in\mathcal{P}} \operatorname{conn}(L \cap \operatorname{conn}(Y, E), E)$.

Let X and nonempty $A, A' \subset X$ be as described in the statement. We show that $\operatorname{conn}(\mathcal{P}, A) = \operatorname{conn}(X, A)$ for any nonempty $\mathcal{P} \subset \mathcal{K}(X)$ such that $\mathcal{P} \ni L_1 \subset L_2 \in \mathcal{K}(X)$ implies $L_2 \in \mathcal{P}$. Let $C = \operatorname{conn}(X, A)$. Since X is locally connected and A is precompact, C is open in X and locally connected. For any $L \in \mathcal{P}$ and $x \in \operatorname{conn}(L \cap C, A)$, there is a compact connected neighborhood F_x of x in C, so then $\operatorname{conn}(L \cap C, A) \cup F_x \subset \operatorname{conn}((L \cup F_x) \cap C, A)$. Therefore $\operatorname{conn}(\mathcal{P}, A)$ is open. For any precompact connected open set $U \subset$

C, we have

$$\left(\begin{aligned} U \cap \operatorname{conn}(\mathcal{P}, A) \neq \varnothing &\implies \exists L \in \mathcal{P}, \quad U \cap \operatorname{conn}(L \cap C, A) \neq \varnothing \\ &\implies U \subset \operatorname{conn}\big((L \cup \overline{U}) \cap C, A\big) \subset \operatorname{conn}(\mathcal{P}, A) \end{aligned}\right).$$

Thus $U \setminus \operatorname{conn}(\mathcal{P}, A) \neq \varnothing$ implies that $U \subset C \setminus \operatorname{conn}(\mathcal{P}, A)$. Since C has a topology base consisting of connected sets, $\operatorname{conn}(\mathcal{P}, A)$ is closed in C. Now $\operatorname{conn}(\mathcal{P}, A)$ is a nonempty and clopen subspace of the connected space C, which is C itself.

Suppose $\operatorname{conn}(X, A) = \operatorname{conn}(X, A') = C$. Let $\mathcal{P} = \{L \in \mathcal{K}(X) \mid L \supset A \cup A'\}$. Since $\operatorname{conn}(\mathcal{P}, A) = \operatorname{conn}(\mathcal{P}, A') = C$, there are $L_1, L_2 \in \mathcal{P}$ such that $\operatorname{conn}(L_1 \cap C, A) \cap \operatorname{conn}(L_2 \cap C, A') \neq \varnothing$. Let $L' = L_1 \cup L_2 \in \mathcal{P}$. Then

$$\begin{aligned} &\operatorname{conn}(L' \cap C, A) \cap \operatorname{conn}(L' \cap C, A') \\ &\qquad \supset \operatorname{conn}(L_1 \cap C, A) \cap \operatorname{conn}(L_2 \cap C, A') \neq \varnothing, \end{aligned}$$

which means $\operatorname{conn}(L' \cap C, A) = \operatorname{conn}(L' \cap C, A')$. The case when either A or A' is empty is trivial. □

2.2. Slicing a manifold by an exhaustion function

Let M be a manifold. An *exhaustion function* f for M is a smooth function $f\colon M \to \mathbb{R}$ such that for any $\alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha])$ is compact. An exhaustion function for M always exists. Let $\operatorname{Reg}(f)$ be the set of regular values of f (including $\mathbb{R} \setminus f(M)$). Fix as a *basepoint* $x_0 \in M$ a minimum point of f. For any $\alpha \in \operatorname{Reg}(f) \cap f(M)$, let C be the connected component of $f^{-1}((-\infty, \alpha])$ containing x_0. Define M_α as the union of C and the precompact connected components of $M \setminus C$. Then M_α is compact and connected, see Figure 1. For $\alpha \in \mathbb{R} \setminus f(M)$, let $M_\alpha = \varnothing$. We call M_α the *saturated slicing* of M by α. For any set $A \subset M$, let $A_\alpha = A \cap M_\alpha$.

We will need the following technical property of precompact subsets in the proof of Lemma 4.1 (which itself is needed to prove the main theorem).

Lemma 2.2. *For any connected precompact set $A \subset M$,*

$$\theta_A \overset{\text{def}}{=} \inf\{\alpha \in f(A) \mid \forall \beta \in \operatorname{Reg}(f), \beta > \alpha, A_\beta \textit{ is connected}\}$$

is finite.

Proof. Fix an $\alpha \in \operatorname{Reg}(f) \cap f(A)$. Since A_α is the interior of a compact manifold with boundary, it can only have finitely many components. By

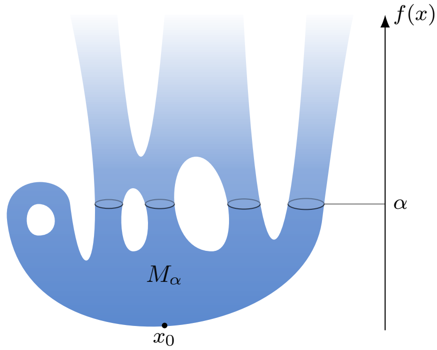

Figure 1: The saturated slicing M_α.

Lemma 2.1, there is $K \in \mathcal{K}(M)$ which is connected and contains x_0 and every component of A_α. Let β be a regular value of f greater than $\max_K f$. Then $A_\beta \supset K$ contains every component of A_α. Note that any component of A_β contains a component of A_α, so A_β is connected. Hence $\theta_A \leqslant \beta < +\infty$. ☐

2.3. A tree structure on a manifold

Consider the following combinatorial notions of trees which will be very useful for the proof of Theorem 1.2.

A *tree* is a strictly partially ordered set $(\mathcal{T}, \prec)$ with the property that for each $x \in \mathcal{T}$, the set $\mathrm{Pre}(x) = \{y \in \mathcal{T} \mid y \prec x\}$ of all *predecessors* of x is well ordered by $\prec$. We write $\mathcal{T}$ for $(\mathcal{T}, \prec)$ when there is no ambiguity. A *branch* in $\mathcal{T}$ is a maximal linearly ordered subset of $\mathcal{T}$. Let $\mathrm{Rt}(\mathcal{T}) = \{x \in T \mid \forall y \in T, y \not\prec x\} \neq \varnothing$ be the set of roots of $\mathcal{T}$. If $\mathrm{Rt}(\mathcal{T})$ is a singleton we call $\mathcal{T}$ *rooted*.

Let $\mathrm{Suc}(x) = \{y \in \mathcal{T} \mid y \succ x\}$ be the set of all *successors* of x. Then $(\mathrm{Suc}(x), \prec)$ is a tree. Let

$$\mathrm{Ch}(x) = \mathrm{Rt}(\mathrm{Suc}(x))$$

be the set of *immediate successors* or *children* of x. If for any $x \in \mathcal{T}$, $\mathrm{Ch}(x)$ is finite, we call $\mathcal{T}$ *locally finite*. Let

$$\mathrm{Gch}(x) = \bigcup_{y \in \mathrm{Ch}(x)} \mathrm{Ch}(y)$$

be the set of *grandchildren* of x. Let $\mathrm{Lf}(\mathcal{T}) = \{x \in T \mid \forall y \in T, x \not< y\}$ be the set of *pendant vertices* or *leaves* of $\mathcal{T}$. If $\mathrm{Lf}(\mathcal{T}) = \varnothing$ we call $\mathcal{T}$ *leafless*.

The *depth* of x is the ordinal of $\mathrm{Pre}(x)$, which we denote by $\mathrm{dpt}(x)$. Let

$$\mathrm{hgt}(\mathcal{T}) = \sup\{\mathrm{dpt}(x) + 1 \mid x \in \mathcal{T}\}$$

be the *height* of $\mathcal{T}$. For any ordinal $\ell < \mathrm{hgt}(\mathcal{T})$, let

$$\mathrm{Lv}(\ell) = \{x \in \mathcal{T} \mid \mathrm{dpt}(x) = \ell\}$$

be the ℓ-th *level* of $\mathcal{T}$.

Let ω denote the smallest infinite ordinal. If $\mathrm{hgt}(\mathcal{T}) = \omega$, then every node in $\mathcal{T}$ has finite depth, but these depths are unbounded. We have the following essential construction for the combinatorial part of the proof of Theorem 1.2.

Lemma 2.3. *Let M be a noncompact manifold, $\alpha_0 = -\infty$ and $\{\alpha_\ell\}_{\ell\in\mathbb{N}} \subset \mathrm{Reg}(f) \cap f(M)$ be an unbounded strictly increasing sequence. Let $\mathcal{L}(\ell)$ be the collection of unbounded connected components of $M \setminus M_{\alpha_{\ell-1}}$. Then there is a tree $(\mathcal{T}, \supsetneq)$ of open subsets of M such that*

$$\mathcal{T} = \coprod_{\ell\in\mathbb{N}\cup\{0\}} \mathcal{L}(\ell).$$

Moreover, $(\mathcal{T}, \supsetneq)$ is a rooted locally finite leafless tree of height ω, and $\mathcal{L}(\ell) = \mathrm{Lv}(\ell)$ for each $\ell \in \mathbb{N} \cup \{0\}$.

Proof. Let $A_i \in \mathcal{L}(\ell_i) \subset \mathcal{T}$ where $\ell_i \in \mathbb{N} \cup \{0\}$, for $i = 1, 2$ and 3. By definition of connected components we have the following: if $A_1 \supsetneq A_2$, then $\ell_1 < \ell_2$; if $A_1, A_2 \supsetneq A_3$ and $\ell_1 < \ell_2$, then $A_1 \supsetneq A_2$. Hence $(\mathcal{T}, \supsetneq)$ is a tree.

The only root of $\mathcal{T}$ is $M \in \mathcal{L}(0)$. By induction $\mathcal{L}(\ell)$ is the ℓ-th level of $\mathcal{T}$, which is finite, so $\mathcal{T}$ is locally finite. For any $A \in \mathrm{Lv}(\ell)$, $A \setminus A_{\alpha_{\ell+1}} \neq \varnothing$, so $\mathcal{T}$ is leafless. Hence $\{\mathrm{dpt}(A) \mid A \in \mathcal{T}\} = \mathbb{N} \cup \{0\}$, and $\mathrm{hgt}(\mathcal{T}) = \omega$. □

3. Geometric-analytic constructions

Throughout this section, M is a noncompact oriented manifold of dimension $m \geqslant 1$. In this section, we present the analytic statements needed to prove the main theorem. The main tool we use is a version of Hodge theory applied to certain noncompact manifolds which is sufficient for the purpose of the present paper. We split the content into several subsections for clarity.

3.1. Forms with compactly supported difference

In this subsection, we prove (using the work of Bueler–Prokhorenkov on Hodge theory [2]) a parametrized Moser stability theorem for two families $\{\omega_p\}_{p\in B}$, $\{\tau_p\}_{p\in B}$ of volume forms whose differences $\omega_p - \tau_p$, $p \in B$ are supported in some compact submanifold with boundary.

Lemma 3.1. *Let f be an exhaustion for M, see Section 2.2. Let N be a compact hypersurface of M through regular points of f. Then there exists $\varepsilon > 0$ and a diffeomorphism $\Phi\colon N \times (-\varepsilon, \varepsilon) \to V_N$ such that V_N is an open neighborhood of $N \subset M$, $\Phi(y, 0) = y$, $\pi(\Phi(y, s)) = \pi(y)$ and $f(\Phi(y, s)) = f(y) + s$ for any $(y, s) \in N \times (-\varepsilon, \varepsilon)$. If N is connected then V_N is connected too.*

Proof. Pick an arbitrary Riemannian metric g on M. Let $\widetilde{V}_N$ be an open neighborhood of $N \subset M$ which consists of regular points of f. Let $X \in \mathfrak{X}(M)$ be such that $X = |\nabla_g f|_g^{-2}\nabla_g f$ in $\widetilde{V}_N$, where $\nabla_g f$ is the gradient of f. Then $X(f) = 1$ in $\widetilde{V}_N$. Take the flow of X, $\Phi\colon N \times (-\varepsilon, \varepsilon) \to M, (y, s) \mapsto x$, that is $\Phi(y, 0) = y$ for all $y \in N$ and $\frac{\partial \Phi}{\partial s}(y, s) = X(\Phi(y, s))$ for all $(y, s) \in N \times (-\varepsilon, \varepsilon)$, for $\varepsilon > 0$ small enough such that the image of Φ is contained in $\widetilde{V}_N$. Then let $V_N = \Phi(N \times (-\varepsilon, \varepsilon))$. Since $X(f) = 1$ in V_N, we have $f(\Phi(y, s)) = f(y) + s$ for any $(y, s) \in N \times (-\varepsilon, \varepsilon)$, and Φ is a diffeomorphism. If N is connected, then V_N is the image of Φ, which is connected. □

Theorem 3.2. *Let W be an open subset of M such that $\overline{W}$ is a submanifold of M with boundary ∂W. Then for any $q \in \mathbb{N}$ with $1 \leqslant q \leqslant m$ there is an operator preserving smooth families of q-forms*

$$I_W^q \colon \bigl\{\xi \in \Omega_{\mathrm{c}}^q(M) \bigm| \operatorname{supp}\xi \subset W, \xi|_W \in \mathrm{d}\Omega_{\mathrm{c}}^{q-1}(W)\bigr\} \\ \to \bigl\{\eta \in \Omega_{\mathrm{c}}^{q-1}(M) \bigm| \operatorname{supp}\eta \subset \overline{W}\bigr\}$$

satisfying $d \circ I_W^q = \mathrm{id}$.

Proof. By [2] there is a weighted Hodge-Laplacian $\Delta_\mu\colon \Omega_{\mathrm{c}}^q(W) \to \Omega_{\mathrm{c}}^q(W)$ on W equipped with a specific metric g and measure μ. Its Green operator $G_\mu\colon \Omega_{\mathrm{c}}^q(W) \to \Omega^q(W)$ and the weighted codifferential $\delta_\mu\colon \Omega_{\mathrm{c}}^q(W) \to \Omega_{\mathrm{c}}^{q-1}(W)$ satisfy the identity $\mathrm{d} \circ \delta_\mu \circ G_\mu \circ \mathrm{d} = \mathrm{d}$. Moreover, any form $\eta \in G_\mu(\Omega_{\mathrm{c}}^q(W))$ has an extension $\tilde{\eta} \in \Omega_{\mathrm{c}}^q(M)$ supported in $\overline{W}$. For any $\xi \in \Omega_{\mathrm{c}}^q(M)$ supported in W such that $\xi|_W \in \mathrm{d}\Omega_{\mathrm{c}}^{q-1}(W)$, we define $I_W^q(\xi)$ as the extension of $(\delta_\mu \circ G_\mu)(\xi|_W)$ to $\Omega_{\mathrm{c}}^{q-1}(M)$, see Figure 2. Then we have $\mathrm{d} \circ I_W^q = \mathrm{id}$.

The operator I_W^q preserves smooth families. Indeed, the p-derivative of a smooth family ξ_p, $p \in B$ of compactly supported forms is still compactly supported. Since the Green's operator G_μ is an integral operator with a singular kernel, we can pass the p-derivative through the operator G_μ, so $\partial_p G_\mu \xi_p$ exists and is a smooth form, for each $p \in B$. By similar arguments for higher order derivatives, $G_\mu \xi_p$, $p \in B$ is a smooth family. The map δ_μ preserves smooth families since it is a differential operator. □

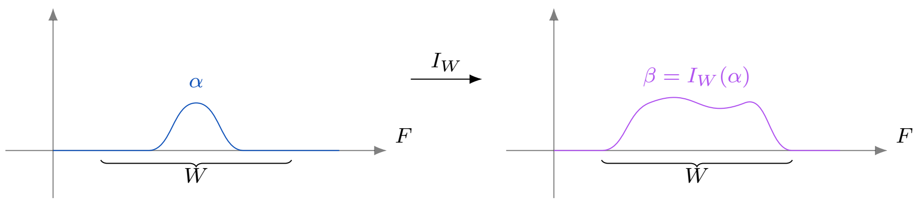

Figure 2: From compactly supported forms to forms with zero extensions.

Let B be a compact manifold. We adopt the following notations.

- $\mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ is the set of smooth families $\omega = \{\omega_p\}_{p \in B}$ of volume forms on M. Similarly, $\mathcal{F}^\infty(B; \Omega^m_{\geqslant 0}(M))$ is the set of smooth families of non-negative m-forms on M. Note that $\mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M)) \subsetneq \mathcal{F}^\infty(B; \Omega^m_{\geqslant 0}(M))$.
- $\mathcal{F}^\infty(B; \mathrm{Diff}(M))$ is the set of smooth families $\varphi = \{\varphi_p\}_{p \in B}$ of diffeomorphisms of M.
- If $\omega \in \mathcal{F}^\infty(B; \Omega^m_{\geqslant 0}(M))$, $\int_M \omega$ is the map $B \to [0, +\infty]$ given by

$$\left(\int_M \omega\right)(p) = \int_M \omega_p.$$

- If $\omega \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$, $\varphi \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$, we define

$$\varphi^*\omega = \left\{\varphi_p^*\omega_p\right\}_{p \in B} \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M)).$$

Figure 4 illustrates the main point of the following lemma, where the shaded region is the support of $\omega_p - \tau_p$.

Lemma 3.3. *Let V be a connected open subset of M such that $\overline{V}$ is a compact submanifold with boundary ∂V. Let $\omega, \tau \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ be such that* $\mathrm{supp}(\omega_p - \tau_p) \subset V, \forall p \in B$ *and $\int_V \omega = \int_V \tau$. Then there is a family $\varphi \in$*

$\mathcal{F}^\infty(B; \mathrm{Diff}(M))$ *such that* $M \setminus V$ *has a neighborhood in which* φ_p *is the identity for* $p \in B$ *and* $\varphi^*\omega = \tau$.

Proof. Let $N = \partial V$. Applying Lemma 3.1 to N there are $\varepsilon > 0$ and V_N a neighborhood of N with the properties stated in the lemma. Since B is compact and $\operatorname{supp}(\omega_p - \tau_p) \subset V, \forall p \in B$, we may decrease ε if necessary so that $\operatorname{supp}(\omega_p - \tau_p) \subset V \setminus \overline{V_N}, \forall p \in B$. Let $W = V \setminus \overline{V_N}$. Since the map $\int_W \colon H^m_{\mathrm{c}}(W) \to \mathbb{R}$ is a linear isomorphism, and $\int_V \omega = \int_V \tau$, we have $(\omega_p - \tau_p)|_W \in \mathrm{d}\Omega^{m-1}_{\mathrm{c}}(W), \forall p \in B$. Therefore by Theorem 3.2 there exists a smooth family $\sigma_p = I^m_W \xi_p \in \Omega^{m-1}_{\mathrm{c}}(M), \forall p \in B$, with $\operatorname{supp} \sigma_p \subset \overline{W}$ such that $\mathrm{d}\sigma_p = \omega_p - \tau_p, \forall p \in B$. Let $\omega_t = (1-t)\omega + t\tau \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ for any $t \in [0,1]$.

Since ω_t is nowhere vanishing there exists a unique smooth family of vector fields $\{X_{t,p}\}_{(t,p)\in[0,1]\times B} \subset \mathfrak{X}(M)$ where each $X_{t,p}$ is supported in $\overline{W}$ and such that $\omega_{t,p}(X_{t,p}, \cdot) = \sigma_p$. Since $\overline{V}$ is compact, for each $p \in B$, the flow $\varphi_{t,p}$, $t \in [0,1]$ in M generated by $X_{t,p}$ exists and is the identity outside of $\overline{W}$. For $t \in [0,1]$, $\varphi_t = \{\varphi_{t,p}\}_{p\in B} \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$. Then $\varphi_t^*\omega_t = \omega$. If $\varphi = \varphi_1^{-1}$ then we have $\varphi^*\omega = \tau$. Since $X_{t,p} = 0$ in $M \setminus W$ for $(t,p) \in [0,1] \times B$, $\varphi_{t,p}$ is the identity outside of $\overline{W}$. □

3.2. The transfer of volumes

In this subsection, we prove a series of lemmas which allow us to transfer volumes of a smooth family of volume forms across the boundaries of compact submanifolds, so as to modify the smooth families $\{\omega_p\}_{p\in B}$, $\{\tau_p\}_{p\in B}$ so that they have the same volume in a certain set of compact submanifolds; then we move the volumes within the compact submanifolds, to pull ω_p back to τ_p for each $p \in B$.

Lemma 3.4. *Let* $\omega \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$, *and let* $V \subset M$ *be a nonempty precompact open set. Then for any* $w \in \mathrm{C}^\infty(B; (0,\infty))$, *there exists* $\tau \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ *such that* $\operatorname{supp}(\omega_p - \tau_p) \subset V, \forall p \in B$ *and* $\int_V \tau = w$.

Proof. Let $\xi \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ be such that $\operatorname{supp}(\xi_p - \omega_p) \subset V, \forall p \in B$ and $\int_V \xi < w$. Let $\eta \in \mathcal{F}^\infty(B; \Omega^m_{\geqslant 0}(M))$ be such that $\operatorname{supp} \eta_p \subset V, \forall p \in B$ and $\int_V \eta > 0$. Then define $\tau = \xi + \frac{w - \int_V \xi}{\int_V \eta}\eta$. □

Lemma 3.5. *Let* N *be a compact hypersurface of* M *and consider* $\omega, \tau \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$. *Then there is* V_N *an open neighborhood of* N *and* $\varphi \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$ *such that, if* V_N^+ *and* V_N^- *are the connected components of*

$V_N \setminus V$, the following hold: $M \setminus V_N$ has a neighborhood in which φ_p is the identity for any $p \in B$; N has a neighborhood in which $\varphi^*\omega = \tau$; $\int_{V_N^+} \varphi^*\omega = \int_{V_N^+} \omega$; and $\int_{V_N^-} \varphi^*\omega = \int_{V_N^-} \omega$.

Proof. By Lemma 3.1, there exists a neighborhood V_N of N in M, $\varepsilon > 0$, and a diffeomorphism $\Phi\colon N \times (-\varepsilon, \varepsilon) \to V_N$ such that $\Phi(y, 0) = y$ and $f(\Phi(y, s)) = f(y) + s$ for any $(y, s) \in N \times (-\varepsilon, \varepsilon)$, see Figure 5. Let $V_N^+ = \Phi(N \times (0, \varepsilon))$ and $V_N^- = \Phi(N \times (-\varepsilon, 0))$.

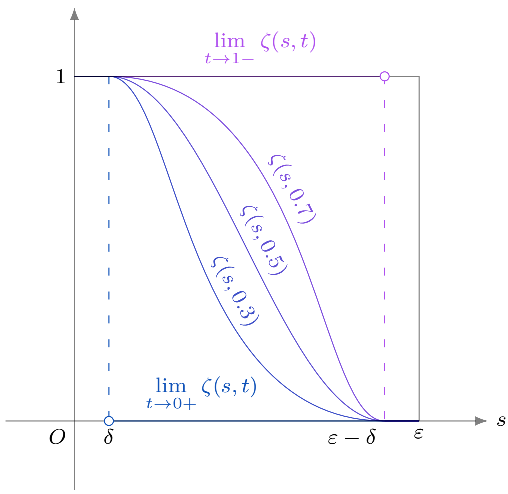

Figure 3: The graph of $\zeta(s, t)$.

First we consider $\Phi(N \times [0, \varepsilon))$. Since B is compact, there is $\delta \in (0, \varepsilon/2)$ such that

$$\int_{\Phi(N\times(0,\varepsilon-\delta))} \tau > \int_{\Phi(N\times(0,\delta))} \omega, \qquad \int_{\Phi(N\times(0,\varepsilon-\delta))} \omega > \int_{\Phi(N\times(0,\delta))} \tau.$$

Let $\zeta\colon (0, \varepsilon) \times (0, 1) \to [0, 1]$ be a smooth function with the properties (see Figure 3):

$$\begin{cases} \zeta(s, \cdot) = 1, & s \in (0, \delta]; \\ \lim_{t\to 0+} \zeta(s, t) = 0, \dfrac{\partial \zeta}{\partial t}(s, \cdot) > 0, \lim_{t\to 1-} \zeta(s, t) = 1, & s \in (\delta, \varepsilon - \delta); \\ \zeta(s, \cdot) = 0, & s \in [\varepsilon - \delta, \varepsilon). \end{cases}$$

Define $\theta\colon B \times (0,1) \to \mathbb{R}$ by

$$\theta(p,t) = \int_{V_N^+} \zeta(s(p),t)\tau_p - \int_{V_N^+} \zeta(s(p),1-t)\omega_p$$

where $s = \mathrm{pr}_2 \circ \Phi^{-1}\colon \Phi(N \times (-\varepsilon,\varepsilon)) \to (-\varepsilon,\varepsilon)$, $\mathrm{pr}_2\colon N \times (-\varepsilon,\varepsilon) \to (-\varepsilon,\varepsilon)$ is the projection to the second factor. Since $\omega, \tau \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ and ζ is smooth it follows that θ is smooth. Furthermore

$$\frac{\partial\theta}{\partial t}(p,t) = \int_{V_N^+} \frac{\partial\zeta}{\partial t}(s(p),t)\tau_p + \int_{V_N^+} \frac{\partial\zeta}{\partial t}(s(p),1-t)\omega_p > 0$$

for any $t \in (0,1)$ and $\lim_{t\to 0+} \theta(p,t) < 0 < \lim_{t\to 1-} \theta(p,t)$ for any $p \in B$. Then for every $p \in B$ there is a unique $t(p)$ solving $\theta(p,t(p)) = 0$. By the implicit function theorem, $t\colon B \to \mathbb{R}$ is smooth.

Define $\lambda(p,x) = \zeta(s(p,x),t(p))$ and $\mu(p,x) = \zeta(s(p,x),1-t(p))$ in $B \times V_N^+$. The functions λ and μ are smooth in x and satisfy $\int_{V_N^+} \mu\omega = \int_{V_N^+} \lambda\tau$. Analogously we can define λ and μ in $B \times V_N^-$, and let $\lambda = \mu = 1$ on N. Notice that $\lambda = \mu = 1$ in $\Phi(N \times [-\delta,\delta])$, so we obtain smooth extensions of λ, μ which we also denote by $\lambda, \mu\colon B \times V_N \to \mathbb{R}$. Hence

$$\int_{V_N^+} ((1-\mu)\omega + \lambda\tau) = \int_{V_N^+} \omega,$$
$$\int_{V_N^-} ((1-\mu)\omega + \lambda\tau) = \int_{V_N^-} \omega.$$

By Lemma 3.3 applied to $(1-\mu)\omega + \lambda\tau$ and ω on V_N^+ and V_N^- respectively, combining the results we obtain $\varphi \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$ such that $\varphi = \mathrm{id}$ in $M \setminus \Phi(N \times (\delta-\varepsilon, \varepsilon-\delta))$ and $\varphi^*\omega = (1-\mu)\omega + \lambda\tau$. □

Lemma 3.6. *Let $\{L_j\}_{j\in\mathbb{N}}$ be a cover of M by connected compact submanifolds with boundary, which have the same dimension as M, and whose interiors are pairwise disjoint. If $\omega, \tau \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ are such that $\int_{L_j} \omega = \int_{L_j} \tau$ for each $j \in \mathbb{N}$ then there is $\varphi \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$ such that $\varphi^*\omega = \tau$.*

Proof. By the construction of $\{L_j\}_{j\in\mathbb{N}}$, any three different L_j's for $j \in \mathbb{N}$ do not intersect. Let $\mathcal{C} = \{N \mid N \in \mathrm{Conn}(L_j \cap L_k), j,k \in \mathbb{N}, j \neq k\}$. Then $\mathcal{C}$ is a collection of pairwise disjoint connected hypersurfaces of M. So for each $N \in \mathcal{C}$, if $N \subset L_j \cap L_k$ where $j,k \in \mathbb{N}$, then by Lemma 3.1, we obtain $\varepsilon_N > 0$ and a diffeomorphism $\Phi_N\colon N \times (-\varepsilon_N, \varepsilon_N) \to V_N$ where V_N is an open neighborhood of $N \subset M$. We require $V_N \subset L_j \cup L_k$.

We apply Lemma 3.5 to V_N to obtain $\varphi_N \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$ such that $\varphi_N = \mathrm{id}$ in a neighborhood of $M \setminus V_N$, $\varphi_N^* \omega = \tau$ in a neighborhood of N, and

$$\int_{V_N^+} \varphi_N^* \omega = \int_{V_N^+} \omega, \qquad \int_{V_N^-} \varphi_N^* \omega = \int_{V_N^-} \omega.$$

Hence $\int_{L_j} \varphi_N^* \omega = \int_{L_j} \omega$, $\int_{L_k} \varphi_N^* \omega = \int_{L_k} \omega$. See Figure 6.

If necessary, choose ε_N small so that $\overline{V_N}$, $N \in \mathcal{C}$, are mutually disjoint. Since replacing ω by $\varphi_N^* \omega$ each time does not change the volume of L_j for any $j \in \mathbb{N}$, we compose these φ_N for $N \in \mathcal{C}$, as they are the identity away from disjoint open sets, to obtain $\varphi' \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$ such that $\omega' = \varphi'^* \omega$ is equal to τ in some neighborhood of $\bigcup_{N \in \mathcal{C}} N$ and $\int_{L_j} \omega' = \int_{L_j} \omega = \int_{L_j} \tau$ for each $j \in \mathbb{N}$. Applying Lemma 3.3 to each L_j for $j \in \mathbb{N}$ we get $\psi_j \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$ such that $\tau = \psi_j^* \omega'$ in L_j and $\psi_j = \mathrm{id}$ in a neighborhood of $M \setminus L_j$. Replacing ω' by $\psi_j^* \omega'$ each time and composing $\{\psi_j\}_{j \in \mathbb{N}}$ we obtain $\psi' \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$ such that $\tau = \psi'^* \omega'$. Let $\varphi = \varphi' \circ \psi'$. □

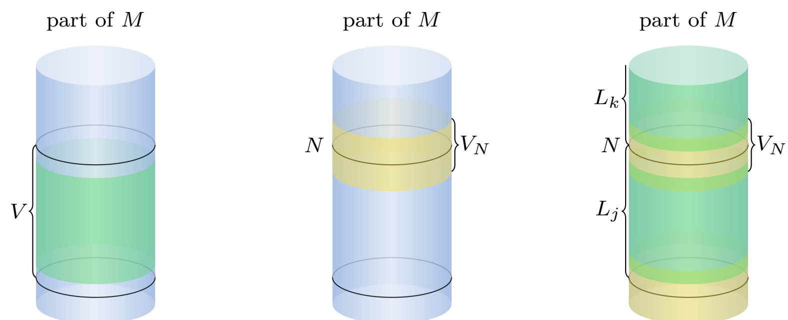

Figure 4: Lemma 3.3. Figure 5: Lemma 3.5. Figure 6: Lemma 3.6.

3.3. Approximation lemma for smooth functions

Here we prove a key technical tool for the proof of Lemma 4.1, in which we often need to express a smooth function as a sum of smooth functions bounded by some continuous functions. The lemma below shows that we can always do that as long as the sum of the bounds is greater than the original smooth function.

In the following lemma, for any $y \in \mathbb{R}$ we let $y^+ = \max(y, 0)$ be its positive part and $y^- = \max(-y, 0)$ be its negative part. For a function $f \colon B \to \mathbb{R}$

we denote $f^+(b) = f(b)^+$, $f^-(b) = f(b)^-$ for $b \in B$, so $f^+, f^- \colon B \to [0, \infty)$ are functions.

Lemma 3.7. *Let B be a connected compact manifold. Let $k \in \mathbb{N}$. Let $a \in \mathrm{C}(B;\mathbb{R})$ and $u \in \mathrm{C}^\infty(B;\mathbb{R})$ be such that $u < a$. Then for any $a_1, \ldots, a_k \in \mathrm{C}(B;\mathbb{R})$, with $\sum_{j=1}^k a_j = a$, there are $u_1, \ldots, u_k \in \mathrm{C}^\infty(B;\mathbb{R})$ such that $u_j < a_j$ for $1 \leqslant j \leqslant k$ and $\sum_{j=1}^k u_j = u$.*

Proof. Without loss of generality we assume $u = 0$. Otherwise, we replace a_j by $a_j - u/k$, u_j by $u_j - u/k$ for $1 \leqslant j \leqslant k$.

Choose $\varepsilon > 0$ with $k\varepsilon < \min a$. Define $h_j = a_j - \varepsilon$ for $1 \leqslant j \leqslant k$. Then $\sum_{j=1}^k h_j = a - k\varepsilon > 0$. So $\sum_{j=1}^k h_j^+ > \sum_{j=1}^k h_j^- \geqslant 0$. Define

$$w_j = \frac{h_j^+}{\sum_{\ell=1}^k h_\ell^+} \sum_{\ell=1}^k h_\ell^- - h_j^-,$$

for $1 \leqslant j \leqslant k$. Then $\sum_{j=1}^k w_j = 0$. Moreover, for $1 \leqslant j \leqslant k$,

$$h_j - w_j = h_j^+ - \frac{\sum_{\ell=1}^k h_\ell^-}{\sum_{\ell=1}^k h_\ell^+} h_j^+ \geqslant 0.$$

By Whitney Approximation Theorem, for $1 \leqslant j \leqslant k$, there is a function $v_j \in \mathrm{C}^\infty(B;\mathbb{R})$ such that $|v_j - w_j| < \varepsilon/2$. Then let $u_j = v_j - \frac{1}{k}\sum_{\ell=1}^k v_\ell \in \mathrm{C}^\infty(B;\mathbb{R})$. So $|u_j - w_j| < \varepsilon$, and $\sum_{j=1}^k u_j = 0$, hence $a_j - u_j > h_j - w_j \geqslant 0$ is as required. □

4. Filtration lemma

Now we combine the topological-combinatorial and geometric-analytic constructions from the previous sections. The objects in the following result are illustrated in Figure 7.

For any tree $\mathcal{T}$ of height ω, for any $X \in \mathrm{Lv}(\ell)$, $\ell \in \mathbb{N} \cup \{0\}$, let

$$\text{Ш}_{\mathcal{T}} X \overset{\text{def}}{=} X_{\alpha_{\ell+1}} = X \setminus \coprod_{Y \in \mathrm{Ch}(X)} Y,$$

$$\text{Ш}_{\mathcal{T}} X \overset{\text{def}}{=} X_{\alpha_{\ell+2}} = X \setminus \coprod_{Z \in \mathrm{Gch}(X)} Z.$$

Lemma 4.1. *Let M be a noncompact oriented connected manifold. Let B be a compact manifold. Suppose $\omega, \tau \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ such that $\int_M \omega = \int_M \tau$, and for any connected component C of the complement of a compact subset of M, either $\int_C \omega = \int_C \tau = +\infty$, or $\int_C \omega$ and $\int_C \tau$ are finite and continuous, and there difference is smooth. Then there is a tree $(\mathcal{T}, \supsetneq)$ of connected open subsets of M and $\{\omega_n\}_{n\in\mathbb{N}\cup\{0\}}, \{\tau_n\}_{n\in\mathbb{N}\cup\{0\}} \subset \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ such that $\omega_0 = \omega, \tau_0 = \tau$ and for any $n \in \mathbb{N}$, $p \in B$, we have that*

$$\operatorname{supp}((\omega_n)_p - (\omega_{n-1})_p) \cup \operatorname{supp}((\tau_n)_p - (\tau_{n-1})_p) \subset \bigcup_{C\in \mathrm{Lv}(2n-2)} (\text{Ш}_{\mathcal{T}} C)^\circ, \tag{4.1}$$

as well as that for each $A \in \mathrm{Lv}(2n-3)$ with $n > 1$, $C \in \mathrm{Lv}(2n-2)$, $E \in \mathrm{Lv}(2n-1)$,

$$\int_{\text{П}_{\mathcal{T}} M} \omega_1 = \int_{\text{П}_{\mathcal{T}} M} \tau_1, \qquad \int_{\text{Ш}_{\mathcal{T}} A} \omega_n = \int_{\text{Ш}_{\mathcal{T}} A} \tau_n \text{ for } n > 1; \tag{4.2}$$

$$\int_{\text{Ш}_{\mathcal{T}} C} \omega_n = \int_{\text{Ш}_{\mathcal{T}} C} \omega_{n-1}, \qquad \int_{\text{Ш}_{\mathcal{T}} C} \tau_n = \int_{\text{Ш}_{\mathcal{T}} C} \tau_{n-1}; \tag{4.3}$$

$$\int_E \omega_n = \int_E \tau_n. \tag{4.4}$$

Proof. The abstract tools we have developed so far in the paper allow us to give an inductive proof of Lemma 4.1 with a minimum of technical fuss.

We aim to find $\alpha_0 = -\infty$ and $\{\alpha_\ell\}_{\ell\in\mathbb{N}} \subset \mathrm{Reg}(f) \cap f(M)$ such that $\mathcal{T}$ is constructed by Lemma 2.3. Note that, if we know $\{\alpha_\ell\}_{0\leqslant\ell\leqslant m}$ for some $m \in \mathbb{N} \cup \{0\}$ for the sequence $\{\alpha_\ell\}_{\ell\in\mathbb{N}\cup\{0\}}$ defining $\mathcal{T}$, then we say $\mathcal{T}$ is constructed up to the m-th level, so we know $\mathrm{Lv}(\ell)$ of $\mathcal{T}$ for any ℓ with $0 \leqslant \ell \leqslant m$.

We proceed by induction on $n \in \mathbb{N} \cup \{0\}$ to find $\alpha_{2n-1}, \alpha_{2n}$ and $\omega_n, \tau_n \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ such that $\int_E \omega_n = \int_E \tau_n$ for any $E \in \mathrm{Lv}(2n-1)$ ($E \in \mathrm{Lv}(0)$ if $n = 0$).

Case 0. Set $\alpha_0 = -\infty$, $M_{\alpha_0} = \varnothing$, and $\mathrm{Lv}(0) = \{M\}$. Since $\omega_0 = \omega$, $\tau_0 = \tau$, we have

$$\int_M \omega_0 = \int_M \tau_0.$$

Case $(n-1)$ *for* $n \in \mathbb{N}$. Assume by induction

$$\int_A \omega_{n-1} = \int_A \tau_{n-1} \tag{4.5}$$

for any $A \in \mathrm{Lv}(2n-3)$ ($A \in \mathrm{Lv}(0)$ when $n = 1$).

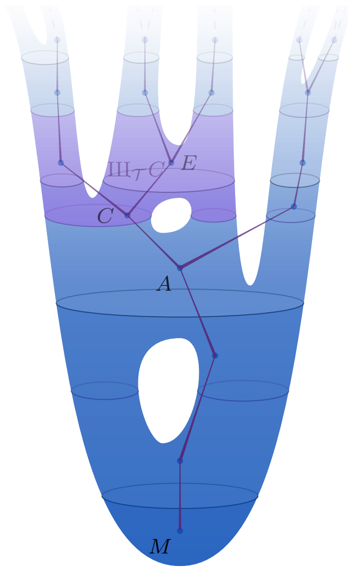

Figure 7: The manifold M sliced by f, and the tree $\mathcal{T}$ in Lemma 4.1.

Case n for $n \in \mathbb{N}$. Let $\alpha_{2n-1} \in \mathrm{Reg}(f)$ such that $\alpha_{2n-1} > \max\{\theta_C \mid C \in \mathrm{Lv}(2n-2)\}$, where θ_C is defined by Lemma 2.2. Then $\mathcal{T}$ is constructed up to the $(2n-1)$-th level. Let $A \in \mathrm{Lv}(2n-3)$ (if $n = 1$ let $A = M$ and replace $\mathrm{Gch}(A)$ by $\mathrm{Ch}(M)$, $Ш_{\mathcal{T}}A$ by $\amalg_{\mathcal{T}}M$ throughout this paragraph). Let $\mathrm{Gch}_0(A)$ (resp. $\mathrm{Gch}_1(A)$) be the subcollection of elements in $\mathrm{Gch}(A)$ with finite (resp. infinite) volume. For any $E \in \mathrm{Gch}(A)$, we define $\delta_E \in \mathrm{C}^\infty(B;\mathbb{R})$ as follows: if E has finite volume, let

$$\delta_E = \int_E \tau_{n-1} - \int_E \omega_{n-1};$$

if E has infinite volume, let

$$\delta_E = \frac{1}{\#\,\mathrm{Gch}_1(A)}\left(\int_{Ш_{\mathcal{T}}A} \tau_{n-1} - \int_{Ш_{\mathcal{T}}A} \omega_{n-1} - \sum_{E_0 \in \mathrm{Gch}_0(A)} \delta_{E_0}\right).$$

Then by (4.5) we have

$$\sum_{E \in \mathrm{Gch}(A)} \delta_E = \int_{Ш_{\mathcal{T}}A} \tau_{n-1} - \int_{Ш_{\mathcal{T}}A} \omega_{n-1}.$$

For any $C \in \mathrm{Ch}(A)$, let $u_C \in \mathrm{C}^\infty(B;\mathbb{R})$ be such that

$$\max\left(-\int_{\amalg_{\mathcal{T}} C} \omega_{n-1}, -\int_{\amalg_{\mathcal{T}} C} \tau_{n-1} + \sum_{E \in \mathrm{Ch}(C)} \delta_E\right) < u_C < \int_C \omega_{n-1} - \int_{\amalg_{\mathcal{T}} C} \omega_{n-1}.$$

Note that if C has finite volume,

$$\begin{aligned}&\left(\int_C \omega_{n-1} - \int_{\amalg_{\mathcal{T}} C} \omega_{n-1}\right) - \left(-\int_{\amalg_{\mathcal{T}} C} \tau_{n-1} + \sum_{E \in \mathrm{Ch}(C)} \delta_E\right)\\ &= \int_C \omega_{n-1} + \left(\int_{\amalg_{\mathcal{T}} C} \tau_{n-1} - \int_{\amalg_{\mathcal{T}} C} \omega_{n-1}\right) + \sum_{E \in \mathrm{Ch}(C)} \left(\int_E \tau_{n-1} - \int_E \omega_{n-1}\right)\\ &= \int_C \tau_{n-1} > 0,\end{aligned}$$

so such u_C exists. Since

$$u_C < \sum_{E \in \mathrm{Ch}(C)} \int_E \omega_{n-1} = \int_C \omega_{n-1} - \int_{\amalg_{\mathcal{T}} C} \omega_{n-1},$$

by Lemma 3.7, we can choose $v_E \in \mathrm{C}^\infty(B;\mathbb{R})$ such that $v_E < \int_E \omega_{n-1}$ and $\sum_{E \in \mathrm{Ch}(C)} v_E = u_C$.

For any $E \in \mathrm{Ch}(C)$, if E has infinite volume, take $\beta_E \in \mathrm{Reg}(f)$ that is greater than θ_E. Otherwise, the function

$$\begin{aligned}&\lambda\colon B \times \mathbb{R} \to \mathbb{R},\\ &\quad (b,\beta) \mapsto \min\left(\left(\int_{E_{(-\infty,\beta]}} \omega_{n-1}\right)(b), \left(\int_{E_{(-\infty,\beta]}} \tau_{n-1} + \delta_E\right)(b)\right) - v_E(b)\end{aligned}$$

is continuous in b and increasing in β. Note that $\lim_{\beta \to +\infty} \rho(b,\beta)(\int_E \omega_{n-1} - v_E)(b) > 0$ for any $b \in B$. Since B is compact there is $\beta_E > \max\{\alpha_{2n-1}, \theta_E\}$ such that $\lambda(\cdot, \beta_E) > 0$. Let $\alpha_{2n} = \max_{E \in \mathrm{Lv}(2n-1)} \beta_E$. Then $\mathcal{T}$ is constructed up to the $2n$-th level. So $\amalg_{\mathcal{T}} E = E_{\alpha_{2n}}$. Then we have $v_E < \int_{\amalg_{\mathcal{T}} E} \omega_{n-1}$, and $v_E - \delta_E < \int_{\amalg_{\mathcal{T}} E} \tau_{n-1}$.

Since all the right hand sides of the next four equations are positive and smooth, by Lemma 3.4, there are $\omega_n, \tau_n \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$ such that

$$\int_{\mathrm{II}_{\mathcal{T}} C} \omega_n = \int_{\mathrm{II}_{\mathcal{T}} C} \omega_{n-1} + u_C, \quad \int_{\mathrm{II}_{\mathcal{T}} C} \tau_n = \int_{\mathrm{II}_{\mathcal{T}} C} \tau_{n-1} + u_C - \sum_{E \in \mathrm{Ch}(C)} \delta_E,$$
$$\int_{\mathrm{II}_{\mathcal{T}} E} \omega_n = \int_{\mathrm{II}_{\mathcal{T}} E} \omega_{n-1} - v_E, \quad \int_{\mathrm{II}_{\mathcal{T}} E} \tau_n = \int_{\mathrm{II}_{\mathcal{T}} E} \omega_{n-1} - (v_E - \delta_E),$$

and

$$\mathrm{supp}((\omega_n)_p - (\omega_{n-1})_p) \cup \mathrm{supp}((\tau_n)_p - (\tau_{n-1})_p) \subset (M_{\alpha_{2n}})^\circ \setminus M_{\alpha_{2n-2}}.$$

Then we have

$$\begin{aligned}\int_{\mathrm{III}_{\mathcal{T}} A} \omega_n &= \int_{\mathrm{III}_{\mathcal{T}} A} \omega_{n-1} + \sum_{C \in \mathrm{Ch}(A)} u_C \\ &= \int_{\mathrm{III}_{\mathcal{T}} A} \tau_{n-1} - \sum_{E \in \mathrm{Gch}(A)} (\delta_E - u_C) = \int_{\mathrm{III}_{\mathcal{T}} A} \tau_n,\end{aligned}$$

and

$$\int_{\mathrm{III}_{\mathcal{T}} C} \omega_n = \int_{\mathrm{II}_{\mathcal{T}} C} \omega_n + \sum_{E \in \mathrm{Ch}(C)} \int_{\mathrm{II}_{\mathcal{T}} E} \omega_n = \int_{\mathrm{III}_{\mathcal{T}} C} \omega_{n-1},$$
$$\int_{\mathrm{III}_{\mathcal{T}} C} \tau_n = \int_{\mathrm{II}_{\mathcal{T}} C} \tau_n + \sum_{E \in \mathrm{Ch}(C)} \int_{\mathrm{II}_{\mathcal{T}} E} \tau_n = \int_{\mathrm{III}_{\mathcal{T}} C} \tau_{n-1},$$

and

$$\begin{aligned}\int_E \omega_n &= \int_{\mathrm{II}_{\mathcal{T}} E} \omega_n + \int_E \omega_{n-1} - \int_{\mathrm{II}_{\mathcal{T}} E} \omega_{n-1} \\ &= \int_E \omega_{n-1} - v_E = \int_E \tau_{n-1} - (v_E - \delta_E) = \int_E \tau_n.\end{aligned}$$

□

5. Proof of main theorem

We apply Lemma 4.1 to M and ω, τ and obtain the tree $\mathcal{T}$ of connected open subsets of M, such that (4.1) to (4.4) are satisfied.

For $n \in \mathbb{N}$ and $C \in \mathrm{Lv}(2n-2)$, applying Lemma 3.3 to $(\text{Ш}_{\mathcal{T}}C)^\circ$, there are $\varphi_n, \psi_n \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$ such that we have $\varphi_n^* \omega_{n-1} = \omega_n$, $\psi_n^* \tau_{n-1} = \tau_n$, and $\varphi_n = \psi_n = \mathrm{id}$ outside of $(M_{\alpha_{2n}})^\circ \setminus M_{\alpha_{2n-2}}$. Let

$$\begin{aligned} \omega_\infty &= \lim_{n\to\infty} \omega_n, & \tau_\infty &= \lim_{n\to\infty} \tau_n, \\ \varphi_\infty &= \varphi_1 \circ \varphi_2 \circ \cdots, & \psi_\infty &= \psi_1 \circ \psi_2 \circ \cdots . \end{aligned} \tag{5.1}$$

Since $\{(\text{Ш}_{\mathcal{T}}C)^\circ\}_{C\in\mathcal{T},2|\mathrm{dpt}(C)}$ is mutually disjoint, the pointwise limits in (5.1) will be stable at a finite n, so $\omega_\infty, \tau_\infty \in \mathcal{F}^\infty(B; \Omega_{\mathrm{vol}}(M))$, $\varphi_\infty, \psi_\infty \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$,

$$\int_{\text{Ш}_{\mathcal{T}}M} \omega_\infty = \int_{\text{Ш}_{\mathcal{T}}M} \tau_\infty, \qquad \int_{\text{Ш}_{\mathcal{T}}A} \omega_\infty = \int_{\text{Ш}_{\mathcal{T}}A} \tau_\infty$$

for each $A \in \mathcal{T}$ with odd depth, $\varphi_\infty^* \omega = \omega_\infty$, and $\psi_\infty^* \tau = \tau_\infty$.

We have left to show that there is $\varphi' \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$ with $\varphi'^* \omega_\infty = \tau_\infty$. Let $\{L_j\}_{j\in\mathbb{N}}$ be $\{\text{Ш}_{\mathcal{T}}M\} \cup \{\overline{\text{Ш}_{\mathcal{T}}A}\}_{A\in\mathcal{T},2\nmid\mathrm{dpt}(A)}$. Then this is the result of Lemma 3.6.

Finally,

$$\varphi = \varphi_\infty \circ \varphi' \circ \psi_\infty^{-1} \in \mathcal{F}^\infty(B; \mathrm{Diff}(M))$$

is as required.

6. Final remarks

We conclude with a few remarks:

1) We have proved Theorem 1.2 using a version of Hodge theory on non-compact manifolds due to Bueler and Prokhorenkov [2]. We believe that there should also be a parametric version of the Greene–Shiohama proof without resorting to Hodge theory. The idea of using Hodge theory is in itself of interest because it can be easily generalized (for instance to symplectic forms [4]).

2) The geometry of volume-preserving diffeomorphisms is much simpler than that of their symplectic counterparts (see [6] and [11]).

3) In the way of applications, we would like to mention that the Moser and Greene–Shiohama results are important in classical mechanics, where understanding the geometry of volume forms is relevant [5, 8].

4) There is a version of Theorem 1.2 for fiber bundles with nontrivial topology. Theorem 1.2 corresponds to the case of trivial bundles over

B. The idea and techniques to prove this more general result are similar, but the statement and proof require the introduction of a significant amount of terminology [10].

5) If $B = [0, 1]$, a version of Theorem 1.2 was given for continuous families as [3, Theorem 1] for the case of manifolds M which are the interior of a compact manifold with boundary. The work relies on a version of Moser's theorem for compact manifolds with boundary due to Banyaga [1].

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