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A correction to "Tamed symplectic forms and strong Kähler with torsion metrics"

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We thank Thomas Madsen and Andrew Swann for pointing out a gap in the proof of Proposition 3.2 in [2] due to the fact that the terms of the form $g([JX,JY],[Z,W])$ in the expression for dc do not correspond directly to terms in $d\hat{c}$. The gap does not affect Theorem 1.3.

Proof of Theorem 1.3. Let $(M = G/\Gamma, J)$ be a nilmanifold with an invariant complex structure and let $(\mathfrak{g}, J)$ be its Lie algebra. Assume that there exists a Hermitian-symplectic structure Ω on (M, J). Using Lemma 3.2 in [2], we may assume that Ω is left-invariant. Hence, Ω can be regarded as a Hermitian-symplectic structure on $(\mathfrak{g}, J)$. Using Proposition 2.1 in [2], we have that Ω induces an SKT inner product on $(\mathfrak{g}, J)$. Since $\mathfrak{g}$ is nilpotent but non abelian, there exists a non-zero vector in the intersection of the center ξ and the commutator $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$. Since the center ξ is J-invariant, then $J\xi \cap \mathfrak{g}^1 \neq \{0\}$ and Theorem 1.1 in [2] implies the statement. $\square$

The 8-dimensional classification of SKT nilmanifolds in [2] is correct, by the results in [6].

The fact that a nilpotent SKT Lie algebra $\mathfrak{g}$ is 2-step has been used in the following results: [1, Th. 2.3], [3, Th. 1.1], [5, Th. 1.1]. In each case the result can be fixed by introducing the extra assumption that $\mathfrak{g}$ is 2-step.

Theorem 1.2 should be changed in the following

Theorem A. *Let $(M = G/\Gamma, J)$ be a nilmanifold (not a torus) endowed with an invariant complex structure J such that $\mathfrak{g}^1_J = \mathfrak{g}^1 + J\mathfrak{g}^1$ is abelian. Assume that there exists a J-Hermitian SKT metric g on M. Then $\mathfrak{g}$ is 2-step nilpotent and M is a total space of a principal holomorphic torus bundle over a torus.*

Proof of Theorem A. Assume that there exists a J-Hermitian SKT metric g on M. By [4, 8] we can assume that g is invariant and (J, g) can be regarded as an SKT structure on $\mathfrak{g}$. Suppose that $\mathfrak{g}$ is s-step with $s > 2$, then $\mathfrak{g}^s = \{0\}$

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and $\{0\} \neq \mathfrak{g}^{s-1}$ is contained in the center ξ of $\mathfrak{g}$. Therefore, there exist non-zero vector $W \in \mathfrak{g}$ and $Y \in \mathfrak{g}^{s-2}$ such that $[W, Y] \neq 0$. We will show using the SKT condition and the assumption that $\mathfrak{g}_J^{s-2} = \mathfrak{g}^1 + J\mathfrak{g}^{s-2}$ is abelian, that $[W, Y] = 0$ and so we will get a contradiction.

Note that since $Y \in \mathfrak{g}^{s-2}$, we have that $[Y, Z] \in \xi$, for every $Z \in \mathfrak{g}$ and by the Jacobi identity $[Y, \mathfrak{g}^1] = 0$. Moreover, since J preserves the center, $J\mathfrak{g}^{s-1} \subseteq \xi$.

Let c be the torsion 3-form of the Bismut connection of $(\mathfrak{g}, J, g)$. Then, for every $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}^{s-2}$ we obtain

$$\begin{aligned}
0 &= dc(X, Y, JX, JY) \\
&= \|[X, Y]\|^2 + \|[X, JY]\|^2 + \|[Y, JX]\|^2 + \|[JX, JY]\|^2 \\
&\quad - 2g([X, JX], [Y, JY]) - g([J[X, JX], JY], JY) + g([Y, J[X, JX]], Y) \\
&\quad + g([J[X, JY], JY], JX) - g([X, J[X, JY]], Y) + g([J[JX, JY], JX], Y) \\
&\quad + g([JY, J[JX, JY]], X).
\end{aligned}$$

Since, $s > 2$, $Y \in \mathfrak{g}^1$ and by using that $\mathfrak{g}_J^1$ is abelian, we get $[Y, JY] = 0$, and therefore

$$\begin{aligned}
0 = {} & \|[X, Y]\|^2 + \|[X, JY]\|^2 + \|[Y, JX]\|^2 + \|[JX, JY]\|^2 \\
& - g([X, J[X, JY]], Y) + g([J[JX, JY], JX], Y).
\end{aligned}$$

Furthermore, taking into account the integrability of J and that $[\mathfrak{g}, Y] \subseteq \xi$, we have

$$[J[JX, JY], JX] = [J[X, Y] - [JX, Y] - [X, JY], JX] = -[[X, JY], JX]$$

and the Jacobi identity yields

$$-[[X, JY], JX] = [[JX, X], JY] + [[JY, JX], X] = [[JY, JX], X].$$

Using again the integrability of J we have

$$[[JY, JX], X] = [J[JY, X] + J[Y, JX] + [X, Y], X] = [J[JY, X], X].$$

Therefore

$$g([J[JX, JY], JX], Y) = g([J[JY, X], X], Y) = g([X, J[X, JY]], Y)$$

and, consequently, we obtain

$$\|[X, Y]\|^2 + \|[X, JY]\|^2 + \|[Y, JX]\|^2 + \|[JX, JY]\|^2 = 0, \quad \forall X \in \mathfrak{g}.$$

So in particular $[X, Y] = 0$, for every $X \in \mathfrak{g}$. Then we get a contradiction and $\mathfrak{g}$ has to be 2-step. Since the center $Z(G)$ of G is J-invariant and $\Gamma \cap Z(G)$ is a uniform discrete subgroup of $Z(G)$, we have that the surjective homomorphism $\pi : \mathfrak{g} \to \mathfrak{g}/\xi$ induces a holomorphic principal torus bundle $\tilde{\pi} : G/\Gamma \to T^{2p}$ over a $2p$-dimensional torus T^{2p} with $2p = \dim \mathfrak{g} - \dim \xi$. $\square$

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