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Stability property of multiplicities of group representations

Paul-Emile Paradan

This paper is dedicated to the study of the stability of multiplicities of group representations.

1	**Introduction**	**1389**
2	**Statement of the results**	**1393**
3	**Reduction of Kähler manifolds**	**1398**
4	**Witten deformation**	**1401**
5	**Examples**	**1411**
	References	**1424**

1. Introduction

In this article we study the asymptotic behavior of multiplicities of Lie group representations arising from group actions on manifolds.

If G is a compact connected Lie group, its irreducible representations are parametrized by a semi-group Λ^+_G of dominant weights. We consider admissible G-representations, namely those admitting a decomposition

$$E = \bigoplus_{\mu\in\Lambda^+_G} \mathbf{m}_E(\mu) V^G_\mu.$$

Here V^G_μ is the irreducible representation of G associated to $\mu \in \Lambda^+_G$, and $\mathbf{m}_E(\mu) \in \mathbb{N}$ is the (finite) multiplicity of V^G_μ in the representation E. The fonction $\mathbf{m}_E : \Lambda^+_G \to \mathbb{N}$ is called a multiplicity map.

Recently, Stembridge [28] has proposed to generalize a classical result of Murnaghan by introducing the notion of **stability**. A weight $\mu \in \Lambda_G^+$ is called

- *semi-stable* if $\mathbf{m}_E(n\mu) = 1$ for all $n \geq 1$.
- *stable* if $\mathbf{m}_E(\mu) > 0$, and if the sequence $\mathbf{m}_E(\lambda + n\mu)$ converges for any $\lambda \in \Lambda_G^+$.

It is natural to consider weaker notions: a weight $\mu \in \Lambda_G^+$ is called

- *weakly semi-stable* if the sequence $\mathbf{m}_E(n\mu)$ is bounded,
- *weakly stable* if the sequence $\mathbf{m}_E(\lambda + n\mu)$ is bounded for any $\lambda \in \Lambda_G^+$.

Obviously we see that *weak stability* $\Longrightarrow$ *weak semi-stability.*

Definition 1.1. The admissible representation E is *fine* if

$$\{\text{weakly stable weights for } \mathbf{m}_E\} = \{\text{weakly semi-stable weights for } \mathbf{m}_E\}$$

and $\{\text{stable weights for } \mathbf{m}_E\} = \{\text{semi-stable weights for } \mathbf{m}_E\}$.

When an admissible representation E is *fine*, we associate a stretched multiplicity map

$$\mathbf{m}_E^\mu : \Lambda_G^+ \to \mathbb{N} \tag{1.1}$$

to any *stable* weight μ, by taking $\mathbf{m}_E^\mu(\lambda) = \lim_{n\to\infty} \mathbf{m}_E(\lambda + n\mu)$.

The main purpose of this paper is to exhibit a large family of fine admissible representations for which we are able to compute the stretched multiplicity maps.

Consider a closed subgroup K of G, not necessarily connected, and a finite dimension K-module V. We assume that the algebra $\mathrm{Sym}(V^*)$ of polynomial functions on V has **finite** K-multiplicities. Let

$$\mathbb{E} = \mathbb{E}_{G,K,V} := \mathrm{Ind}_K^G(\mathrm{Sym}(V^*))$$

be the representation of G which is induced by the K-module $\mathrm{Sym}(V^*)$. We have $\mathbb{E} = \sum_\mu \mathbf{m}_\mathbb{E}(\mu) V_\mu^G$ where each multiplicity

$$\mathbf{m}_\mathbb{E}(\mu) = \dim \left[\mathrm{Sym}(V^*) \otimes (V_\mu^G)^*|_K\right]^K .$$

is finite.

The main result of this paper is the following

Theorem 1.2.
- *The admissible representations* $\mathbb{E}_{G,K,V}$ *are fine.*
- *Let* μ *be a stable weight for* $\mathbf{m}_{\mathbb{E}}$. *The stretched multiplicity map* $\mathbf{m}_{\mathbb{E}}^{\mu}$ *has the following expression:*

$$\mathbf{m}_{\mathbb{E}}^{\mu} = \mathbf{m}_{\mathbb{E}'},$$

where $\mathbb{E}' = \mathbb{E}_{G_\mu,H,V'}$. *Here* G_μ *is the stabilizer subgroup of* μ, H *is a closed subgroup of* G_μ *and* V' *is a* H*-module such that the algebra* $\mathrm{Sym}((V')^*)$ *has finite* H*-multiplicities.*

The following important example is concerned with the branching laws.

Example 1.3. Consider a morphism $\rho : K \to \tilde{K}$ between two connected compact Lie groups. Let us work with the groups $G := K \times \tilde{K}$, $K \hookrightarrow G$ embedded diagonally, and with the trivial K-module $V = 0$. In this setting the multiplicity function $\mathbf{m}_{\mathbb{E}}$ corresponds to the branching laws[1] between the representations of K and $\tilde{K}$:

$$\mathbf{m}_{\mathbb{E}}(\lambda, \tilde{\lambda}) = \dim \left[V_\lambda^K \otimes V_{\tilde{\lambda}}^{\tilde{K}}|_K \right]^K, \tag{1.2}$$

for $(\lambda, \tilde{\lambda}) \in \Lambda_K^+ \times \Lambda_{\tilde{K}}^+$.

So Theorem 1.2 shows that any branching law defines *fine* multiplicity map. This fact generalizes previous results obtained by Stembridge [28] and Sam-Snowden [26] for the Kronecker coefficients (see Section 5.3 for more details). Notice that Pelletier has also obtained a geometric proof of the equivalence *stability* $\simeq$ *semi-stability* for the Kronecker coefficients [25].

Our computation of the stretched multiplicity maps extends some results obtained by Brion [9], Manivel [15] and Montagard [19] in the plethysm case. In fact, when μ is weakly stable, we get a formula for $\mathbf{m}_{\mathbb{E}}(\lambda + n\mu)$ when n is large enough.

Another interesting question is to produce examples of stable weights. In the case of Kronecker coefficients, Vallejo [31] and Manivel [16] introduced a notion of "additive matrix" that permits them to parametrize a large family of stable elements. In Section 5 we show that this notion can be adapted

[1] In Section 2.2 we will use another convention for branching coefficients, taking the dual of V_λ^K in (1.2).

to any branching laws (see Definition 5.1), and we compute the stretched multiplicity maps associated to the corresponding stable weights.

We finish this introduction by explaining a geometric result that we use to obtain Theorem 1.2 and which is interesting for itself.

Let M be a compact complex manifold acted on by a compact Lie group G. Let $\mathcal{L} \to M$ be a G-equivariant holomorphic line bundle that is assumed to be **ample**: the group G acts by holomorphic transformations on $\mathcal{L}$. In this context, we are interested in the family of vector spaces $\Gamma(M, \mathcal{L}^{\otimes n})^G$ consisting of G-invariant holomorphic sections, and more particularly to the sequence

$$\mathbf{H}(n) := \dim \Gamma(M, \mathcal{L}^{\otimes n})^G, n \geq 1.$$

For any holomorphic G-vector bundle $\mathcal{E} \to M$, we consider also the sequence

$$\mathbf{H}_{\mathcal{E}}(n) := \dim \Gamma(M, \mathcal{E} \otimes \mathcal{L}^{\otimes n})^G, \ n \geq 1.$$

We obtain the following geometric stability result.

Theorem 1.4. *If $\mathbf{H}(n)$ is bounded, then the sequence $\mathbf{H}_{\mathcal{E}}(n)$ is bounded and can be computed for large values of n.*

Let us explain the contents of the different sections of the article.

• In Section 2.1 the precise statement of Theorem 1.4 is given in Theorem A.
• In Section 2.2 we apply Theorem A to the case of branching law coefficients. See Theorem B.
• In Section 2.3 the precise statement of Theorem 1.2 is given in Theorem C.
• In Section 3 we recall some basic properties that follows from the $[Q, R] = 0$ theorem.
• Section 4 is dedicated to the proof of Theorems A and C.
• The final section is devoted to some examples.

Notations. Throughout the paper:

- G denotes a compact connected Lie group with Lie algebra $\mathfrak{g}$.
- T is a maximal torus in G with Lie algebra $\mathfrak{t}$.
- $\Lambda \subset \mathfrak{t}^*$ is the weight lattice of T : every $\mu \in \Lambda$ defines a 1-dimensional T-representation, denoted by $\mathbb{C}_\mu$, where $t = \exp(X)$ acts by $t^\mu := e^{i\langle \mu, X\rangle}$.
- We denote by $R(G)$ the representation ring of G : an element $E \in R(G)$ can be represented as finite sum $E = \sum_{\mu \in \Lambda_G^+} \mathbf{m}_\mu V_\mu^G$, with $\mathbf{m}_\mu \in \mathbb{Z}$. The multiplicity $\mathbf{m}_0$ of the trivial representation is also denoted $[E]^G$.

- We denote by $\hat{R}(G)$ the space of $\mathbb{Z}$-valued functions on $\hat{G}$. An element $E \in \hat{R}(G)$ can be represented as an infinite sum $E = \sum_{\mu\in\Lambda_G^+} \mathbf{m}(\mu)V_\mu^G$, with $\mathbf{m}(\mu) \in \mathbb{Z}$.
- If K is a closed subgroup of G, the induction map $\mathrm{Ind}_K^G : \hat{R}(K) \to \hat{R}(G)$ is the dual of the restriction morphism $R(G) \to R(K)$.
- When G acts on a set X, the stabilizer subgroup of $x \in G$ is denoted by $G_x := \{g \in G \mid g \cdot x = x\}$. The Lie algebra of G_x is denoted by $\mathfrak{g}_x$.

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2. Statement of the results

In this section, we consider the action of a compact connected Lie group G on a complex manifold M.

2.1. Geometric stability

We assume here that M is compact and is equipped with a G-equivariant holomorphic line bundle $\mathcal{L}$ that is assumed to be **ample**. Then there exists an Hermitian metric h on $\mathcal{L}$ such that the curvature $\Omega := i(\nabla^h)^2$ of its Chern connection ∇^h is a Kähler class : Ω is a symplectic form on M that is compatible with the complex structure. By an averaging process we can assume that the G-action leaves the metric and connection invariant.

The moment map $\Phi : M \to \mathfrak{g}^*$ is defined by Kostant's relations

$$L(X) - \iota(X_M)\nabla^h = i\langle\Phi, X\rangle \quad \text{for all} \quad X \in \mathfrak{g}. \tag{2.3}$$

Here $L(X)$ is the Lie derivative on the sections of $\mathcal{L}$, and $X_M(m) := \frac{d}{ds}e^{-sX} \cdot m|_{s=0}$ is the vector field generated by $X \in \mathfrak{g}$.

An important object here is the Marsden-Weinstein symplectic reduced space

$$M_0 := \Phi^{-1}(0)/G.$$

The first important result is that M_0 is homeomorphic to the Mumford GIT quotient $M/\!\!/G_{\mathbb{C}} = \mathrm{Proj}\left(\oplus_{n\geq 0}\Gamma(M, \mathcal{L}^{\otimes n})^G\right)$ [12, 27]. We can then deduce the following basic fact.

Lemma 2.1. *The sequence* $\mathbf{H}(n) := \dim \Gamma(M, \mathcal{L}^{\otimes n})^G, n \geq 1$ *satisfies the following equivalences:*

- $\mathbf{H}(n) = 0, \ \forall n \geq 1 \iff M_0 = \emptyset$,
- $\mathbf{H}(n)$ *is non-zero and bounded* $\iff M_0 = \{pt\}$.

We take $m_o \in \Phi^{-1}(0)$ and we denote by H the stabilizer subgroup of m_o. Kostant's relations implies that the action of the connected component H^o on $\mathcal{L}|_{m_o}$ is trivial and so the H-module $\mathcal{L}^{\otimes n}|_{m_o}$ is periodic.

The following result is a particular case of the $[Q, R] = 0$ theorem of Guillemin-Sternberg [10, 27, 29].

Proposition 2.2. *When* $M_0 = \{pt\}$, *we have* $\mathbf{H}(n) := \dim[\mathcal{L}^{\otimes n}|_{m_o}]^H$. *In particular if* $\mathbf{H}(1) \neq 0$, *the* H*-module* $\mathcal{L}|_{m_o}$ *is trivial and then* $\mathbf{H}(n) = 1$ *for all* $n \geq 1$.

Let us recall the geometric criterion that characterizes the fact that the reduced space M_0 is a singleton. The tangent space $\mathrm{T}_{m_o}M$ at m_o is a H-module and we consider the sub-module $\mathfrak{g}_{\mathbb{C}} \cdot m_o \subset \mathrm{T}_{m_o}M$ consisting of the tangent vectors at m_o of the complex orbit $G_{\mathbb{C}} \cdot m_o$.

The following H-module is important for our purpose:

$$\mathbb{W} := \mathrm{T}_{m_o}M / \mathfrak{g}_{\mathbb{C}} \cdot m_o. \tag{2.4}$$

Let $\mathrm{Sym}(\mathbb{W}^*)$ be the H-module consisting of polynomial functions on $\mathbb{W}$. The following standard fact is explained in Section 3.

Proposition 2.3. *We have* $\Phi^{-1}(0) = Gm_o$ *if and only if the* H*-multiplicities of* $\mathrm{Sym}(\mathbb{W}^*)$ *are finite.*

Our "geometric stability" result takes the following form.

Theorem A. *Let* $\mathcal{E} \to M$ *be an holomorphic* G*-vector bundle, and consider the sequence* $\mathbf{H}_{\mathcal{E}}(n) := \dim \Gamma(M, \mathcal{E} \otimes \mathcal{L}^{\otimes n})^G, \ n \geq 1$.

- *If* $\mathbf{H}(n) = 0, \forall n \geq 1$, *then* $\mathbf{H}_{\mathcal{E}}(n) = 0$ *if* n *is large enough.*
- *If* $\mathbf{H}(n)$ *is bounded and non-zero, then*

$$\mathbf{H}_{\mathcal{E}}(n) = \dim \left[\mathrm{Sym}(\mathbb{W}^*) \otimes \mathcal{E}|_{m_o} \otimes \mathcal{L}^{\otimes n}|_{m_o}\right]^H$$

for n *large enough. Thus, the sequence* $\mathbf{H}_{\mathcal{E}}(n)$ *is periodic from a certain rank, and accordingly it is bounded.*

- *If* $\mathbf{H}(n)$ *is bounded and* $\mathbf{H}(1) \neq 0$*, the sequence* $\mathbf{H}_{\mathcal{E}}(n)$ *is increasing and converging to* $\dim\,[\mathrm{Sym}(\mathbb{W}^*) \otimes \mathcal{E}|_{m_o}]^H$.

In the next section we apply Theorem **A** to the branching laws between compact Lie groups.

2.2. Stability of branching law coefficients

Let $\rho : G \to \tilde{G}$ be a morphism between two connected compact Lie groups. We denote by $d\rho : \mathfrak{g} \to \tilde{\mathfrak{g}}$ the induced Lie algebras morphism, and by $\pi : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$ the dual map.

Select maximal tori T in G and $\tilde{T}$ in $\tilde{G}$, and Weyl chambers $\tilde{\mathfrak{t}}^*_{\geq 0}$ in $\tilde{\mathfrak{t}}^*$ and $\mathfrak{t}^*_{\geq 0}$ in $\mathfrak{t}^*$, where $\mathfrak{t}$ and $\tilde{\mathfrak{t}}$ denote respectively the Lie algebras of T and $\tilde{T}$. Let $\Lambda^+_{\tilde{G}} \subset \tilde{\mathfrak{t}}^*_{\geq 0}$, $\Lambda^+_G \subset \mathfrak{t}^*_{\geq 0}$ be the set of dominant weights.

For any $(\mu, \tilde{\mu}) \in \Lambda^+_G \times \Lambda^+_{\tilde{G}}$, we denote by V^G_μ, $V^{\tilde{G}}_{\tilde{\mu}}$ the corresponding irreducible representations of G and $\tilde{G}$, and we define

$$\mathbf{m}_\rho(\mu, \tilde{\mu}) \in \mathbb{N} \tag{2.5}$$

as the multiplicity of V^G_μ in $V^{\tilde{G}}_{\tilde{\mu}}|_G$.

To $(\mu, \tilde{\mu}) \in \Lambda^+_G \times \Lambda^+_{\tilde{G}}$ we associate the coadjoint orbits $G\mu$ and $\tilde{G}\tilde{\mu}$, viewed as Kähler manifolds, and the ample line bundles $\mathcal{L}_\mu \to G\mu$ and $\tilde{\mathcal{L}}_{\tilde{\mu}} \to \tilde{G}\tilde{\mu}$ that are defined by $\mathcal{L}_\mu \simeq G \times_{G_\mu} \mathbb{C}_\mu$ and $\tilde{\mathcal{L}}_{\tilde{\mu}} \simeq \tilde{G} \times_{\tilde{G}_{\tilde{\mu}}} \mathbb{C}_{\tilde{\mu}}$. The G-invariant complex structure on the homogeneous manifold $G\mu$ is such that the tangent space $\mathrm{T}_\mu(G\mu)$ is isomorphic to $\sum_{(\alpha,\mu)>0} (\mathfrak{g} \otimes \mathbb{C})_\alpha$.

By Borel-Weil theorem, we have $V^G_\mu = \Gamma(G\mu, \mathcal{L}_\mu)$ and $V^{\tilde{G}}_{\tilde{\mu}} = \Gamma(\tilde{G}\tilde{\mu}, \tilde{\mathcal{L}}_{\tilde{\mu}})$, so that

$$\mathbf{m}_\rho(n\mu, n\tilde{\mu}) = \dim \Gamma(M_{\mu,\tilde{\mu}}, \mathcal{L}^{\otimes n}_{\mu,\tilde{\mu}})^G, \quad n \geq 1,$$

where $M_{\mu,\tilde{\mu}} = (G\mu)^- \times \tilde{G}\tilde{\mu}$ is a G-compact complex manifold[2] and $\mathcal{L}_{\mu,\tilde{\mu}} := (\mathcal{L}_\mu)^{-1} \boxtimes \tilde{\mathcal{L}}_{\tilde{\mu}}$ is a G-equivariant ample line bundle on $M_{\mu,\tilde{\mu}}$.

Another version of Borel-Weil theorem[3] says that

$$V^G_{\lambda+n\mu} = \Gamma(G\mu, \mathcal{E}_\lambda \otimes \mathcal{L}^{\otimes n}_\mu), \quad n \geq 0,$$

where $\mathcal{E}_\lambda \simeq G \times_{G_\mu} V^{G_\mu}_\lambda$ is the holomorphic G-vector bundle associated to the irreducible representation $V^{G_\mu}_\lambda$ of G_μ with highest weight λ. Finally we

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[2] $(G\mu)^-$ denotes the manifold $G\mu$ with the opposite complex structure.

[3] See Section 1 of [19] for an explanation.

see that

$$\mathbf{m}_\rho(\lambda + n\mu, \tilde{\lambda} + n\tilde{\mu}) = \dim \Gamma(M_{\mu,\tilde{\mu}}, \mathcal{E}_{\lambda,\tilde{\lambda}} \otimes \mathcal{L}_{\mu,\tilde{\mu}}^{\otimes n})^G \quad n \geq 1,$$

with $\mathcal{E}_{\lambda,\tilde{\lambda}} := (\mathcal{E}_\lambda)^* \boxtimes \tilde{\mathcal{E}}_{\tilde{\lambda}}$.

For any couple of weights $(\mu, \tilde{\mu})$, we denote by $(\tilde{G}\tilde{\mu})_\mu$ the reduction of the G-Hamiltonian manifold $\tilde{G}\tilde{\mu}$ at μ : in other words $(\tilde{G}\tilde{\mu})_\mu := \tilde{G}\tilde{\mu} \cap \pi^{-1}(G\mu)/G$. Thanks to the shifting trick, we notice that the symplectic reduction of the G-manifold $M_{\mu,\tilde{\mu}}$ at 0 coincides with $(\tilde{G}\tilde{\mu})_\mu$.

In this setting Lemma 2.1 gives the following

Lemma 2.4. *We have the following equivalences*

- $\mathbf{m}_\rho(n\mu, n\tilde{\mu}) = 0, \ \forall n \geq 1 \iff (\tilde{G}\tilde{\mu})_\mu = (M_{\mu,\tilde{\mu}})_0 = \emptyset$
- $\mathbf{m}_\rho(n\mu, n\tilde{\mu})$ *is bounded and non-zero* $\iff (\tilde{G}\tilde{\mu})_\mu = (M_{\mu,\tilde{\mu}})_0 = \{pt\}$.

When $(\tilde{G}\tilde{\mu})_\mu = \emptyset$, Theorem **A** tell us that for any dominant weight $(\lambda, \tilde{\lambda})$, $\mathbf{m}_\rho(\lambda + n\mu, \tilde{\lambda} + n\tilde{\mu}) = 0$ when n is large enough.

Let us concentrate on the case where $(\tilde{G}\tilde{\mu})_\mu = (M_{\mu,\tilde{\mu}})_0 = \{pt\}$. Let $\xi_o \in \tilde{G}\tilde{\mu}$ such that $\pi(\xi_o) = \mu$. We consider the point $m_o = (\mu, \xi_o) \in M_{\mu,\tilde{\mu}}$ and its stabilizer subgroup $H = G_{m_o}$ that is contained in G_μ.

We consider the following H-modules associated to $m_o = (\mu, \xi_o)$:

1) $\mathbb{D}_{\mu,\tilde{\mu}} := \mathcal{L}_{\mu,\tilde{\mu}}|_{m_o} = (\mathbb{C}_\mu)^*|_H \otimes \tilde{\mathcal{L}}_{\tilde{\mu}}|_{\xi_o}$,
2) $\mathbb{E}_{\lambda,\tilde{\lambda}} := \mathcal{E}_{\lambda,\tilde{\lambda}}|_{m_o} = (V_\lambda^{G_\mu})^*|_H \otimes \mathcal{E}_{\tilde{\lambda}}|_{\xi_o}$,
3) $\mathbb{W} := \mathrm{T}_{m_o} M_{\mu,\tilde{\mu}}/\mathfrak{g}_\mathbb{C} \cdot m_o$ that is isomorphic to $\mathrm{T}_{\xi_o}\tilde{G}\tilde{\mu}/\rho(\mathfrak{p}_\mu) \cdot \xi_o$. Here

$$\mathfrak{p}_\mu := \mathfrak{t} \otimes \mathbb{C} \ \oplus \bigoplus_{(\alpha,\mu) \geq 0} (\mathfrak{g} \otimes \mathbb{C})_\alpha \tag{2.6}$$

is the parabolic subalgebra of $\mathfrak{g} \otimes \mathbb{C}$ associated to μ.

Note that H^o acts trivially on the 1-dimensional H-module $\mathbb{D}_{\mu,\tilde{\mu}}$ (it is a consequence of Kostant's relations). Thus the sequence $(\mathbb{D}_{\mu,\tilde{\mu}}^{\otimes n})_{n\geq 1}$ of H-modules is periodic.

In this setting Proposition 2.3 says that $(\tilde{G}\tilde{\mu})_\mu = \{pt\}$ if and only if the H-module $\mathrm{Sym}(\mathbb{W}^*)$ has finite H-multiplicities. Theorem **A** becomes

Theorem B. *Let* $(\mu, \tilde{\mu})$ *be a dominant weight such that* $\mathbf{m}_\rho(n\mu, n\tilde{\mu})$ *is bounded and non-zero.*

- *We have* $\mathbf{m}_\rho(n\mu, n\tilde{\mu}) = \dim[\mathbb{D}_{\mu,\tilde{\mu}}^{\otimes n}]^H$, $n \geq 1$, *and for any dominant weight* $(\lambda, \tilde{\lambda})$ *the equality*

$$\mathbf{m}_\rho(\lambda + n\mu, \tilde{\lambda} + n\tilde{\mu}) = \dim[\mathrm{Sym}(\mathbb{W}^*) \otimes \mathbb{E}_{\lambda,\tilde{\lambda}} \otimes \mathbb{D}_{\mu,\tilde{\mu}}^{\otimes n}]^H$$

holds for n *large enough. In particular the sequence* $\mathbf{m}_\rho(\lambda + n\mu, \tilde{\lambda} + n\tilde{\mu})$ *is bounded.*

- *If* $\mathbf{m}_\rho(\mu, \tilde{\mu}) \neq 0$, *we have* $\mathbf{m}_\rho(n\mu, n\tilde{\mu}) = 1, \forall n \geq 1$. *Moreover the sequence* $\mathbf{m}_\rho(\lambda + n\mu, \tilde{\lambda} + n\tilde{\mu})$ *is increasing and constant for large enough* n, *equal to* $\dim[\mathrm{Sym}(\mathbb{W}^*) \otimes \mathbb{E}_{\lambda,\tilde{\lambda}}]^H$.

In Section 5 we give some examples where Theorem **B** applies.

2.3. Stability in a non-compact case

We consider here a closed subgroup K of G, not necessarily connected, and a Hermitian K-module V. We denote by $\Phi_V : V \to \mathfrak{k}^*$ the (moment) map defined by $\langle \Phi_V(v), X \rangle = \frac{1}{i}(v, Xv)$. In this section we assume that the algebra $\mathrm{Sym}(V^*)$ of polynomial functions on V has finite K-multiplicities.

Let $\mathbb{E}$ be the G-representation that is induced by the K-module $\mathrm{Sym}(V^*)$. We have $\mathbb{E} = \sum_\mu \mathbf{m}_\mathbb{E}(\mu) V_\mu^G$ where $\mathbf{m}_\mathbb{E}(\mu) = \dim[\mathrm{Sym}(V^*) \otimes (V_\mu^G)^*|_K]^K$.

The study of the asymptotic behavior of the multiplicity function $\mu \mapsto \mathbf{m}_\mathbb{E}(\mu)$ uses that the representation space $\mathbb{E}$ can be constructed as the "geometric quantization" of the Hamiltonian G-manifold

$$M := G \times_K (\mathfrak{k}^\perp \oplus V). \tag{2.7}$$

The moment map on M is defined by the relation

$$\Phi([g; \xi \oplus v]) := g\,(\xi + \Phi_V(v))\,,$$

and the complex structure on M comes from the natural isomorphism $M \simeq G_\mathbb{C} \times_{K_\mathbb{C}} V$.

We denote by $M_\mu := \Phi^{-1}(G\mu)/G$ the symplectic reduction of M at μ. Here the $[Q, R] = 0$ theorem gives the following

Proposition 2.5. *We have the following equivalences:*

- $\mathbf{m}_\mathbb{E}(n\mu) = 0,\ \forall n \geq 1 \iff M_\mu = \emptyset$,
- $\mathbf{m}_\mathbb{E}(n\mu)$ *is non-zero and bounded* $\iff M_\mu = \{pt\}$.

We fix a dominant weight μ. Let $x_o \in M$ such that $\Phi(x_o) = \mu$. Its stabilizer subgroup $H \subset G$ is contained in G_μ. Hence the 1-dimensional representation $\mathbb{C}_\mu$ of the group G_μ can be restricted to H. It is not difficult to see that the connected component H^o acts trivially on $\mathbb{C}_\mu$. Hence the sequence $\mathbb{C}_{n\mu}|_H$ of H-modules is periodic.

Let $m_o = (\mu, x_o) \in P = (G\mu)^- \times M$. The H-module $\mathbb{W} := \mathrm{T}_{m_o} P/\mathfrak{g}_\mathbb{C} \cdot m_o$ is canonically isomorphic to $\mathrm{T}_{x_o} M/\mathfrak{p}_\mu \cdot x_o$, where $\mathfrak{p}_\mu$ is the parabolic subalgebra of $\mathfrak{g} \otimes \mathbb{C}$ associated to μ (see (2.6)).

Recall that the H-multiplicities in $\mathrm{Sym}(\mathbb{W}^*)$ are finite if and only if $\Phi^{-1}(G\mu) = Gx_o$.

In this non-compact setting, we obtain the following stability result.

Theorem C.

- *If* $\mathbf{m}_\mathbb{E}(n\mu) = 0,\ \forall n \geq 1,$ *then for any dominant weight* λ *we have* $\mathbf{m}_\mathbb{E}(\lambda + n\mu) = 0$ *if* n *is large enough.*
- *If* $\mathbf{m}_\mathbb{E}(n\mu)$ *is bounded and non-zero, then* $\mathbf{m}_\mathbb{E}(n\mu) = \dim[\mathbb{C}_{n\mu}|_H]^H, n \geq 0,$ *and for any dominant weight* λ
$$\mathbf{m}_\mathbb{E}(\lambda + n\mu) = \dim\left[\mathrm{Sym}(\mathbb{W}^*) \otimes (V_\lambda^{G_\mu})^*|_H \otimes \mathbb{C}_{-n\mu}|_H\right]^H$$
for n *large enough. In particular the sequence* $\mathbf{m}_\mathbb{E}(\lambda + n\mu)$ *is bounded.*
- *If* $\mathbf{m}_\mathbb{E}(n\mu)$ *is bounded and* $\mathbf{m}_\mathbb{E}(\mu) = 1,$ *the sequence* $\mathbf{m}_\mathbb{E}(\lambda + n\mu)$ *is increasing and constant for large enough* n*. This constant limit value is equal to*
$$\dim\left[\mathrm{Sym}(\mathbb{W}^*) \otimes (V_\lambda^{G_\mu})^*|_H\right]^H.$$

3. Reduction of Kähler manifolds

We consider a complex manifold M, not necessarily compact, and a holomorphic Hermitian line bundle $(\mathcal{L}, \mathrm{h})$ on it. We assume that the curvature $\Omega = i(\nabla^h)^2$ of its Chern connexion ∇^h is a Kähler class (we say that the line bundle $\mathcal{L}$ prequantizes the symplectic form Ω).

We suppose furthermore that a compact connected Lie group G acts on $\mathcal{L} \to M$ leaving the metric and connection invariant. Hence we have a moment map $\Phi : M \to \mathfrak{g}^*$ defined by Kostant's relations (see (2.3)). Let us assume that the G-action on M extends to a $G_\mathbb{C}$-action and that the momentum map Φ is **proper**. Then the G-actions on $\mathcal{L}$ and on its smooth sections can both be uniquely extended to actions of $G_{\mathbb{C}}$, and the projection $\mathcal{L} \to M$ is equivariant [27].

When 0 is a regular value of Φ, the symplectic reduced space

$$M_0 := \Phi^{-1}(0)/G$$

is an orbifold equipped with an induced Kähler structure form (Ω_0, J_0), and the line orbibundle $\mathcal{L}_0 := \mathcal{L}|_{\Phi^{-1}(0)}/G$ prequantizes (M_0, Ω_0).

In general the reduced space M_0 has a natural structure of a singular Kähler manifold that is defined as follows. A point $m \in M$ is (analytically) semi-stable if the closure of the $G_{\mathbb{C}}$-orbit through m intersects the zero level set $\Phi^{-1}(0)$, and we denote the set of semi-stable points by M^{ss}.

On M^{ss}, we have a natural equivalence relation : $x \sim y \iff \overline{G_{\mathbb{C}}x} \cap \overline{G_{\mathbb{C}}y} \cap M^{\mathrm{ss}} \neq \emptyset$. The Mumford GIT quotient $M/\!\!/G_{\mathbb{C}}$ is the quotient of M^{ss} by this equivalence relation (see [12, 20, 27]).

We have the following crucial fact

Theorem 3.1. *The set $M/\!\!/G_{\mathbb{C}}$ has a canonical structure of a complex analytic space, and the inclusion $\Phi^{-1}(0) \hookrightarrow M^{\mathrm{ss}}$ induces an homeomorphism $M_0 \simeq M/\!\!/G_{\mathbb{C}}$.*

To get a genuine line bundle on M_0, we have to replace $\mathcal{L}$ by a suitable power $\mathbb{L} := \mathcal{L}^{\otimes q}$ such that for any $m \in \Phi^{-1}(0)$ the stabilizer subgroup G_m acts trivially on $\mathbb{L}|_m$. Then $\mathbb{L}_0 := \mathcal{L}^{\otimes q}|_{\Phi^{-1}(0)}/G$ is an holomorphic line bundle on M_0.

We need the following result (see Theorem 2.14 in [27]).

Theorem 3.2. *The line bundle $\mathbb{L}_0$ is positive in the sense of Grauert. The reduced space M_0 is a complex projective variety, a projective embedding can be given by the Kodaira map $M_0 \to \mathbb{P}(\Gamma(M_0, \mathbb{L}_0^{\otimes k}))$ for some sufficiently large k.*

The following theorem is the first instance of the $[Q, R] = 0$ phenomenon. It was proved by Guillemin-Sternberg [10] in the case where 0 is a regular value of Φ and M is compact. In [27] Sjamaar extends their result by dealing with the non-smoothness of M_0 and the non-compactness of M.

Theorem 3.3. *The quotient map $M^{\mathrm{ss}} \to M_0$ and the inclusion $M^{\mathrm{ss}} \subset M$ induce the isomorphisms $\Gamma(M, \mathcal{L})^G \simeq \Gamma(M^{\mathrm{ss}}, \mathcal{L})^G \simeq \Gamma(M_0, q_*^G\mathcal{L})$, where $q_*^G\mathcal{L}$ is the sheaf of invariant sections induces by the line bundle $\mathcal{L}$.*

In this paper we will use Theorems 3.2 and 3.3 to get basic results concerning the sequence $\mathbf{H}(n) := \dim \Gamma(M, \mathcal{L}^{\otimes n})^G,\ n \geq 1$.

Proposition 3.4. *For n large enough, the sequence $\mathbf{H}(nq)$ is polynomial with a dominant term of the form cn^α where α is the complex dimension of the (smooth part of the) irreducible variety M_0.*

Proof. It is direct consequence of two facts: $\mathbf{H}(nq) := \dim \Gamma(M_0, \mathbb{L}_0^{\otimes n})$ thanks to Theorem 3.3 and the Kodaira map $M_0 \to \mathbb{P}(\Gamma(M_0, \mathbb{L}_0^{\otimes n}))$ is a projective embedding for n large enough. □

We get then the following useful result.

Lemma 3.5. • $\mathbf{H}(n) = 0,\ n \geq 1 \iff M_0 = \emptyset$.
- $\mathbf{H}(n)$ *is non-zero and bounded* $\iff M_0 = \{pt\}$.
- *If* $\mathbf{H}(n)$ *is bounded and* $\mathbf{H}(1) \neq 0$, *then* $\mathbf{H}(n) = 1$ *for all* $n \geq 1$.

Proof. The implications $\Longrightarrow$ are a consequence of Proposition 3.4, and the implications $\Longleftarrow$ are a consequence of Theorem 3.3. For the last point we use first the $[Q, R] = 0$ theorem when $M_0 = \{pt\}$: we have

$$\mathbf{H}(n) := \dim \left[\mathcal{L}^{\otimes n}|_{m_o}\right]^H$$

where $m \in \Phi^{-1}(0)$ and H is the stabilizer subgroup of m_o. The H-module $\mathcal{L}|_{m_o}$ is trivial if and only if $\mathbf{H}(1) = 1$. The third point follows then. □

We can now state the corresponding result that relates the multiplicities

$$\mathbf{m}^{\mathcal{L}}(\mu, n) := \dim \left[\Gamma(M, \mathcal{L}^{\otimes n}) \otimes (V_\mu^G)^*\right]^G.$$

with the reduced spaces $M_\mu := \Phi^{-1}(G\mu)/G$.

Lemma 3.6. • $\mathbf{m}^{\mathcal{L}}(n\mu, n) = 0,\ n \geq 1 \iff M_\mu = \emptyset$.
- $\mathbf{m}^{\mathcal{L}}(n\mu, n)$ *is non-zero and bounded* $\iff M_\mu = \{pt\}$.

Proof. It is a direct consequence of the shifting trick. We apply Lemma 3.5 to the Kähler manifold $M \times (G\mu)^-$ prequantized by the holomorphic line bundle $\mathcal{L} \boxtimes \mathcal{L}_\mu^{-1}$. □

We finish this section by recalling the following basic facts.

Lemma 3.7. • *Suppose that* $\mathbf{H}(1) \neq 0$. *Then for any holomorphic vector bundle* $\mathcal{E} \to M$, *the sequence* $\mathbf{H}_{\mathcal{E}}(n) = \dim \Gamma(M, \mathcal{E} \otimes \mathcal{L}^{\otimes n})^G$ *is increasing.*

- *Let* $m_o \in \Phi^{-1}(0)$ *with stabilizer subgroup* H. *We consider the* H*-module* $\mathbb{W} := \mathrm{T}_{m_o}M/\mathfrak{g}_{\mathbb{C}} \cdot m_o$. *Then* $\Phi^{-1}(0) = Gm_o$ *if and only if the algebra* $\mathrm{Sym}(\mathbb{W}^*)$ *has finite* H*-multiplicities.*

Proof. The first point follows from the fact that for any non-zero section $s \in \Gamma(M, \mathcal{L})^G$, the linear map $w \mapsto w \otimes s$ defines a one to one map from $\Gamma(M, \mathcal{E} \otimes \mathcal{L}^{\otimes n})^G$ into $\Gamma(M, \mathcal{E} \otimes \mathcal{L}^{\otimes n+1})^G$.

Let us check the second point. The vector space $\mathfrak{g} \cdot m_o \subset \mathrm{T}_{m_o}M$ is totally isotropic, since $\Omega_{m_o}(X \cdot m_o, Y \cdot m_o) = \langle \Phi(m_o), [X, Y] \rangle = 0$. Hence we can consider the vector space $E_{m_o} := (\mathfrak{g} \cdot m_o)^\perp / \mathfrak{g} \cdot m_o$ that is equipped with a H-equivariant symplectic structure $\Omega_{E_{m_o}}$: we denote by $\Phi_{E_{m_o}} : E_{m_o} \to \mathfrak{h}^*$ the corresponding moment map. A local model for a symplectic neighborhood of Gm_o is $G \times_H (\mathfrak{h}^\perp \times E_{m_o})$ where the moment map is $\Phi_{m_o}[g; \xi, v] = g(\xi + \Phi_{E_{m_o}}(v))$. We see then that $\Phi^{-1}(0) = Gm_o$ if and only if the set $\Phi^{-1}_{E_{m_o}}(0)$ is reduced to $\{0\}$, and it is a standard fact that $\Phi^{-1}_{E_{m_o}}(0) = \{0\}$ if and only if the algebra $\mathrm{Sym}(E^*_{m_o})$ has finite H-multiplicities.

We are left to prove that $E_{m_o} \simeq \mathbb{W}$. Let J be a complex structure on $\mathrm{T}_{m_o}M$ compatible with the symplectic form Ω_{m_o}. Since the vector space $\mathfrak{g}_{\mathbb{C}} \cdot m_o$ is equal to the symplectic subspace $\mathfrak{g} \cdot m_o \oplus J(\mathfrak{g} \cdot m_o)$, the H-module $\mathbb{W}$ has a canonical identification with its (symplectic) orthogonal $(\mathfrak{g} \cdot m_o \oplus J(\mathfrak{g} \cdot m_o))^\perp$. Finally the orthogonal decomposition

$$(\mathfrak{g} \cdot m_o \oplus J(\mathfrak{g} \cdot m_o))^\perp \oplus \mathfrak{g} \cdot m_o = (\mathfrak{g} \cdot m_o)^\perp$$

shows that the H-modules $\mathbb{W}$ and E_{m_o} are equal. □

4. Witten deformation

Let us recall the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. We refer to [8, 23] for more details.

4.1. Elliptic and transversally elliptic symbols

Let M be a compact G-manifold with cotangent bundle T^*M. Let $p : \mathrm{T}^*M \to M$ be the projection. If $\mathcal{E}$ is a vector bundle on M, we may still denote by $\mathcal{E}$ the vector bundle $p^*\mathcal{E}$ on the cotangent bundle T^*M. If $\mathcal{E}^+, \mathcal{E}^-$ are G-equivariant vector bundles over M, a G-equivariant morphism $\sigma \in \mathcal{C}^\infty(\mathrm{T}^*M, \mathrm{Hom}(\mathcal{E}^+, \mathcal{E}^-))$ is called a *symbol* on M. For $x \in M$, and $\nu \in T^*_xM$, thus $\sigma(x, \nu) : \mathcal{E}|^+_x \to \mathcal{E}|^-_x$ is a linear map. The subset of all $(x, \nu) \in \mathrm{T}^*M$

where the map $\sigma(x,\nu)$ is not invertible is called the *characteristic set* of σ, and is denoted by $\mathrm{Char}(\sigma)$. A symbol is elliptic if its characteristic set is compact.

An elliptic symbol σ on M defines an element $[\sigma]$ in the equivariant **K**-theory of T^*M with compact support, which is denoted by $\mathbf{K}^0_G(\mathrm{T}^*M)$. The index of σ is a virtual finite dimensional representation of G, that we denote by $\mathrm{Index}^M_G(\sigma) \in R(G)$ [3–6].

Recall the notion of *transversally elliptic symbol.* Let $\mathrm{T}^*_G M$ be the following G-invariant closed subset of T^*M

$$\mathrm{T}^*_G M \;= \{(x,\nu) \in \mathrm{T}^*M,\ \langle \nu, X\cdot x\rangle = 0 \quad \text{for all } X \in \mathfrak{g}\}\,.$$

Its fiber over a point $x \in M$ consists of the cotangent vectors $v \in T^*_x M$ which vanish on the tangent space to the orbit of x under G, at the point x. Thus each fiber $(\mathrm{T}^*_G M)_x$ is a linear subspace of $T^*_x M$. In general the dimension of $(\mathrm{T}^*_G M)_x$ is not constant and this space is not a vector bundle. A symbol σ is *G-transversally elliptic* if the restriction of σ to $\mathrm{T}^*_G M$ is invertible outside a compact subset of $\mathrm{T}^*_G M$ (i.e. $\mathrm{Char}(\sigma)\cap \mathrm{T}^*_G M$ is compact).

A *G-transversally elliptic* symbol σ defines an element of $\mathbf{K}^0_G(\mathrm{T}^*_G M)$, and the index of σ defines an element $\mathrm{Index}^M_G(\sigma)$ of $\hat{R}(G)$.

Any elliptic symbol is G-transversally elliptic, hence we have a restriction map $\mathbf{K}^0_G(\mathrm{T}^*M) \to \mathbf{K}^0_G(\mathrm{T}^*_G M)$, and a commutative diagram

$$\begin{array}{ccc} \mathbf{K}^0_G(\mathrm{T}^*M) & \longrightarrow & \mathbf{K}^0_G(\mathrm{T}^*_G M) \\ \downarrow{\scriptstyle \mathrm{Index}^M_G} & & \downarrow{\scriptstyle \mathrm{Index}^M_G} \\ R(G) & \longrightarrow & \hat{R}(G)\ . \end{array} \tag{4.8}$$

Using the *excision property*, one can easily show that the index map $\mathrm{Index}^{\mathcal{U}}_G : \mathbf{K}^0_G(\mathrm{T}^*_G\mathcal{U}) \to \hat{R}(G)$ is still defined when $\mathcal{U}$ is a G-invariant open subset of a G-manifold (see [21, 24]).

Remark. In the following the manifold M will carry a G-invariant Riemannian metric and we will denote by $\nu \in \mathrm{T}^*M \mapsto \tilde{\nu} \in \mathrm{T}M$ the corresponding identification.

4.2. Localization of the Riemann-Roch character

Let M be a G-manifold equipped with an invariant almost complex structure J. Let $p : \mathrm{T}M \to M$ be the projection. The vector bundle $(\mathrm{T}^*M)^{0,1}$ is G-equivariantly identified with the tangent bundle $\mathrm{T}M$ equipped with the complex structure J. Let h_M be an Hermitian structure on $(\mathrm{T}M, J)$. The symbol $\mathrm{Thom}(M,J) \in \mathcal{C}^\infty\left(\mathrm{T}^*M, \mathrm{Hom}(p^*(\wedge_{\mathbb{C}}^{even}\mathrm{T}M),\, p^*(\wedge_{\mathbb{C}}^{odd}\mathrm{T}M))\right)$ at $(m,\nu) \in \mathrm{T}M$ is equal to the Clifford map

$$\mathbf{c}_m(\nu) \;:\; \wedge_{\mathbb{C}}^{even}\mathrm{T}_mM \longrightarrow \wedge_{\mathbb{C}}^{odd}\mathrm{T}_mM, \tag{4.9}$$

where $\mathbf{c}_m(\nu).w = \tilde{\nu} \wedge w - \iota(\tilde{\nu})w$ for $w \in \wedge_{\mathbb{C}}^{\bullet}\mathrm{T}_mM$. Here $\iota(\tilde{\nu}) : \wedge_{\mathbb{C}}^{\bullet}\mathrm{T}_mM \to \wedge_{\mathbb{C}}^{\bullet-1}\mathrm{T}_mM$ denotes the contraction map relative to h_M. Since $\mathbf{c}_m(\nu)^2 = -\|\nu\|^2\mathrm{Id}$, the map $\mathbf{c}_m(\nu)$ is invertible for all $\nu \neq 0$. Hence the symbol $\mathrm{Thom}(M,J)$ is elliptic when the manifold M is compact.

Definition 4.1. Suppose that M is compact. To any G-equivariant vector bundle $\mathcal{E} \to M$, we associate its Riemann-Roch character

$$\mathrm{RR}_G^J(M,\mathcal{E}) := \mathrm{Index}_G^M(\mathrm{Thom}(M,J) \otimes \mathcal{E}) \in R(G).$$

If the complex structure J is understood we simply denote by $\mathrm{RR}_G(M,-)$ the Riemann-Roch character.

Remark 4.2. The character $\mathrm{RR}_G(M,\mathcal{E})$ is equal to the equivariant index of the Dolbeault-Dirac operator $\mathcal{D}_\mathcal{E} := \sqrt{2}(\overline{\partial}_\mathcal{E} + \overline{\partial}_\mathcal{E}^*)$, since $\mathrm{Thom}(M,J) \otimes \mathcal{E}$ corresponds to the principal symbol of $\mathcal{D}_\mathcal{E}$ (see [7][Proposition 3.67]).

Let us briefly explain how we perform the "Witten deformation" of the symbol $\mathrm{Thom}(M,J)$ with the help of an equivariant map $\phi : M \to \mathfrak{g}^*$ [14, 21, 24]. Consider the identification $\xi \mapsto \widetilde{\xi}, \mathfrak{g}^* \to \mathfrak{g}$ defined by a G-invariant scalar product on $\mathfrak{g}^*$. We define the *Kirwan vector field*:

$$\kappa_\phi(m) = \left(\widetilde{\phi(m)}\right)_M(m), \quad m \in M. \tag{4.10}$$

We denote by $Z_\phi \subset M$ the subset where κ_ϕ vanishes.

Definition 4.3. The symbol $\mathrm{Thom}(M,J)$ pushed by the vector field κ_ϕ is the symbol $\mathbf{c}_\phi$ defined by the relation

$$\mathbf{c}_\phi|_m(\nu) = \mathrm{Thom}(M,J)|_m(\tilde{\nu} - \kappa_\phi(m))$$

for any $(m,v) \in \mathrm{T}^*M$.

Note that $\mathbf{c}_\phi|_m(\nu)$ is invertible except if $\tilde{\nu} = \kappa_\phi(m)$. If furthermore ν belongs to the subset $\mathrm{T}^*_G M$ of cotangent vectors orthogonal to the G-orbits, then $\nu = 0$ and $m \in Z_\phi = \{\kappa_\phi = 0\}$. Indeed $\kappa_\phi(m)$ is tangent to $G \cdot m$ while ν is orthogonal. Finally we have $\mathrm{Char}(\mathbf{c}_\phi) \cap \mathrm{T}^*_G M \simeq Z_\phi$.

Definition 4.4. When the critical set Z_ϕ is compact, we define $\mathrm{RR}_G(M, \mathcal{E}, \phi) \in \hat{R}(G)$ as the equivariant index of the transversally elliptic symbol $\mathbf{c}_\phi \otimes \mathcal{E} \in \mathbf{K}^0_G(\mathrm{T}^*_G M)$.

When M is compact, it is clear that the classes of the symbols $\mathbf{c}_\phi \otimes \mathcal{E}$ and $\mathrm{Thom}(M, J) \otimes \mathcal{E}$ are equal in $\mathbf{K}^0_G(\mathrm{T}^*_G M)$, hence the equivariant indices $\mathrm{RR}_G(M, \mathcal{E})$ and $\mathrm{RR}_G(M, \mathcal{E}, \phi)$ are equal.

For any G-invariant open subset $U \subset M$ such that $U \cap Z_\phi$ is compact in M, we see that the restriction $\mathbf{c}_\phi|_{\mathrm{T}^*U}$ is a transversally elliptic symbol on U, and so its equivariant index is a well defined element in $\hat{R}(G)$.

Definition 4.5. • A closed invariant subset $Z \subset Z_\phi$ is called a component if it is a union of connected components of Z_ϕ.

• For a compact component Z of Z_ϕ, we denote by

$$\mathrm{RR}_G(M, \mathcal{E}, Z, \phi) \;\in\; \hat{R}(G)$$

the equivariant index of $\mathbf{c}_\phi \otimes \mathcal{E}|_{\mathrm{T}^*U}$, where U is any G-invariant open subset such that $U \cap \{\kappa_\phi = 0\} = Z$. By definition, $\mathrm{RR}_G(M, \mathcal{E}, Z, \phi) = 0$ when $Z = \emptyset$.

In this paper we will be particularly interested in the character

$$\mathrm{RR}_G(M, \mathcal{E}, \phi^{-1}(0), \phi) \in \hat{R}(G),$$

that is defined when $\phi^{-1}(0)$ is a compact component of Z_ϕ.

4.3. $[Q, R] = 0$ theorem

When (M, Ω, Φ) is a compact Hamiltonian G-manifold, the Riemann-Roch character $\mathrm{RR}_G(M, -)$ is computed with an invariant almost complex structure J that is compatible with Ω. Here the Kirwan vector field κ_Φ is the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi\|^2$. Hence the set Z_Φ of zeros of κ_Φ coincides with the set of critical points of $\|\Phi\|^2$. When M is non compact but the critical set Z_Φ is compact, we can define the localized Riemann-Roch character $\mathrm{RR}_G(M, -, \Phi)$. If moreover the map Φ is proper, the set $\Phi^{-1}(0)$ will be a compact component of Z_Φ, so we can consider the localized Riemann-Roch character $\mathrm{RR}_G(M, -, \Phi^{-1}(0), \Phi)$.

Let $\mathcal{L} \to M$ be a Hermitian line bundle that prequantizes the data (M, Ω, Φ). In this setting we are interested in the dimension of the trivial G-representation in $\mathrm{RR}_G(M, \mathcal{L}^{\otimes n})$ that we simply denote by $[\mathrm{RR}_G(M, \mathcal{L}^{\otimes n})]^G \in \mathbb{Z}$.

The main facts of this localization procedure is summarized in the following.

Theorem 4.6 ([21, 24]). *Let (M, Ω, Φ) be a Hamiltonian G-manifold prequantized by a line bundle $\mathcal{L}$. Let $\mathcal{E}$ be an equivariant vector bundle on M.*

- *When M is* **compact**, *we have*

$$\big[\mathrm{RR}_G(M, \mathcal{L}^{\otimes n})\big]^G = \big[\mathrm{RR}_G(M, \mathcal{L}^{\otimes n}, \Phi^{-1}(0), \Phi)\big]^G, \text{ for } n \geq 1,$$
$$\big[\mathrm{RR}_G(M, \mathcal{L}^{\otimes n} \otimes \mathcal{E})\big]^G = \big[\mathrm{RR}_G(M, \mathcal{L}^{\otimes n} \otimes \mathcal{E}, \Phi^{-1}(0), \Phi)\big]^G, \text{ for } n \gg 1.$$

- *If Φ is* **proper** *and the critical set Z_Φ is* **compact**, *we have*

$$\big[\mathrm{RR}_G(M, \mathcal{L}^{\otimes n}, \Phi)\big]^G = \big[\mathrm{RR}_G(M, \mathcal{L}^{\otimes n}, \Phi^{-1}(0), \Phi)\big]^G, \text{ for } n \geq 1,$$
$$\big[\mathrm{RR}_G(M, \mathcal{L}^{\otimes n} \otimes \mathcal{E}, \Phi)\big]^G = \big[\mathrm{RR}_G(M, \mathcal{L}^{\otimes n} \otimes \mathcal{E}, \Phi^{-1}(0), \Phi)\big]^G, \text{ for } n \gg 1.$$

Let us finish this section by explaining the case where the quantity $\big[\mathrm{RR}_G(M, \mathcal{E}, \Phi^{-1}(0), \Phi)\big]^G$ can be computed as an index on the reduced space M_0.

First suppose that 0 is a regular value of Φ. The reduced space M_0 is a symplectic orbifold, and we can define in this context a Riemann-Roch character $\mathrm{RR}(M_0, -)$ with the help of a compatible almost complex structure. For any equivariant vector bundle $\mathcal{F}$ on M we define the orbibundle $\mathcal{F}_0 := \mathcal{F}|_{\Phi^{-1}(0)}/G$ on M_0, and we have

$$\big[\mathrm{RR}_G(M, \mathcal{F}, \Phi^{-1}(0), \Phi)\big]^G = \mathrm{RR}(M_0, \mathcal{F}_0). \tag{4.11}$$

Suppose now that 0 is a quasi-regular value of Φ. By definition, this is the case when there exists a sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $Z := \Phi^{-1}(0)$ is contained in the sub-manifold $M_{(\mathfrak{h})} = GM_{\mathfrak{h}}$ where $M_{\mathfrak{h}} = \{m \in M, \mathfrak{g}_m = \mathfrak{h}\}$. Let N be the normalizer subgroup of $\mathfrak{h}$ in G, and let H^o be the closed connected subgroup of G with Lie algebra $\mathfrak{h}$. Thus $M_{(\mathfrak{h})} \simeq G \times_N M_{\mathfrak{h}}$ and $Z \simeq G \times_N Z_{\mathfrak{h}}$ where $Z_{\mathfrak{h}} := \Phi^{-1}(0) \cap M_{\mathfrak{h}}$ is a compact N-submanifold of M

with a locally free action of N/H^o. Then the reduced space

$$M_0 := \Phi^{-1}(0)/G \simeq Z_{\mathfrak{h}}/(N/H^o)$$

is a compact connected symplectic orbifold.

Let $\mathcal{W} \to Z$ be the symplectic normal bundle of the submanifold Z in M: for $x \in Z$,

$$\mathcal{W}|_x = (\mathrm{T}_x Z)^\perp/(\mathrm{T}_x Z)^\perp \cap \mathrm{T}_x Z,$$

were we have denoted by $(\mathrm{T}_x Z)^\perp$ the orthogonal with respect to the symplectic form. We can equip $\mathcal{W}$ with an H-invariant Hermitian structure h such that the symplectic structure on the fibers of $\mathcal{W} \to Z$ is equal to $-\mathrm{Im}(\mathrm{h})$.

The sub-algebra $\mathfrak{h}$ acts fiber-wise on the vector bundle $\mathcal{W}|_{Z_{\mathfrak{h}}}$. We consider the action of $\mathfrak{h}$ on the fibers of the bundle $\mathrm{Sym}(\mathcal{W}^*|_{Z_{\mathfrak{h}}})$. We will use the following result ([24][Section 12.2]).

Lemma 4.7. *The sub-bundle* $[\mathrm{Sym}(\mathcal{W}^*|_{Z_{\mathfrak{h}}})]^{\mathfrak{h}}$ *is reduced to the trivial bundle* $[\mathbb{C}] \to Z_{\mathfrak{h}}$.

Thanks to Lemma 4.7, we can introduce the following notion of reduction in the quasi-regular case.

Definition 4.8. If $\mathcal{F} \to M$ is a K-equivariant vector bundle, we define on M_0 the (finite dimensional) orbibundle

$$\mathcal{F}_0 := \left[\mathcal{F}|_{Z_{\mathfrak{h}}} \otimes \mathrm{Sym}(\mathcal{W}^*|_{Z_{\mathfrak{h}}})\right]^{\mathfrak{h}}/(N/H^o).$$

If $\mathfrak{h}$ acts trivially on the fibers of $\mathcal{F}|_{Z_{\mathfrak{h}}}$, the bundle $\mathcal{F}_0$ is equal to $\mathcal{F}|_{Z_{\mathfrak{h}}}/(N/H^o)$.

The following result is proved in [24][Section 12.2].

Theorem 4.9. *Assume that* $\Phi^{-1}(0) \subset M_{(\mathfrak{h})}$. *For any G-equivariant vector bundle* $\mathcal{F} \to M$, *we have*

$$\left[\mathrm{RR}_G(M, \mathcal{F}, \Phi^{-1}(0), \Phi)\right]^G = \mathrm{RR}(M_0, \mathcal{F}_0).$$

With Theorem 4.9 in hand, we can restate Theorem 4.6 when 0 is a quasi-regular value of Φ.

Theorem 4.10. *Let* (M, Ω, Φ) *be a Hamiltonian G-manifold prequantized by a line bundle* $\mathcal{L}$. *Let* $\mathcal{E}$ *be an equivariant vector bundle on* M. *Suppose that* 0 *is a quasi-regular value of* Φ.

- *When M is* **compact***, we have*

$$\big[\mathrm{RR}_G(M,\mathcal{L}^{\otimes n})\big]^G = \mathrm{RR}(M_0,\mathcal{L}_0^{\otimes n}),\ \text{for } n\geq 1,$$
$$\big[\mathrm{RR}_G(M,\mathcal{L}^{\otimes n}\otimes\mathcal{E})\big]^G = \mathrm{RR}(M_0,\mathcal{L}_0^{\otimes n}\otimes\mathcal{E}_0),\ \text{for } n\gg 1.$$

- *If Φ is* **proper** *and the critical set Z_Φ is* **compact***, we have*

$$\big[\mathrm{RR}_G(M,\mathcal{L}^{\otimes n},\Phi)\big]^G = \mathrm{RR}(M_0,\mathcal{L}_0^{\otimes n}),\ \text{for } n\geq 1,$$
$$\big[\mathrm{RR}_G(M,\mathcal{L}^{\otimes n}\otimes\mathcal{E},\Phi)\big]^G = \mathrm{RR}(M_0,\mathcal{L}_0^{\otimes n}\otimes\mathcal{E}_0),\ \text{for } n\gg 1.$$

The famous identity $[\mathrm{RR}_G(M,\mathcal{L})]^G = \mathrm{RR}(M_0,\mathcal{L}_0)$, commonly called the *"quantization commutes with reduction"* theorem, was first obtained by Meinrenken [17] and Meinrenken-Sjamaar [18].

A case of particular interest for us is when the reduced space $M_0 := \Phi^{-1}(0)/G$ is reduced to a point : we are in the quasi-regular case. Let H be the stabilizer subgroup of $m_o\in Z:=\Phi^{-1}(0)$: note that the group H is not necessarily connected. Then $Z = G\cdot m_o\simeq G/H$ is contained in $GM_{\mathfrak{h}}$.

By definition, the fiber of the vector bundle $\mathcal{W}\to Z$ at m_o is $\mathcal{W}|_{m_o} = (\mathfrak{g}\cdot m_o)^\perp/\mathfrak{g}\cdot m_o$. We have checked in the proof of Lemma 3.7 that the H-module $\mathcal{W}|_{m_o}$ coincides with $\mathbb{W}:=\mathrm{T}_{m_o}M/\mathfrak{g}_{\mathbb{C}}\cdot m_o$. Recall that the equality $\Phi^{-1}(0) = G\cdot m_o$ is equivalent to the fact that the H-module $\mathrm{Sym}(\mathbb{W}^*)$ has finite multiplicities.

In this case Theorem 4.10 gives the following result.

Corollary 4.11. *Let (M,Ω,Φ) be a Hamiltonian G-manifold prequantized by a line bundle $\mathcal{L}$. Let $\mathcal{E}$ be an equivariant vector bundle on M. Suppose that $\Phi^{-1}(0) = G\cdot m_o$ with $G_{m_o} = H$.*

- *When M is* **compact***, we have*

$$\big[\mathrm{RR}_G(M,\mathcal{L}^{\otimes n})\big]^G = \big[\mathcal{L}^{\otimes n}|_{m_o}\big]^H,\ \text{for } n\geq 1,$$
$$\big[\mathrm{RR}_G(M,\mathcal{L}^{\otimes n}\otimes\mathcal{E})\big]^G = \big[\mathrm{Sym}(\mathbb{W}^*)\otimes\mathcal{E}|_{m_o}\otimes\mathcal{L}^{\otimes n}|_{m_o}\big]^H,\ \text{for } n\gg 1.$$

- *If Φ is* **proper** *and the critical set Z_Φ is* **compact***, we have*

$$\big[\mathrm{RR}_G(M,\mathcal{L}^{\otimes n},\Phi)\big]^G = \big[\mathcal{L}^{\otimes n}|_{m_o}\big]^H,\ \text{for } n\geq 1,$$
$$\big[\mathrm{RR}_G(M,\mathcal{L}^{\otimes n}\otimes\mathcal{E},\Phi)\big]^G = \big[\mathrm{Sym}(\mathbb{W}^*)\otimes\mathcal{E}|_{m_o}\otimes\mathcal{L}^{\otimes n}|_{m_o}\big]^H,\ \text{for } n\gg 1.$$

4.4. Main proofs

4.4.1. Proof of Theorem A. Consider a G-compact complex manifold M endowed with an ample holomorphic G-line bundle $\mathcal{L} \to M$ with curvature the symplectic two-form Ω. Let $\Phi : M \to \mathfrak{g}^*$ be the moment map associated to the G-action on $\mathcal{L}$ (see (2.3)).

Let $\mathcal{E} \to M$ be an holomorphic G-vector bundle. In this context, we are interested in the family of G-modules $\Gamma(M, \mathcal{E} \otimes \mathcal{L}^{\otimes n})$ consisting of the holomorphic sections. We denote by $\mathbf{H}_{\mathcal{E}}(n)$ the dimension of $\Gamma(M, \mathcal{E} \otimes \mathcal{L}^{\otimes n})^G$. When we take $\mathcal{E} = \mathbb{C}$, we denote by $\mathbf{H}(n) = \dim \Gamma(M, \mathcal{L}^{\otimes n})^G$.

By Kodaira vanishing theorem, we know that

$$\mathbf{H}_{\mathcal{E}}(n) = [\mathrm{RR}_G(M, \mathcal{E} \otimes \mathcal{L}^{\otimes n})]^G$$

when n is sufficiently large.

Two cases are considered in Theorem **A**.

• Suppose that $\mathbf{H}(n) = 0$ for all $n \geq 1$. We have seen in Lemma 3.5 that it means that $\Phi^{-1}(0) = \emptyset$. In this case Corollary 4.11 says that $\mathbf{H}_{\mathcal{E}}(n) = 0$ if n is large enough.

• Suppose that the sequence $\mathbf{H}(n)$ is non-zero and bounded: here we have that $\Phi^{-1}(0) = G \cdot m_o$ for some $m_o \in M$. Corollary 4.11 tell us that

$$\left[\mathrm{RR}_G(M, \mathcal{E} \otimes \mathcal{L}^{\otimes n})\right]^G = \left[\mathrm{Sym}(\mathbb{W}^*) \otimes \mathcal{E}|_{m_o} \otimes \mathcal{L}^{\otimes n}|_{m_o}\right]^H,$$

for n large enough.

The proof of Theorem **A** is then completed.

4.4.2. Proof of Theorem C. Here K is a closed subgroup of G, and we use a K-invariant decomposition : $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. Let V be a K-Hermitian vector space such that the K-module $\mathrm{Sym}(V^*)$ has finite multiplicities. The proof of Theorem **C** is an adaptation of the previous arguments to the case where we work with the non-compact manifold $M := G \times_K (\mathfrak{q}^* \oplus V) \simeq G_{\mathbb{C}} \times_{K_{\mathbb{C}}} V$.

The symplectic structure on M is defined as follows. Let $\theta \in \mathcal{A}^1(G) \otimes \mathfrak{g}$ the canonical connection relatively to right translation : $\theta(\frac{d}{dt}|_{t=0} g e^{tX}) = X$. Let Ω_V be the symplectic structure on V which is -1 times the imaginary part of the hermitian structure of V. Let λ_V the invariant 1-form on V defined by $\lambda_V(v) = \frac{1}{2}\Omega_V(v, -)$: we have $\Omega_V = d\lambda_V$. The moment map $\Phi_V : V \to \mathfrak{k}^*$ associated to the K-action on (V, Ω_V) is defined by $\langle \Phi_V(v), X \rangle = \frac{1}{2}\Omega_V(Xv, v)$. We will use the following basic Lemma.

Lemma 4.12. *The followings statements are equivalent.*

1) *The K-module* $\mathrm{Sym}(V^*)$ *has finite multiplicities.*
2) *The map Φ_V is proper.*
3) *One has the relation $\|\Phi_V(v)\| \geq c\|v\|^2$, $\forall v \in V$, for some $c > 0$.*

We consider the 1-form $\lambda := \lambda_V - \langle \xi \oplus \Phi_V, \theta \rangle$ on $G \times (\mathfrak{q}^* \oplus V)$, which is $G \times K$-equivariant and K-basic. It induces a 1-form λ_M on M.

We have the standard fact.

Proposition 4.13.

- *The 2-form $\Omega_M := d\lambda_M$ defines a G-invariant symplectic form on M. The corresponding moment map is $\Phi([g; \xi \oplus v]) = g(\xi \oplus \Phi_V(v))$.*
- *The moment map Φ is proper and $Z_\Phi \simeq G/K$ is compact.*
- *The trivial line bundle $\mathbb{C}$ on M prequantizes the 2-form Ω_M.*

We equip M with an invariant almost complex structure compatible with Ω_M. Since the critical set Z_Φ is compact, one can define the localized Riemann-Roch character $\mathrm{RR}_G(M, -, \Phi)$. The following result is proved in [22][Proposition 2.18].

Proposition 4.14. *We have* $\mathrm{RR}_G(M, \mathbb{C}, \Phi) = \mathrm{Ind}_K^G(\mathrm{Sym}(V^*))$.

In order to compute geometrically $\mathbf{m}(\mu) = \dim[\mathrm{Sym}(V^*) \otimes (V_\mu^G)^*|_K]^K$ we have to adapt the shifting trick to this non-compact setting. Let us fix two dominant weights μ and λ. The G-manifold $P = M \times (G\mu)^-$ is equipped with the following data:

- the symplectic form $\Omega_P := \Omega_M \times -\Omega_{G\mu}$,
- the line bundle $\mathcal{L}_P := \mathbb{C} \boxtimes \mathcal{L}_\mu^{-1}$ that prequantizes Ω_P,
- the proper moment map $\Phi_P : P \to \mathfrak{g}^*$, $\Phi_P(m, \xi) = \Phi(m) - \xi$,
- the vector bundle $\mathcal{E}_\lambda := \mathbb{C} \boxtimes G \times_{G_\mu} V_\lambda^{G_\mu}$.

For any $R \geq 0$, let $M_{\leq R}$ be the compact subset of points $[g; \xi \oplus v]$ such that $\|\xi\| \leq R$ and $\|v\| \leq R$. We start with the following basic fact whose proof is left to the reader.

Lemma 4.15. *There exists $c > 0$, such that for any μ the critical set $Z_{\Phi_P} \subset P = M \times G\mu$ is contained in the compact set $M_{\leq c\|\mu\|} \times G\mu$.*

Since Z_{Φ_P} is compact we can consider the localized Riemann-Roch character $\mathrm{RR}_G(P, -, \Phi_P)$.

Lemma 4.16. *We have* $\mathbf{m}(\lambda + n\mu) = \left[\mathrm{RR}_G(P, \mathcal{E}_\lambda^* \otimes \mathcal{L}_P^{\otimes n}, \Phi_P)\right]^G$ *for any* $n \geq 0$.

Proof. We consider the family of equivariant maps $\phi^t : P \to \mathfrak{g}^*, t \in [0,1]$ defined by the relation $\phi^t(m, \xi) = \Phi(m) - t\xi$. Let κ^t be the Kirwan vector field attached to ϕ^t, and let Z_{ϕ^t} be the vanishing set of κ^t: thanks to Lemma 4.15 we know that Z_{ϕ^t} is a compact subset included in $M_{\leq c\|\mu\|} \times G\mu$ for any $t \in [0,1]$.

We know then that the family of pushed symbols $\mathbf{c}_{\phi^t}$ is an homotopy of transversally elliptic symbols on P. We get then that

$$\begin{aligned}\mathrm{RR}_G(P, \mathcal{E}_\lambda^* \otimes \mathcal{L}_P^{\otimes n}, \Phi_P) &= \mathrm{RR}_G(P, \mathcal{E}_\lambda^* \otimes \mathcal{L}_P^{\otimes n}, \phi^0)\\ &= \mathrm{RR}_G(M, \mathbb{C}, \Phi) \otimes \mathrm{RR}_G((G\mu)^-, \mathcal{E}_\lambda^* \otimes \mathcal{L}_\mu^{\otimes -n})\\ &= \mathrm{RR}_G(M, \mathbb{C}, \Phi) \otimes (V_{\lambda+n\mu}^G)^*.\end{aligned}$$

As $\mathrm{RR}_G(M, \mathbb{C}, \Phi) = \mathrm{Ind}_K^G(\mathrm{Sym}(V^*))$, the proof of the Lemma is completed. □

Like in the previous section, thanks to Corollary 4.11, we know that the term $\left[\mathrm{RR}_G(P, \mathcal{E}_\lambda^* \otimes \mathcal{L}_P^{\otimes n}, \Phi_P)\right]^G$ can be computed explicitly when the reduced space $\Phi_P^{-1}(0)/G \simeq M_\mu$ is empty or a point:

• If $\Phi_P^{-1}(0) = \emptyset$, we have $\left[\mathrm{RR}_G(P, \mathcal{E}_\lambda^* \otimes \mathcal{L}_P^{\otimes n}, \Phi_P)\right]^G = 0$ when n is large enough.

• If $\Phi_P^{-1}(0) = G \cdot (x_o, \mu)$ for some $x_o \in M$, we have

$$\left[\mathrm{RR}_G(P, \mathcal{E}_\lambda^* \otimes \mathcal{L}_P^{\otimes n}, \Phi_P)\right]^G = \left[\mathrm{Sym}(\mathbb{W}^*) \otimes \mathbb{C}_{-n\mu} \otimes (V_\lambda^{G_\mu})^*|_H\right]^H$$

when n is large enough.

Let us summarize what we have just demonstrated.

- If $M_\mu = \emptyset$, we have $\mathbf{m}(\lambda + n\mu) = 0$ if n is large enough, for any dominant weight λ,
- If $M_\mu = \{pt\}$, we have $\mathbf{m}(\lambda + n\mu) = [\mathrm{Sym}(\mathbb{W}^*) \otimes \mathbb{C}_{-n\mu} \otimes (V_\lambda^{G_\mu})^*|_H]^H$ if n is large enough, for any dominant weight λ.

The last thing that we need to check is the following

Proposition 4.17. • $\mathbf{m}(n\mu) = 0, n \geq 1 \iff M_\mu = \emptyset$,

• $\mathbf{m}(n\mu)$ *is non-zero and bounded* $\iff M_\mu = \{pt\}$.

Proof. The symplectic manifold $M = G \times_K (\mathfrak{q} \oplus V)$ admits a natural identification with the complex manifold $G_{\mathbb{C}} \times_{K_{\mathbb{C}}} V$, through the map $[g; X \oplus v] \mapsto [ge^{iX}; v]$. Hence M inherits a $G_{\mathbb{C}}$-action and a $G_{\mathbb{C}}$-invariant (integrable) complex structure J_M: it is not difficult to check that J_M is compatible with the symplectic form Ω_M.

We are in the setting of Section 3, where the trivial line bundle $\mathbb{C} \to M$ prequantizes Ω_M. In this context, the space $\Gamma(M, \mathbb{C}^{\otimes n})$ of holomorphic section does not depends on $n \geq 1$ and is equal to the vector space $\mathcal{C}^{hol}(M)$ of holomorphic functions on M.

According to Lemma 3.6, the sequence

$$\mathbf{m}^{hol}(n\mu) = \dim[\mathcal{C}^{hol}(M) \otimes (V^G_{n\mu})^*]^G$$

is related to the reduced space M_μ as follows:

- $\mathbf{m}^{hol}(n\mu) = 0,\ n \geq 1 \iff M_\mu = \emptyset$,
- $\mathbf{m}^{hol}(n\mu)$ is non-zero and bounded $\iff M_\mu = \{pt\}$.

Since the vector space $\mathcal{C}^{hol}(G_{\mathbb{C}} \times_{K_{\mathbb{C}}} V)$ admits the vector space

$$\bigoplus_{\lambda \in \Lambda^+_G} V^G_\lambda \otimes \left[(V^G_\lambda)^*|_K \otimes \mathrm{Sym}(V^*)\right]^K$$

as a dense subspace, we know that the multiplicities $\mathbf{m}^{hol}(\mu)$ and $\mathbf{m}(\mu)$ coincide. The proof is then completed. □

5. Examples

Let $\rho : G \to \tilde{G}$ be a morphism between two connected compact Lie groups. The purpose of this section is to give examples of stable weights for the multiplicity function $\mathbf{m}_\rho$.

5.1. Basic examples of stable weights

We denote by $d\rho : \mathfrak{g} \to \tilde{\mathfrak{g}}$ the induced Lie algebras morphism, and $\pi : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$ the dual map. Select maximal tori T in G and $\tilde{T}$ in $\tilde{G}$, such that $\rho(T) \subset \tilde{T}$. We still denote by $d\rho : \mathfrak{t} \to \tilde{\mathfrak{t}}$ the induced map, and $\pi : \tilde{\mathfrak{t}}^* \to \mathfrak{t}^*$ the dual map. Let $\Lambda_{\tilde{G}} \subset \tilde{\mathfrak{t}}^*$, $\Lambda_G \subset \mathfrak{t}^*$ be the set of weights for the torus $\tilde{T}$ and T: we have naturally that $\pi(\Lambda_{\tilde{G}}) \subset \Lambda_G$.

Let $\tilde{\mathfrak{R}} := \mathfrak{R}(\tilde{G}, \tilde{T})$ (resp. $\mathfrak{R} := \mathfrak{R}(G, T)$) be the set of roots for the group $\tilde{G}$ (resp. G). Recall that an element $\tilde{\xi} \in \tilde{\mathfrak{t}}^*$ defines a parabolic sub-algebra

$$\tilde{\mathfrak{p}}_{\tilde{\xi}} := \tilde{\mathfrak{t}}_{\mathbb{C}} \oplus \sum_{\alpha \in \tilde{\mathfrak{R}}, (\alpha, \tilde{\xi}) \geq 0} (\tilde{\mathfrak{g}}_{\mathbb{C}})_\alpha$$

of the reductive Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$. Its nilpotent radical is $\tilde{\mathfrak{n}}_{\tilde{\xi}} := \sum_{\alpha \in \tilde{\mathfrak{R}}, (\alpha, \tilde{\xi}) > 0} (\tilde{\mathfrak{g}}_{\mathbb{C}})_\alpha$.

Definition 5.1. An element $\tilde{\xi} \in \tilde{\mathfrak{t}}^*$ is G-**adapted** if the image of $\{\alpha \in \tilde{\mathfrak{R}}, (\alpha, \tilde{\xi}) > 0\}$ by the projection π is contained in an open half space, i.e. if there exists $\xi_o \in \mathfrak{t}^*$ such that $\forall \alpha \in \tilde{\mathfrak{R}},\ (\alpha, \tilde{\xi}) > 0 \Longrightarrow (\pi(\alpha), \xi_o) > 0$.

Let $\tilde{\mathcal{O}}$ be a coadjoint orbit of the group $\tilde{G}$. The moment map $\tilde{\mathcal{O}} \to \mathfrak{g}^*$ relative to the action of G on $\tilde{\mathcal{O}}$ is the restriction of π on $\tilde{\mathcal{O}}$. Hence for any $\xi \in \mathfrak{g}^*$, the G-reduction of $\tilde{\mathcal{O}}$ at ξ is $\tilde{\mathcal{O}}_\xi := \tilde{\mathcal{O}} \cap \pi^{-1}(G\xi)/G$.

The main tool used in this section is the following

Proposition 5.2. *Let $\tilde{\xi} \in \tilde{\mathfrak{t}}^*$ and $\xi = \pi(\tilde{\xi})$. If $\tilde{\xi}$ is G-**adapted**, then*

- *the G-reduction of $\tilde{G}\tilde{\xi}$ at ξ is reduced to a point,*
- $\rho(G_\xi) \subset \tilde{G}_{\tilde{\xi}}$,
- $\rho(\mathfrak{p}_\xi) \subset \tilde{\mathfrak{p}}_{\tilde{\xi}}$,
- *The linear map $\rho : \mathfrak{p}_\xi \to \tilde{\mathfrak{p}}_{\tilde{\xi}}$ factorizes to a linear map $\overline{\rho} : \mathfrak{n}_\xi \to \tilde{\mathfrak{n}}_{\tilde{\xi}}$.*

Proof. Let $\tilde{\mathcal{O}} := \tilde{G}\tilde{\xi}$. It is immediate to see that the first two points are a consequence of the following equality

$$\tilde{\mathcal{O}} \cap \pi^{-1}(\xi) = \{\tilde{\xi}\}. \tag{5.12}$$

Let us denote by $\pi_{\tilde{\mathfrak{t}}} : \tilde{\mathfrak{g}}^* \to \tilde{\mathfrak{t}}^*$ the projection. Since $\tilde{\mathcal{O}} \cap \pi_{\tilde{\mathfrak{t}}}^{-1}(\tilde{\xi})$ is reduced to the singleton $\{\tilde{\xi}\}$, the identity (5.12) follows from

$$\pi_{\tilde{\mathfrak{t}}}(\tilde{\mathcal{O}}) \cap \pi_{\tilde{\mathfrak{t}}}\left(\pi^{-1}(\xi)\right) = \{\tilde{\xi}\}. \tag{5.13}$$

Thanks to the Convexity Theorem [13] we know that $\pi_{\tilde{\mathfrak{t}}}(\tilde{\mathcal{O}})$ is equal to the convex hull $\mathrm{Conv}(\tilde{W}\tilde{\xi})$, where $\tilde{W}$ is the Weyl group of $(\tilde{G}, \tilde{T})$. On the other hand the set $\pi_{\tilde{\mathfrak{t}}}\left(\pi^{-1}(\xi)\right)$ is equal to the affine subspace $\tilde{\xi} + E$ where $E \subset \tilde{\mathfrak{t}}^*$ is equal to the kernel of $\pi : \tilde{\mathfrak{t}}^* \to \mathfrak{t}^*$. Let $\mathcal{A} \subset \tilde{\mathfrak{t}}^*$ be the tangent cone at $\tilde{\xi}$ of the convex set $\mathrm{Conv}(\tilde{W}\tilde{\xi})$: by standard computation we know that $-\mathcal{A}$ is the cone generated by $\alpha \in \tilde{\mathfrak{R}}, (\alpha, \tilde{\xi}) > 0$. Since $\pi_{\tilde{\mathfrak{t}}}(\tilde{\mathcal{O}}) \subset \tilde{\xi} + \mathcal{A}$ we see that (5.13) is a consequence of

$$\mathcal{A} \cap E = \{0\}. \tag{5.14}$$

Our proof of (5.12) is now completed since (5.14) follows immediately from the fact that for some $\xi_o \in \mathfrak{t}$ we have: $\forall \alpha \in \tilde{\mathfrak{R}},\ (\alpha, \tilde{\xi}) > 0 \Longrightarrow (\pi(\alpha), \xi_o) > 0$.

Let us concentrate on the third point. We know already that $\rho(G_\xi) \subset \tilde{G}_{\tilde{\xi}}$. Hence to get the inclusion $\rho(\mathfrak{p}_\xi) \subset \tilde{\mathfrak{p}}_{\tilde{\xi}}$ we have just to check that

$$\rho((\mathfrak{g}_{\mathbb{C}})_\beta) \subset \tilde{\mathfrak{p}}_{\tilde{\xi}} \tag{5.15}$$

for any $\beta \in \mathfrak{R}$ such that $(\beta, \xi) > 0$. A small computation shows that (5.15) is a consequence of

$$\left\{\alpha \in \tilde{\mathfrak{R}}, (\alpha, \tilde{\xi}) < 0\right\} \bigcap \pi^{-1}(\beta) = \emptyset. \tag{5.16}$$

It is proved in [11][Lemma 8.3], that

$$\{\beta \in \mathfrak{R}, (\beta, \xi) > 0\} \subset \pi\left(\left\{\alpha \in \tilde{\mathfrak{R}}, (\alpha, \tilde{\xi}) > 0\right\}\right). \tag{5.17}$$

Since $\tilde{\xi} \in \tilde{\mathfrak{t}}^*$ is adapted to the group G, we have

$$\pi\left(\left\{\alpha \in \tilde{\mathfrak{R}}, (\alpha, \tilde{\xi}) > 0\right\}\right) \bigcap \pi\left(\left\{\alpha \in \tilde{\mathfrak{R}}, (\alpha, \tilde{\xi}) < 0\right\}\right) = \emptyset. \tag{5.18}$$

Hence (5.16) follows from identities (5.17) and (5.18).

For the last point we just use that the linear map $\rho : \mathfrak{p}_\xi \to \tilde{\mathfrak{p}}_{\tilde{\xi}}$ sends $(\mathfrak{g}_\xi)_{\mathbb{C}}$ into $(\tilde{\mathfrak{g}}_{\tilde{\xi}})_{\mathbb{C}}$. Then it factorizes to a map $\overline{\rho}$ from $\mathfrak{n}_\xi \simeq \mathfrak{p}_\xi/(\mathfrak{g}_\xi)_{\mathbb{C}}$ into $\tilde{\mathfrak{n}}_{\tilde{\xi}} \simeq \tilde{\mathfrak{p}}_{\tilde{\xi}}/(\tilde{\mathfrak{g}}_{\tilde{\xi}})_{\mathbb{C}}$. □

Let us fix the sets of dominant weights $\Lambda^+_{\tilde{G}}$, Λ^+_G for the groups $\tilde{G}$ and G. For any $(\mu, \tilde{\mu}) \in \Lambda^+_G \times \Lambda^+_{\tilde{G}}$, we denote by V^G_μ, $V^{\tilde{G}}_{\tilde{\mu}}$ the corresponding irreducible representations of G and $\tilde{G}$, and we define $\mathbf{m}_\rho(\mu, \tilde{\mu})$ as the multiplicity of V^G_μ in $V^{\tilde{G}}_{\tilde{\mu}}|_G$.

Here is a first type of examples of stable weights for the multiplicity map $\mathbf{m}_\rho$. Let $\tilde{W} = N_{\tilde{G}}(\tilde{T})/\tilde{T}$ be the Weyl group of $\tilde{G}$.

Theorem 5.3. *Let $(\tilde{\mu}, \tilde{w}) \in \Lambda^+_{\tilde{G}} \times \tilde{W}$ such that $\tilde{w}\tilde{\mu}$ is* **adapted** *to G. Up to the conjugation by an element of the Weyl group of G we can assume that $\mu := \pi(\tilde{w}\tilde{\mu})$ is a dominant weight. Then*

- *$(\mu, \tilde{\mu})$ is a stable weight for $\mathbf{m}_\rho$.*
- *For any dominant weight $(\lambda, \tilde{\lambda})$ the sequence $\mathbf{m}_\rho(\lambda + n\mu, \tilde{\lambda} + n\tilde{\mu})$ is increasing and equal to*

$$\dim \left[\mathrm{Sym}(\mathbb{W}^*) \otimes (V_\lambda^H)^* \otimes V_{\tilde{w}\tilde{\lambda}}^{\tilde{H}}|_H\right]^H$$

for n large enough. Here $H \subset G$ and $\tilde{H} \subset \tilde{G}$ are the respective stabilizers[4] of μ and $\tilde{w}\tilde{\mu}$, and $\mathbb{W}$ corresponds to the H-module

$$\tilde{\mathfrak{n}}_{\tilde{w}\tilde{\mu}}/\overline{\rho}(\mathfrak{n}_\mu). \tag{5.19}$$

Proof. The first point is due to the fact that the stabilizer of $\tilde{w}\tilde{\mu}$ relative to the G-action is equal to the connected subgroup H, hence the H-module $\mathbb{D}$ is trivial. For the second point we have just to check the computation of the H-module $\mathbb{W}$. Let $a = \tilde{w}\tilde{\mu} \in \tilde{\mathcal{O}} := \tilde{G}\tilde{\mu}$. Here $\mathrm{T}_a\tilde{\mathcal{O}} \simeq \tilde{\mathfrak{p}}_{\tilde{w}\tilde{\mu}}/\tilde{\mathfrak{h}}_\mathbb{C}$. As $\rho(\mathfrak{p}_\mu) \subset \tilde{\mathfrak{p}}_{\tilde{w}\tilde{\mu}}$ one sees directly that $\mathbb{W} = \mathrm{T}_a\tilde{\mathcal{O}}/\rho(\mathfrak{p}_\mu) \cdot a$ is equal to (5.19). □

We have another specialization of Theorem **B** that will be used in the plethysm case. We suppose here that the sets of positive roots $\tilde{\mathfrak{R}}^+$ and $\mathfrak{R}^+$ are chosen so that the corresponding Borel subgroups $B \subset G_\mathbb{C}$ and $\tilde{B} \subset \tilde{G}_\mathbb{C}$ satisfy

$$\rho(B) \subset \tilde{B}. \tag{5.20}$$

Let $\Lambda_{\tilde{G}}^+$, Λ_G^+ be the corresponding set of dominants weight. When we work with this parametrization we have the following classical fact.

Lemma 5.4. *Let $\tilde{\mu} \in \Lambda_{\tilde{G}}^+$ and $\mu = \pi(\tilde{\mu})$. We have*

- *$\mu \in \Lambda_G^+$ and $\mathbf{m}_\rho(\mu, \tilde{\mu}) \neq 0$,*
- *$\rho(\mathfrak{p}_\mu) \subset \tilde{\mathfrak{p}}_{\tilde{\mu}}$ and $\rho(G_\mu) \subset \tilde{G}_{\tilde{\mu}}$.*

Proof. Let $\tilde{V}_{\tilde{\mu}}$ be an irreducible representation of $\tilde{G}$ with highest weight $\tilde{\mu}$. There exists a non-zero vector $v_o \in \tilde{V}_{\tilde{\mu}}$ such that the line $\mathbb{C}v_o$ is fixed by $\tilde{B}$ and the maximal torus $\tilde{T}$ acts on $\mathbb{C}v_o$ through the character $\tilde{t} \mapsto \tilde{t}^{\tilde{\mu}}$.

Let V be the vector space generated by $\rho(g)v_o, g \in G$. It is an irreducible representation of G and v_o is still a highest weight vector for the G-action : the line $\mathbb{C}v_o$ is fixed by B and the maximal torus T acts on $\mathbb{C}v_o$ through the character $t \mapsto t^\mu$. This forces μ to be a dominant weight for G (relatively

[4]Observe that $\rho(H) \subset \tilde{H}$.

to B) and then $V \subset \tilde{V}_{\tilde{\mu}}$ is G-representation with highest weight μ : the first point is proved.

For the second point we look at the $\tilde{G}_{\mathbb{C}}$-action (resp. $G_{\mathbb{C}}$-action) on the projective space $\mathbb{P}(\tilde{V}_{\tilde{\mu}})$ (resp. $\mathbb{P}(V)$), the stabilizer subgroup of the line $\mathbb{C}v_o$ is equal to the parabolic subgroup $\tilde{P}_{\tilde{\mu}} \subset \tilde{G}_{\mathbb{C}}$ (resp. $P_\mu \subset G_{\mathbb{C}}$) : hence $\rho(P_\mu) \subset \tilde{P}_{\tilde{\mu}}$. If we work with the actions of the compact groups G and $\tilde{G}$ we get similarly that $\rho(G_\mu) \subset \tilde{G}_{\tilde{\mu}}$. □

Like in Proposition 5.2, the linear map $\rho : \mathfrak{p}_\mu \to \tilde{\mathfrak{p}}_{\tilde{\mu}}$ factorizes to a linear map $\overline{\rho} : \mathfrak{n}_\mu \to \tilde{\mathfrak{n}}_{\tilde{\mu}}$. We have another specialization of Theorem **B**.

Theorem 5.5. *Suppose that (5.20) holds. Let $\tilde{\mu} \in \Lambda^+_{\tilde{G}}$ and $\mu := \pi(\tilde{\mu}) \in \Lambda^+_G$. We denote by $H \subset G$ and $\tilde{H} \subset \tilde{G}$ the respective stabilizers*[5] *of μ and $\tilde{\mu}$. Let $\mathbb{W} := \tilde{\mathfrak{n}}_{\tilde{\mu}}/\overline{\rho}(\mathfrak{n}_\mu)$.*

The following statements are equivalent:

a) $\mathbf{m}(n\mu, n\tilde{\mu}) = 1$, *for all* $n \geq 1$.

b) *For any dominant weight* $(\lambda, \tilde{\lambda})$ *the sequence* $\mathbf{m}(\lambda + n\mu, \tilde{\lambda} + n\tilde{\mu})$ *is increasing and converging.*

c) *The algebra* $\mathrm{Sym}(\mathbb{W}^*)$ *has finite H-multiplicities.*

If these statements hold the limit of the sequence $\mathbf{m}(\lambda + n\mu, \tilde{\lambda} + n\tilde{\mu})$ *is equal to the multiplicity of* V^H_λ *in the H-module* $\mathrm{Sym}(\mathbb{W}^*) \otimes V^{\tilde{H}}_{\tilde{\lambda}}$.

Proof. We have constructed $(\mu, \tilde{\mu})$ so that $\mathbf{m}(\mu, \tilde{\mu}) \neq 0$. In this case Lemma 2.4 and Theorem **B** tells us that the following equivalences hold $\mathbf{m}(n\mu, n\tilde{\mu}) = 1, \forall n \geq 1 \iff \mathbf{m}(n\mu, n\tilde{\mu})$ is bounded $\iff (\tilde{G}\tilde{\mu})_\mu = \{pt\}$. Hence we have proved that $a) \Leftrightarrow c)$ and $b) \Rightarrow a)$. The other implication $a) \Rightarrow b)$ is also a consequence of Theorem **B**. □

5.2. The Littlewood-Richardson coefficients

Here we work with G embedded diagonally in $\tilde{G} := G \times G$. The map $\pi : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by $(\xi_1, \xi_2) \mapsto \xi_1 + \xi_2$.

[5]Note that $\rho(H) \subset \tilde{H}$.

Here the multiplicity function $\mathbf{m} : \Lambda^+_G \times \Lambda^+_G \times \Lambda^+_G \to \mathbb{N}$ is defined by

$$\mathbf{m}(a,b,c) := \dim \left[(V_a^G)^* \otimes V_b^G \otimes V_c^G\right]^G$$

We fix an element $(\mu_1,\mu_2) \in (\Lambda^+_G)^2$. It is easy to see that (μ_1,μ_2) is adapted to G. Let $\mu = \mu_1 + \mu_2$. The stabilizer subgroup G_μ is equal to $G_{\mu_1} \cap G_{\mu_2}$. We work with the G_μ-module

$$\mathbb{W}_{\mu_1,\mu_2} := \sum_{\substack{(\alpha,\mu_1)>0 \\ (\alpha,\mu_2)>0}} (\mathfrak{g}_\mathbb{C})_\alpha. \tag{5.21}$$

In this case Theorem 5.3 gives

Proposition 5.6. *Let $(\mu_1,\mu_2) \in (\Lambda^+_G)^2$ and $\mu = \mu_1 + \mu_2$.*

- *We have $\mathbf{m}(n\mu, n\mu_1, n\mu_2) = 1$ for any $n \geq 1$.*
- *For any $(a,b,c) \in (\Lambda^+_G)^3$, the sequence $\mathbf{m}(a+n\mu, b+n\mu_1, c+n\mu_2)$ is increasing and equal to*

$$\dim \left[\mathrm{Sym}(\mathbb{W}^*_{\mu_1,\mu_2}) \otimes (V_a^{G_\mu})^* \otimes V_b^{G_{\mu_1}}|_{G_\mu} \otimes V_c^{G_{\mu_2}}|_{G_\mu}\right]^{G_\mu}.$$

  for n large enough.

Proof. In the notations of Theorem 5.3, we have $\tilde{\mu} = (\mu_1,\mu_2)$, $\tilde{w} = 1$, $\mu = \mu_1 + \mu_2$, the parabolic subgroups $\tilde{\mathfrak{p}}_{\tilde{w}\tilde{\mu}}, \mathfrak{p}_\mu$ are respectively equal to $\mathfrak{p}_{\mu_1} \times \mathfrak{p}_{\mu_2}$ and $\mathfrak{p}_{\mu_1} \cap \mathfrak{p}_{\mu_2}$ and the subgroup $\tilde{H}$ is equal to $G_{\mu_1} \times G_{\mu_2}$. We check then easily that the G_μ-module $\tilde{\mathfrak{n}}_{\tilde{w}\tilde{\mu}}/\overline{\rho}(\mathfrak{n}_{\mu_{\tilde{w}}})$ is equal to $\mathbb{W}_{\mu_1,\mu_2}$. □

5.3. The Kronecker coefficients

Let $\mathrm{U}(E), \mathrm{U}(F)$ be the unitary groups of two hermitian vector spaces E, F. The aim of this section is to detail our results for the canonical morphism

$$\rho : G := \mathrm{U}(E) \times \mathrm{U}(F) \to \tilde{G} := \mathrm{U}(E \otimes F).$$

This problem is equivalent to the question on the decomposition of tensor products of representations for the symmetric group.

A partition λ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of weakly decreasing non-negative integers. By convention, we allow partitions with some zero parts, and two partitions that differ by zero parts are the same. For any partition

λ, we define $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ and $l(\lambda)$ as the number of non-zero parts of λ.

Recall that the $\mathrm{U}(E)$ irreducible polynomial representations are in bijection with the partitions λ such that $l(\lambda) \leq \dim E$. We denote by $S_\lambda(E)$ the representation associated to λ.

We consider the groups $G := \mathrm{U}(E) \times \mathrm{U}(F)$ and $\tilde{G} := \mathrm{U}(E \otimes F)$. Let γ be a partition such that $l(\gamma) \leq \dim E \cdot \dim F$. We can decompose the irreducible representation $S_\gamma(E \otimes F)$ as a G-representation:

$$S_\gamma(E \otimes F) = \sum_{\alpha,\beta} g(\alpha, \beta, \gamma)\ S_\alpha(E) \otimes S_\gamma(F)$$

where the sum is taken over partitions α, β such that $|\alpha| = |\beta| = |\gamma|$, $l(\alpha) \leq \dim E$ and $l(\beta) \leq \dim F$.

We fix an orthonormal basis (e_i) for E, (f_j) for F: let $(e_i \otimes f_j)$ the corresponding orthonormal basis of $E \otimes F$. We denote by T_E (resp. T_F) the maximal tori of $\mathrm{U}(E)$ (resp. $\mathrm{U}(F)$) consisting of the endomorphisms that are diagonal over (e_i) (resp. (f_j)). Let $T = T_E \times T_F$ be the maximal torus of G. Similarly let $\tilde{T}$ be the maximal tori of $\tilde{G}$ associated to the endomorphisms that diagonalize the basis $(e_i \otimes f_j)$. At the level of tori, the morphism ρ induces a map $\rho : T \to \tilde{T}$ sending $((t_i), (s_j))$ to $(t_i s_j)$. At the level of Lie algebra the map $\rho : \mathfrak{t} \to \tilde{\mathfrak{t}}$ is defined by

$$\rho(x, y) = (x_i + y_j)_{i,j}$$

for $x = (x_1, \ldots, x_{\dim E}) \in \mathbb{R}^{\dim E} \simeq \mathrm{Lie}(T_E)$ and $y = (y_1, \ldots, y_{\dim F}) \in \mathbb{R}^{\dim F} \simeq \mathrm{Lie}(T_F)$.

Let $\theta_{kl} \in \tilde{\mathfrak{t}}^*$ be the linear form that send an element $(a_{i,j}) \in \tilde{\mathfrak{t}}$ to a_{kl}. Then $\tilde{\mathfrak{t}}^*$ is canonically identified with the vector space of matrices of size $\dim E \times \dim F$ through the use of the basis θ_{kl}, and the dual map $\pi : \tilde{\mathfrak{t}}^* \to \mathfrak{t}^*$ is given by $\pi((\xi_{ij})) = ((\sum_j \xi_{ij})_i, (\sum_i \xi_{ij})_j)$.

Recall the following definition [16, 31].

Definition 5.7. Let $A = (a_{i,j})$ be a matrix of size $\dim E \times \dim F$. Then, A is called *additive* if there exist real numbers $x_1, \ldots, x_{\dim E}$, $y_1, \ldots, y_{\dim F}$ such that

$$a_{i,j} > a_{k,l} \Longrightarrow x_i + y_j > x_k + y_l,$$

for all $i, k \in [1, \ldots, \dim E]$ and all $j, l \in [1, \ldots, \dim F]$.

The following easy fact is important.

Lemma 5.8. *Let $\xi \in \tilde{\mathfrak{t}}^*$ that is represented by a matrix (ξ_{ij}). Then ξ is adapted to the group G if and only if the matrix (ξ_{ij}) is additive.*

Proof. The system of roots for $\tilde{G}$ is $\tilde{\mathfrak{R}} = \{\theta_{ij} - \theta_{kl}, (i,j) \neq (k,l)\}$. By definition $\xi \in \tilde{\mathfrak{t}}^*$ is adapted to G if and only if there exists $(x,y) \in \mathbb{R}^{\dim E} \times \mathbb{R}^{\dim F} \simeq \mathfrak{t}^*$ such that

$$(\theta_{ij} - \theta_{kl}, \xi) > 0 \Longrightarrow (\pi(\theta_{ij} - \theta_{kl}), (x,y)).$$

Our proof is completed since $(\theta_{ij} - \theta_{kl}, \xi) = \xi_{ij} - \xi_{kl}$ and $(\pi(\theta_{ij} - \theta_{kl}), (x,y)) = x_i + y_j - (x_k + y_l)$. □

Definition 5.9. If $A = (a_{i,j})$ is a matrix of size $\dim E \times \dim F$ with non negative integral coefficients, we define the partition $\alpha_A, \beta_A, \gamma_A$ where $\alpha_A \simeq (\sum_j a_{ij})_i$, $\beta_A \simeq (\sum_i a_{ij})_j$ and $\gamma_A \simeq (a_{i,j})$. Note that $|\alpha_A| = |\beta_A| = |\gamma_A|$.

The first part of Theorem 5.3 permits us to recover the following result of Vallejo [31] and Manivel [16].

Proposition 5.10. *Let $A = (a_{i,j})$ is a matrix of size* $\dim E \times \dim F$ *with non negative integral coefficients. If the matrix A is* additive *then*

- *$g(n\alpha_A, n\beta_A, n\gamma_A) = 1$ for all $n \geq 1$,*
- *the sequence $g(a + n\alpha_A, b + n\beta_A, c + n\gamma_A)$ is increasing and stationary for any partition a, b, c such that $|a| = |b| = |c|$, $l(a) \leq \dim E$, $l(b) \leq \dim F$ and $l(c) \leq \dim E \cdot \dim F$.*

Now we apply the second part of Theorem 5.3 to obtain a formula for the stretched multiplicities.

Definition 5.11. Let $A = (a_{i,j})$ is an *additive* matrix of size $\dim E \times \dim F$ with non negative integral coefficients. For any partition a, b, c such that $|a| = |b| = |c|$, we define $g_A(a,b,c) \in \mathbb{N}$ as the limit of the sequence $g(a + n\alpha_A, b + n\beta_A, c + n\gamma_A)$ when $n \to \infty$.

Let E_i^k (resp. F_j^l) be the orthogonal projection of rank 1 of E (resp. F) that sends e_i to e_k (resp. f_j to f_l).

To an additive matrix A, we attach :

- The stabilizer $\tilde{H}_A \subset \tilde{G}$ of the element $A \in \tilde{\mathfrak{t}}^*$, with Lie algebra $\tilde{\mathfrak{h}}_A$.
- The stabilizer $H_A \subset G$ of the element $\pi(A)$. We have $H_A = H_A^E \times H_A^F$ with $H_A^E = \mathrm{U}(E)_{\alpha_A}$ and $H_A^F = \mathrm{U}(F)_{\beta_A}$.

• The $\tilde{H}_A$-module

$$\tilde{\mathfrak{p}}_A := \sum_{a_{ij} \geq a_{kl}} \mathbb{C}\, E_i^k \otimes F_j^l$$

that corresponds to the parabolic sub-algebra of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ attached to A. Its nil-radical is $\tilde{\mathfrak{n}}_A = \sum_{a_{ij} > a_{kl}} \mathbb{C}\, E_i^k \otimes F_j^l$.

• the sub-algebras $\mathfrak{n}_{\pi(A)} \subset \mathfrak{p}_{\pi(A)} \subset \mathfrak{g}_{\mathbb{C}}$ and their images by ρ:

$$\rho(\mathfrak{p}_{\pi(A)}) = \sum_{\alpha_i \geq \alpha_k} \mathbb{C}\, E_i^k \otimes \mathrm{Id}_F \oplus \sum_{\beta_j \geq \beta_l} \mathbb{C}\, \mathrm{Id}_E \otimes F_j^l$$
$$\rho(\mathfrak{n}_{\pi(A)}) = \sum_{\alpha_i > \alpha_k} \mathbb{C}\, E_i^k \otimes \mathrm{Id}_F \oplus \sum_{\beta_j > \beta_l} \mathbb{C}\, \mathrm{Id}_E \otimes F_j^l$$

Thanks to proposition 5.2 we know that $\rho(H_A) \subset \tilde{H}_A$ and that $\rho(\mathfrak{p}_{\pi(A)}) \subset \tilde{\mathfrak{p}}_A$. We denote by $\overline{\rho}(\mathfrak{n}_{\pi(A)})$ the projection of $\rho(\mathfrak{n}_{\pi(A)}) \subset \tilde{\mathfrak{p}}_A$ on $\tilde{\mathfrak{p}}_A/(\tilde{\mathfrak{h}}_A)_{\mathbb{C}} \simeq \tilde{\mathfrak{n}}_A$.

We define the H_A-module

$$\mathbb{W}_A = \tilde{\mathfrak{n}}_A / \overline{\rho}(\mathfrak{n}_{\pi(A)}) \tag{5.22}$$

and we know that $\mathrm{Sym}(\mathbb{W}_A^*)$ has finite H_A-multiplicities.

For a partition $a = (a_1, a_2, \ldots, a_{\dim E})$, we define $V_a^{H_A^E}$ as the irreducible representation of H_A^E with highest weight a. If $\alpha_A = (l_1^{n_1}, l_2^{n_2}, \ldots, l_r^{n_r})$ with $l_1 > l_2 > \cdots > l_r$, the subgroup H_A^E is isomorphic to $\mathrm{U}(E_1) \times \cdots \times \mathrm{U}(E_r)$ with $\dim E_k = n_k$, and the representation $V_a^{H_A^E}$ is equal to the tensor product $S_{a[1]}(E_1) \otimes S_{a[2]}(E_r) \otimes \cdots \otimes S_{a[r]}(E_r)$ where $a[k]$ is the partition

$$(a_{n_1+\cdots+n_r+1}, \ldots, a_{n_1+\cdots+n_{r+1}}).$$

We can define similarly the representations $V_c^{\tilde{H}_A}$ and $V_b^{H_A^F}$. Theorem 5.3 give us the following

Theorem 5.12. *Let $A = (a_{i,j})$ be a* additive *matrix of size* $\dim E \times \dim F$ *with non negative integral coefficients. For any partition a, b, c such that $|a| = |b| = |c|$, $l(a) \leq \dim E$, $l(b) \leq \dim F$ and $l(c) \leq \dim E \cdot \dim F$, we have*

$$g_A(a,b,c) = \dim \left[\mathrm{Sym}(\mathbb{W}_A^*) \otimes (V_a^{H_A^E})^* \otimes (V_b^{H_A^F})^* \otimes V_c^{\tilde{H}_A}|_{H_A^E \times H_A^F} \right]^{H_A^E \times H_A^F}$$

5.3.1. The partition (1^{pq}). Let us work out the example of the partition $A = (1^{pq})$ where $1 \leq p \leq \dim E$ and $1 \leq q \leq \dim F$.

We see $A = (1^{pq})$ as an additive matrix (a_{ij}) of type $\dim E \times \dim F$: a_{ij} is non-zero, equal to 1, only if $1 \leq i \leq p$ and $1 \leq j \leq q$. Let g_{pq} be the corresponding stretched Kronecker coefficients.

We use an orthogonal decomposition of our vector spaces : $E = E_p \oplus E'$ and $F = F_q \oplus F'$ with $\dim E_p = p$ and $\dim F_q = q$. For the tensor product we have $E \otimes F = E_p \otimes F_q \oplus (E_p \otimes F_q)^\perp$ where $(E_p \otimes F_q)^\perp = E_p \otimes F' \oplus E' \otimes F_q \oplus E' \otimes F'$.

The stabiliser subgroup of A in $\tilde{G}$ is $\tilde{H}_{pq} := \mathrm{U}(E_p \otimes F_q) \times \mathrm{U}((E_p \otimes F_q)^\perp)$ and the stabiliser subgroup of $\pi(A)$ in G is $H_{pq} := H_p^E \times H_q^F$ where $H_p^E = \mathrm{U}(E_p) \times \mathrm{U}(E')$ and $H_q^F = \mathrm{U}(F_q) \times \mathrm{U}(F')$.

If $A = (1^{pq})$, we denote by $\mathbb{W}_A = \mathbb{W}_{pq}$ the H_{pq}-module introduced in (5.22). A direct computation shows that

$$\mathbb{W}_{pq} = \hom(E_p, E') \otimes \mathfrak{sl}(F_q) \bigoplus \mathfrak{sl}(E_p) \otimes \hom(F_q, F') \bigoplus \hom(E_p, E') \otimes \hom(F_q, F').$$

A partition $a = (a_1, \ldots, a_{\dim E})$ defines the partitions $a(p) := (a_1, \ldots, a_p)$ and $a' := (a_{p+1}, \ldots, a_{\dim E})$. Similarly a partition $b = (b_1, \ldots, b_{\dim F})$ defines the partitions $b(q) := (b_1, \ldots, b_q)$ and $b' := (a_{q+1}, \ldots, a_{\dim F})$.

A partition c of length $\dim E \times \dim F$ is represented by a matrix (c_{ij}). We define then the partition $c(pq)$ of length pq represented by the coefficients c_{ij} when $1 \leq i \leq p$ and $1 \leq j \leq q$, and the partition c' which is the complement of $c(pq)$ in c.

Theorem 5.12 tell us that the stretched Kronecker coefficient $g_{pq}(a, b, c)$ is equal to the multiplicity of the irreducible representation

$$S_{a(p)}(E_p) \otimes S_{a'}(E') \otimes S_{b(q)}(F_q) \otimes S_{b'}(F')$$

in

$$\mathrm{Sym}(\mathbb{W}_{pq}^*) \otimes S_{c(pq)}(E_p \otimes F_q) \otimes S_{c'}((E_p \otimes F_q)^\perp).$$

When $q = 1$ the following expression for the stretched coefficient was obtained by Manivel [16], extending the case $p = q = 1$ treated by Brion [9].

5.3.2. The triple $(22), (22), (22)$. In this section we explain how our technique allows us to recover the result of Stembridge [28] concerning the stability of the triple $(22), (22), (22)$. Moreover we compute the stretched multiplicty map associated to this triple. Notice that the triple $(22), (22), (22)$ is not attached to an additive matrix.

First we work with the morphism $\rho : \mathrm{U}(\mathbb{C}^2) \times \mathrm{U}(\mathbb{C}^2) \to \mathrm{U}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. The matrix

$$\tilde{\mu} := i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

represents a weight of the maximal torus $\tilde{T}$ of $\tilde{G} = \mathrm{U}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. The stabilizer subgroup $\tilde{G}_{\tilde{\mu}}$ is canonically isomorphic with $U(V_1) \times U(V_2)$ where $V_1 = \mathrm{Vect}(e_1 \otimes f_1, e_2 \otimes f_2)$ and $V_2 = \mathrm{Vect}(e_1 \otimes f_2, e_2 \otimes f_1)$. The character $\chi_{\tilde{\mu}}$ on $\tilde{G}_{\tilde{\mu}}$ defined by the weight $\tilde{\mu}$ is the morphism $(g_1, g_2) \in U(V_1) \times U(V_2) \mapsto \det(g_1)\det(g_2)$.

The restriction of $\tilde{\mu}$ to the maximal torus T of $G = \mathrm{U}(\mathbb{C}^2) \times \mathrm{U}(\mathbb{C}^2)$ defines a weight $\mu = \pi(\tilde{\mu})$. We see that μ is the differential of the character $\chi_\mu := \det \times \det$.

The Kronecker coefficient $g(n(1,1), n(1,1), n(1,1))$ corresponds to the multiplicity of the character $\chi_\mu^{\otimes n}$ in $V_{n\tilde{\mu}}^{\tilde{G}}|_G$. Let us check that the sequence $g(n(1,1), n(1,1), n(1,1))$ is bounded.

We consider the point $m_o = (\mu, \tilde{\mu}) \in G\mu \times \tilde{G}\tilde{\mu}$. The stabilizer subgroup G_{m_o} is equal to $H = G \cap \rho^{-1}(\tilde{G}_{\tilde{\mu}})$ since μ is G-invariant. A small computation shows that the connected component H^o is equal to the torus T.

Here we work with the H-module $\mathbb{W} := \mathrm{T}_{\tilde{\mu}}(\tilde{G}\tilde{\mu})/\mathfrak{g}_{\mathbb{C}} \cdot \tilde{\mu}$.

Lemma 5.13. 1) *The H-module $\mathbb{W}$ is reduced to $\{0\}$.*

2) *The reduced space $(\tilde{G}\tilde{\mu})_\mu$ is a singleton.*

3) *The character $\chi_{\tilde{\mu}}\chi_\mu^{-1}$ is trivial on $H^o = T$ and defines an isomorphism between H/H^o and $\{\pm 1\}$.*

4) $g(n(1,1), n(1,1), n(1,1)) = \frac{1+(-1)^n}{2}$.

Proof. If we compute the real dimensions we have

$$\dim \tilde{G}\tilde{\mu} = \dim \mathrm{U}(4) - 2\dim \mathrm{U}(2) = 8.$$

On the other hand,

$$\dim \mathfrak{g}_{\mathbb{C}} \cdot \tilde{\mu} = 2 \dim \mathfrak{g} \cdot \tilde{\mu} = 2(\dim G - \dim H).$$

As $H^o = T$ we have $\dim H = 4$ and then $\dim \mathfrak{g}_{\mathbb{C}} \cdot \tilde{\mu} = \dim \tilde{G}\tilde{\mu}$. The first point is proved.

The second point is a consequence of the first point (see Lemma 2.4). At this stage we know that $g(n(1,1), n(1,1), n(1,1)) = \dim[(\chi_{\tilde{\mu}}\chi_\mu^{-1})^{\otimes n}]^H$. The last point is a consequence of the third one. The easy checking of the third point is left to the reader. □

We have proved that $\tau := \{(22),(22),(22)\}$ is a stable triple. We will now compute the associated stretched multiplicity map. Consider partitions a, b, c such that $|a| = |b| = |c|$, $2 \leq l(a) \leq p$, $2 \leq l(b) \leq q$ and $2 \leq l(c) \leq pq$. We define

$$g_\tau(a,b,c) := \lim_{n\to\infty} g(a+n(2,2), b+n(2,2), c+n(2,2)).$$

Here we consider the morphism $\rho_{p,q} : \mathrm{U}(\mathbb{C}^p) \times \mathrm{U}(\mathbb{C}^q) \to \mathrm{U}(\mathbb{C}^p \otimes \mathbb{C}^q)$.

The subgroups of $U(\mathbb{C}^p)$, $U(\mathbb{C}^q)$ and $U(\mathbb{C}^p \otimes \mathbb{C}^q)$ that stabilizes the weights $(2,2,0,\ldots,0)$ are denoted respectively $K_p \simeq U(\mathbb{C}^2) \times U(\mathbb{C}^{p-2})$, $K_q \simeq U(\mathbb{C}^2) \times U(\mathbb{C}^{q-2})$ and $K_{pq} \simeq U(\mathbb{C}^2) \times U(\mathbb{C}^{pq-2})$.

We work with the subgroup

$$H := \rho_{p,q}^{-1}(K_{pq}) \subset K_p \times K_q.$$

A small computation shows that the connected component H^o is isomorphic with $U(\mathbb{C}) \times U(\mathbb{C}) \times U(\mathbb{C}^{p-2}) \times U(\mathbb{C}) \times U(\mathbb{C}) \times U(\mathbb{C}^{q-2})$.

We associate to the partition $a = (a_1, \ldots, a_p)$ the partitions $a(2) := (a_1, a_2)$ and $a' := (a_3, \ldots, a_p)$. Similarly we associate to the partitions b and c the partitions $b(2), c(2)$ and b', c'.

We consider the following irreducible representations.

- $\mathbf{V}_{a,b} := S_{a(2)}(\mathbb{C}^2) \otimes S_{a'}(\mathbb{C}^{p-2}) \otimes S_{b(2)}(\mathbb{C}^2) \otimes S_{b'}(\mathbb{C}^{q-2})$ is a irreducible representation of $K_p \times K_q$.
- $\mathbf{W}_c := S_{c(2)}(\mathbb{C}^2) \otimes S_{c'}(\mathbb{C}^{pq-2})$ is a irreducible representation of K_{pq}.

In this setting Theorem **B** gives that

$$g_\tau(a,b,c) = \left[\mathbf{W}_c|_H \otimes (\mathbf{V}_{a,b})^*|_H\right]^H .$$

5.4. Plethysm

Let $\rho : G \to \tilde{G} := \mathrm{U}(V)$ be an irreducible representation of the group G. Let $N = \dim V$. Let T be a maximal torus of G. The T-action on V can be diagonalized: there exists an orthonormal basis $(v_j)_{j\in J}$ and a family of weights $(\alpha_j)_{j\in J}$ such that $\rho(t)v_j = t^{\alpha_j} v_j$ for all $t \in T$. Let $\tilde{T}$ be the maximal torus of $\tilde{G}$ consisting of the unitary endomorphisms that are diagonalized by the basis $(v_j)_{j\in J}$: we have then $\rho(T) \subset \tilde{T}$. We denote by $\pi : \tilde{\mathfrak{t}}^* \to \mathfrak{t}^*$ the projection, and by $e_k \in \tilde{\mathfrak{t}}^*$ the linear form that sends $(x_j)_{j\in J}$ to x_k.

Let B be a Borel subgroup of G: there exists a Borel subgroup $\tilde{B} \subset \tilde{G}$ such that $\rho(B) \subset \tilde{B}$. We work with the set of dominant weights $\Lambda^+_{\tilde{G}}$, Λ^+_G defined by this choice: the Borel subgroup $\tilde{B}$ fix an ordering $>$ on the elements of J, and a weight $\tilde{\xi} = \sum_{j\in J} a_j e_j$ belongs to $\Lambda^+_{\tilde{G}}$ only if $j > k \Longrightarrow a_j \geq a_k$. For simplicity we write $J = \{1, \ldots, N\}$ with the canonical ordering.

For the remaining part of this section we work with a fixed partition $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$, and we denote by $S_\sigma(V)$ the corresponding irreducible representation of $\mathrm{U}(V)$. We can represent σ by the element $\sum_{j=1}^N \sigma_j e_j \in \tilde{\mathfrak{t}}^*$ (that we still denote by σ). Let $\mu = \pi(\sigma) = \sum_{j=1} \sigma_j \alpha_j \in \Lambda^+_G$.

Let $\{0 = j_0 > j_2 > \cdots > j_p = N\}$ be the set of element $j \in [0, \ldots, N]$ such that $\sigma_{j+1} > \sigma_j$ or $j \in \{0, N\}$. We have an orthogonal decomposition $V = \oplus_{k=1}^p V_{[k]}$ where $V_{[k]}$ is the vector space generated by the v_j for $j \in [j_{k-1}+1, \ldots, j_k]$. The nilradical $\tilde{\mathfrak{n}}_\sigma$ of the parabolic subgroup $\tilde{\mathfrak{p}}_\sigma \subset \mathfrak{gl}(V)$ corresponds to the set of endomorphisms f such that $f(V_{[k]}) \subset \oplus_{j<k} V_{[j]}$.

The following Lemma is proved in [19].

Lemma 5.14. *Let $\mathfrak{n}_\mu$ the nilradical of the parabolic subgroup $\mathfrak{p}_\mu \subset \mathfrak{g}_\mathbb{C}$. The morphism $d\rho : \mathfrak{g}_\mathbb{C} \to \mathfrak{gl}(V)$ defines an injective map from $\mathfrak{n}_\mu$ into $\tilde{\mathfrak{n}}_\sigma$.*

We define $\mathbb{W}_\sigma$ as the quotient $\tilde{\mathfrak{n}}_\sigma/\rho(\mathfrak{n}_\mu)$. Recall that the image by ρ of the stabilizer subgroup G_μ is contained in the stabilizer subgroup of σ: hence $\mathbb{W}_\sigma$ is a G_μ-module.

For any partition $\theta = (\theta_1, \ldots, \theta_N)$, we associate the partition of length $\dim V_{[k]}$, $\theta_{[k]} := (\theta_{j_{k-1}+1}, \ldots, V_{j_k})$, and the irreducible representation $S_{\theta_{[k]}}(V_{[k]})$ of the unitary group $\mathrm{U}(V_{[k]})$.

For any partition θ of length N and any dominant weight of $\lambda \in \Lambda^+_G$, let

$$\left[V^G_{\lambda+n\mu} : S_{\theta+n\sigma}(V)\right]$$

be the multiplicity of the irreducible representation $V^G_{\lambda+n\mu}$ in the restriction $S_{\theta+n\sigma}(V)|_G$.

The following theorem, which is a particular case of Theorem 5.5, was first obtained by Manivel [15] when $G = \mathrm{U}(E)$ and by Brion [9] when $\sigma = (1)$. The following version was obtained by Montagard [19]: the only improvement that we obtain here is condition a).

Theorem 5.15. *Let σ a partition of length* $\dim V$ *and $\mu = \pi(\sigma)$.*

The following statements are equivalent:

a) $[V^G_{n\mu} : S_{n\sigma}(V)] = 1$, *for all $n \geq 1$.*

b) *For any couple* (λ, θ) *the increasing sequence* $[V^G_{\lambda+n\mu} : S_{\theta+n\sigma}(V)]$ *has a limit.*

c) *The algebra* $\mathrm{Sym}(\mathbb{W}_\sigma{}^*)$ *has finite* G_μ*-multiplicities.*

If these statements hold the limit of the sequence $[V^G_{\lambda+n\mu} : S_{\theta+n\sigma}(V)]$ *is equal to the multiplicity of* $V^{G_\mu}_\lambda$ *in the* G_μ*-module*

$$\mathrm{Sym}(\mathbb{W}^*_\sigma) \otimes S_{\theta_{[1]}}(V_{[1]}) \otimes S_{\theta_{[2]}}(V_{[2]}) \otimes \cdots \otimes S_{\theta_{[p]}}(V_{[p]}).$$

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Institut Montpelliérain Alexander Grothendieck
CNRS, Université de Montpellier
34095 Montpellier Cedex 5, France
E-mail address: paul-emile.paradan@umontpellier.fr

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