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Futaki invariant for Fedosov star products

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We study obstructions to the existence of closed Fedosov star products on a given Kähler manifold (M, ω, J). In our previous paper [14], we proved that the Levi-Civita connection of a Kähler manifold will produce a closed Fedosov star product (closed in the sense of Connes–Flato–Sternheimer [4]) only if it is a zero of a moment map μ on the space of symplectic connections. By analogy with the Futaki invariant obstructing the existence of constant scalar curvature Kähler metric, we build an obstruction for the existence of zero of μ and hence for the existence of closed Fedosov star product on a Kähler manifold.

1. Introduction

In [3], a moment map μ on the space of symplectic connections is introduced. The study of zeroes of μ and of the so-called critical symplectic connections was first proposed by D.J. Fox [9] in analogy with the moment map picture for the Hermitian scalar curvature on almost-Kähler manifolds [6]. Recently [14], we give additional motivations for the study of μ, and its zeroes on Kähler manifolds, coming from the formal deformation quantization of symplectic manifolds.

We exhibit an obstruction to the existence of zeroes of μ on closed Kähler manifolds in the spirit of Futaki invariants [10]. It is a character on $\mathfrak{h}$ the Lie algebra of holomorphic vector fields in $T^{(1,0)}M$ having a zero on M, see [15]. Recall that on a Kähler manifold (M, ω, J), elements in $\mathfrak{h}$ are $(1,0)$-part of vector fields on M of the form $Z = X_F + JX_H$, for $F, H \in C^\infty(M)$ with zero mean, X_F (resp. X_H) is the Hamiltonian vector field defined by $i(X_F)\omega = dF$ (resp. $i(X_H)\omega = dH$) so that F and H depend on ω.

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Theorem 1. *Let (M, ω, J) be a closed Kähler manifold with Kähler class Θ, Levi-Civita connection ∇ and fixed complex structure J. Then, the map*

$$\mathcal{F}^{\omega} : \mathfrak{h} \to \mathbb{R} : Z \mapsto \int_M H\mu(\nabla)\frac{\omega^n}{n!},$$

for $Z = X_F + JX_H$, μ the Cahen–Gutt moment map on $\mathcal{E}(M, \omega)$, is a character that does not depend on the choice of a Kähler form in the Kähler class Θ.

Deformation quantization as defined in [2] is a formal associative deformation of the Poisson algebra $(C^{\infty}(M), ., \{\cdot, \cdot\})$ of a Poisson manifold (M, π) in the direction of the Poisson bracket. The deformed algebra is the space $C^{\infty}(M)[[\nu]]$ of formal power series of smooth functions with composition law $*$ called star product.

On a symplectic manifold (M, ω) endowed with a symplectic connection ∇ (i.e. torsion-free connection leaving ω parallel), one can associate the Fedosov star product $*_{\nabla}$, [7]. The moment map μ evaluated at ∇ is the first non-trivial term in the expression of a trace density for the star product $*_{\nabla}$, see [14]. So that, if the star product $*_{\nabla}$ is closed (in the sense of Connes–Flato–Sternheimer [4]), then $\mu(\nabla)$ is the zero function which implies the following result.

Corollary 1.1. *Let (M, ω, J) be a closed Kähler manifold with Kähler class Θ, such that $\mathcal{F}^{\omega}$ is not identically zero, then, given any Kähler form $\widetilde{\omega} \in \mathcal{M}_{\Theta}$ with Levi-Civita connection $\widetilde{\nabla}$, the Fedosov star product $*_{\widetilde{\nabla}}$ is not closed.*

Finally, we identify the character $\mathcal{F}^{\omega}$ with one of the so-called higher Futaki invariants [11]. It enables us to exhibit an example of Kähler manifold [17, 18] admitting non-zero values of $\mathcal{F}^{\omega}$ and hence no closed Fedosov star products as considered in Corollary 1.1.

2. The moment map and Fedosov star products

Consider a closed symplectic manifold (M, ω) of dimension $2n$. A symplectic connection ∇ on (M, ω) is a torsion-free connection such that $\nabla\omega = 0$. There always exists a symplectic connection on a symplectic manifold. If we denote by $A(\cdot)$ a field of 1-form with values in $\mathrm{End}(TM)$ and if ∇ is a symplectic connection then the connection $\nabla + A(\cdot)$ is symplectic if and only if the 3-tensor field $\omega(A(\cdot)\cdot, \cdot)$ is symmetric. The space $\mathcal{E}(M, \omega)$ of symplectic connections is the affine space

$$\mathcal{E}(M,\omega) = \nabla + \Gamma(S^3T^*M) \text{ for some } \nabla \in \mathcal{E}(M,\omega),$$

where $S^3T^*M := \{A \in \Lambda^1(M) \otimes \mathrm{End}(TM) \mid \omega(A(\cdot)\cdot,\cdot) \text{ is symmetric}\}$. For $A \in S^3T^*M$, we set $\underline{A}(\cdot,\cdot,\cdot)$ for the symmetric 3-tensor $\omega(A(\cdot)\cdot,\cdot)$.

There is a natural symplectic form on $\mathcal{E}(M,\omega)$. For $A, B \in T_\nabla\mathcal{E}(M,\omega)$, seen as sections of $\Lambda^1(M) \otimes \mathrm{End}(TM,\omega)$, one defines

$$\Omega^{\mathcal{E}}_\nabla(A,B) := \int_M \mathrm{tr}(A \overset{\circ}{\wedge} B) \wedge \frac{\omega^{n-1}}{(n-1)!} = -\int_M \Lambda^{kl}\mathrm{tr}(A(e_k)B(e_l))\frac{\omega^n}{n!},$$

where $\overset{\circ}{\wedge}$ is the product on $\Lambda^1(M) \otimes \mathrm{End}(TM,\omega)$ induced by the usual $\wedge$-product on forms and the composition on the endomorphism part, Λ^{kl} is defined by $\Lambda^{kl}\omega_{lt} = \delta^k_t$ for $\omega_{lt} := \omega(e_l, e_t)$ for a frame $\{e_k\}$ of T_xM and using, as for the rest of the paper, Einstein summation convention on repeated indices. The 2-form $\Omega^{\mathcal{E}}$ is a symplectic form on $\mathcal{E}(M,\omega)$.

Remark 2.1. The symplectic form $\Omega^{\mathcal{E}}$ can be written in coordinate as :

$$\Omega^{\mathcal{E}}_\nabla(A,B) := \int_M \Lambda^{i_1j_1}\Lambda^{i_2j_2}\Lambda^{i_3j_3}\underline{A}_{i_1i_2i_3}\underline{B}_{j_1j_2j_3}\frac{\omega^n}{n!},$$

for $A, B \in T_\nabla\mathcal{E}(M,\omega)$.

There is a natural symplectic action of the group of symplectomorphisms on $\mathcal{E}(M,\omega)$. For φ, a symplectic diffeomorphism, we define an action

$$(\varphi.\nabla)_X Y := \varphi_*(\nabla_{\varphi_*^{-1}X}\varphi_*^{-1}Y), \tag{2.1}$$

for all $X, Y \in TM$ and $\nabla \in \mathcal{E}(M,\omega)$.

Recall that a Hamiltonian vector field is a vector field X_F for $F \in C^\infty(M)$ such that $i(X_F)\omega = dF$. We denote by $\mathrm{Ham}(M,\omega)$ the group of Hamiltonian diffeomorphisms of the symplectic manifold (M,ω) with Lie algebra the space $C^\infty_0(M)$ of smooth functions F such that $\int_M F\frac{\omega^n}{n!} = 0$.

The action defined in Equation (2.1) restricts to an action of the group $\mathrm{Ham}(M,\omega)$. Let X_F be a Hamiltonian vector field with $F \in C^\infty_0(M)$, the fundamental vector field on $\mathcal{E}(M,\omega)$ associated to X_F is the Lie derivative:

for $Y, Z \in \Gamma(TM)$,

$$(\mathcal{L}_{X_F}\nabla)(Y)Z = \nabla^2_{(Y,Z)}X_F + R^\nabla(X_F, Y)Z,$$

where $\nabla^2_{(U,V)}W := \nabla_U\nabla_V W - \nabla_{\nabla_U V}W$ is the second covariant derivative and $R^\nabla(U,V)W := [\nabla_U, \nabla_V]W - \nabla_{[U,V]}W$ is the curvature tensor of ∇, for $U, V, W \in \Gamma(TM)$.

Let $Ric^\nabla(X,Y) := \mathrm{tr}[V \mapsto R^\nabla(V,X)Y]$ for all $X, Y \in TM$ be the Ricci tensor of ∇. Let $P(\nabla)$ be the function defined by

$$P(\nabla)\frac{\omega^n}{n!} := \frac{1}{2}\mathrm{tr}(R^\nabla(.,.) \overset{\circ}{\wedge} R^\nabla(.,.)) \wedge \frac{\omega^{n-2}}{(n-2)!},$$

with integral $\mu_0 := \int_M P(\nabla)\frac{\omega^n}{n!}$, note that μ_0 is a topological constant depending on the first Pontryagin class of M and $[\omega]$, hence not depending on ∇. Define the map $\mu : \mathcal{E}(M,\omega) \to C^\infty_0(M)$ by

$$\mu(\nabla) := (\nabla^2_{(e_p,e_q)}Ric^\nabla)(e^p, e^q) + P(\nabla) - \mu_0$$

where $\{e_k\}$ is a frame of T_xM and $\{e^l\}$ is the symplectic dual frame of $\{e_k\}$ (that is $\omega(e_k, e^l) = \delta^l_k$).

Theorem 2.2 (Cahen–Gutt [3]). *The map $\mu : \mathcal{E}(M,\omega) \to C^\infty_0(M)$ is an equivariant moment map for the action of* $\mathrm{Ham}(M,\omega)$ *on $\mathcal{E}(M,\omega)$, i.e.*

$$\left.\frac{d}{dt}\right|_0 \int_M \mu(\nabla + tA)F\frac{\omega^n}{n!} = \Omega^{\mathcal{E}}_\nabla(\mathcal{L}_{X_F}\nabla, A). \tag{2.2}$$

In [14], the moment map μ is related to the notion of trace density for Fedosov star products. Also, the closedness (closedness in the sense of Connes–Flato–Sternheimer [4]) of a Fedosov star product implies $\mu = 0$. Let us recall briefly all those notions and results.

A **star product**, as defined in [2], on (M,ω) is a $\mathbb{R}[[\nu]]$-bilinear associative law on the space $C^\infty(M)[[\nu]]$ of formal power series of smooth functions:

$$* : (C^\infty(M)[[\nu]])^2 \to C^\infty(M)[[\nu]] : (H,K) \mapsto H * K := \sum_{r=0}^\infty \nu^r C_r(H,K)$$

where the C_r's are bidifferential operators null on constants such that for all H, $K \in C^\infty(M)[[\nu]] : C_0(H,K) = HK$ and $C_1(H,K) - C_1(K,H) = \{H,K\}$.

In [7], Fedosov gave a geometric construction of star products on symplectic manifolds using a symplectic connection ∇ and a formal series of closed 2-forms $\Omega \in \nu\Omega^2(M)[[\nu]]$. We will only consider Fedosov star products built with $\Omega = 0$ and denote them by $*_\nabla$.

Let $*$ be a star product on a symplectic manifold. A **trace** for $*$ is a $\mathbb{R}[[\nu]]$-linear map

$$\mathrm{tr} : C^\infty(M)[[\nu]] \to \mathbb{R}[[\nu]],$$

satisfying $\mathrm{tr}(F * H) = \mathrm{tr}(H * F)$ for all $F, H \in C^\infty(M)[[\nu]]$.

Any star product $*$ on a symplectic manifold (M, ω) admits a trace [8, 13, 16]. More precisely, there exists $\kappa \in C^\infty(M)[[\nu]]$ such that

$$\mathrm{tr}(F) := \int_M F\kappa \frac{\omega^n}{n!}$$

for all $F \in C^\infty(M)[[\nu]]$. The function κ is called a **trace density**. Moreover, any two traces for $*$ differ from each other by multiplication with a formal constant $C \in \mathbb{R}[\nu^{-1}, \nu]]$.

A star product is called **closed** [4] if the map $F \mapsto \int_M F\frac{\omega^n}{n!}$ satisfies the trace property:

$$\int_M F * H\frac{\omega^n}{n!} = \int_M H * F\frac{\omega^n}{n!}, \quad \text{for all } F, H \in C^\infty(M)[[\nu]].$$

In [14], we linked the moment map with the trace density κ^∇ of the Fedosov star product $*_\nabla$ by the formula :

$$\kappa^\nabla := 1 + \frac{\nu^2}{24}\mu(\nabla) + O(\nu^3). \tag{2.3}$$

So that, if $*_\nabla$ is closed, then $\mu(\nabla) = 0$.

3. Futaki invariant for μ

3.1. Definition and main Theorem

We consider a closed Kähler manifold (M, ω, J). Let Θ be the Kähler class of ω and denote by $\mathcal{M}_\Theta$ the set of Kähler forms in the class Θ:

$$\mathcal{M}_\Theta := \{\omega_\phi = \omega + dd^c\phi \text{ s.t. } \phi \in C^\infty_0(M),\ \omega_\phi(\cdot, J\cdot) \text{ is positive definite }\},$$

where $d^cF := -dF \circ J$ for $F \in C^\infty(M)$.

Consider the functional

$$\omega_\phi \in \mathcal{M}_\Theta \mapsto \mu^\phi(\nabla^\phi) \in C^\infty(M),$$

where μ^ϕ is the moment map on $\mathcal{E}(M, \omega_\phi)$ and ∇^ϕ is the Levi-Civita connection of $g_\phi(\cdot,\cdot) := \omega_\phi(\cdot, J\cdot)$. Using the second Bianchi identity, one can write:

$$\mu^\phi(\nabla^\phi) = -\frac{1}{2}\Delta^\phi Scal^{\nabla^\phi} + P(\nabla^\phi) - \mu_0,$$

where $Scal^{\nabla^\phi}$ denotes the scalar curvature of ∇^ϕ. Recall that μ_0 is a topological constant so that $\mu^\phi(\nabla^\phi)$ is normalised with respect to the integral with volume form $\frac{\omega_\phi^n}{n!}$. Finally, remark that one uses the Kähler metric to define the scalar curvature, for a general symplectic connection there is no notion of scalar curvature [12].

Let $\mathfrak{h}$ the Lie algebra of holomorphic vector fields in $T^{(1,0)}M$ having a zero on M. For any $\omega_\phi \in \mathcal{M}_\Theta$, elements in $\mathfrak{h}$ can be represented as vector fields on M by $Z = X^{\omega_\phi}_{F^\phi} + JX^{\omega_\phi}_{H^\phi}$ for unique $F^\phi, H^\phi \in C^\infty(M)$ (depending on ω_ϕ) whose integral with respect to $\frac{\omega_\phi^n}{n!}$ is zero and where $X^{\omega_\phi}_K$ denotes the Hamiltonian vector field of $K \in C^\infty(M)$ with respect to the symplectic form ω_ϕ.

Definition 3.1. *For $\omega_\phi \in \mathcal{M}_\Theta$, we define the map*

$$\mathcal{F}^{\omega_\phi} : \mathfrak{h} \mapsto \mathbb{R} : Z \mapsto \int_M H^\phi \mu^\phi(\nabla^\phi)\frac{\omega_\phi^n}{n!}, \tag{3.1}$$

for $Z = X^{\omega_\phi}_{F^\phi} + JX^{\omega_\phi}_{H^\phi}$ as above.

Though the definition of $\mathcal{F}^{\omega_\phi}$ seems a priori to depend on the choice of a point in $\mathcal{M}_\Theta$, we will prove it is not the case.

Theorem 1. *Let (M, ω, J) be a closed Kähler manifold with Kähler class Θ, Levi-Civita connection ∇ and fixed complex structure J. Then, the map*

$$\mathcal{F}^\omega : \mathfrak{h} \to \mathbb{R} : Z \mapsto \int_M H\mu(\nabla)\frac{\omega^n}{n!},$$

for $Z = X_F + JX_H$, μ the Cahen–Gutt moment map on $\mathcal{E}(M, \omega)$, is a character that does not depend on the choice of a Kähler form in the Kähler class Θ.

The Theorem 1 implies that the non-vanishing of $\mathcal{F}^{\omega}$ is an obstruction to the existence of $\omega_{\phi} \in \mathcal{M}_{\Theta}$ such that $\mu^{\phi}(\nabla^{\phi}) = 0$.

Proof of Corollary 1.1. For $\widetilde{\omega} \in \mathcal{M}_{\Theta}$ with Levi-Civita connection $\widetilde{\nabla}$, assume the Fedosov star product $*_{\widetilde{\nabla}}$ is closed. Then $\mu^{\widetilde{\omega}}(\widetilde{\nabla}) = 0$ and hence $\mathcal{F}^{\omega} = 0$. □

3.2. The space $\mathcal{J}_{int}(M, \omega)$

The goal of this subsection is to state the formulas coming from [14] we will use to prove Theorem 1.

Definition 3.2. *We denote by $\mathcal{J}_{int}(M, \omega)$ the space of integrable complex structures on M compatible with ω, that is $J \in \mathcal{J}_{int}(M, \omega)$ is a complex structure such that $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$ and $\omega(\cdot, J\cdot)$ is a Riemannian metric.*

For $J_t \in \mathcal{J}_{int}(M, \omega)$ a smooth path of complex structures at $J := J_0$ and $A := \frac{d}{dt}\big|_0 J_t \in T_J\mathcal{J}_{int}(M, \omega)$. Then, $A \in \Gamma(\mathrm{End}(TM))$ satisfies $AJ + JA = 0$ and the 1-form $(\nabla A)(\cdot)$ with values in $\mathrm{End}(TM)$ satisfies:

$$J(\nabla A)(X)Y - (\nabla A)(JX)Y \text{ is symmetric in } X, Y.$$

Consider the map

$$\mathrm{lc} : \mathcal{J}_{int}(M, \omega) \to \mathcal{E}(M, \omega) : J \mapsto \nabla^J$$

which associates to an integrable complex structure J compatible with ω, the Levi-Civita connection ∇^J of the Kähler metric $g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$.

The map lc is equivariant with respect to the group of symplectic diffeomorphisms of (M, ω). That is: for all $\varphi \in \mathrm{Symp}(M, \omega)$ and $J \in \mathcal{J}_{int}(M, \omega)$ with $\varphi \cdot J := \varphi_* J \varphi_*^{-1}$:

$$\mathrm{lc}(\varphi \cdot J) = \varphi \cdot \mathrm{lc}(J).$$

Proposition 3.3. *Let $A \in T_J\mathcal{J}_{int}(M, \omega)$ and write $B \in T_{\nabla}\mathcal{E}(M, \omega)$ such that $B = \mathrm{lc}_{*J}(A)$. Then B is the unique solution to the equation*

$$B(X)Y + JB(X)JY = -J(\nabla A)(X)Y.$$

and if $JA \in T_J\mathcal{J}_{int}(M, \omega)$, then :

$$\mathrm{lc}_{*J}(JA)(X)Y = JB(JX)JY + \frac{1}{2}\left(J(\nabla A)(JX)Y) + (\nabla A)(X)Y\right).$$

From those equations we obtain [14]:

Lemma 3.4. *If A, A' and $JA, JA' \in T_J\mathcal{J}_{int}(M,\omega)$ then*

$$(lc^*\Omega^{\mathcal{E}})_J(JA, JA') = (lc^*\Omega^{\mathcal{E}})_J(A, A').$$

3.3. Proof of Theorem 1

Consider a smooth map $\phi :]-\epsilon, \epsilon[\to C_0^\infty(M) : t \mapsto \phi(t)$ for some $\epsilon \in \mathbb{R}_0^+$ such that the 2-form $\omega_{\phi(t)} := \omega + dd^c\phi(t)$ is a smooth path in $\mathcal{M}_\Theta$ passing through $\omega = \omega_{\phi(0)}$. To prove the independence of $\mathcal{F}^{\omega_\phi}$, we will show that for all $Z \in \mathfrak{h}$:

$$\left.\frac{d}{dt}\right|_0 \mathcal{F}^{\omega_{\phi(t)}}(Z) = 0.$$

All the forms $\omega_{\phi(t)}$ are symplectomorphic to each other. Indeed, consider the one parameter family of diffeomorphisms f_t integrating the time-dependent vector field $-JX_{\dot{\phi}}^{\omega_{\phi(t)}}$. Then,

$$f_t^*\omega_{\phi(t)} = \omega. \tag{3.2}$$

Consider f_t as in the above equation (3.2). Then, the natural action of f_t^{-1} on J produces a path

$$J_t := f_t^{-1} \cdot J := f_{t*}^{-1} J f_{t*} \in \mathcal{J}_{int}(M,\omega).$$

Define the associated Kähler metric $g_{J_t}(\cdot,\cdot) := \omega(\cdot, J_t\cdot)$ and denote by ∇^{J_t} its Levi-Civita connection. Then, ∇^{J_t} and $\nabla^{\phi(t)}$ are related by the following formula :

$$\nabla^{J_t} = f_t^{-1} \cdot \nabla^{\phi(t)},$$

where $(f_t^{-1} \cdot \nabla^{\phi(t)})_Y Z = f_{t*}^{-1} \nabla^{\phi(t)}_{f_{t*}Y} f_{t*} Z$. Then, their image by the moment map is related by :

$$\mu(\nabla^{J_t}) = f_t^* \mu^{\phi(t)}(\nabla^{\phi(t)}). \tag{3.3}$$

Note that on the LHS the moment map is taken with respect to a fixed symplectic form while on the RHS $\mu^{\phi(t)}$ is a function on $\mathcal{E}(M, \omega_{\phi(t)})$.

Proof of Theorem 1. We will use the notations introduced above. First, using Equations (3.1), (3.2) and (3.3), we have:

$$\mathcal{F}^{\omega_{\phi(t)}}(Z) = \int_M H^{\phi(t)}\mu^{\phi(t)}(\nabla^{\phi(t)})\frac{\omega^n_{\phi(t)}}{n!} = \int_M f_t^*(H^{\phi(t)})\mu(\nabla^{J_t})\frac{\omega^n}{n!}.$$

We will differentiate at $t=0$. Using $\frac{d}{dt}\big|_0 H^{\phi(t)} = Z(\dot\phi(0))$ (see for example [20]) and writing H for $H^{\phi(0)}$, we have:

$$\frac{d}{dt}\bigg|_0 f_t^*(H^{\phi(t)}) = -JX_{\dot\phi}(H) + Z(\dot\phi(0)).$$

Using $-JX_{\dot\phi}(H) = -\omega(X_H, JX_{\dot\phi}) = -JX_H(\dot\phi(0))$, we obtain:

$$\frac{d}{dt}\bigg|_0 f_t^*(H^{\phi(t)}) = X_F(\dot\phi(0)).$$

Now, applying $\frac{d}{dt}\big|_0 \int_M H\mu(\nabla^{J_t})\frac{\omega^n}{n!} = \Omega^{\mathcal{E}}_\nabla(\mathcal{L}_{X_H}\nabla, \frac{d}{dt}\big|_0 \nabla^{J_t})$ by the moment map equation (2.2), we get

$$\frac{d}{dt}\bigg|_0 \mathcal{F}^{\omega_{\phi(t)}}(Z) = \int_M X_F(\dot\phi(0))\mu(\nabla)\frac{\omega^n}{n!} + \Omega^{\mathcal{E}}_\nabla\left(\mathcal{L}_{X_H}\nabla, \frac{d}{dt}\bigg|_0 \nabla^{J_t}\right).$$

Now, the first term of the above equation becomes $\int_M X_F(\dot\phi(0))\mu(\nabla)\frac{\omega^n}{n!} = -\int_M \dot\phi(0)X_F(\mu(\nabla))\frac{\omega^n}{n!}$ and using the equivariance of μ and again the moment map equation (2.2), we get

$$\frac{d}{dt}\bigg|_0 \mathcal{F}^{\omega_{\phi(t)}}(Z) = -\Omega^{\mathcal{E}}_\nabla(\mathcal{L}_{X_{\dot\phi(0)}}\nabla, \mathcal{L}_{X_F}\nabla) + \Omega^{\mathcal{E}}_\nabla\left(\mathcal{L}_{X_H}\nabla, \frac{d}{dt}\bigg|_0 \nabla^{J_t}\right).$$

To finish the proof, we will make use of the map lc. Recall that lc is equivariant, hence

$$\frac{d}{dt}\bigg|_0 \mathcal{F}^{\omega_{\phi(t)}}(Z) = -(\mathrm{lc}^*\Omega^{\mathcal{E}})_J(\mathcal{L}_{X_{\dot\phi(0)}}J, \mathcal{L}_{X_F}J) + (\mathrm{lc}^*\Omega^{\mathcal{E}})_J\left(\mathcal{L}_{X_H}J, \frac{d}{dt}\bigg|_0 J_t\right).$$

Now, we compute $\frac{d}{dt}\big|_0 J_t = -\mathcal{L}_{JX_{\dot\phi(0)}}J$. The vanishing of the Nijenhuis tensor implies $\mathcal{L}_{JV}J = J\mathcal{L}_V J$, for any vector field V, so that

$$\begin{aligned}\frac{d}{dt}\bigg|_0 \mathcal{F}^{\omega_{\phi(t)}}(Z) &= -(\mathrm{lc}^*\Omega^{\mathcal{E}})_J(\mathcal{L}_{X_{\dot\phi(0)}}J, \mathcal{L}_{X_F}J) - (\mathrm{lc}^*\Omega^{\mathcal{E}})_J(\mathcal{L}_{X_H}J, J\mathcal{L}_{X_{\dot\phi(0)}}J),\\ &= -(\mathrm{lc}^*\Omega^{\mathcal{E}})_J(\mathcal{L}_{X_{\dot\phi(0)}}J, \mathcal{L}_{X_F}J) + (\mathrm{lc}^*\Omega^{\mathcal{E}})_J(J\mathcal{L}_{X_H}J, \mathcal{L}_{X_{\dot\phi(0)}}J),\end{aligned}$$

where we use Lemma 3.4 in the last equality.

Finally, $Z \in \mathfrak{h}$ means that $\mathcal{L}_Z J = 0$. As $Z = X_F + JX_H$, then $J\mathcal{L}_{X_H} J = -\mathcal{L}_{X_F} J$ (where we use again $\mathcal{L}_{JV} J = J\mathcal{L}_V J$, for any vector field V). Consequently,

$$\left.\frac{d}{dt}\right|_0 \mathcal{F}^{\omega_{\phi(t)}}(Z) = 0,$$

which implies that $\mathcal{F}^{\omega_\phi}$ does not depend on the choice of $\omega_\phi \in \mathcal{M}_\Theta$.

We finish by showing $\mathcal{F}^\omega$ is a character. It is a consequence of the fact that $\mathcal{F}^\omega$ is an invariant of the Kähler class, indeed, for $Y, Z \in \mathfrak{h}$, one has $[Y, Z] = \left.\frac{d}{dt}\right|_0 \varphi^Y_{-t*} Z$, for φ^Y_{t*} the flow of Y so that

$$\mathcal{F}^\omega([Y, Z]) = \left.\frac{d}{dt}\right|_0 \mathcal{F}^\omega(\varphi^Y_{-t*} Z) = \left.\frac{d}{dt}\right|_0 \mathcal{F}^{\varphi_t^{Y*}\omega}(\varphi^Y_{-t*} Z).$$

Now, when $Z = X^\omega_F + JX^\omega_H$, one computes $\varphi^Y_{-t*} Z = X^{\varphi_t^{Y*}\omega}_{\varphi_t^{Y*}F} + JX^{\varphi_t^{Y*}\omega}_{\varphi_t^{Y*}H}$. Then

$$\left.\frac{d}{dt}\right|_0 \mathcal{F}^{\varphi_t^{Y*}\omega}(\varphi^Y_{-t*} Z) = 0.$$

□

4. Generalised Futaki invariants

4.1. $\mathcal{F}^\omega$ is a generalised Futaki invariant

In [11], Futaki generalised the Futaki invariant obstructing the existence of Kähler-Einstein metrics. One of these so-called generalised Futaki invariants is the invariant we define using the moment map.

Futaki's construction goes as follows. On a Kähler manifold (M, ω, J), consider the holomorphic bundle $T^{(1,0)}M$ of tangent vectors of type $(1, 0)$. Choose any $(1, 0)$-connection $\overline{\nabla}$ on $T^{(1,0)}M$ with curvature $R^{\overline{\nabla}}$. For $Z \in \mathfrak{h}$, define $L(Z^{(1,0)}) := \overline{\nabla}_{Z^{(1,0)}} - \mathcal{L}_{Z^{(1,0)}}$, it is a 0-form with values in $\mathrm{End}(T^{(1,0)}M)$. Let q be a $\mathrm{Gl}(n, \mathbb{C})$-invariant polynomial on $\mathfrak{gl}(n, \mathbb{C})$ of degree p, Futaki defined in [11], the map $\mathfrak{F}_q : \mathfrak{h} \to \mathbb{C}$ by

$$\mathfrak{F}_q(Z) := \int_M (n - p + 1) u_Z q(R^{\overline{\nabla}}) \wedge \omega^{(n-p)} + q(L(Z^{(1,0)}) + R^{\overline{\nabla}}) \wedge \omega^{(n-p+1)},$$

where $u_Z = F + iH \in C^\infty_0(M, \mathbb{C})$ for $Z = X_F + JX_H \in \mathfrak{h}$. Remark that as $L(Z^{(1,0)}) + R^{\overline{\nabla}}$ is a form of mixed degree, the form $q(L(Z^{(1,0)}) + R^{\overline{\nabla}})$ is also of mixed degree but in the second term of $\mathfrak{F}_q$ only the component of degree $p - 1$ will contribute to the integral.

Futaki shows $\mathfrak{F}_q$ depends neither on the choice of the $(1,0)$-connection nor on the choice of the Kähler form in $\mathcal{M}_\Theta$, see [11]. Moreover, if you take $q = c_k$ the polynomials defining the k-th Chern form, it is proved in [11] that one recovers Bando's obstruction [1] to the harmonicity of the kth Chern form:

$$\mathfrak{F}_{c_k}(Z) = (n-k+1)\int_M u_Z c_k(R^\nabla) \wedge \omega^{(n-k)}. \tag{4.1}$$

Proposition 4.1. *We have that $\mathcal{F}^\omega$ is the imaginary part of $\mathfrak{F}_{\frac{8\pi^2}{(n-1)!}(c_2 - \frac{1}{2}c_1 \cdot c_1)}$*

Proof. The key of the computation is that the Pontryagin 4-form defining $P(\nabla)$ satisfies:

$$\mathrm{tr}(R^\nabla \overset{\circ}{\wedge} R^\nabla) = 16\pi^2 \left(c_2 - \frac{1}{2}c_1 \cdot c_1\right)(R^\nabla).$$

Then, for $Z = X_F + JX_H \in \mathfrak{h}$,

$$\begin{aligned}\mathcal{F}^\omega(Z) &= -\frac{1}{2}\int_M H\Delta Scal^\nabla \frac{\omega^n}{n!} + 8\pi^2 \int_M Hc_2(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &\quad -4\pi^2 \int_M Hc_1 \cdot c_1(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!}.\end{aligned}$$

As $u_Z = F + iH$, Equation (4.1) tells us that the imaginary part of $\mathfrak{F}_{\frac{8\pi^2}{(n-1)!}c_2}(Z)$ is:

$$8\pi^2 \int_M Hc_2(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!}.$$

It remains to compute $\mathfrak{F}_{\frac{4\pi^2}{(n-1)!}c_1 c_1}$:

$$\begin{aligned}\mathfrak{F}_{\frac{4\pi^2}{(n-1)!}c_1 \cdot c_1}(Z) &= 4\pi^2 \int_M u_Z c_1 \cdot c_1(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &\quad + 4\pi^2 \int_M c_1 \cdot c_1(L(Z^{(1,0)}) + R^\nabla) \wedge \frac{\omega^{n-1}}{(n-1)!} \\ &= 4\pi^2 \int_M u_Z c_1 \cdot c_1(R^\nabla) \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &\quad + 2i \int_M \mathrm{tr}^{\mathbb{C}}(L(Z^{(1,0)}))\rho^\nabla \wedge \frac{\omega^{n-1}}{(n-1)!},\end{aligned} \tag{4.2}$$

where we used $c_1(\cdot) := \frac{1}{2\pi}\mathrm{tr}^{\mathbb{C}}(\cdot)$ and $c_1(R^\nabla) = \frac{i}{2\pi}\rho^\nabla$ for $\rho^\nabla := \mathrm{Ric}^\nabla(J\cdot,\cdot)$ the Ricci form. Since $\mathrm{tr}^{\mathbb{C}}(L(Z^{(1,0)})) = \frac{-i}{2}(\Delta F + i\Delta H)$, we have:

$$2i\int_M \mathrm{tr}^{\mathbb{C}}(L(Z^{(1,0)}))\rho^\nabla \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2}\int_M (\Delta F + i\Delta H)\, Scal^\nabla \frac{\omega^n}{n!},$$
$$= \frac{1}{2}\int_M (F + iH)\, \Delta Scal^\nabla \frac{\omega^n}{n!}.$$

So, $\mathcal{F}^\omega$ is the imaginary part of $\mathfrak{F}_{\frac{8\pi^2}{(n-1)!}(c_2-\frac{1}{2}c_1\cdot c_1)}$. □

Remark 4.2. From Equation (2.3), we see that for $Z := X_F + JX_H \in \mathfrak{h}$:

$$\mathrm{tr}^{*\nabla}(H) = \frac{\nu^2}{24}\mathcal{F}^\omega(Z) + O(\nu^3).$$

A natural question is: what is hidden behind the higher order terms of $\mathrm{tr}^{*\nabla}(H)$? As the index theorem for deformation quantization [8, 16] shows that $\mathrm{tr}^{*\nabla}(1)$ writes in term of characteristic classes of the manifold, one should check if other generalised Futaki invariants show up in higher order terms of $\mathrm{tr}^{*\nabla}(H)$.

4.2. Example

For $q = \mathrm{Td}_p$, the invariant polynomials defining the p^{th} Todd class, methods are developped to compute $\mathfrak{F}_{\mathrm{Td}_p}$, see [5, 18, 19], in order to study the asymptotic semi-stability [11] of the manifold. Those methods and this notion of asymptotic semi-stability are beyond the scope of this paper. However, when the manifold is Kähler-Einstein, as it is the case in [18], $\mathfrak{F}_{\mathrm{Td}_2}$ determines completely $\mathcal{F}^\omega$.

Observation 4.3. When (M, ω, J) is Kähler-Einstein, $\mathcal{F}^\omega$ is the imaginary part of $\frac{8\pi^2}{(n-1)!}\mathfrak{F}_{\mathrm{Td}_2}$.

Proof. Recall that $\mathrm{Td}_2 = c_2 + c_1 \cdot c_1$. From Equation (4.2), we have for $Z = X_F + JX_H \in \mathfrak{h}$ that $\mathfrak{F}_{\frac{4\pi^2}{(n-1)!}c_1\cdot c_1}(Z)$ equals:

$$-\int_M (F + iH)\rho^\nabla \wedge \rho^\nabla \wedge \frac{\omega^{n-2}}{(n-2)!} + \frac{1}{2}\int_M (F + iH)\, \Delta Scal^\nabla \frac{\omega^n}{n!},$$

where ρ^∇ denotes the Ricci form. Since the manifold is Kähler-Einstein $\rho^\nabla = \lambda\omega$, then $\mathfrak{F}_{c_1\cdot c_1} = 0$. So that, $\frac{8\pi^2}{(n-1)!}F_{\mathrm{Td}_2} = \mathfrak{F}_{\frac{8\pi^2}{(n-1)!}(c_2-\frac{1}{2}c_1\cdot c_1)}$ and its imaginary part is $\mathcal{F}^\omega$ by Proposition 4.1. □

In [17], a 7-dimensional (complex dimension) smooth Kähler manifold (V, ω, J) is constructed, the so-called Nill–Paffenholz example. V is a toric Fano manifold that is Kähler-Einstein, [17]. Moreover, Ono, Sano and Yotsutani [18] showed that, on V, $\mathfrak{F}_{\mathrm{Td}_p} \neq 0$ for $2 \leq p \leq 7$. Combined with the above Observation 4.3, it means $\mathcal{F}^{\omega} \neq 0$. Consequently, Corollary 1.1 implies:

Theorem 4.4. *Let (V, ω, J) be the Nill–Paffenholz example [17] and $\Theta = [\omega]$, then there is no closed Fedosov star products of the form $*_{\widetilde{\nabla}}$ for $\widetilde{\nabla}$ the Levi-Civita connection of some $\widetilde{\omega} \in \mathcal{M}_{\Theta}$.*

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