JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 17, Number 4, 1083–1158, 2019

Real Gromov-Witten theory in all genera and real enumerative geometry: Properties

Penka Georgieva[†] and Aleksey Zinger[‡]

The first part of this work constructs positive-genus real Gromov-Witten invariants of real-orientable symplectic manifolds of odd "complex" dimensions; the present part focuses on their properties that are essential for actually working with these invariants. We determine the compatibility of the orientations on the moduli spaces of real maps constructed in the first part with the standard node-identifying immersion of Gromov-Witten theory. We also compare these orientations with alternative ways of orienting the moduli spaces of real maps that are available in special cases. In a sequel, we use the properties established in this paper to compare real Gromov-Witten and enumerative invariants, to describe equivariant localization data that computes the real Gromov-Witten invariants of odd-dimensional projective spaces, and to establish vanishing results for these invariants in the spirit of Walcher's predictions.

[†]Partially supported by ERC grant STEIN-259118.
[†]Partially supported by NSF grants DMS 0846978 and 1500875.

1. Introduction

The theory of J-holomorphic maps plays prominent roles in symplectic topology, algebraic geometry, and string theory. The foundational work of [6, 15, 19, 21, 25] has established the theory of (closed) Gromov-Witten invariants, i.e. counts of J-holomorphic maps from closed Riemann surfaces to symplectic manifolds. The two main obstacles to defining real Gromov-Witten invariants, i.e. counts of J-holomorphic maps from symmetric Riemann surfaces commuting with the involutions on the domain and the target, are the potential non-orientability of the moduli space of real J-holomorphic maps and the existence of real codimension-one boundary strata. These obstacles are overcome in many genus 0 situations in [2, 5, 9, 26, 28, 29]; see [12, Section 1.3] for some comparisons. In the first part of this work, we introduce the notion of real orientation on a real symplectic $2n$-manifold (X,ω,ϕ) and overcome both obstacles in *all* genera for real-orientable symplectic manifolds of odd "complex" dimension n.

A real orientation on a real symplectic $2n$-manifold (X,ω,ϕ) with $n\notin 2\mathbb{Z}$ induces orientations on the moduli spaces of real J-holomorphic maps from arbitrary genus g symmetric surfaces to (X,ϕ). Theorems 1.3 and 1.4 compare these orientations with the natural complex orientations and with the orientations induced by the corresponding spin and relative spin structures whenever the latter three make sense. By Theorem 1.2, the orientations on the moduli spaces of real J-holomorphic maps induced by a real orientation on (X,ω,ϕ) are "anti-compatible" with the node-identifying immersion (1.4) which is central to much of "classical" Gromov-Witten theory. This theorem is instrumental for any study of the structure of the real Gromov-Witten invariants that depends on a splitting property at a conjugate pair of nodes in the spirit of [18, 2.2.6] and in particular for interpreting real Gromov-Witten theory in terms of integrable systems in the spirit of [3]. A similar comparison of orientations in Lagrangian Floer theory is key to establishing the renown A_∞-relations of [7]. Theorems 1.2, 1.3, and 1.4 are likewise essential for studying the properties of real GW-invariants constructed in [12]. For example, they play crucial roles in determining the normal bundles to the torus-fixed loci in [13] and the contributions from the degenerate loci in [23].

1.1. Real-orientable symplectic manifolds

An involution on a topological space X is a homeomorphism $\phi\colon X\longrightarrow X$ such that $\phi\circ\phi=\mathrm{id}_X$. By an involution on a manifold, we will mean a smooth

involution. Let

$$X^\phi = \{x \in X : \ \phi(x) = x\}$$

denote the fixed locus. An anti-symplectic involution ϕ on a symplectic manifold (X, ω) is an involution $\phi: X \longrightarrow X$ such that $\phi^*\omega = -\omega$. A real symplectic manifold is a triple (X, ω, ϕ) consisting of a symplectic manifold (X, ω) and an anti-symplectic involution ϕ.

Let (X, ϕ) be a topological space with an involution. A conjugation on a complex vector bundle $V \longrightarrow X$ lifting an involution ϕ is a vector bundle homomorphism $\varphi: V \longrightarrow V$ covering ϕ (or equivalently a vector bundle homomorphism $\varphi: V \longrightarrow \phi^*V$ covering id_X) such that the restriction of φ to each fiber is anti-complex linear and $\varphi \circ \varphi = \mathrm{id}_V$. A real bundle pair $(V, \varphi) \longrightarrow (X, \phi)$ consists of a complex vector bundle $V \longrightarrow X$ and a conjugation φ on V lifting ϕ. For example,

$$(X \times \mathbb{C}, \phi \times \mathfrak{c}) \longrightarrow (X, \phi),$$

where $\mathfrak{c}: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is the standard conjugation on $\mathbb{C}^n$, is a real bundle pair. If X is a smooth manifold, then $(TX, \mathrm{d}\phi)$ is also a real bundle pair over (X, ϕ). For any real bundle pair $(V, \varphi) \longrightarrow (X, \phi)$, we denote by

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}(V, \varphi) = (\Lambda^{\mathrm{top}}_{\mathbb{C}} V, \Lambda^{\mathrm{top}}_{\mathbb{C}} \varphi)$$

the top exterior power of V over $\mathbb{C}$ with the induced conjugation. Direct sums, duals, and tensor products over $\mathbb{C}$ of real bundle pairs over (X, ϕ) are again real bundle pairs over (X, ϕ).

Definition 1.1 ([12, Definition 5.1]). Let (X, ϕ) be a topological space with an involution and (V, φ) be a real bundle pair over (X, ϕ). A real orientation on (V, φ) consists of

(RO1) a rank 1 real bundle pair $(L, \widetilde{\phi})$ over (X, ϕ) such that

$$w_2(V^\varphi) = w_1(L^{\widetilde{\phi}})^2 \quad \text{and} \quad \Lambda^{\mathrm{top}}_{\mathbb{C}}(V, \varphi) \approx (L, \widetilde{\phi})^{\otimes 2}, \tag{1.1}$$

(RO2) a homotopy class of isomorphisms of real bundle pairs in (1.1), and

(RO3) a spin structure on the real vector bundle $V^\varphi \oplus 2(L^*)^{\widetilde{\phi^*}}$ over X^ϕ compatible with the orientation induced by (RO2).

An isomorphism in (1.1) restricts to an isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}} V^{\varphi} \approx (L^{\widetilde{\phi}})^{\otimes 2} \tag{1.2}$$

of real line bundles over X^{ϕ}. Since the vector bundles $(L^{\widetilde{\phi}})^{\otimes 2}$ and $2(L^*)^{\widetilde{\phi}^*}$ are canonically oriented, (RO2) determines orientations on the vector bundles V^{φ} and $V^{\varphi} \oplus 2(L^*)^{\widetilde{\phi}^*}$. By the first assumption in (1.1), the real vector bundle $V^{\varphi} \oplus 2(L^*)^{\widetilde{\phi}^*}$ over X^{ϕ} admits a spin structure.

Let (X, ω, ϕ) be a real symplectic manifold. A real orientation on (X, ω, ϕ) is a real orientation on the real bundle pair $(TX, \mathrm{d}\phi)$. We call (X, ω, ϕ) real-orientable if it admits a real orientation.

1.2. Compatibility with node-identifying immersion

A symmetric surface (Σ, σ) is a closed oriented surface Σ (manifold of real dimension 2) with an orientation-reversing involution σ. The fixed locus of σ is a disjoint union of circles. If in addition (X, ϕ) is a manifold with an involution, a real map

$$u \colon (\Sigma, \sigma) \longrightarrow (X, \phi)$$

is a smooth map $u \colon \Sigma \longrightarrow X$ such that $u \circ \sigma = \phi \circ u$. We denote the space of such maps by $\mathfrak{B}_g(X)^{\phi,\sigma}$. The main focus of [12] is on smooth and one-nodal connected symmetric surfaces, but in the present paper we also need to consider disconnected and two-nodal symmetric surfaces. Throughout this paper, the term symmetric surface will thus refer to smooth connected surfaces unless explicitly stated otherwise.

For a symplectic manifold (X, ω), we denote by $\mathcal{J}_{\omega}$ the space of ω-compatible almost complex structures on X. If ϕ is an anti-symplectic involution on (X, ω), let

$$\mathcal{J}^{\phi}_{\omega} = \big\{ J \in \mathcal{J}_{\omega} \colon \phi^* J = -J \big\}.$$

For a genus g symmetric surface (Σ, σ), possibly nodal and disconnected, we similarly denote by $\mathcal{J}^{\sigma}_{\Sigma}$ the space of complex structures $\mathfrak{j}$ on Σ compatible with the orientation such that $\sigma^* \mathfrak{j} = -\mathfrak{j}$. For $J \in \mathcal{J}^{\phi}_{\omega}$, $\mathfrak{j} \in \mathcal{J}^{\sigma}_{\Sigma}$, and $u \in \mathfrak{B}_g(X)^{\phi,\sigma}$, let

$$\bar{\partial}_{J,\mathfrak{j}} u = \frac{1}{2} \big(\mathrm{d}u + J \circ \mathrm{d}u \circ \mathfrak{j} \big)$$

be the $\bar{\partial}_{J,\mathfrak{j}}$-operator on $\mathfrak{B}_g(X)^{\phi,\sigma}$.

Let (X,ω,ϕ) be a real-orientable symplectic $2n$-manifold with $n\notin 2\mathbb{Z}$, $g,l\in\mathbb{Z}^{\geq 0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}_\omega^\phi$. We denote by $\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$ the moduli space of equivalence classes of stable real degree B J-holomorphic maps from genus g symmetric (possibly nodal) surfaces with l pairs of conjugate marked points. By [12, Theorem 1.4], a real orientation on (X,ω,ϕ) determines an orientation on this compact space, endows it with a virtual fundamental class, and thus gives rise to genus g real GW-invariants of (X,ω,ϕ) that are independent of the choice of $J\in\mathcal{J}_\omega^\phi$.

We denote by $\overline{\mathfrak{M}}^\bullet_{g,l}(X,B;J)^\phi$ the moduli space of stable real degree B morphisms from possibly disconnected nodal symmetric surfaces of holomorphic Euler characteristic $1-g$ with l pairs of conjugate marked points. For each $i=1,\ldots,l$, let

$$\mathrm{ev}_i\colon \overline{\mathfrak{M}}^\bullet_{g,l}(X,B;J)^\phi\longrightarrow X,\quad \big[u,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-)\big]\longrightarrow u(z_i^+),$$

be the evaluation at the first point in the i-th pair of conjugate points. If $l\geq 2$, let

$$\overline{\mathfrak{M}}'^\bullet_{g,l}(X,B;J)^\phi=\big\{[\mathbf{u}]\in\overline{\mathfrak{M}}^\bullet_{g,l}(X,B;J)^\phi\colon\ \mathrm{ev}_{l-1}([\mathbf{u}])=\mathrm{ev}_l([\mathbf{u}])\big\}.$$

The short exact sequence

$$0\longrightarrow T\overline{\mathfrak{M}}'^\bullet_{g,l}(X,B;J)^\phi\longrightarrow T\overline{\mathfrak{M}}^\bullet_{g,l}(X,B;J)^\phi|_{\overline{\mathfrak{M}}'^\bullet_{g,l}(X,B;J)^\phi}\longrightarrow \mathrm{ev}^*TX\longrightarrow 0$$

induces an isomorphism

$$\begin{aligned}(1.3)\qquad &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\overline{\mathfrak{M}}^\bullet_{g,l}(X,B;J)^\phi|_{\overline{\mathfrak{M}}'^\bullet_{g,l}(X,B;J)^\phi}\big)\\ &\approx\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\overline{\mathfrak{M}}'^\bullet_{g,l}(X,B;J)^\phi\big)\otimes\mathrm{ev}_l^*\big(\Lambda^{\mathrm{top}}_{\mathbb{R}}(TX)\big)\end{aligned}$$

of real line bundles over $\overline{\mathfrak{M}}'^\bullet_{g,l}(X,B;J)^\phi$.

The identification of the last two pairs of conjugate marked points induces an immersion

$$(1.4)\qquad \iota\colon\overline{\mathfrak{M}}'^\bullet_{g-2,l+2}(X,B;J)^\phi\longrightarrow\overline{\mathfrak{M}}^\bullet_{g,l}(X,B;J)^\phi\,.$$

This immersion takes the main stratum of the domain, i.e. the subspace consisting of real morphisms from smooth symmetric surfaces, to the subspace of the target consisting of real morphisms from symmetric surfaces with one

pair of conjugate nodes. There is a canonical isomorphism

$$\mathcal{N}\iota \equiv \frac{\iota^* T\overline{\mathfrak{M}}^{\bullet}_{g,l}(X,B;J)^{\phi}}{T\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}} \approx \mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$$

of the normal bundle of ι with the tensor product of the universal tangent line bundles for the first points in the last two conjugate pairs. It induces an isomorphism

$$\begin{aligned}(1.5)\qquad &\iota^*\big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\overline{\mathfrak{M}}^{\bullet}_{g,l}(X,B;J)^{\phi}\big)\big)\\ &\approx \Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}\big)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\big)\end{aligned}$$

of real line bundles over $\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}$. Along with (1.3) with (g,l) replaced by $(g-2,l+2)$, it determines an isomorphism

$$\begin{aligned}(1.6)\qquad &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\overline{\mathfrak{M}}^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}|_{\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}}\big)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\big)\\ &\approx \iota^*\big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\overline{\mathfrak{M}}^{\bullet}_{g,l}(X,B;J)^{\phi}\big)\big)\otimes \mathrm{ev}_{l+1}^*\big(\Lambda^{\mathrm{top}}_{\mathbb{R}}(TX)\big)\end{aligned}$$

of real line bundles over $\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}$.

Theorem 1.2. *Let (X,ω,ϕ) be a real-orientable $2n$-manifold with $n\notin 2\mathbb{Z}$, $g,l\in\mathbb{Z}^{\geq 0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}^{\phi}_{\omega}$. The isomorphism (1.6) is orientation-reversing with respect to the orientations on the moduli spaces determined by a real orientation on (X,ω,ϕ) and the canonical orientations on the bundles $\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$ and TX.*

The substance of this statement is that the orientations on

$$\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B,J)^{\phi}$$

induced from the orientations of $\overline{\mathfrak{M}}^{\bullet}_{g-2,l+2}(X,B,J)^{\phi}$ and $\overline{\mathfrak{M}}^{\bullet}_{g,l}(X,B,J)^{\phi}$ via the isomorphisms (1.3) and (1.5) are opposite. This may seem surprising from the point of view of the classical (closed) GW-theory, where moduli spaces have canonical orientations and signs do not appear. On the other hand, systematic orientations of moduli spaces in open and real GW-theories depend on orienting conventions and additional topological data (such as a spin structure or a real orientation) on the target manifold. The appearance of signs is then a fairly common occurrence, and it is notoriously difficult to determine them correctly in most cases. The orientation conventions in [12]

are natural from mathematical considerations and conform with the mirror symmetry expectations described in [27]; see [12, Section 3] and [13, Section 1.3], respectively, for details. While it is possible to adjust the orientations on moduli spaces of real maps to make the isomorphism (1.6) orientation-preserving, this would be artificial from the geometric standpoint and undesirable based on mirror symmetry considerations.

We note that the statement of Theorem 1.2 is invariant under interchanging the points within the last two conjugate pairs simultaneously (this corresponds to reordering the nodes of a nodal map). This interchange reverses the orientation of the last factor on the left-hand side of (1.6), because the complex rank of $\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$ is 1, and the orientation of the last factor on the right-hand side of (1.6), because the complex rank of TX is odd.

An analogue of Theorem 1.2 with $n\in 2\mathbb{Z}$ is described in Remark 4.20.

1.3. Comparison with complex orientation

Let $g_0\in\mathbb{Z}^{\geq 0}$. We define a g_0-doublet to be a two-component smooth symmetric surface (Σ,σ) of the form

$$\Sigma\equiv\Sigma_1\sqcup\Sigma_2\equiv\{1\}\times\Sigma_0\sqcup\{2\}\times\overline{\Sigma}_0,\quad \sigma(i,z)=\big(3-i,z\big)\ \forall\ (i,z)\in\Sigma, \tag{1.7}$$

where Σ_0 is a connected smooth oriented genus g_0 surface and $\overline{\Sigma}_0$ denotes Σ_0 with the opposite orientation. The holomorphic Euler characteristic of a g_0-doublet is $1-g$ with $g=2g_0-1$.

Suppose (X,ω,ϕ) is a real-orientable $2n$-manifold, $l\in\mathbb{Z}^{\geq 0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}_\omega^\phi$. With (Σ,σ) as in (1.7), let

$$\mathfrak{M}^\bullet_{2g_0-1,l}(X,B;J)^{\phi,\sigma}\subset\overline{\mathfrak{M}}^\bullet_{2g_0-1,l}(X,B;J)^\phi$$

denote the open subspace of real J-holomorphic maps from (Σ,σ). For each $\mathfrak{s}\subset\{1,\ldots,l\}$, let

$$\mathfrak{M}^\bullet_{2g_0-1,l}(X,B;J)^{\phi,\sigma}_{\mathfrak{s}}\subset\mathfrak{M}^\bullet_{2g_0-1,l}(X,B;J)^{\phi,\sigma}$$

be the open subspace consisting of marked maps so that the second point in the i-th conjugate pair lies on Σ_1 if and only if $i\in\mathfrak{s}$. In particular,

$$\mathfrak{M}^\bullet_{2g_0-1,l}(X,B;J)^{\phi,\sigma}_{\mathfrak{s}}\subset\bigsqcup_{\substack{B_0\in H_2(X;\mathbb{Z})\\ B_0-\phi_*B_0=B}}\big(\mathfrak{M}_{g_0,l}(X,B_0;J)\times\mathfrak{M}_{g_0,l}(X,-\phi_*B_0;J)\big), \tag{1.8}$$

where $\mathfrak{M}_{g_0,l}(X,B_0;J)$ is the usual moduli space of degree B_0 J-holomorphic maps from smooth genus g_0 curves with l marked points. The projection

$$\mathfrak{M}^\bullet_{2g_0-1,l}(X,B;J)^{\phi,\sigma}_{\mathfrak{s}} \longrightarrow \bigsqcup_{\substack{B_0\in H_2(X;\mathbb{Z})\\ B_0-\phi_*B_0=B}} \mathfrak{M}_{g_0,l}(X,B_0;J) \tag{1.9}$$

to the first factor in (1.8) is an isomorphism (in the sense of Kuranishi structures, i.e. (1.9) identifies the deformation-obstruction theories of the two moduli spaces). The moduli space on the right-hand side of (1.9) carries a natural orientation obtained by homotoping the linearization of the $\bar\partial$-operator to a $\mathbb{C}$-linear Fredholm operator; see [22, Section 3.2]. We will call the orientation on the left-hand side of (1.9) induced by this orientation the complex orientation of $\mathfrak{M}^\bullet_{2g_0-1,l}(X,B;J)^{\phi,\sigma}_{\mathfrak{s}}$.

Theorem 1.3. *Let (X,ω,ϕ) be a real-orientable $2n$-manifold with $n\notin 2\mathbb{Z}$, $g_0,l\in\mathbb{Z}^{\ge 0}$, (Σ,σ) a g_0-doublet, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}^\phi_\omega$. The orientation on $\mathfrak{M}^\bullet_{2g_0-1,l}(X,B;J)^{\phi,\sigma}_{\mathfrak{s}}$ induced by a real orientation on (X,ω,ϕ) and its complex orientation differ by $(-1)^{g_0+1+|\mathfrak{s}|}$.*

Since the orientation on $\mathfrak{M}^\bullet_{g,l}(X,B;J)^\phi$ induced by a real orientation on (X,ω,ϕ) is compatible with orienting the fibers of the forgetful morphisms

$$\overline{\mathfrak{M}}^\bullet_{g,l+1}(X,B;J)^\phi \longrightarrow \overline{\mathfrak{M}}^\bullet_{g,l}(X,B;J)^\phi \tag{1.10}$$

by the first marked point in the last conjugate pair, the statement of this theorem is compatible with the forgetful morphisms. Under the assumptions of this theorem, the "complex" dimension of the right-hand side of (1.9) in the $l=0$ case, i.e.

$$\dim^{\mathrm{vrt}}_{\mathbb{C}}\mathfrak{M}_{g_0,0}(X,B_0;J) = \big\langle c_1(TX),B_0\big\rangle + (n-3)(1-g_0),$$

is even by the second condition in (1.1). Thus, the "conjugation" diffeomorphism

$$\bigsqcup_{\substack{B_0\in H_2(X;\mathbb{Z})\\ B_0-\phi_*B_0=B}} \mathfrak{M}_{g_0,0}(X,B_0;J) \longrightarrow \bigsqcup_{\substack{B_0\in H_2(X;\mathbb{Z})\\ B_0-\phi_*B_0=B}} \mathfrak{M}_{g_0,0}(X,B_0;J), \quad [u,\mathfrak{j}]\longrightarrow \big[\phi\circ u,-\mathfrak{j}\big],$$

is orientation-preserving. This implies that the validity of Theorem 1.3 is independent of the ordering of the topological components of Σ.

An illustration of Theorems 1.2 and 1.3 in the genus 0 case is [11, Lemma 5.2]. It describes the normal bundle to a stratum of genus 0 maps consisting of a central component with a pair of conjugate bubbles, i.e. a 0-doublet, attached. This boundary stratum is oriented by choosing one of the nodes and taking the complex orientation associated with the corresponding bubble. The claim of [11, Lemma 5.2] is that the normal bundle is then oriented by the complex orientation of the smoothings of this node. According to Theorem 1.3, the "canonical" orientation of this boundary stratum is obtained by taking the opposite of the complex orientation on the distinguished bubble. According to Theorem 1.2, the orientation of the normal bundle is then opposite to the complex orientation of the smoothings of the distinguished node. Thus, [11, Lemma 5.2] is a consequence of Theorems 1.2 and 1.3.

1.4. Comparison with spin and relative spin orientations

Let X be a topological space, $Y \subset X$ be a subspace, and $F \longrightarrow Y$ be a real oriented vector bundle. A relative spin structure on F as in [7] consists of a real oriented vector bundle $E \longrightarrow X$ and a spin structure on $F \oplus E|_Y$. If (X, ϕ) is a topological space with an involution and $(L, \widetilde{\phi})$ is a real bundle pair over (X, ϕ), the map

$$2(L^*)^{\widetilde{\phi}^*} \longrightarrow L^*|_{X^\phi}, \quad (v, w) \longrightarrow v + \mathfrak{i}w, \tag{1.11}$$

is an isomorphism of real oriented vector bundles over X^ϕ. Thus, a real orientation on a real bundle pair (V, φ) as in Definition 1.1 determines a relative spin structure on the real oriented vector bundle $V^\varphi \longrightarrow X^\phi$ with $E = L^*$ in the above notation; we will call this structure the associated relative spin structure on V^φ. If in addition $L^{\widetilde{\phi}} \longrightarrow X^\phi$ is orientable, $2(L^*)^{\widetilde{\phi}^*}$ has a canonical homotopy class of trivializations as in the proof of [12, Corollary 5.6]. Such a real orientation on (V, φ) thus determines a spin structure on V^φ; we will call the latter the associated spin structure on V^φ.

Let τ be the standard involution on $\mathbb{P}^1$; we take it to be given by $z \longrightarrow 1/\bar{z}$ on $\mathbb{C}$. For $l \geq 2$, we denote by $\mathcal{M}^\tau_{0,l}$ the uncompactified moduli space of equivalence classes of $(\mathbb{P}^1, \tau)$ with l pairs of conjugate marked points. The Deligne-Mumford compactification $\overline{\mathcal{M}}^\tau_{0,2}$ of $\mathcal{M}^\tau_{0,2}$ includes 3 additional stable real two-component nodal curves. A diffeomorphism of $\overline{\mathcal{M}}^\tau_{0,2}$ with a closed

interval is given by

$$\begin{aligned}(1.12)\qquad &\overline{\mathcal{M}}^{\tau}_{0,2} \longrightarrow \overline{\mathbb{R}}^{+} \equiv [0,\infty],\\ &\big[(z_1^+,z_1^-),(z_2^+,z_2^-)\big] \longrightarrow \frac{z_2^+-z_1^+}{z_2^--z_1^+} : \frac{z_2^+-z_1^-}{z_2^--z_1^-} = \frac{|z_1^+-z_2^+|^2}{|1-z_1^+/z_2^-|^2}.\end{aligned}$$

It takes the two-component curve with z_1^+ and z_2^+ on the same component to 0 and the two-component curve with z_1^+ and z_2^- on the same component to ∞. For $l\ge 2$, the fibers of the forgetful morphism

$$\overline{\mathcal{M}}^{\tau}_{0,l+1} \longrightarrow \overline{\mathcal{M}}^{\tau}_{0,l}$$

are oriented by the canonical complex orientation of the tangent space at the first marked point in the last conjugate pair. It follows that the moduli space $\overline{\mathcal{M}}^{\tau}_{0,l}$ is orientable.

Let (X,ω,ϕ) be a real symplectic manifold. By [12, Theorem 1.3], a real orientation on (X,ω,ϕ) and an orientation on $\overline{\mathcal{M}}^{\tau}_{0,2}$ determine an orientation on each moduli space $\mathfrak{M}_{0,l}(X,B;J)^{\phi,\tau}$ of real J-holomorphic maps from $(\mathbb{P}^1,\tau)$ to (X,ϕ). The standard approach [2, 8, 26] to orienting $\mathfrak{M}_{0,l}(X,B;J)^{\phi,\tau}$ involves orienting the associated moduli space of disk maps from a relative spin structure on $TX^{\phi}\longrightarrow X^{\phi}$; in some cases, the resulting orientation on the disk space descends to an orientation on $\mathfrak{M}_{0,l}(X,B;J)^{\phi,\tau}$. Theorem 1.4 below compares the orientations on $\mathfrak{M}_{0,l}(X,B;J)^{\phi,\tau}$ resulting from the two approaches to orienting it. Both approaches involve some sign conventions, which we specify next.

The construction of the orientation on the real line bundle (2.12) in the proof of [12, Proposition 5.9] involves a somewhat arbitrary sign choice for the Serre duality isomorphism [12, (5.21)]. The (real) dimensions of its domain and target are $3(g-1)+2l$. Thus, this choice has no effect on the homotopy class of this isomorphism or the resulting orientation of the real line bundle (2.12) if $g\notin 2\mathbb{Z}$. If $g\in 2\mathbb{Z}$, changing this choice changes the resulting orientation of (2.12) and the orientation on the moduli spaces $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ of real maps. In light of Proposition 4.18, the above sign choice is determined by a choice of orientation of the real line bundle (2.12) over $\overline{\mathcal{M}}^{\tau}_{0,2}$. In this case, the operator $\bar\partial_{\mathbb{C}}$ is surjective and its kernel consists of constant $\mathbb{R}$-valued functions. Thus, an orientation on (2.12) over $\overline{\mathcal{M}}^{\tau}_{0,2}$ is determined by an orientation on $\overline{\mathcal{M}}^{\tau}_{0,2}$. As in [11, Section 3], we orient $\overline{\mathcal{M}}^{\tau}_{0,2}$ by the diffeomorphism (1.12).

Let G_τ denote the group of holomorphic automorphisms of $(\mathbb{P}^1,\tau)$. The exact sequence

$$0 \longrightarrow T_{\mathrm{id}}S^1 \longrightarrow T_{\mathrm{id}}G_\tau \longrightarrow T_0\mathbb{C} \longrightarrow 0$$

and the standard orientations of S^1 and $\mathbb{C}$ determine an orientation on G_τ. Let $\mathfrak{P}_0(X,B;J)$ denote the space of (parametrized) degree B J-holomorphic real maps from $(\mathbb{P}^1,\tau)$ to (X,ϕ); thus,

$$\mathfrak{M}_{0,0}(X,B;J)^{\phi,\tau} = \mathfrak{P}_0(X,B;J)\big/G_\tau\,. \tag{1.13}$$

An orientation on the left-hand side of (1.13) determines an orientation on $\mathfrak{P}_0(X,B;J)$ via the canonical isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_u\mathfrak{P}_0(X,B;J)\big) \approx \Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[u]}\mathfrak{M}_{0,0}(X,B;J)^{\phi,\tau}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\mathrm{id}}G_\tau\big). \tag{1.14}$$

An orientation on the marked moduli spaces $\mathfrak{M}_{0,l}(X,B;J)^{\tau,\phi}$ is then determined by orienting the fibers of the forgetful morphisms (1.10) by the first marked point in the last conjugate pair. Since G_τ has two topological components, an orientation on $\mathfrak{P}_0(X,B;J)$ may not descend to the quotient (1.14). By [9, Theorem 6.6] with $(E,\widetilde{\tau})=(L,\widetilde{\phi})^*$, a real orientation on (X,ω,ϕ) induces an orientation on $\mathfrak{P}_0(X,B;J)$ that descends to this quotient and extends to the stable map compactification.

The (virtual) tangent space of $\mathfrak{P}_0(X,B;J)$ is the index (as a K-theory class) of the linearization of the $\bar\partial_J$-operator at u. An orientation on this index, or equivalently on $\det D_{(TX,\mathrm{d}\phi)}|_u$, is determined by a relative spin structure on $TX^\phi\longrightarrow X^\phi$; see the proof of [7, Theorem 8.1.1] or [20, Theorem 6.36]. If this orientation descends to the quotient (1.13), the induced orientation on the latter depends on the ordering of the two lines on the right-hand side of (1.14) if

$$\dim^{\mathrm{vrt}}_{\mathbb{R}}\mathfrak{M}_{0,0}(X,B;J)^{\phi,\tau} = \big\langle c_1(TX),B\big\rangle + n-3,$$

is odd. If (X,ω,ϕ) is real-orientable, this is the case if and only if $n\in 2\mathbb{Z}$.

The marked moduli space $\mathfrak{M}_{0,l}(X,B;J)^{\phi,\tau}$ can also be oriented by first orienting the marked parametrized space $\mathfrak{P}_l(X,B;J)$ from the orientation of $\mathfrak{P}_0(X,B;J)$ via the forgetful morphism as in (1.10) and then taking the quotient as in (1.13). If $l\ge 2$, we can then take $(X,B)=(\mathrm{pt},0)$ and obtain

an orientation on

$$\mathcal{M}_{0,2}^{\tau} = \mathfrak{M}_{0,2}(\mathrm{pt},0)^{\mathrm{id},\tau}\,.$$

With the orienting convention (1.14), this orientation agrees with the orientation on $\overline{\mathcal{M}}_{0,2}^{\tau}$ determined by the diffeomorphism (1.12).

Theorem 1.4. *Suppose (X,ω,ϕ) is a real-orientable manifold, $l\in\mathbb{Z}^{\geq 0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}_{\omega}^{\phi}$. The orientations on $\mathfrak{M}_{0,l}(X,B;J)^{\phi,\tau}$ induced by a real orientation on (X,ω,ϕ) as in Definition 1.1 and by the associated relative spin structure on $TX^{\phi}\longrightarrow X^{\phi}$ differ by $(-1)^{\varepsilon(B)}$, where*

$$\varepsilon(B)\equiv\left\lfloor\frac{\langle c_1(X),B\rangle+2}{4}\right\rfloor.$$

If in addition $L^{\widetilde{\phi}}\longrightarrow X^{\phi}$ is orientable, then the orientations on

$$\mathfrak{M}_{0,l}(X,B;J)^{\phi,\tau}$$

induced by the real orientation on (X,ω,ϕ) and by the associated spin structure on TX^{ϕ} are the same.

A key step in the proof of this theorem in Section 3.2 is Proposition 3.5; it obtains an explicit comparison of orientations of determinants of Fredholm operators. This comparison is in the spirit of the undetermined sign of [26, Proposition 8.4]. As indicated in Section 3.3 and illustrated in [13], Proposition 3.5 makes it possible to determine the equivariant weights of vector bundles along torus fixed loci in settings such as in [17, Section 5], [24, Section 4], and [5, Section 6.4]. We in fact give three proofs of Proposition 3.5, a direct computation and as a consequence of the equivariant computations in [5].

Remark 1.5. The approach to orienting the moduli spaces of real maps from $(\mathbb{P}^1,\tau)$ to (X,ϕ) by "stabilizing" the real bundle pair $(TX,\mathrm{d}\phi)$ with two copies of a real bundle pair $(E,\widetilde{\tau})$ over (X,ϕ) is introduced in [9]. For these moduli spaces, the orienting procedure of [12, Theorem 1.3] specializes to the orienting procedure of [9]. While the stabilizing real bundle pair $(E,\widetilde{\tau})$ in [9] can be of any rank, the purpose of $(E,\widetilde{\tau})$ is also fulfilled by $\Lambda_{\mathbb{C}}^{\mathrm{top}}(E,\widetilde{\tau})$ and so it is sufficient to restrict to the rank 1 real bundle pairs. On the other hand, the proof of Theorem 1.4 readily extends to real bundle pairs $(L,\widetilde{\phi})$ of any rank. In sharp contrast to the relative spin orienting procedure of [7, Theorem 8.1.1], the orientation from the approach of [9] with a rank 1 real

bundle $(E,\widetilde{\tau})$ depends only on $w_1(E^{\widetilde{\tau}})$ and the spin structure on $TX^{\phi}\oplus 2E^{\widetilde{\tau}}$, not on $(E,\widetilde{\tau})$ itself; see Remark 3.10.

1.5. Outline of the paper and acknowledgments

Section 2 sets up the notation necessary for the remainder of this paper and summarizes the orientation construction of [12]. Theorems 1.3 and 1.4 are proved in Sections 3.1 and 3.2, respectively. Section 3.3 obtains a number of computationally useful statements concerning orientations of the determinants of real Cauchy-Riemann operators on real bundle pairs. Theorem 1.2 is established in Section 3.

We would like to thank E. Brugallé, R. Crétois, E. Ionel, S. Lisi, M. Liu, J. Solomon, J. Starr, M. Tehrani, G. Tian, and J. Welschinger for related discussions. The second author is very grateful to the IAS School of Mathematics for its hospitality during the initial stages of our project on real GW-theory.

2. Notation and review

We set up the necessary notation involving moduli spaces of stable maps and curves in Section 2.1. We then recall standard facts concerning determinant lines of Fredholm operators in Section 2.2. Section 3 reviews some of the key statements from [12].

2.1. Moduli spaces of symmetric surfaces and real maps

Let (Σ,σ) be a genus g symmetric surface. We denote by $\mathcal{D}_\sigma$ the group of orientation-preserving diffeomorphisms of Σ commuting with the involution σ. If (X,ϕ) is a smooth manifold with an involution, $l\in\mathbb{Z}^{\geq 0}$, and $B\in H_2(X;\mathbb{Z})$, let

$$\mathfrak{B}_{g,l}(X,B)^{\phi,\sigma}\subset\mathfrak{B}_g(X)^{\phi,\sigma}\times\Sigma^{2l}$$

denote the space of real maps $u\colon(\Sigma,\sigma)\longrightarrow(X,\phi)$ with $u_*[\Sigma]_{\mathbb{Z}}=B$ and l pairs of conjugate non-real marked distinct points. We define

$$\mathcal{H}_{g,l}(X,B)^{\phi,\sigma}=\big(\mathfrak{B}_{g,l}(X,B)^{\phi,\sigma}\times\mathcal{J}_\Sigma^\sigma\big)/\mathcal{D}_\sigma.$$

If $J \in \mathcal{J}_\omega^\phi$, the moduli space of marked real J-holomorphic maps in the class $B \in H_2(X;\mathbb{Z})$ is the subspace

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} = \big\{[u,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-),\mathfrak{j}] \in \mathcal{H}_{g,l}(X,B)^{\phi,\sigma} :\ \bar\partial_{J,\mathfrak{j}} u = 0\big\},$$

where $\bar\partial_{J,\mathfrak{j}}$ is the usual Cauchy-Riemann operator with respect to the complex structures J on X and $\mathfrak{j}$ on Σ. If $g+l \ge 2$,

$$\mathcal{M}_{g,l}^\sigma \equiv \mathfrak{M}_{g,l}(\mathrm{pt},0)^{\mathrm{id},\sigma} \equiv \mathcal{H}_{g,l}(\mathrm{pt},0)^{\mathrm{id},\sigma}$$

is the moduli space of marked symmetric domains. There is a natural forgetful morphism

$$\mathfrak{f} : \mathcal{H}_{g,l}(X,B)^{\phi,\sigma} \longrightarrow \mathcal{M}_{g,l}^\sigma ; \tag{2.1}$$

it drops the map component u from each element of the domain.

We denote by

$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma} \supset \mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$$

Gromov's convergence compactification of $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ obtained by including stable real maps from nodal symmetric surfaces. The (virtually) codimension-one boundary strata of

$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma} - \mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} \subset \overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$$

consist of real J-holomorphic maps from one-nodal symmetric surfaces to (X,ϕ). Each stratum is either a (virtual) hypersurface in $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ or a (virtual) boundary of the spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ for precisely two topological types of orientation-reversing involutions σ on Σ. Let

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi} = \bigsqcup_\sigma \mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} \quad \text{and}$$
$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi} = \bigcup_\sigma \overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$$

denote the (disjoint) union of the uncompactified real moduli spaces and the union of the compactified real moduli spaces, respectively, taken over all topological types of orientation-reversing involutions σ on Σ. If $g+l \ge 2$, we

denote by

$$\overline{\mathcal{M}}^{\sigma}_{g,l} \equiv \overline{\mathfrak{M}}_{g,l}(\mathrm{pt},0)^{\mathrm{id},\sigma} \supset \mathcal{M}^{\sigma}_{g,l}\,, \quad \mathbb{R}\overline{\mathcal{M}}_{g,l} \equiv \overline{\mathfrak{M}}_{g,l}(\mathrm{pt},0)^{\mathrm{id}} = \bigcup_{\sigma} \overline{\mathcal{M}}^{\sigma}_{g,l}$$

the real Deligne-Mumford moduli spaces. The forgetful morphism (2.1) extends to a morphism

$$\mathfrak{f}\colon \overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi} \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,l} \tag{2.2}$$

between the compactifications.

2.2. Determinant line bundles

Let (V,φ) be a real bundle pair over a symmetric surface (Σ,σ). A real Cauchy-Riemann (or CR-) operator on (V,φ) is a linear map of the form

$$\begin{aligned} D = \bar\partial + A\colon \Gamma(\Sigma;V)^{\varphi} &\equiv \big\{\xi\in\Gamma(\Sigma;V)\colon\ \xi\circ\sigma = \varphi\circ\xi\big\} \\ &\longrightarrow \Gamma^{0,1}_{\mathfrak{j}}(\Sigma;V)^{\varphi} \equiv \big\{\zeta\in\Gamma(\Sigma;(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}V)\colon\ \zeta\circ\mathrm{d}\sigma = \varphi\circ\zeta\big\}, \end{aligned} \tag{2.3}$$

where $\bar\partial$ is the holomorphic $\bar\partial$-operator for some $\mathfrak{j}\in\mathcal{J}^{\sigma}_{\Sigma}$ and a holomorphic structure in V and

$$A \in \Gamma\big(\Sigma;\mathrm{Hom}_{\mathbb{R}}(V,(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}V)\big)^{\varphi}$$

is a zeroth-order deformation term. A real CR-operator on a real bundle pair is Fredholm in the appropriate completions.

If X,Y are Banach spaces and $D\colon X\longrightarrow Y$ is a Fredholm operator, let

$$\det D \equiv \Lambda^{\mathrm{top}}_{\mathbb{R}}(\ker D)\otimes\big(\Lambda^{\mathrm{top}}_{\mathbb{R}}(\mathrm{cok}\, D)\big)^{*}$$

denote the determinant line of D. A continuous family of such Fredholm operators D_t over a topological space $\mathcal{H}$ determines a line bundle over $\mathcal{H}$, called the determinant line bundle of $\{D_t\}$ and denoted $\det D$; see [22, Section A.2] and [30] for a construction. A short exact sequence of Fredholm operators

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle D'} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D''} & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0 \end{array}$$

determines a canonical isomorphism

$$\det D \cong (\det D') \otimes (\det D''). \tag{2.4}$$

For a continuous family of short exact sequences of Fredholm operators, the isomorphisms (2.4) give rise to a canonical isomorphism between determinant line bundles.

Families of real CR-operators often arise by pulling back data from a target manifold by smooth maps as follows. Suppose (X, J, ϕ) is an almost complex manifold with an anti-complex involution and (V, φ) is a real bundle pair over (X, ϕ). Let ∇ be a φ-compatible connection in V and

$$A \in \Gamma\big(X; \mathrm{Hom}_{\mathbb{R}}(V, (T^*X, J)^{0,1} \otimes_{\mathbb{C}} V)\big)^{\varphi}.$$

For any real map $u\colon (\Sigma, \sigma) \longrightarrow (X, \phi)$ and $\mathfrak{j} \in \mathcal{J}_{\Sigma}^{\sigma}$, let ∇^u denote the induced connection in u^*V and

$$A_{\mathfrak{j};u} = A \circ \partial_{\mathfrak{j}} u \in \Gamma(\Sigma; \mathrm{Hom}_{\mathbb{R}}(u^*V, (T^*\Sigma, \mathfrak{j})^{0,1} \otimes_{\mathbb{C}} u^*V))^{u^*\varphi}.$$

The homomorphisms

$$\begin{aligned} \bar\partial_u^{\nabla} &= \frac{1}{2}(\nabla^u + \mathfrak{i} \circ \nabla^u \circ \mathfrak{j}), \\ D_{(V,\varphi);u} \equiv \bar\partial_u^{\nabla} + A_{\mathfrak{j};u}\colon \Gamma(\Sigma; u^*V)^{u^*\varphi} &\longrightarrow \Gamma_{\mathfrak{j}}^{0,1}(\Sigma; u^*V)^{u^*\varphi} \end{aligned}$$

are real CR-operators on $u^*(V, \varphi) \longrightarrow (\Sigma, \sigma)$ that form families of real CR-operators over families of maps. If $g, l \in \mathbb{Z}^{\geq 0}$ and $B \in H_2(X; \mathbb{Z})$, let

$$\det D_{(V,\varphi)} \longrightarrow \mathfrak{B}_{g,l}(X, B)^{\phi,\sigma} \times \mathcal{J}_{\Sigma}^{\sigma}$$

denote the determinant line bundle of such a family. It descends to a fibration

$$\det D_{(V,\varphi)} \longrightarrow \mathcal{H}_{g,l}(X, B)^{\phi,\sigma},$$

which is a line bundle over the open subspace of the base consisting of marked maps with no non-trivial automorphisms.

Example 2.1. Let $(V, \varphi) = (\mathbb{C}, \mathfrak{c})$; this is a real bundle pair over $(\mathrm{pt}, \mathrm{id})$. If $g + l \geq 2$, the induced family of operators $\bar\partial_{\mathbb{C}} \equiv D_{(\mathbb{C},\mathfrak{c})}$ on $\mathcal{M}_{g,l}^{\sigma}$ defines a line

bundle

$$\det \bar\partial_{\mathbb{C}} \longrightarrow \mathcal{M}^{\sigma}_{g,l}\,.$$

If (X,ϕ) is an almost complex manifold with an anti-complex involution ϕ and

$$(V,\varphi) = (X\times\mathbb{C}, \phi\times\mathfrak{c}) \longrightarrow (X,\phi),$$

then there is a canonical isomorphism

$$\det D_{(\mathbb{C},\mathfrak{c})} \approx \mathfrak{f}^*\big(\det\bar\partial_{\mathbb{C}}\big)$$

of line bundles over $\mathcal{H}_{g,l}(X,B)^{\phi,\sigma}$.

For a real CR-operator D on a rank n real bundle pair (V,φ) over a symmetric surface (Σ,σ), we define the relative determinant of D to be the tensor product

$$\widehat{\det}\, D \equiv \big(\det D\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes n}\,, \tag{2.5}$$

where $\det\bar\partial_{\Sigma;\mathbb{C}}$ is the standard real CR-operator on (Σ,σ) with values in $(\mathbb{C},\mathfrak{c})$. This notion plays a central role in the construction of real GW-theory in [12].

Let (X,ω,ϕ) be a real symplectic $2n$-manifold, $g,l\in\mathbb{Z}^{\ge 0}$, $B\in H_2(X;\mathbb{Z})$, $J\in\mathcal{J}^{\phi}_{\omega}$, and

$$[\mathbf{u}] \equiv \big[u,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-),\mathfrak{j}\big] \in \overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}\,.$$

Denote by Σ_u the domain of u. If

$$\mathcal{C} \equiv \big(\Sigma_u,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-),\mathfrak{j}\big)$$

is a stable curve, then the forgetful morphism (2.2) induces an isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\mathbf{u}]}\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}\big) \approx \big(\det D_{(TX,\mathrm{d}\phi);u}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\mathcal{C}]}\overline{\mathcal{M}}^{\sigma}_{g,l}\big)\,. \tag{2.6}$$

Orientations on the two lines on the right-hand side of (2.6) thus determine an orientation on the left-hand side of (2.6). If (X,ω,ϕ) is real-orientable and n is odd, as in the cases relevant to the present paper, the index of $D_{(TX,\mathrm{d}\phi);u}$ is odd if and only if $g\in 2\mathbb{Z}$. The induced orientation on the left-hand side of (2.6) then depends on the specified order of the factors on the right-hand side of (2.6).

2.3. Real orientations and relative determinants

Let (X,ϕ) be a topological space with an involution and (V,φ) be a real bundle pair over (X,ϕ). An isomorphism Θ in (1.1) determines orientations on V^{φ} and $V^{\varphi}\oplus 2(L^*)^{\widetilde{\phi}^*}$. Given a real orientation on (V,φ) as in Definition 1.1, we will call these orientations the orientations determined by (RO2) if Θ lies in the chosen homotopy class. An isomorphism Θ in (1.1) also induces an isomorphism

$$\begin{aligned}\Lambda^{\mathrm{top}}_{\mathbb{C}}\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big) &\approx \Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\varphi)\otimes(L^*,\widetilde{\phi}^*)^{\otimes 2}\\ &\approx (L,\widetilde{\phi})^{\otimes 2}\otimes(L^*,\widetilde{\phi}^*)^{\otimes 2}\approx\big(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c}\big),\end{aligned}\tag{2.7}$$

where the last isomorphism is the canonical pairing. We will call the homotopy class of isomorphisms (2.7) induced by the isomorphisms Θ in (RO2) the homotopy class determined by (RO2).

Proposition 2.2 ([14, Proposition 7.3]). *Suppose (Σ,σ) is a symmetric surface, possibly disconnected and nodal, and (V,φ) is a rank n real bundle pair over (Σ,σ). A real orientation on (V,φ) as in Definition 1.1 determines a homotopy class of isomorphisms*

$$\Psi\colon\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big)\approx\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big)\tag{2.8}$$

of real bundle pairs over (Σ,σ). An isomorphism Ψ belongs to this homotopy class if and only if the restriction of Ψ to the real locus induces the chosen spin structure (RO3) and the isomorphism

$$\begin{aligned}\Lambda^{\mathrm{top}}_{\mathbb{C}}\Psi\colon\Lambda^{\mathrm{top}}_{\mathbb{C}}\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big)&\longrightarrow\Lambda^{\mathrm{top}}_{\mathbb{C}}\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big)\\ &=\big(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c}\big)\end{aligned}\tag{2.9}$$

lies in the homotopy class determined by (RO2).

The only cases of this proposition relevant to [12] are for Σ smooth and with one real node; the only cases relevant to Theorem 1.2 are for Σ smooth and with one pair of conjugate nodes. The proof of [12, Proposition 5.2] establishes Proposition 2.2 under the assumption that Σ is connected and smooth, but it applies without the first restriction. The proof of [12, Proposition 6.2] extends [12, Proposition 5.2] to one-nodal symmetric surfaces. The analogue of this extension for symmetric surfaces with one pair of conjugate nodes is carried out in the proof of Lemma 4.4 of the present paper. The

principles behind the two extensions are used in [14] to establish the full statement of Proposition 2.2.

Corollary 2.3. *Suppose (Σ,σ) is a symmetric surface, possibly disconnected and nodal, and D is a real CR-operator on a rank n real bundle pair (V,φ) over (Σ,σ). Then a real orientation on (V,φ) as in Definition 1.1 induces an orientation on the relative determinant $\widehat{\det}\,D$ of D.*

For Σ smooth or one-nodal, this corollary is deduced from the corresponding cases of Proposition 2.2 in the proofs of [12, Corollary 5.7] and [12, Corollary 6.6], respectively. The proof of the latter readily extends to all symmetric surfaces (Σ,σ).

Corollary 2.3 implies that a real orientation on a real symplectic manifold (X,ω,ϕ) determines an orientation on the line

$$\widehat{\det}\,D_{(TX,\mathrm{d}\phi);u} \equiv \big(\det D_{(TX,\mathrm{d}\phi);u}\big)\otimes\big(\det\bar\partial_{\mathbb{C}|\Sigma_u}\big)^{\otimes n}. \tag{2.10}$$

By [12, Corollary 6.7] and Corollary 4.6, this orientation varies continuously with $[\mathbf{u}]$.

Corollary 2.4. *Suppose (Σ,σ) is a symmetric surface, possibly disconnected and nodal, and $(L,\widetilde\phi)\longrightarrow(\Sigma,\sigma)$ is a rank 1 real bundle pair. If the line bundle $L^{\widetilde\phi}\longrightarrow\Sigma^\sigma$ is orientable, there exists a canonical homotopy class of isomorphisms*

$$\big(L^{\otimes2}\oplus2L^*,\widetilde\phi^{\otimes2}\oplus2\widetilde{\phi^*}\big)\approx\big(\Sigma\times\mathbb{C}^3,\sigma\times\mathfrak{c}\big) \tag{2.11}$$

of real bundle pairs over (Σ,σ).

As explained in the proof of [12, Corollary 5.6], there is a canonical real orientation on the real bundle $(L,\widetilde\phi)^{\otimes2}$ over (Σ,σ) if $L^{\widetilde\phi}\longrightarrow\Sigma^\sigma$ is orientable. In particular, there is a canonical homotopy class of isomorphisms

$$\big(T^*\Sigma^{\otimes2}\oplus2\,T\Sigma,(\mathrm{d}\sigma^*)^{\otimes2}\oplus2\mathrm{d}\sigma\big)\approx\big(\Sigma\times\mathbb{C}^3,\sigma\times\mathfrak{c}\big)$$

of real bundle pairs over (Σ,σ) if Σ contains no real nodes (of type (H2) or (H3) in the terminology of [20, Section 3] and [12, Section 3.2]).

Let $g,l\in\mathbb{Z}^{\geq0}$ be such that $g+l\geq2$ and (Σ,σ) be a smooth connected symmetric surface of genus g. Combining the Kodaira-Spencer isomorphism, Dolbeault Isomorphism, Serre Duality, and Corollaries 2.3 and 2.4, we find

that the real line bundle

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathcal{M}^{\sigma}_{g,l}\big)\otimes\big(\det\bar\partial_{\mathbb{C}}\big)\longrightarrow\mathcal{M}^{\sigma}_{g,l} \tag{2.12}$$

is canonically oriented; see the proof of [12, Proposition 5.9]. If $n\notin 2\mathbb{Z}$ and the domain Σ_u of $\mathbf{u}$ in (2.6) is smooth, the canonical orientation on (2.12) and an orientation on (2.10) determine an orientation on the line (2.6) which varies continuously with $\mathbf{u}$. Thus, a real orientation on a real symplectic manifold (X,ω,ϕ) determines orientations on the uncompactified moduli spaces $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ of real J-holomorphic maps from (Σ,σ) to (X,ϕ).

By [12, Proposition 6.1], the canonical orientations of the real line bundle (2.12) extend across a codimension-one boundary stratum of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ if and only if the parity of the number $|\sigma|_0$ of connected components of the fixed locus Σ^{σ} of Σ remains unchanged. By construction, the same is the case of the orientations on $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ induced by a real orientation on (X,ω,ϕ) if $n\notin 2\mathbb{Z}$. In order to orient the compactified moduli spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$, we multiply the orientation on $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ induced by a real orientation on (X,ω,ϕ) by $(-1)^{g+|\sigma|_0+1}$. This does not change the orientations whenever the fixed locus Σ^{σ} of Σ is separating.

3. Comparison of orientations

There are now standard ways of imposing orientations on the moduli spaces $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ for certain types of symmetric surfaces (Σ,σ). Theorems 1.3 and 1.4 compare such orientations with the orientations constructed in [12] and briefly described in Section 2.3.

3.1. Canonical vs. complex

We continue with the notation and setup of Section 1.3. In the setting of Theorem 1.3, each of the factors in (2.10) and (2.12) has a natural complex orientation. By Lemma 3.1 below, the orientations of the tensor product in (2.10) induced by a real orientation on (X,ω,ϕ) and by the complex orientations on the two factors are the same. By Lemma 3.2, the canonical orientation of the tensor product in (2.12) and the orientation induced by the complex orientations on the two factors differ by $(-1)^{g_0+1+|\mathfrak{s}|}$.

The exponent of g_0+1 above arises for the following reason. Let V be a complex vector space of dimension k. The map

$$\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})\longrightarrow\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}),\quad\theta\longrightarrow\mathrm{Re}\,\theta, \tag{3.1}$$

is then an isomorphism of real vector spaces. Its domain is a complex vector space and thus has a canonical complex orientation; its image has an orientation induced from the complex orientation of V. The isomorphism (3.1) is orientation-preserving with respect to these orientations if and only if k is even. The only step in the proof of [12, Proposition 5.9] not compatible with the natural complex orientations is taking the (real) dual in [12, (5.21)]. The sign discrepancy of (3.1) for the twists by the marked points is taken into account earlier in the proof. The "remaining" vector space in [12, (5.21)] has complex dimension $3g_0-3$ and accounts for the exponent of g_0+1 in the sign of Theorem 1.3.

A rank n real bundle pair (V,φ) over a doublet (Σ,σ) as in (1.7) corresponds to a complex vector bundle $V_0\longrightarrow\Sigma_0$ with

$$V=V_1\sqcup V_2\equiv\{1\}\times V_0\sqcup\{2\}\times\overline{V}_0,\quad \varphi(i,z)=\big(3-i,v\big)\ \forall\ (i,v)\in V,$$

where $\overline{V}_0$ denotes V_0 with the opposite complex structure. With these identifications,

$$\begin{aligned}\Gamma(\Sigma;V)^{\varphi}&\subset\Gamma(\Sigma_1;V_1)\oplus\Gamma(\Sigma_2;V_2),\\ \Gamma^{0,1}_{\mathfrak{j}}(\Sigma;V)^{\varphi}&\subset\Gamma^{0,1}_{\mathfrak{j}}(\Sigma_1;V_1)\oplus\Gamma^{0,1}_{-\mathfrak{j}}(\Sigma_2;V_2),\end{aligned}$$

and the projections

$$\Gamma(\Sigma;V)^{\varphi}\longrightarrow\Gamma(\Sigma_0;V_0)\quad\text{and}\quad\Gamma^{0,1}_{\mathfrak{j}}(\Sigma;V)^{\varphi}\subset\Gamma^{0,1}_{\mathfrak{j}}(\Sigma_0;V_0)\tag{3.2}$$

to the first component are isomorphisms of real vector spaces. Via these projections, every real CR-operator D on the real bundle pair (V,φ) corresponds to an operator

$$D_0\colon\Gamma(\Sigma_0;V_0)\longrightarrow\Gamma^{0,1}_{\mathfrak{j}}(\Sigma_0;V_0)\,.$$

The projections (3.2) induce isomorphisms between the kernels and cokernels of D and D_0 and thus an isomorphism

$$\det D\approx\det D_0\,.\tag{3.3}$$

Since D_0 is a real linear CR-operator on V_0 in the sense of [22, Definition C.1.5], $\det D_0$ has a canonical "complex" orientation obtained by homotoping D_0 to a $\mathbb{C}$-linear Fredholm operator; see [22, Section 3.2]. We will call the orientation on $\det D$ induced from this orientation via the isomorphism (3.3) the complex orientation of $\det D$.

Lemma 3.1. *Let (Σ,σ), (V,φ), and D be as above. The orientations of the relative determinant $\widehat{\det}\, D$ of D induced by a real orientation on (V,φ) as in Corollary 2.3 and by the complex orientations on the two factors are the same.*

Proof. The homotopy class of isomorphisms as in (2.8) determined by a real orientation on (V,φ) determines an orientation on the line

$$\begin{aligned}(3.4)\qquad &\big(\det D_{(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*)}\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes(n+2)}\\ &\approx\big(\det\big(D_{(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*)}\big)_0\big)\otimes\big(\det\big(\bar\partial_{\Sigma;\mathbb{C}}\big)_0\big)^{\otimes(n+2)}.\end{aligned}$$

Any isomorphism Ψ in (2.8) corresponds to an isomorphism

$$\Psi_0\colon V_0\oplus 2L_0^*\longrightarrow\Sigma_0\times\mathbb{C}^{n+2}$$

of complex vector bundles over Σ_0 by the restriction to $\Sigma_1\subset\Sigma$. The isomorphism in (3.4) is orientation-preserving with respect to the orientation on the left-hand side induced by Ψ and the orientation on the right-hand side induced by Ψ_0. Since Ψ_0 is a $\mathbb{C}$-linear isomorphism, the operator on $\Sigma_0\times\mathbb{C}^{n+2}$ induced by $(D_{(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*)})_0$ via Ψ_0 is a real linear CR-operator. Since any two such operators are homotopic, the orientation on the last factor in (3.4) induced from the complex orientation of the third factor in (3.4) is the complex orientation. Thus, the orientation on the left-hand side of (3.4) induced by a real orientation on (V,φ) is the orientation induced by the complex orientations on the two factors.

By (2.4), there are horizontal canonical isomorphisms

(3.5)

$$\begin{array}{ccc}\det D_{(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*)} & \xrightarrow[\approx]{(2.4)} & \big(\det D_{(V,\varphi)}\big)\otimes\big(\det D_{(L^*,\widetilde{\phi}^*)}\big)^{\otimes 2}\\ {\scriptstyle(3.3)}\big\downarrow\approx & & {\scriptstyle(3.3)}\big\downarrow\approx\\ \det\big(D_{(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*)}\big)_0 & \xrightarrow[\approx]{(2.4)} & \big(\det\big(D_{(V,\varphi)}\big)_0\big)\otimes\big(\det\big(D_{(L^*,\widetilde{\phi}^*)}\big)_0\big)^{\otimes 2}\end{array}$$

making the diagram commute. Thus, the top isomorphism in (3.5) is orientation-preserving with respect to the complex orientations on the three determinants. The orientation of $\widehat{\det}\, D$ induced by a real orientation on (V,φ) as in Corollary 2.3 is obtained by combining

(1) the orientation on LHS of (3.4) induced by,

(2) the top isomorphism in (3.5), and

(3) the canonical orientations of $(\det D_{(L^*,\widetilde{\phi}^*)})^{\otimes 2}$ and $(\det \bar\partial_{\Sigma;\mathbb{C}})^{\otimes 2}$.

By the last sentence of the previous paragraph and the sentence after (3.5), this is the orientation induced by the complex orientations on the two factors. □

For $g\in\mathbb{Z}$ and $l\in\mathbb{Z}^{\ge 0}$ with $g+l\ge 2$, we denote by

$$\mathbb{R}\overline{\mathcal{M}}^\bullet_{g,l}\supset\mathbb{R}\mathcal{M}^\bullet_{g,l}$$

the Deligne-Mumford moduli space of possibly disconnected stable nodal symmetric surfaces of Euler characteristic $2(1-g)$ with l pairs of conjugate marked points and its subspace consisting of smooth curves. If $g_0,l\in\mathbb{Z}^{\ge 0}$ with $2g_0+l\ge 3$ and (Σ,σ) is a g_0-doublet as in (1.7), let

$$\mathcal{M}^\sigma_{2g_0-1,l}=\mathfrak{M}^\bullet_{2g_0-1,l}(\mathrm{pt},0)^{\mathrm{id},\sigma}\subset\mathbb{R}\mathcal{M}^\bullet_{2g_0-1,l}\,.$$

For each $\mathfrak{s}\subset\{1,\ldots,l\}$, let

$$\mathcal{M}^\sigma_{2g_0-1,l;\mathfrak{s}}=\mathfrak{M}^\bullet_{2g_0-1,l}(\mathrm{pt},0)^{\mathrm{id},\sigma}_{\mathfrak{s}}\subset\mathcal{M}^\sigma_{2g_0-1,l}$$

be the open subspace consisting of marked curves so that the second point in the i-th conjugate pair lies on Σ_1 if and only if $i\in\mathfrak{s}$. In particular,

$$\mathcal{M}^\sigma_{2g_0-1,l;\mathfrak{s}}\subset\mathcal{M}_{g_0,l}\times\mathcal{M}_{g_0,l}\,, \tag{3.6}$$

where $\mathcal{M}_{g_0,l}$ is the usual Deligne-Mumford moduli space of smooth genus g_0 curves with l marked points. The projection

$$\mathcal{M}^\sigma_{2g_0-1,l;\mathfrak{s}}\longrightarrow\mathcal{M}_{g_0,l} \tag{3.7}$$

to the first factor in (3.6) is a diffeomorphism. The moduli space on the right-hand side of (3.7) carries a natural complex orientation. We will call the orientation on the left-hand side of (3.7) induced by this orientation the complex orientation of $\mathcal{M}^\sigma_{2g_0-1,l;\mathfrak{s}}$.

Lemma 3.2. *Let $g_0,l\in\mathbb{Z}^{\ge 0}$ with $2g_0+l\ge 3$ and (Σ,σ) be a g_0-doublet. The canonical orientation on the real line bundle*

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathcal{M}^\sigma_{2g_0-1,l;\mathfrak{s}}\big)\otimes\big(\det\bar\partial_{\mathbb{C}}\big)\longrightarrow\mathcal{M}^\sigma_{2g_0-1,l;\mathfrak{s}} \tag{3.8}$$

constructed as in the proof of [12, Proposition 5.9] and the orientation induced by the complex orientations of the factors differ by $(-1)^{g_0+1+|\mathfrak{s}|}$.

Proof. Since the interchange of the points within a conjugate pair reverses the canonical orientation of (3.8), it is sufficient to establish the claim for $\mathfrak{s}=\emptyset$. Let

$$\begin{aligned}[\mathcal{C}_0] &= \big[\Sigma_0, z_1^+, \ldots, z_l^+, \mathfrak{j}\big] \in \mathcal{M}_{g_0,l} \quad \text{and}\\ [\mathcal{C}] &= \big[\Sigma, (z_1^+, z_1^-), \ldots, (z_l^+, z_l^-), \mathfrak{j}\sqcup(-\mathfrak{j})\big] \in \mathcal{M}^\sigma_{2g_0-1,l}.\end{aligned}$$

Similarly to the proof of [12, Proposition 5.9], we define

$$\begin{aligned}T\mathcal{C}_0 &= T\Sigma_0\big(-z_1^+-\cdots-z_l^+\big),\\ T^*\mathcal{C}_0 &= T^*\Sigma_0\big(z_1^++\cdots+z_l^+\big),\\ T\mathcal{C} &= T\Sigma\big(-z_1^+-z_1^--\cdots-z_l^+-z_l^-\big),\\ T^*\mathcal{C} &= T^*\Sigma\big(z_1^++z_1^-+\cdots+z_l^++z_l^-\big).\end{aligned}$$

Denote by $S\mathcal{C}_0$ the skyscraper sheaf over Σ_0 and by $S\mathcal{C}^+$, $S\mathcal{C}^-$, and $S\mathcal{C}$ the skyscraper sheaves over Σ given by

$$\begin{aligned}S\mathcal{C}_0 &= T^*\Sigma_0|_{z_1^++\cdots+z_l^+}, \quad S\mathcal{C}^+ = T^*\Sigma|_{z_1^++\cdots+z_l^+},\\ S\mathcal{C}^- &= T^*\Sigma|_{z_1^-+\cdots+z_l^-}, \quad S\mathcal{C} = S\mathcal{C}^+\oplus S\mathcal{C}^-.\end{aligned}$$

The projection

$$\begin{aligned}\pi_1\colon H^0(\Sigma;S\mathcal{C})^\sigma &= \big(H^0(\Sigma;S\mathcal{C}^+)\oplus H^0(\Sigma;S\mathcal{C}^-)\big)^\sigma\\ &\longrightarrow H^0(\Sigma;S\mathcal{C}^+) = H^0(\Sigma_0;S\mathcal{C}_0)\end{aligned} \tag{3.9}$$

is an isomorphism of real vector spaces. In the proof of [12, Proposition 5.9], we orient the domain of this isomorphism and its dual, i.e. the space of homomorphisms into $\mathbb{R}$, via the isomorphism

$$\pi_1^*\colon H^0(\Sigma;S\mathcal{C}^+)^* = T_{z_1^+}\Sigma\oplus\cdots\oplus T_{z_l^+}\Sigma \longrightarrow \big(H^0(\Sigma;S\mathcal{C})^\sigma\big)^*$$

from the complex orientations of $T_{z_1^+}\Sigma,\ldots,T_{z_l^+}\Sigma$. Thus, the isomorphism

$$\mathrm{Hom}_{\mathbb{C}}\big(H^0(\Sigma_0;S\mathcal{C}_0),\mathbb{C}\big) \approx H^0(\Sigma;S\mathcal{C}^+)^* \xrightarrow{\pi_1^*} \big(H^0(\Sigma;S\mathcal{C})^\sigma\big)^* \tag{3.10}$$

is orientation-preserving with respect to the complex orientation on the left-hand side and the orientation in the proof of [12, Proposition 5.9] on the right-hand side.

The Kodaira-Spencer map, Dolbeault isomorphism, and Serre Duality for $[\mathcal{C}]\in\mathcal{M}^{\sigma}_{2g_0-1,l}$ as in [12, (5.20),(5.21)] and $[\mathcal{C}_0]\in\mathcal{M}_{g_0,l}$ form a commutative diagram

$$\begin{array}{ccccccc}
T_{[\mathcal{C}]}\mathcal{M}^{\sigma}_{2g_0-1,l} & \xrightarrow[\approx]{\mathrm{KS}} & \check{H}^1(\Sigma;T\mathcal{C})^{\sigma} & \xrightarrow[\approx]{\mathrm{DI}} & H^1(\Sigma;T\mathcal{C})^{\sigma} & \xrightarrow[\approx]{\mathrm{SD}} & \big(H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)^{\sigma}\big)^* \\
\downarrow\approx & & \downarrow\approx & & \downarrow\approx & & \pi_1^*\uparrow\approx \\
T_{[\mathcal{C}_0]}\mathcal{M}_{g_0,l} & \xrightarrow[\approx]{\mathrm{KS}} & \check{H}^1(\Sigma_0;T\mathcal{C}_0) & \xrightarrow[\approx]{\mathrm{DI}} & H^1(\Sigma_0;T\mathcal{C}_0) & \xrightarrow[\approx]{\mathrm{SD}} & \mathrm{Hom}_{\mathbb{C}}\big(H^0(\Sigma_0;T^*\mathcal{C}_0\otimes T^*\Sigma_0),\mathbb{C}\big)
\end{array}$$

with the vertical arrows given by the restrictions to $\Sigma_1=\Sigma_0$. Since the isomorphisms in the bottom row of the above diagram are $\mathbb{C}$-linear, the natural isomorphism

$$\begin{aligned}
&\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\mathcal{C}]}\mathcal{M}^{\sigma}_{2g_0-1,l}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\big(H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)^{\sigma}\big)^*\big)\\
&\approx\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\mathcal{C}_0]}\mathcal{M}_{g_0,l}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\mathrm{Hom}_{\mathbb{C}}\big(H^0(\Sigma_0;T^*\mathcal{C}_0\otimes T^*\Sigma_0),\mathbb{C}\big)\big)
\end{aligned}\tag{3.11}$$

is orientation-preserving with respect to the orientation on the left-hand side in the proof of [12, Proposition 5.9] and the orientation on the right-hand side induced by the complex orientations on the factors.

Since $2g_0+l\ge 3$,

$$\begin{aligned}
&H^1(\Sigma_0;T^*\mathcal{C}_0\otimes T^*\Sigma_0)=0,\\
&\dim_{\mathbb{C}}H^0(\Sigma_0;T^*\mathcal{C}_0\otimes T^*\Sigma_0)=3g_0-3+l.
\end{aligned}\tag{3.12}$$

The short exact sequence of sheaves [12, (5.22)] over Σ and its analogue over Σ_0 induce a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
H^0(\Sigma;T^*\Sigma\otimes T^*\Sigma)^{\sigma} & \longrightarrow & H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)^{\sigma} & \longrightarrow & H^0(\Sigma;S\mathcal{C})^{\sigma} & \longrightarrow & H^1(\Sigma;T^*\Sigma\otimes T^*\Sigma)^{\sigma}\\
\downarrow\approx & & \downarrow\approx & & \pi_1\downarrow\approx & & \downarrow\approx\\
H^0(\Sigma_0;T^*\Sigma_0\otimes T^*\Sigma_0) & \longrightarrow & H^0(\Sigma_0;T^*\mathcal{C}_0\otimes T^*\Sigma_0) & \longrightarrow & H^0(\Sigma_0;S\mathcal{C}_0) & \longrightarrow & H^1(\Sigma_0;T^*\Sigma_0\otimes T^*\Sigma_0),
\end{array}$$

where we omit the zero vector spaces on the ends of the two rows. Since the isomorphisms in the bottom row of the above diagram are $\mathbb{C}$-linear, the natural isomorphism

$$\begin{aligned}
&\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)^{\sigma}\big)\otimes\det\bar{\partial}_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}}\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma_0;S\mathcal{C})^{\sigma}\big)\\
&\approx\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma_0;T^*\mathcal{C}_0\otimes T^*\Sigma_0),\mathbb{C}\big)\otimes\det\big(\bar{\partial}_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}}\big)_0\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma_0;S\mathcal{C}_0)\big)
\end{aligned}$$

is orientation-preserving with respect to the orientation on the left-hand side in the proof of [12, Proposition 5.9] and the orientation on the right-hand side induced by the complex orientations on the factors.

By the choice of the orientation on $H^0(\Sigma_0;S\mathcal{C})^\sigma$, the isomorphism π_1^* in (3.10) is orientation-preserving with respect to the complex orientation on its domain. Since the complex dimension of the last vector space is l, it follows that the sign of the vertical isomorphism π_1 in the last commutative diagram is $(-1)^l$. Thus, the sign of the natural isomorphism

$$\begin{aligned}(3.13)\qquad &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)^\sigma\big)\otimes\det\bar\partial_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}}\\ &\approx\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma_0;T^*\mathcal{C}_0\otimes T^*\Sigma_0),\mathbb{C}\big)\otimes\det\big(\bar\partial_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}}\big)_0\end{aligned}$$

with respect to the orientation on the left-hand side in the proof of [12, Proposition 5.9] and the orientation on the right-hand side induced by the complex orientations on the factors is $(-1)^l$.

By Lemma 3.1, the natural isomorphism

$$(3.14)\qquad \det\bar\partial_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}}\otimes\det\bar\partial_{\Sigma;\mathbb{C}}\approx\det\big(\bar\partial_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}}\big)_0\otimes\det\big(\bar\partial_{\Sigma;\mathbb{C}}\big)_0$$

is orientation-preserving with respect to the orientation on the left-hand side induced by a real orientation on $(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}$ and the orientation on the right-hand side induced by the complex orientations on the factors. The canonical orientation on the real line bundle (3.8) is obtained by combining the canonical orientations of the left-hand sides of (3.11), (3.13), and (3.14).

By the second statement in (3.12), the sign of the canonical isomorphism

$$\begin{aligned}&\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\big(H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)^\sigma\big)^*\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)^\sigma\big)\\ \approx\;&\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\mathrm{Hom}_{\mathbb{C}}\big(H^0(\Sigma_0;T^*\mathcal{C}_0\otimes T^*\Sigma_0),\mathbb{C}\big)\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma_0;T^*\mathcal{C}_0\otimes T^*\Sigma_0)\big)\end{aligned}$$

with respect to the canonical orientation on the left-hand side and the orientation on the right-hand side induced by the complex orientations on the factors is $(-1)^{3g_0-3+l}$. Combining this with the sign of the isomorphism (3.13), we obtain the claim. □

Proof of Theorem 1.3. Throughout this argument, we will refer to the orientation on the moduli space $\mathfrak{M}^\bullet_{2g_0-1,l}(X,B;J)^\phi$ determined by a fixed real orientation on (X,ω,ϕ) as the canonical orientation. Since the canonical orientation is compatible with orienting the fibers of the forgetful morphism (1.10) by the first point in the last conjugate pair, we can assume that $2g_0+l\ge 3$. Let $[\mathbf{u}]$ be an element of $\mathfrak{M}^\bullet_{2g_0-1,l}(X,B;J)^{\phi,\sigma}_{\mathfrak{s}}$, $[\mathbf{u}_0]$ be its image under (1.9), and $[\mathcal{C}]\in\mathcal{M}^\sigma_{2g_0-1,l;\mathfrak{s}}$ and $[\mathcal{C}_0]\in\mathcal{M}_{g_0,l}$ be their images under the forgetful morphisms to the corresponding Deligne-Mumford moduli spaces.

The canonical orientation of the tangent space at $[\mathbf{u}]$ is obtained from the canonical isomorphism

$$\begin{aligned}(3.15)\qquad &\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T_{[\mathbf{u}]}\mathfrak{M}^{\bullet}_{2g_0-1,l}(X,B;J)^{\phi}_{\mathfrak{s}}\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes(n+1)}\\ &\approx\Big(\big(\det D_{(TX,\mathrm{d}\phi)}|_u\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes n}\Big)\\ &\quad\otimes\Big(\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T_{[\mathcal{C}]}\mathcal{M}^{\sigma}_{2g_0-1,l;\mathfrak{s}}\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)\Big)\end{aligned}$$

determined by the forgetful morphism (2.2) and the canonical orientation of $(\det\bar\partial_{\Sigma;\mathbb{C}})^{\otimes(n+1)}$ for $n\notin 2\mathbb{Z}$. The orientation of the first tensor product on the right-hand side of (3.15) is determined by the real orientation on (X,ω,ϕ) as in Corollary 2.3. The orientation of the last tensor product on the right-hand side of (3.15) is the canonical orientation of [12, Proposition 5.9]. The standard complex orientation of the tangent space at $[\mathbf{u}_0]$ is obtained from the canonical isomorphism

$$\begin{aligned}(3.16)\qquad &\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T_{[\mathbf{u}_0]}\mathfrak{M}_{g_0,l}(X,B_0;J)\big)\otimes\big(\det\big(\bar\partial_{\Sigma;\mathbb{C}}\big)_0\big)^{\otimes(n+1)}\\ &\approx\Big(\big(\det\big(D_{(TX,\mathrm{d}\phi)}|_u\big)_0\big)\otimes\big(\det\big(\bar\partial_{\Sigma;\mathbb{C}}\big)_0\big)^{\otimes n}\Big)\\ &\quad\otimes\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T_{[\mathcal{C}_0]}\mathcal{M}_{g_0,l}\big)\otimes\big(\det\big(\bar\partial_{\Sigma;\mathbb{C}}\big)_0\big)\end{aligned}$$

determined by the forgetful morphism to the Deligne-Mumford space and the standard complex orientation of $\det(\bar\partial_{\mathbb{C}}|_{\Sigma_u})_0$. The orientations of all four factors on the right-hand side of (3.16) are the standard complex orientations.

The restriction to $\Sigma_1=\Sigma_0$ intertwines the isomorphisms (3.15) and (3.16) and respects the four factors on the right-hand sides. By Lemma 3.1, the isomorphism between the first pairs of factors on the right-hand sides is orientation-preserving. By Lemma 3.2, the sign of the isomorphism between the last pairs of factors is $(-1)^{g_0+1+|\mathfrak{s}|}$. This establishes the claim. □

3.2. Canonical vs. spin and relative spin

We establish Theorem 1.4 and similar statements by relating the orientations arising from Corollary 2.3 to the orienting procedure for the determinants of Fredholm operators over oriented symmetric half-surfaces described in [10].

An oriented symmetric half-surface (or simply oriented sh-surface) is a pair (Σ^b,c) consisting of an oriented bordered smooth surface Σ^b and an involution $c\colon\partial\Sigma^b\longrightarrow\partial\Sigma^b$ preserving each component and the orientation of $\partial\Sigma^b$. The restriction of c to a boundary component $(\partial\Sigma^b)_i$ is either the

identity or the antipodal map

$$\mathfrak{a}\colon S^1 \longrightarrow S^1, \quad z \longrightarrow -z,$$

for a suitable identification of $(\partial\Sigma^b)_i$ with $S^1 \subset \mathbb{C}$; the latter type of boundary structure is called crosscap in the string theory literature. We denote by

$$\partial_0^c\Sigma^b, \partial_1^c\Sigma^b \subset \partial\Sigma^b \subset \Sigma^b$$

the unions of the standard boundary components of (Σ^b, c) and of the crosscaps, respectively. If $\partial_1^c\Sigma^b = \emptyset$, (Σ^b, c) is a bordered surface in the usual sense. An oriented sh-surface (Σ^b, c) doubles to a symmetric surface (Σ, σ) so that σ restricts to c on the cutting circles (the boundary of Σ^b); see [10, (1.6)]. In particular, $\Sigma^\sigma = \partial_0^c\Sigma^b$. Since this doubling construction covers all topological types of orientation-reversing involutions σ on Σ, for every symmetric surface (Σ, σ) there is an oriented sh-surface (Σ^b, c) which doubles to (Σ, σ).

A real bundle pair $(V^b, \widetilde{c})$ over an oriented sh-surface (Σ^b, c) consists of a complex vector bundle $V^b \longrightarrow \Sigma^b$ with a conjugation $\widetilde{c}$ on $V^b|_{\partial\Sigma^b}$ lifting c. Via the doubling construction after [10, Remark 3.4], such a pair $(V^b, \widetilde{c})$ corresponds to a real bundle pair (V, φ) over the associated symmetric surface (Σ, σ) so that $V^b = V|_{\Sigma^b}$ and $\widetilde{c}$ is the restriction of φ to $V^b|_{\partial\Sigma^b}$. In particular,

$$V^\varphi = \big(V^b\big)^{\widetilde{c}} \subset V|_{\Sigma^\sigma} = V^b\big|_{\partial_0^c\Sigma^b}$$

is a totally real subbundle.

By [5, Lemma 2.4], the homotopy classes of trivializations of the real bundle pair (V, φ) over $\partial_1^c\Sigma^b$ correspond to the homotopy classes of trivializations of its top exterior power $\Lambda^{\mathrm{top}}_{\mathbb{C}}(V, \varphi)$. If $(L, \widetilde{\phi})$ is a rank 1 real bundle pair over (Σ, σ), the real bundle pair $2(L, \widetilde{\phi})$ has a canonical homotopy class of trivializations over $\partial_1^c\Sigma^b$; see the proof of [5, Theorem 1.3]. Thus, a homotopy class of trivializations of (V, φ) over $\partial_1^c\Sigma^b$ corresponds to a homotopy of trivializations of $(V, \varphi)\oplus 2(L, \widetilde{\phi})$. Furthermore, a homotopy class of isomorphisms of real bundle pairs as in (1.1) determines a homotopy class of trivializations of the restriction of (V, φ) to $\partial_1^c\Sigma^b$. It also induces an orientation on the real vector bundle $V^\varphi \longrightarrow \partial_0^c\Sigma^b$.

If the real vector bundle $V^\varphi \longrightarrow \partial_0^c\Sigma^b$ is oriented, a relative spin structure on V^φ consists of an oriented vector bundle $L \longrightarrow \Sigma$ and a homotopy class of trivializations of the oriented vector bundle

$$V^\varphi \oplus L|_{\partial_0^c\Sigma^b} \longrightarrow \partial_0^c\Sigma^b = \Sigma^\sigma\,. \tag{3.17}$$

Since every oriented vector bundle over Σ^b is trivializable, the vector bundle $L|_{\Sigma^b} \longrightarrow \Sigma^b$ admits a trivialization Ψ_L^b. Along with a trivialization of (3.17), the restriction of Ψ_L^b to $L|_{\partial_0^c\Sigma^b}$ induces a trivialization of V^φ. If $\partial_1^c\Sigma^b=\emptyset$ and the rank n of V is at least 3, the homotopy classes of the trivializations of V^φ induced by two trivializations of $L|_{\Sigma^b}$ differ on an even number of components of $\partial_0^c\Sigma^b=\partial\Sigma^b$.

A real CR-operator on a real bundle pair $(V^b,\tilde{c})$ over an oriented sh-surface (Σ^b,c) is a linear map of the form

$$D^b=\bar\partial^b+A\colon \Gamma(\Sigma^b;V^b)^{\tilde c} \equiv \big\{\xi\in\Gamma(\Sigma^b;V^b)\colon\ \xi\circ c=\tilde c\circ\xi|_{\partial\Sigma^b}\big\} \longrightarrow \Gamma^{0,1}_{\mathfrak{j}^b}(\Sigma^b;V^b)\equiv\Gamma\big(\Sigma^b;(T^*\Sigma^b,\mathfrak{j}^b)^{0,1}\otimes_{\mathbb{C}}V^b\big),$$

where $\bar\partial^b$ is the holomorphic $\bar\partial$-operator for some complex structure $\mathfrak{j}^b$ on Σ^b and holomorphic structure in V^b and

$$A\in\Gamma\big(\Sigma^b;\mathrm{Hom}_{\mathbb{R}}(V^b,(T^*\Sigma^b,\mathfrak{j}^b)^{0,1}\otimes_{\mathbb{C}}V^b)\big)$$

is a zeroth-order deformation term. By [10, Corollary 3.3], $\mathfrak{j}^b$ doubles to some $\mathfrak{j}\in\mathcal{J}_\Sigma^\sigma$ if and only if c is real-analytic with respect to $\mathfrak{j}^b$. In such a case, D^b is Fredholm in appropriate completions and corresponds to a real CR-operator D on the associated real bundle pair (V,φ) over (Σ,σ); see [10, Proposition 3.6]. In particular, there is a canonical isomorphism

$$\begin{aligned}\widehat{\det}\,D &\equiv \big(\det D\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes n}\\ &\approx \big(\det D^b\big)\otimes\big(\det\bar\partial^b_{\Sigma;\mathbb{C}}\big)^{\otimes n}\equiv\widehat{\det}\,D^b\,,\end{aligned}\tag{3.18}$$

where $n=\mathrm{rk}_{\mathbb{C}}V$, $\bar\partial_{\Sigma;\mathbb{C}}$ is the standard real CR-operator on the trivial real bundle pair $(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c})$ over (Σ,σ) as in Example 2.1, and $\bar\partial^b_{\Sigma;\mathbb{C}}\equiv\bar\partial_{\Sigma^b;\mathbb{C}}$ is the standard real CR-operator on the relative bundle pair $(\Sigma^b\times\mathbb{C},c\times\mathfrak{c})$ over (Σ^b,c).

An orientation on the right-hand side of (3.18) thus determines an orientation on the left-hand side of (3.18). By the proofs of [20, Lemma 6.37] and [10, Theorem 1.1], an orientation on the former is determined by a collection consisting of

(OC1) a homotopy class of trivializations of V^φ over $\partial_0^c\Sigma^b$;

(OC2) a homotopy class of trivializations of the real bundle pair (V,φ) over $\partial_1^c\Sigma^b$.

If $n\ge 3$, changing the homotopy class in (OC1) within its orientation class over precisely one topological component of $\partial^c_0\Sigma^b$ changes the induced orientation on the right-hand side of (3.18). Changing the homotopy class in (OC2) class over precisely one topological component of $\partial^c_1\Sigma^b$ also changes the induced orientation on the right-hand side of (3.18).

Let $(L,\widetilde{\phi})$ be a rank 1 real bundle pair over (Σ,σ) and D_L be a real CR-operator on $(L,\widetilde{\phi})$. By the sentence above containing (OC1) and (OC2) applied with V replaced by $V\oplus 2L$, an orientation on

$$\begin{aligned}(3.19)\qquad &\big(\det(D^b\oplus D^b_{2L})\big)\otimes\big(\det\bar\partial^b_{\Sigma;\mathbb{C}}\big)^{\otimes(n+2)}\\ &\approx\big(\det D^b\big)\otimes\big(\det\bar\partial^b_{\Sigma;\mathbb{C}}\big)^{\otimes n}\otimes\big(\det D^b_L\big)^{\otimes 2}\otimes\big(\det\bar\partial^b_{\Sigma;\mathbb{C}}\big)^{\otimes 2}\end{aligned}$$

is determined by a trivialization $\psi_{V\oplus 2L}$ of the real vector bundle $V^\varphi\oplus 2L^{\widetilde{\phi}}$ over $\partial^c_0\Sigma^b$ and a trivialization $\psi'_{V\oplus 2L}$ of the real bundle pair $(V\oplus 2L,\varphi\oplus 2\widetilde{\phi})$ over $\partial^c_1\Sigma^b$. Since the last two factors in (3.19) are canonically oriented, $\psi_{V\oplus 2L}$ and $\psi'_{V\oplus 2L}$ thus determine an orientation on the right-hand side of (3.18). We will call it the stabilization orientation induced by $\psi_{V\oplus 2L}$ and $\psi'_{V\oplus 2L}$, omitting $\psi'_{V\oplus 2L}$ if $\partial^c_1\Sigma^b=\emptyset$ and $\psi_{V\oplus 2L}$ if $\partial^c_0\Sigma^b=\emptyset$.

Via (1.11) with L^* replaced by L, $\psi_{V\oplus 2L}$ also induces a trivialization of (3.17). If $\partial^c_1\Sigma^b=\emptyset$, $\psi_{V\oplus 2L}$ thus determines a relative spin structure on V^φ, and another orientation on the right-hand side of (3.18). We will call the latter the associated relative spin (or simply ARS) orientation. If $L^{\widetilde{\phi}}\longrightarrow\Sigma^\sigma$ is orientable (but $\partial^c_1\Sigma^b$ is not necessarily empty), then

- $\psi_{V\oplus 2L}$ and the canonical homotopy class of trivializations of $2L^{\widetilde{\phi}}$ determine a homotopy class of trivializations of V^φ over $\partial^c_0\Sigma^b$, and
- $\psi'_{V\oplus 2L}$ and the canonical homotopy class of trivializations of $2(L,\widetilde{\phi})$ determine a homotopy class of trivializations of (V,φ) over $\partial^c_1\Sigma^b$.

Thus, $\psi_{V\oplus 2L}$ and $\psi'_{V\oplus 2L}$ determine another orientation on the right-hand side of (3.18) in this case; we will call it the associated spin (or simply AS) orientation. Lemmas 3.3-3.7 and Corollary 3.8 below compare these three orientations on the right-hand side of (3.18).

In the case of the involutions

$$\tau\colon\mathbb{P}^1\longrightarrow\mathbb{P}^1,\quad z\longrightarrow 1/\bar z,\quad\text{and}\quad \eta\colon\mathbb{P}^1\longrightarrow\mathbb{P}^1,\quad z\longrightarrow -1/\bar z,$$

we can take Σ^b to be the unit disk around the origin in $\mathbb{C}\subset\mathbb{P}^1$. This will be our default choice in these settings.

Lemma 3.3. *With notation as above, suppose* $(\Sigma,\sigma)=(\mathbb{P}^1,\tau)$. *If* $L^{\widetilde{\phi}}\longrightarrow S^1$ *is orientable, the stabilization and AS orientations on the right-hand side of (3.18) induced by a trivialization* $\psi_{V\oplus 2L}$ *of* $V^{\varphi}\oplus 2L^{\widetilde{\phi}}$ *are the same.*

Proof. Fix a trivialization $\psi_L\colon L^{\widetilde{\phi}}\longrightarrow S^1\times\mathbb{R}$; the canonical homotopy class of trivializations of $2L^{\widetilde{\phi}}$ is the class containing $2\psi_L$. A trivialization ψ_V of V^{φ} lies in the associated homotopy class of trivializations of V^{φ} if and only if $\psi_{V\oplus 2L}$ and $\psi_V\oplus 2\psi_L$ lie in the same homotopy class of trivializations of $V^{\varphi}\oplus 2L^{\widetilde{\phi}}$. In this case, the natural isomorphism (3.19) is orientation-preserving with respect to the orientation on the left-hand side induced by $\psi_{V\oplus 2L}$ and the orientations on

$$\big(\det D^b\big)\otimes\big(\det\bar\partial^b_{\Sigma;\mathbb{C}}\big)^{\otimes n}\quad\text{and}\quad\big((\det D^b_L)\otimes(\det\bar\partial^b_{\Sigma;\mathbb{C}})\big)^{\otimes 2}\tag{3.20}$$

induced by ψ_V and ψ_L, respectively. Since the last orientation is the same as the orientation induced by the canonical orientations of $(\det D^b_L)^{\otimes 2}$ and $(\det\bar\partial^b_{\Sigma;\mathbb{C}})^{\otimes 2}$, the stabilization orientation on the first tensor product in (3.20) induced by $\psi_{V\oplus 2L}$ and the AS orientation (i.e. the orientation induced by ψ_V) are the same. □

Lemma 3.4. *With notation as above, suppose* $(\Sigma,\sigma)=(\mathbb{P}^1,\tau)$. *If* $L^{\widetilde{\phi}}\longrightarrow S^1$ *is orientable, the stabilization and ARS orientations on the right-hand side of (3.18) induced by a trivialization* $\psi_{V\oplus 2L}$ *of* $V^{\varphi}\oplus 2L^{\widetilde{\phi}}$ *are the same if and only if* $\deg L\in 4\mathbb{Z}$.

Proof. Let $d=\deg L$. By [1, Proposition 4.1], we can assume that $(L,\widetilde{\phi})$ is the holomorphic line $\mathcal{O}_{\mathbb{P}^1}(d)$ with the standard lift of τ. Since $L^{\widetilde{\phi}}\longrightarrow S^1$ is orientable, $d\in 2\mathbb{Z}$. By [22, Theorem C.3.6], there exists a trivialization Ψ^b_L of $L|_{\Sigma^b}$ so that

$$\Psi^b_L\big(L^{\widetilde{\phi}}\big)=\big\{\big(\mathrm{e}^{\mathrm{i}\theta},a\mathrm{e}^{\mathrm{i}d\theta/2}\big)\colon\,\mathrm{e}^{\mathrm{i}\theta}\in S^1,\,a\in\mathbb{R}\big\}\subset S^1\times\mathbb{C}\,.\tag{3.21}$$

Let ψ_L be the trivialization of $L^{\widetilde{\phi}}$ given by

$$\psi_L\big(\{\Psi^b_L\}^{-1}(\mathrm{e}^{\mathrm{i}\theta},a\mathrm{e}^{\mathrm{i}d\theta/2})\big)=\big(\mathrm{e}^{\mathrm{i}\theta},a\big)\in S^1\times\mathbb{R}\,.$$

The trivialization Ψ^{∂}_L of $L|_{\partial\Sigma^b}$ induced by $2\psi_L$ via (1.11) with L^* replaced by L is then described by

$$\begin{aligned}&\Psi^{\partial}_L\colon L|_{\partial\Sigma^b}\longrightarrow S^1\times\mathbb{C},\\&\Psi^{\partial}_L\big(\{\Psi^b_L\}^{-1}(\mathrm{e}^{\mathrm{i}\theta},c)\big)=\big(\mathrm{e}^{\mathrm{i}\theta},c\mathrm{e}^{-\mathrm{i}d\theta/2}\big)\;\forall\,\big(\mathrm{e}^{\mathrm{i}\theta},c\big)\in S^1\times\mathbb{C}.\end{aligned}\tag{3.22}$$

Thus, the homotopy classes of Ψ_L^{∂} and $\Psi_L^b|_{\partial\Sigma^b}$ differ by $d/2$ times a generator of $\pi_1(\mathrm{SO}(2))\approx\mathbb{Z}$.

Let ψ_V and ψ'_V be trivializations of V^{φ} so that $\psi_V\oplus 2\psi_L$ and $\psi'_V\oplus\Psi_L^b|_{\partial\Sigma^b}$ lie in the same homotopy class of trivializations of $V^{\varphi}\oplus 2L^{\widetilde{\phi}}$ as $\psi_{V\oplus 2L}$. By Lemma 3.3, the stabilization orientation on the right-hand side of (3.18) induced by $\psi_{V\oplus 2L}$ is the orientation induced by ψ_V as in the proof of [7, Theorem 8.1.1]. By definition, the ARS orientation on the right-hand side of (3.18) induced by $\psi_{V\oplus 2L}$ is the orientation induced by ψ'_V. By (3.22), ψ_V and ψ'_V are homotopic (and thus the two induced orientations are the same) if and only if $d/2\in 2\mathbb{Z}$. □

Let $(\Sigma,\sigma)=(\mathbb{P}^1,\tau)$ and $\deg L=1$. Similarly to the proof of Lemma 3.4, [1, Proposition 4.1] and [22, Theorem C.3.6] imply that there exists a trivialization Ψ_L^b of $L|_{\Sigma^b}$ so that (3.21) holds with $d=1$. Let ψ_0 be the trivialization of $2L^{\widetilde{\phi}}$ given by

$$\begin{aligned}(3.23)\qquad &\psi_0\big(\{\Psi_L^b\}^{-1}(\mathrm{e}^{\mathrm{i}\theta},a_1\mathrm{e}^{\mathrm{i}\theta/2}),\\ &\{\Psi_L^b\}^{-1}(\mathrm{e}^{\mathrm{i}\theta},a_2\mathrm{e}^{\mathrm{i}\theta/2})\big)=\big(\mathrm{e}^{\mathrm{i}\theta},(a_1+\mathrm{i}a_2)\mathrm{e}^{\mathrm{i}\theta/2}\big)\in S^1\times\mathbb{C}\end{aligned}$$

for all $a_1,a_2\in\mathbb{R}$.

Proposition 3.5. *The orientation on* $\det D_{2L}^b=(\det D_L^b)^{\otimes 2}$ *induced by the trivialization* ψ_0 *as in the proof of [7, Theorem 8.1.1] agrees with the canonical square orientation.*

We give three proofs. In the first one, we write out the real holomorphic sections and the relevant trivializations explicitly. In the second proof, we use the comparisons of different orientations on the moduli spaces of real lines obtained in [5]. The last argument deduces the claim directly from the fixed-edge equivariant contribution determined in [5]. In all three arguments, we take D_L to be the standard $\bar\partial$-operator in $\mathcal{O}_{\mathbb{P}^1}(1)$.

Proof 1. Let $\mathbb{P}^1_\bullet=\mathbb{P}^1-\{1\}$. The holomorphic map

$$h\colon B\equiv\big\{t\in\mathbb{C}\colon\,|t|<1\big\}\longrightarrow\mathbb{P}^1,\quad t\longrightarrow\mathrm{e}^{\mathrm{i}t},$$

is injective and intertwines the standard conjugation on B with τ on $\mathbb{P}^1$. We can assume that

$$\begin{aligned}&L=\big(h(B)\times\mathbb{C}\sqcup\mathbb{P}^1_\bullet\times\mathbb{C}\big)/\!\sim,\quad \big(h(t),tc\big)\sim\big(h(t),c\big)\ \forall\ (t,c)\in(B-0)\times\mathbb{C},\\ &\widetilde{\phi}\big([t,c]\big)=\big[\bar t,\bar c\big]\ \forall\ (t,c)\in B\times\mathbb{C},\quad \widetilde{\phi}\big([z,c]\big)=\big[\tau(z),\bar c\big]\ \forall\ (z,c)\in\mathbb{P}^1_\bullet\times\mathbb{C}.\end{aligned}$$

The space of real holomorphic sections of L is then generated by the sections s_1 and s_2 described by

$$s_1\big([z]\big)=1, \quad s_2\big([z]\big)=\mathfrak{i}\frac{1+z}{1-z} \quad \forall\, z\in\mathbb{P}^1_\bullet.$$

The canonical orientation for $\det D^b_{2L}$ is then determined by the basis

$$s_{11}\equiv(s_1,0), \quad s_{12}\equiv(s_2,0), \quad s_{21}\equiv(0,s_1), \quad s_{22}\equiv(0,s_2),$$

for the kernel of the surjective operator D^b_{2L}.

We define a trivialization Ψ^b_L of L over the unit disk Σ^b around $z=0$ in $\mathbb{C}\subset\mathbb{P}^1$ by

$$\begin{aligned}\Psi^b_L\big([h(t),c]\big)&=\big(\mathrm{e}^{\mathfrak{i}t},2\mathfrak{i}\frac{\mathrm{e}^{\mathfrak{i}t}-1}{t}c\big) \quad \forall\,(t,c)\in B\times\mathbb{C},\\ \Psi^b_L\big([z,c]\big)&=\big(z,2\mathfrak{i}(z-1)c\big) \quad \forall\,(z,c)\in\big(\mathbb{P}^1_\bullet-\{\infty\}\big)\times\mathbb{C}.\end{aligned}$$

This trivialization satisfies (3.21) with $d=1$. The trivialization ψ_0 of $2L^{\widetilde{\phi}}$ over S^1 extends to the trivialization

$$\begin{aligned}&\Psi_0\colon 2L|_{\mathbb{P}^1-\{0,\infty\}}\longrightarrow\big(\mathbb{P}^1-\{0,\infty\}\big)\times\mathbb{C}^2,\\ &\Psi_0\big([z,c_1],[z,c_2]\big)=\big(z,\mathfrak{i}(z-z^{-1})c_1-z^{-1}(1-z)^2c_2,\\ &\qquad\qquad z^{-1}(1-z)^2c_1+\mathfrak{i}(z-z^{-1})c_2\big).\end{aligned}$$

This trivialization intertwines $2\widetilde{\phi}$ with the standard lift of $\tau|_{\mathbb{P}^1_\bullet-\{0,\infty\}}$ to a conjugation on the trivial bundle $(\mathbb{P}^1_\bullet-\{0,\infty\})\times\mathbb{C}^2$.

We note that

$$\begin{aligned}\big\{\Psi_0 s_{11}\big\}(z)&=\big(\mathfrak{i}z^{-1}(z^2-1),z^{-1}(1-z)^2\big),\\ \big\{\Psi_0 s_{12}\big\}(z)&=\big(z^{-1}(1+z)^2,\mathfrak{i}z^{-1}(1-z^2)\big),\\ \big\{\Psi_0 s_{21}\big\}(z)&=\big(-z^{-1}(1-z)^2,\mathfrak{i}z^{-1}(z^2-1)\big),\\ \big\{\Psi_0 s_{22}\big\}(z)&=\big(-\mathfrak{i}z^{-1}(1-z^2),z^{-1}(1+z)^2\big).\end{aligned}$$

The orientation on $\det D^b_{2L}$ induced by the trivialization ψ_0 is obtained from the isomorphism

$$\begin{aligned}\ker D^b_{2L}&\longrightarrow\mathbb{R}\oplus\mathbb{R}\oplus\big\{\mathrm{Res}_{z=0}(\Psi_0\xi)\colon\,\xi\in\ker D^b_{2L}\big\},\\ \xi&\longrightarrow\big(\{\Psi_0\xi\}(1),\mathrm{Res}_{z=0}(\Psi_0\xi)\big).\end{aligned}$$

The last space above is a complex subspace of $\mathbb{C}^2$. Under this isomorphism, the basis $s_{11}, s_{12}, s_{21}, s_{22}$ is sent to

$$(0,0;-\mathfrak{i},1), \quad (4,0;1,\mathfrak{i}), \quad (0,0;-1,-\mathfrak{i}), \quad (0,4;-\mathfrak{i},1).$$

Thus, an oriented basis for the target of the above isomorphism is given by

$$(4,0;0,0), \quad (0,4;0,0), \quad (0,0;-\mathfrak{i},1), \quad (0,0;1,\mathfrak{i}).$$

The change of basis matrix from the first basis to this one is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

The determinant of this matrix is $+1$. □

Proof 2. Define

$$\tau_3 \colon \mathbb{P}^3 \longrightarrow \mathbb{P}^3, \quad [Z_1,Z_2,Z_3,Z_4] \longrightarrow \big[\overline{Z}_2,\overline{Z}_1,\overline{Z}_4,\overline{Z}_3\big],$$
$$\mathfrak{M}_1(\mathbb{P}^1) = \mathfrak{M}_1(\mathbb{P}^1,1)^{\tau,\tau}, \quad \mathfrak{M}_1(\mathbb{P}^3) = \mathfrak{M}_1(\mathbb{P}^3,1)^{\tau_3,\tau}.$$

The inclusion $\iota \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ as the first two coordinates induces an embedding of $\mathfrak{M}_1(\mathbb{P}^1)$ into $\mathfrak{M}_1(\mathbb{P}^3)$. Let

$$\mathcal{N}_{\iota(0)}\mathbb{P} = \frac{T_{\iota(0)}\mathbb{P}^3}{T_{\iota(0)}\mathbb{P}^1} \quad \text{and} \quad \mathcal{N}_{[\iota,0]}\mathfrak{M} \equiv \frac{T_{[\iota,0]}\mathfrak{M}_1(\mathbb{P}^3)}{T_{[\iota,0]}\mathfrak{M}_1(\mathbb{P}^1)}$$

denote the normal bundle of $\mathbb{P}^1$ in $\mathbb{P}^3$ at $[1,0,0,0]$ and the normal bundle of $\mathfrak{M}_1(\mathbb{P}^1)$ in $\mathfrak{M}_1(\mathbb{P}^3)$ at ι with the positive marked point at $z=0$, respectively. The former is a complex vector space and thus is canonically oriented. The differential of the evaluation map ev_1 induces an isomorphism

$$\mathrm{d}_{[\iota,0]}\mathrm{ev}_1 \colon \mathcal{N}_{[\iota,0]}\mathfrak{M} \longrightarrow \mathcal{N}_{\iota(0)}\mathbb{P}\,. \tag{3.24}$$

By [5, Lemma 5.3], this isomorphism is orientation-*reversing* with respect to the algebraic orientations on $\mathfrak{M}_1(\mathbb{P}^1)$ in $\mathfrak{M}_1(\mathbb{P}^3)$ defined in [5, Section 5.2].

Since the normal bundle of $(\mathbb{P}^1,\tau)$ in $(\mathbb{P}^3,\tau_3)$ is isomorphic to $2(L,\widetilde{\phi})$, the composition

$$\ker D_{2L}^b \longrightarrow T_{[\iota,0]}\mathfrak{M}_1(\mathbb{P}^3) \longrightarrow \mathcal{N}_{[\iota,0]}\mathfrak{M}$$

is an isomorphism. Combining it with (3.24), we obtain an isomorphism

$$\ker D_{2L}^b \longrightarrow \mathcal{N}_{[\iota,0]}\mathfrak{M} \longrightarrow \mathcal{N}_{\iota(0)}\mathbb{P}\,. \tag{3.25}$$

Since the canonical orientation on $\det D_{2L}^b$ is obtained from the isomorphism

$$\ker D_{2L}^b \longrightarrow 2L_0, \quad \xi \longrightarrow \xi(0),$$

and the complex orientation on L_0, the isomorphism (3.25) is orientation-*preserving* with respect to the canonical orientation on the left-hand side.

The real vector bundle

$$4L^{\widetilde{\phi}} \longrightarrow S^1 = \mathbb{R}\mathbb{P}^1 \subset \mathbb{P}^1 \tag{3.26}$$

carries a canonical spin structure; see [5, Section 5.5]. Along with Euler's sequence for $\mathbb{P}^3$, it determines an orientation on $\mathfrak{M}_1(\mathbb{P}^3)$; we will call it the spin orientation. It agrees with the orientation induced by the trivialization $2\psi_0$ over S^1. Along with Euler's sequence for $\mathbb{P}^1$ and the relative spin orienting procedure of [7, Theorem 8.1.1], the canonical spin structure on (3.26) determines an orientation on $\mathfrak{M}_1(\mathbb{P}^1)$; we will call it the relative spin orientation. Along with the spin orientation on $\mathfrak{M}_1(\mathbb{P}^3)$, it induces an orientation on $\mathcal{N}_{[\iota,0]}\mathfrak{M}$; we will call it the spin orientation. Since ψ_0 extends over the disk $\Sigma^b \subset \mathbb{P}^1$, the first isomorphism in (3.25) is orientation-*preserving* with respect to the orientation on the left-hand side induced by ψ_0 and the spin orientation on $\mathcal{N}_{[\iota,0]}\mathfrak{M}$.

As summarized in the paragraph above [5, Remark 6.9], the algebraic orientations on $\mathfrak{M}_1(\mathbb{P}^1)$ and $\mathfrak{M}_1(\mathbb{P}^3)$ are the same as the relative spin orientation and the opposite of the spin orientation, respectively. Therefore, the spin orientation on $\mathcal{N}_{[\iota,0]}\mathfrak{M}$ is the opposite of the algebraic orientation. Since the second isomorphism in (3.25) is orientation-reversing with respect to the latter, it follows that the composite isomorphism in (3.25) is orientation-preserving with respect to the orientation on the left-hand side induced by ψ_0. Since this is also the case with respect to the canonical orientation on the left-hand side, these two orientations on $\ker D_{2L}^b$ agree. □

Proof 3. Under a change of coordinate on $2(L,\widetilde{\phi})$ which is homotopic to the identity, the trivialization ψ_0 is equivalent to the trivialization [5, (6.13)].

By [5, Section 6.4], there are natural S^1-actions on $(\mathbb{P}^1,\tau)$ and $2(L,\widetilde{\phi})$ so that the evaluation isomorphism

$$\ker D^b_{2L} \xrightarrow{\mathrm{ev}_{2L,i0}} 2L|_0, \quad \xi \longrightarrow \xi(0), \tag{3.27}$$

is S^1-equivariant. By the $d_0=1$, $\bar{i}\in 2\mathbb{Z}$ case of [5, (6.21)], the S^1-equivariant Euler class of $\ker D^b_{2L}$ with respect to the orientation induced by ψ_0 is given by

$$\mathbf{e}\big(\ker D^b_{2L}\big) = -\big(\lambda_i-\lambda_j\big)\big(-\lambda_i-\lambda_j\big) = \big(\lambda_i-\lambda_j\big)\big(\lambda_i+\lambda_j\big) = \mathbf{e}(2L|_0)\,.$$

This establishes the claim. □

Corollary 3.6. *With notation as above, let $(\Sigma,\sigma)=(\mathbb{P}^1,\tau)$. If $L^{\widetilde{\phi}}\longrightarrow S^1$ is not orientable, the stabilization and ARS orientations on the right-hand side of (3.18) induced by a trivialization $\psi_{V\oplus 2L}$ of $V^\varphi\oplus 2L^{\widetilde{\phi}}$ are the same if and only if* $\deg L-1\in 4\mathbb{Z}$.

Proof. Let $d=\deg L$. Since $L^{\widetilde{\phi}}\longrightarrow S^1$ is not orientable, $d\notin 2\mathbb{Z}$. Similarly to the proof of Lemma 3.4, [1, Proposition 4.1] and [22, Theorem C.3.6] imply that there exists a trivialization Ψ^b_L of $L|_{\Sigma^b}$ so that (3.21) holds. Let ψ_{2L} be the trivialization of $2L^{\widetilde{\phi}}$ given by

$$\begin{aligned}&\psi_{2L}\big(\{\Psi^b_L\}^{-1}(\mathrm{e}^{\mathrm{i}\theta},a_1\mathrm{e}^{\mathrm{i}d\theta/2}),\{\Psi^b_L\}^{-1}(\mathrm{e}^{\mathrm{i}\theta},a_2\mathrm{e}^{\mathrm{i}d\theta/2})\big)\\ &=\psi_0\big((\mathrm{e}^{\mathrm{i}\theta},a_1\mathrm{e}^{\mathrm{i}\theta/2}),(\mathrm{e}^{\mathrm{i}\theta},a_2\mathrm{e}^{\mathrm{i}\theta/2})\big)\in S^1\times\mathbb{C}\,.\end{aligned}$$

The trivialization Ψ^∂_L of $L|_{\partial\Sigma^b}$ induced by ψ_{2L} via (1.11) with L^* replaced by L is then described by

$$\begin{aligned}&\Psi^\partial_L\colon L|_{\partial\Sigma^b}\longrightarrow S^1\times\mathbb{C},\\ &\Psi^\partial_L\big(\{\Psi^b_L\}^{-1}(\mathrm{e}^{\mathrm{i}\theta},c)\big)=\big(\mathrm{e}^{\mathrm{i}\theta},c\mathrm{e}^{-\mathrm{i}(d-1)\theta/2}\big)\ \forall\,\big(\mathrm{e}^{\mathrm{i}\theta},c\big)\in S^1\times\mathbb{C}.\end{aligned} \tag{3.28}$$

Thus, the homotopy classes of Ψ^∂_L and $\Psi^b_L|_{\partial\Sigma^b}$ differ by $(d-1)/2$ times a generator of $\pi_1(\mathrm{SO}(2))\approx\mathbb{Z}$.

Let ψ_V and ψ'_V be trivializations of V^φ so that $\psi_V\oplus\psi_{2L}$ and $\psi'_V\oplus\Psi^b_L|_{\partial\Sigma^b}$ lie in the same homotopy class of trivializations of $V^\varphi\oplus 2L^{\widetilde{\phi}}$ as $\psi_{V\oplus 2L}$. By Proposition 3.5, the stabilization orientation on the right-hand side of (3.18) induced by $\psi_{V\oplus 2L}$ via the isomorphism (3.19) is the orientation induced by ψ_V as in the proof of [7, Theorem 8.1.1]. By definition, the ARS orientation on the right-hand side of (3.18) induced by $\psi_{V\oplus 2L}$ is the orientation

induced by ψ'_V. By (3.28), ψ_V and ψ'_V are homotopic (and thus the two induced orientations are the same) if and only if $(d-1)/2 \in 2\mathbb{Z}$. □

Lemma 3.7. *With notation as above, suppose* $(\Sigma,\sigma)=(\mathbb{P}^1,\eta)$. *The stabilization and AS orientations on the right-hand side of (3.18) induced by a trivialization* $\psi'_{V\oplus 2L}$ *of* $(V\oplus 2L,\varphi\oplus 2\widetilde{\phi})$ *are the same.*

Proof. Fix a trivialization ψ'_L of $(L,\widetilde{\phi})$ over $(S^1,\mathfrak{a})$; the canonical homotopy class of trivializations of $2(L,\widetilde{\phi})$ is the class containing $2\psi'_L$. A trivialization ψ'_V of (V,φ) over $(S^1,\mathfrak{a})$ lies in the associated homotopy class of trivializations of (V,φ) over $(S^1,\mathfrak{a})$ if and only if $\psi'_{V\oplus 2L}$ and $\psi'_V\oplus 2\psi'_L$ lie in the same homotopy class of trivializations of $(V\oplus 2L,\varphi\oplus 2\widetilde{\phi})$ over $(S^1,\mathfrak{a})$. In this case, the natural isomorphism (3.19) is orientation-preserving with respect to the orientation on the left-hand side induced by $\psi'_{V\oplus 2L}$ and the orientations on (3.20) induced by ψ'_V and ψ'_L, respectively. Since the last orientation is the same as the orientation induced by the canonical orientations of $(\det D^b_L)^{\otimes 2}$ and $(\det\bar\partial^b_{\mathbb{C}})^{\otimes 2}$, the stabilization orientation on the first tensor product in (3.20) induced by $\psi'_{V\oplus 2L}$ and the AS orientation (i.e. the orientation induced by ψ'_V) are the same. □

Corollary 3.8. *Let* (Σ^b,c), (Σ,σ), (V,φ), $(L,\widetilde{\phi})$, D, *and* D^b *be as above Lemma 3.3.*

(1) If $\partial^c_1\Sigma^b=\emptyset$ *and* $(\partial\Sigma^b)_1,\ldots,(\partial\Sigma^b)_m$ *are the components of* $\partial^c_0\Sigma^b=\partial\Sigma^b$, *then the stabilization and ARS orientations on the right-hand side of (3.18) induced by a trivialization of* $V^\varphi\oplus 2L^{\widetilde{\phi}}$ *are the same if and only if*

$$\deg L - \big|\big\{i=1,\ldots,m\colon\ w_1(L^{\widetilde{\phi}})|_{(\partial\Sigma^b)_i}\neq 0\big\}\big| \in 4\mathbb{Z}.$$

(2) If $L^{\widetilde{\phi}}\longrightarrow\partial^c_0\Sigma^b$ *is orientable, then the stabilization and AS orientations on the right-hand side of (3.18) induced by a trivialization of* $V^\varphi\oplus 2L^{\widetilde{\phi}}$ *and a trivialization of* $(V\oplus 2L,\varphi\oplus 2\widetilde{\phi})|_{\partial^c_1\Sigma^b}$ *are the same.*

Proof. For each $i=1,\ldots,m$, let

$$\varepsilon_i(L)=\begin{cases}0, & \text{if } w_1(L^{\widetilde{\phi}})|_{(\partial\Sigma^b)_i}=0,\\ 1, & \text{if } w_1(L^{\widetilde{\phi}})|_{(\partial\Sigma^b)_i}\neq 0.\end{cases}$$

As in the proofs of [20, Lemma 6.37] and [10, Theorem 1.1], we pinch off a circle near each boundary component $(\partial\Sigma^b)_i$ to form a closed surface Σ'

with m disks $B_1, \ldots, B_m$ attached. We deform the bundles V and L to bundles V_0 and L_0 over the resulting nodal surface Σ_0 so that $\deg L_0|_{\Sigma'} = 0$. Thus, a trivialization of $L_0|_{\partial\Sigma^b}$ that extends over each disk extends over Σ_0. The two determinants on the right-hand side of (3.18) are canonically isomorphic to the determinants of the induced real linear CR-operators D_0 and $\bar\partial_0$ on V_0 and $\Sigma_0 \times \mathbb{C}$, respectively. An orientation on $(\det D_0) \otimes (\det \bar\partial_0)^{\otimes n}$ is determined by orientations of the analogous tensor products over Σ_0 and the m disks. The former have canonical complex orientations. If $\partial_1^c \Sigma^b = \emptyset$, the stabilization and ARS orientations of the tensor products of the determinant lines over B_i induced by a trivialization of $V^\varphi \oplus 2L^{\widetilde\phi}$ are the same if and only if

$$\deg L_0|_{B_i} - \varepsilon_i(L) \in 4\mathbb{Z}; \tag{3.29}$$

see Lemma 3.4 and Corollary 3.6. Summing up (3.29) over $i = 1, \ldots, m$, we obtain the first claim. The second claim follows similarly from Lemmas 3.3 and 3.7. □

Proof of Theorem 1.4. Since the fibers of the forgetful morphism (1.10) are canonically oriented, it is sufficient to establish the claims for $l = 2$. In this case, the moduli space is oriented via the canonical isomorphism (2.6) with $(g, l) = (0, 2)$ and $\sigma = \tau$. By the paragraph above Theorem 1.4, the orientation of the last factor in (2.6) is the same in all three approaches to orienting the moduli space. The orientations of the first factor on the right-hand side of (2.6) are compared by Corollary 3.8 with L replaced by L^*. Taking into account that $c_1(TX) = 2c_1(L)$, we obtain Theorem 1.4. □

Remark 3.9. It is not necessary to require that the rank n of the real bundle pair (V, φ) being stabilized be at least 3, since lower-rank real bundle pairs can first be stabilized with the trivial rank 2 real bundle pair. The proof of Theorem 1.4 requires only the $(\Sigma, \sigma) = (\mathbb{P}^1, \tau)$ case of Corollary 3.8, but it is natural to formulate it for arbitrary symmetric surfaces (Σ, σ).

Remark 3.10. Two real line bundles $L_1^{\mathbb{R}}, L_2^{\mathbb{R}} \longrightarrow Y$ are isomorphic if and only if $w_1(L_1^{\mathbb{R}}) = w_1(L_2^{\mathbb{R}})$, provided Y is paracompact. In such a case, there is a canonical homotopy class of isomorphisms between $2L_1^{\mathbb{R}}$ and $2L_2^{\mathbb{R}}$. If $V^{\mathbb{R}} \longrightarrow Y$ is an oriented vector bundle, a spin structure on $V^{\mathbb{R}} \oplus 2L_1^{\mathbb{R}}$ thus corresponds to a spin structure on $V^{\mathbb{R}} \oplus 2L_2^{\mathbb{R}}$. The proofs of Proposition 3.5 and Corollaries 3.6 and 3.8(1) imply that the stabilization orientation on the right-hand side of (3.18) induced by a spin structure on $V^\varphi \oplus 2L^{\widetilde\phi}$ depends only on $w_1(L^{\widetilde\phi})$ and this spin structure, and not on $(L, \widetilde\phi)$ itself.

3.3. Some applications

We now make a number of explicit statements concerning orientations of the determinants of real CR-operators on real bundle pairs over $(\mathbb{P}^1,\tau)$ and $(\mathbb{P}^1,\eta)$. The proofs of these statements, which are useful for computational purposes and are applied in [13], are in the spirit of Section 3.2.

Let $\gamma_1^{\mathbb{R}}\longrightarrow\mathbb{RP}^1$ denote the tautological line bundle. For $f:\mathbb{RP}^1\longrightarrow\mathrm{GL}_k\mathbb{R}$, define

$$\Psi_f:\mathbb{RP}^1\times\mathbb{R}^k\longrightarrow\mathbb{RP}^1\times\mathbb{R}^k \quad\text{by}\quad \Psi_f(z,v)=\big(z,f(z)v\big).$$

Denote by $I_k^-\in\mathrm{O}(k)$ the diagonal matrix with the first diagonal entry equal to -1 and the remaining diagonal entries equal to 1.

Lemma 3.11. *Let $k,m\in\mathbb{Z}^{\ge0}$. If $k\ge2$, every automorphism Ψ of the real vector bundle*

$$V_{k,m}\equiv\big(\mathbb{RP}^1\times\mathbb{R}^k\big)\oplus m\gamma_1^{\mathbb{R}}\longrightarrow\mathbb{RP}^1$$

is homotopy equivalent to an automorphism of the form $\Psi_f\oplus\mathrm{Id}_{m\gamma_1^{\mathbb{R}}}$ for some $f:\mathbb{RP}^1\longrightarrow\mathrm{O}(k)$; any two such maps f differ by an even multiple of a generator of $\pi_1(\mathrm{SO}(k))$. If $m\ge1$, the automorphism Ψ negating a $\gamma_1^{\mathbb{R}}$ component is not homotopic to $\Psi_f\oplus\mathrm{Id}_{m\gamma_1^{\mathbb{R}}}$ for any constant map f. If $m\ge2$, the interchange Ψ of two of the $\gamma_1^{\mathbb{R}}$ components is not homotopic to $\Psi_f\oplus\mathrm{Id}_{m\gamma_1^{\mathbb{R}}}$ for any constant map f.

Proof. Let $I_k^+=I_k$, $x_0\in\mathbb{RP}^1$ be any point, and

$$\mathrm{Aut}^{\pm}_{x_0}(V_{k,m})\equiv\big\{\Psi\in\mathrm{Aut}(V_{k,m}):\ \Psi_{x_0}=I_k^{\pm}\oplus I_{m\gamma_1^{\mathbb{R}}|_{x_0}}\big\}.$$

Since $\mathrm{O}(k+m)$ has two connected components, one containing I_{k+m}^+ and the other I_{k+m}^-, it is sufficient to establish the first two claims of this lemma for an automorphism $\Psi\in\mathrm{Aut}^{\pm}_{x_0}(V_{k,m})$.

Since every line bundle over the interval $\mathbb{I}\equiv[0,1]$ is trivial,

$$\mathrm{Aut}^{\pm}_{x_0}(V_{k,m})\approx\big\{f\in C(\mathbb{I};\mathrm{O}(k+m)):\ f(0),f(1)=I_{k+m}^{\pm}\big\}. \tag{3.30}$$

The first claim thus follows from the map

$$\pi_1\big(\mathrm{O}(k),I_k^{\pm}\big)\longrightarrow\pi_1\big(\mathrm{O}(k+m),I_{k+m}^{\pm}\big)$$

induced by the natural inclusion $\mathrm{O}(k)\longrightarrow\mathrm{O}(k+m)$ being surjective for $k\ge2$. The second claim follows from the kernel of this map being the even multiples of a generator of $\pi_1(\mathrm{SO}(k))$.

By rotating in the fibers of $2\gamma_1^{\mathbb{R}}$, the interchange of the two components of $2\gamma_1^{\mathbb{R}}$ can be homotoped to the automorphism negating the first component and leaving the second component unchanged. Thus, the last claim of the lemma follows from the third. It is sufficient to establish the latter for $k \geq 1$.

We first consider the $(k,m)=(1,1)$ case of the third claim. Since every line bundle over $\mathbb{I}$ is trivial,

$$V_{1,1} = \big(\mathbb{I}\times\mathbb{C}\big)/\sim, \quad (1,c) \sim \big(0,\bar{c}\big)\ \forall\ c\in\mathbb{C}.$$

With respect to this identification, the relevant automorphism Ψ is given by

$$\Psi\colon V_{1,1} \longrightarrow V_{1,1}, \quad \Psi\big([t,c]\big) = \big[t,\bar{c}\big].$$

For each $s\in\mathbb{R}$, define an automorphism Ψ_s of $V_{1,1}$ by

$$\Psi_s\colon V_{1,1} \longrightarrow V_{1,1}, \quad \Psi_s\big([t,c]\big) = \big[t,\mathrm{e}^{\mathrm{i}\pi(1-2t)s}\bar{c}\big].$$

The family $(\Psi_s)_{s\in[0,1]}$ is a homotopy from the automorphism Ψ of $V_{1,1}$ to the element of $\mathrm{Aut}_{x_0}^-(V_{1,1})$ corresponding to the map

$$f\colon (\mathbb{I},0,1) \longrightarrow \big(\mathrm{O}(2), I_2^-, I_2^-\big), \quad t \longrightarrow \mathrm{e}^{-2\pi\mathrm{i}t} I_2^-\,,$$

under the identification (3.30). Since f is a generator of $\pi_1(\mathrm{O}(2), I_2^-)\approx\mathbb{Z}$, its image under the homomorphism

$$\pi_1\big(\mathrm{O}(2), I_2^-\big) \longrightarrow \pi_1\big(\mathrm{O}(k+m), I_{k+m}^-\big)$$

induced by the natural inclusion $\mathrm{O}(2)\longrightarrow\mathrm{O}(k+m)$ is non-trivial. This implies the last claim. □

Let $a\in\mathbb{Z}^{\geq 0}$, $(L,\widetilde{\phi})$ be a rank 1 degree $1+2a$ real bundle pair over $(\mathbb{P}^1,\tau)$, and D_L be a real CR-operator on $(L,\widetilde{\phi})$. Fix a nonzero vector $e\in T_0\mathbb{P}^1$. The homomorphism

$$\mathrm{ev}_{L;0} : \ker D_L \longrightarrow (1+a)L|_0, \quad \mathrm{ev}_{L;0}(\xi) = \big(\xi(0), \nabla_e\xi, \ldots, \nabla_e^{\otimes a}\xi\big),$$

is then an isomorphism. It thus induces an orientation on $\det D_L$ from the complex orientation of $L|_0$; we will call the former the complex orientation of $\det D_L$.

Let $(L_0,\widetilde{\phi}_0)$ be a rank 1 real bundle pair over $(\mathbb{P}^1,\tau)$ of degree 1. If $(L_1,\widetilde{\phi}_1)$ and $(L_2,\widetilde{\phi}_2)$ are rank 1 real bundle pairs over $(\mathbb{P}^1,\tau)$ of odd degrees,

the composition of the isomorphism ψ_0 in (3.23) with the isomorphism

$$L_1^{\widetilde{\phi}_1} \oplus L_2^{\widetilde{\phi}_2} \approx L_0^{\widetilde{\phi}_0} \oplus L_0^{\widetilde{\phi}_0}$$

induced by isomorphisms on each component determines an orientation on

$$\det D_{L_1 \oplus L_2} \approx \big(\det D_{L_1}\big) \otimes \big(\det D_{L_2}\big)$$

via the isomorphism (3.18) with Σ^b being the unit disk around $0 \in \mathbb{C}$. By the third statement of Lemma 3.11, changing the homotopy class of a component isomorphism would change the orientation and the spin of the induced trivialization and thus would have no effect on the induced orientation. This is also implied by the next statement.

Corollary 3.12. *Suppose $a_1, a_2 \in \mathbb{Z}^{\geq 0}$, $(L_1, \widetilde{\phi}_1)$ and $(L_2, \widetilde{\phi}_2)$ are rank 1 real bundle pairs over $(\mathbb{P}^1, \tau)$ of degrees $1+2a_1$ and $1+2a_2$, respectively, and D_{L_1} and D_{L_2} are real CR-operators on $(L_1, \widetilde{\phi}_1)$ and $(L_2, \widetilde{\phi}_2)$. The orientations on $\det(D_{L_1} \oplus D_{L_2})$ induced by the isomorphism ψ_0 in (3.23) and by the complex orientations on $\det(D_{L_1})$ and $\det(D_{L_2})$ are the same.*

Proof. The construction of the orientation on the determinant line induced by a trivialization of the real part of the bundle in the proofs of [7, Theorem 8.1.1] and [20, Lemma 6.37] commutes with the evaluations at the interior points; these can be used to reduce the degree of the bundle. Thus, it is sufficient to consider the case $a_1, a_2 = 0$. The latter is Proposition 3.5. □

Suppose

$$0 \longrightarrow (V, \varphi) \longrightarrow (V_\bullet, \varphi_\bullet) \oplus (V_c, \varphi_c) \longrightarrow (\mathcal{L}, \widetilde{\phi}) \longrightarrow 0 \tag{3.31}$$

is an exact sequence of real bundle pairs over $(\mathbb{P}^1, \tau)$ such that $V_\bullet^{\varphi_\bullet} \longrightarrow S^1$ is orientable of rank $k \geq 2$ and

$$(V_c, \varphi_c) = \bigoplus_{i=1}^m \big(V_{c;i}, \phi_{c;i}\big) \quad \text{and} \quad (\mathcal{L}, \widetilde{\phi}) = \bigoplus_{i=1}^m \big(L_i, \widetilde{\phi}_i\big)$$

are direct sums of rank 1 real vector bundle pairs of odd positive degrees. By Lemma 3.11, the short exact sequence (3.31) and a trivialization of $V_\bullet^{\varphi_\bullet}$ determine a homotopy class of trivializations of V^φ up to

(1) simultaneous flips of the orientation and the spin,

(2) composition with an even multiple of a generator of $\pi_1(\mathrm{SO}(k))$.

Via the isomorphism (3.18) with Σ^b being the unit disk around $0\in\mathbb{C}$, a trivialization of $V_\bullet^{\varphi_\bullet}$ thus determines an orientation of the determinant of a real CR-operator D_V on the real bundle pair (V,φ). It also determines an orientation of the determinant of a real CR-operator $D_{V_\bullet}$ on the real bundle pair $(V_\bullet,\varphi_\bullet)$. A short exact sequence

$$0\longrightarrow D_V\longrightarrow D_{V_\bullet}\oplus D_{V_c}\longrightarrow D_{\mathcal{L}}\longrightarrow 0 \tag{3.32}$$

of real CR-operators on the real bundle pairs in (3.31) gives rise to an isomorphism

$$\det\big(D_V\big)\otimes\det\big(D_{\mathcal{L}}\big)\approx\det\big(D_{V_\bullet}\big)\otimes\det\big(D_{V_c}\big)\,. \tag{3.33}$$

Corollary 3.13. *The isomorphism (3.33) is orientation-preserving with respect to*

- *the orientations on* $\det(D_V)$ *and* $\det(D_{V_\bullet})$ *induced by a trivialization of* $V_\bullet^{\varphi_\bullet}$ *and*
- *the complex orientations on* $\det(D_{\mathcal{L}})$ *and* $\det(D_{V_c})$.

Proof. Since the claim is invariant under augmenting (V_c,φ_c) and $(\mathcal{L},\widetilde{\phi})$ by the same rank 1 real bundle pair of odd positive degree, we can assume that $m=2m'$ for some $m'\in\mathbb{Z}^{\geq 0}$. By Corollary 3.12, the complex orientations on $\det(D_{\mathcal{L}})$ and $\det(D_{V_c})$ are then induced by the trivializations $m'\psi_0$ of $\mathcal{L}^{\widetilde{\phi}}$ and $V_c^{\varphi_c}$. The short exact sequence (3.31) determines a homotopy class of isomorphisms of real bundle pairs

$$(V,\varphi)\oplus(\mathcal{L},\widetilde{\phi})\approx(V_\bullet,\varphi_\bullet)\oplus(V_c,\varphi_c) \tag{3.34}$$

over $(\mathbb{P}^1,\tau)$. By the above, the orientations on

$$\begin{aligned}\det\big(D_V\oplus D_{\mathcal{L}}\big)&=\det\big(D_V\big)\otimes\det\big(D_{\mathcal{L}}\big)\quad\text{and}\\ \det\big(D_{V_\bullet}\oplus D_{V_c}\big)&=\det\big(D_{V_\bullet}\big)\otimes\det\big(D_{V_c}\big)\end{aligned} \tag{3.35}$$

specified in the statement of this corollary are induced by homotopy classes of trivializations of the real bundles

$$V^{\varphi}\oplus\mathcal{L}^{\widetilde{\phi}},V_\bullet^{\varphi_\bullet}\oplus V_c^{\varphi_c}\longrightarrow S^1$$

that are identified under the isomorphism (3.34) restricted to the real parts of the bundles. The isomorphism (3.33) is orientation-preserving with respect to these orientations. □

We will next obtain an analogue of Corollary 3.13 for real bundle pairs over $(\mathbb{P}^1,\eta)$. Define a $\mathbb{C}$-antilinear automorphism of $\mathbb{C}^2$ by

$$\mathfrak{c}_\eta\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \quad \mathfrak{c}_\eta\big(v_1,v_2\big) = \big(\bar v_2, -\bar v_1\big);$$

it has order 4. Let

$$\gamma = \mathcal{O}_{\mathbb{P}^1}(-1) \equiv \big\{(\ell,v)\in\mathbb{P}^1\times\mathbb{C}^2\colon\ v\in\ell\subset\mathbb{C}^2\big\}$$

denote the tautological line bundle. For $a\in\mathbb{Z}^+$, the involution η lifts to a conjugation on $2\gamma^{\otimes a}$ as

$$\widetilde\eta_{1,1}^{(-a)}\big(\ell, v^{\otimes a}, w^{\otimes a}\big) = \big(\eta(\ell), (\mathfrak{c}_\eta(w))^{\otimes a}, (-\mathfrak{c}_\eta(v))^{\otimes a}\big).$$

We denote the induced conjugations on

$$2\mathcal{O}_{\mathbb{P}^1}(a) = \big(2\gamma^{\otimes a}\big)^* \quad\text{and}\quad \mathcal{O}_{\mathbb{P}^1}(2a) \equiv \Lambda^2_{\mathbb{C}}\big(2\mathcal{O}_{\mathbb{P}^1}(a)\big)$$

by $\widetilde\eta_{1,1}^{(a)}$ and $\widetilde\eta_1^{(2a)}$, respectively. We note that $\widetilde\eta_{1,1}^{(2a)} \approx 2\widetilde\eta_1^{(2a)}$.

Let $a\in\mathbb{Z}^{\ge 0}$ and D_a be the real CR-operator on $(2\mathcal{O}_{\mathbb{P}^1}(1+2a), \widetilde\eta_{1,1}^{(1+2a)})$ induced by the standard $\bar\partial$-operator on $2\mathcal{O}_{\mathbb{P}^1}(1+2a)$. Fix a holomorphic connection ∇ on $\mathcal{O}_{\mathbb{P}^1}(1+2a)$ and a nonzero vector $e\in T_0\mathbb{P}^1$. The homomorphism

$$\begin{aligned}\mathrm{ev}_{a;0} : \ker D_a &\longrightarrow \big((1+a)\mathcal{O}_{\mathbb{P}^1}(1+2a)|_0\big)\oplus\big((1+a)\mathcal{O}_{\mathbb{P}^1}(1+2a)|_0\big),\\ \mathrm{ev}_{a;0}(\xi_1,\xi_2) &= \big((\xi_1(0),\nabla_e\xi_1,\ldots,\nabla_e^{\otimes a}\xi_1),(\xi_2(0),\nabla_e\xi_2,\ldots,\nabla_e^{\otimes a}\xi_2)\big),\end{aligned}$$

is then an isomorphism. It thus induces an orientation on $\det D_a$ from the complex orientation of $\mathcal{O}_{\mathbb{P}^1}(1+2a)|_0$; we will call the former the complex orientation of $\det D_a$.

As before, denote by $S^1\subset\mathbb{P}^1$ and $\Sigma^b\subset\mathbb{P}^1$ the unit circle and the unit disk around $0\in\mathbb{P}^1$, respectively. Let ψ_0' be the trivialization of $(2\mathcal{O}_{\mathbb{P}^1}(1),\widetilde\eta_{1,1}^{(1)})$ over S^1 given by

$$\begin{aligned}\psi_0'\big(\alpha_1,\alpha_2\big) &= \begin{pmatrix} \mathfrak{i}\alpha_1(1,z) - \mathfrak{i}z^{-1}\alpha_2(1,z)\\ \alpha_1(1,z)+z^{-1}\alpha_2(1,z)\end{pmatrix}\in\mathbb{C}^2\\ &\forall\ (\alpha_1,\alpha_2)\in 2\mathcal{O}_{\mathbb{P}^1}(1)|_z,\ z\in S^1.\end{aligned}\tag{3.36}$$

This is a component of the composite trivialization appearing in the proof of [5, Proposition 6.2]. The next statement is the analogue of Proposition 3.5 in this setting.

Corollary 3.14. *The orientation on* $\det D_0^b$ *induced by the trivialization* ψ_0' *as in the proof of [5, Lemma 2.5] agrees with the complex orientation.*

We give three proofs of this statement; they correspond to the three proofs of Proposition 3.5.

Proof 1. We denote by p_1 and p_2 the two standard holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(1)$:

$$p_1\big(\ell,(v_1,v_2)\big)=v_1, \quad p_2\big(\ell,(v_1,v_2)\big)=v_2 \quad \forall\ \big(\ell,(v_1,v_2)\big)\in\gamma.$$

The complex orientation for $\det D_0^b$ is determined by the basis

$$s_{11}\equiv(p_1,p_2), \quad s_{12}\equiv(\mathfrak{i}p_1,-\mathfrak{i}p_2), \quad s_{21}\equiv(-p_2,p_1), \quad s_{22}\equiv(\mathfrak{i}p_2,\mathfrak{i}p_1),$$

for the kernel of the surjective operator D_0^b.

The trivialization ψ_0' extends as a trivialization Ψ_0' of $(2\mathcal{O}_{\mathbb{P}^1}(1),\widetilde{\eta}_{1,1}^{(1)})$ over $\mathbb{P}^1-\{0,\infty\}$ by the same formula. We note that

$$\begin{aligned}
\big\{\Psi_0's_{11}\big\}(z)&=(0,2),\\
\big\{\Psi_0's_{12}\big\}(z)&=(-2,0),\\
\big\{\Psi_0's_{21}\big\}(z)&=\big(-\mathfrak{i}z^{-1}(1+z^2),z^{-1}(1-z^2)\big),\\
\big\{\Psi_0's_{22}\big\}(z)&=\big(z^{-1}(1-z^2),\mathfrak{i}z^{-1}(1+z^2)\big).
\end{aligned}$$

The orientation on $\det D_0^b$ induced by the trivialization ψ_0' is obtained from the isomorphism

$$\begin{aligned}
\ker D_0^b&\longrightarrow \mathbb{R}\oplus\mathbb{R}\oplus\big\{\mathrm{Res}_{z=0}(\Psi_0'\xi)\colon\ \xi\in\ker D_0^b\big\},\\
\xi&\longrightarrow \big(\mathrm{Re}(\{\Psi_0'\xi\}(1)),\mathrm{Res}_{z=0}(\Psi_0'\xi)\big).
\end{aligned}$$

The last space above is a complex subspace of $\mathbb{C}^2$. Under this isomorphism, the basis $s_{11},s_{12},s_{21},s_{22}$ is sent to

$$(0,2;0,0), \quad (-2,0;0,0), \quad (0,0;-\mathfrak{i},1), \quad (0,0;1,\mathfrak{i}).$$

Thus, an oriented basis for the target of the above isomorphism is given by

$$(2,0;0,0), \quad (0,2;0,0), \quad (0,0;-\mathfrak{i},1), \quad (0,0;1,\mathfrak{i}).$$

The change of basis matrix from the first basis to this one is given by

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The determinant of this matrix is $+1$. □

Proof 2. Define

$$\eta_3 \colon \mathbb{P}^3 \longrightarrow \mathbb{P}^3, \quad [Z_1, Z_2, Z_3, Z_4] \longrightarrow \big[\overline{Z}_2, -\overline{Z}_1, \overline{Z}_4, -\overline{Z}_3\big],$$
$$\mathfrak{M}_1(\mathbb{P}^1) = \mathfrak{M}_1(\mathbb{P}^1, 1)^{\eta,\eta}, \quad \mathfrak{M}_1(\mathbb{P}^3) = \mathfrak{M}_1(\mathbb{P}^3, 1)^{\eta_3,\eta}.$$

We now proceed through the first two paragraphs of the second proof of Proposition 3.5 replacing τ, τ_3, and D^b_{2L} by η, η_3, and D^b_0, respectively. By [5, Lemma 5.3], the isomorphism (3.24) is still orientation-*reversing* with respect to the algebraic orientations on $\mathfrak{M}_1(\mathbb{P}^1)$ in $\mathfrak{M}_1(\mathbb{P}^3)$ defined in [5, Section 5.2]. The isomorphism (3.25) is now orientation-*preserving* with respect to the complex orientation on the left-hand side.

Along with Euler's sequence for $\mathbb{P}^1$ and the orienting procedure of [5, Lemma 2.5], the trivialization ψ'_0 determines an orientation on $\mathfrak{M}_1(\mathbb{P}^1)$; we will call it the ψ'_0-orientation. Since the top exterior power of the real bundle pair

$$2\big((2\mathcal{O}_{\mathbb{P}^1}(1), \widetilde{\eta}^{(1)}_{1,1})\big|_{S^1} \longrightarrow \big(S^1, \eta|_{S^1}\big) \subset (\mathbb{P}^1, \eta) \tag{3.37}$$

is canonically a square, it admits a canonical homotopy class of trivializations; see Lemma 2.4 and Section 5.5 in [5]. Along with Euler's sequence for $\mathbb{P}^3$, it determines an orientation on $\mathfrak{M}_1(\mathbb{P}^3)$; we will call it the square root orientation. Along with the ψ'_0-orientation on $\mathfrak{M}_1(\mathbb{P}^1)$, it induces an orientation on $\mathcal{N}_{[\iota,0]}\mathfrak{M}$; we will call it the ψ'_0-orientation. Since the square root orientation on $\mathfrak{M}_1(\mathbb{P}^3)$ agrees with the orientation induced by the trivialization $2\psi'_0$ of (3.37), the first isomorphism in (3.25) is orientation-*preserving* with respect to the orientation on the left-hand side induced by ψ'_0 and the ψ'_0-orientation on $\mathcal{N}_{[\iota,0]}\mathfrak{M}$.

As summarized in the paragraph above [5, Remark 6.9], the algebraic orientations on $\mathfrak{M}_1(\mathbb{P}^1)$ and $\mathfrak{M}_1(\mathbb{P}^3)$ are the same as the ψ'_0-orientation and the opposite of the square root orientation, respectively. Therefore, the ψ'_0-orientation on $\mathcal{N}_{[\iota,0]}\mathfrak{M}$ is the opposite of the algebraic orientation. Since the second isomorphism in (3.25) is orientation-reversing with respect to the

latter, it follows that the composite isomorphism in (3.25) is orientation-preserving with respect to the orientation on the left-hand side induced by ψ_0'. Since this is also the case with respect to the complex orientation on the left-hand side, these two orientations on $\ker D_0^b$ agree. □

Proof 3. The reasoning in the third proof of Proposition 3.5 with τ and D_{2L}^b replaced by η and D_0^b, respectively, applies without any changes, except [5, (6.13)] is no longer relevant. □

By [5, Lemma 2.4], the homotopy classes of trivializations of

$$(2\mathcal{O}_{\mathbb{P}^1}(1+2a), \widetilde{\eta}_{1,1}^{(1+2a)})$$

over S^1 correspond to the homotopy classes of trivializations of

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}\big(2\mathcal{O}_{\mathbb{P}^1}(1+2a), \widetilde{\eta}_{1,1}^{(1+2a)}\big) \approx \Lambda_{\mathbb{C}}^{\mathrm{top}}\big(2\mathcal{O}_{\mathbb{P}^1}(1), \widetilde{\eta}_{1,1}^{(1)}\big) \otimes \big(\mathcal{O}_{\mathbb{P}^1}(2a), \widetilde{\eta}_1^{(2a)}\big)^{\otimes 2} \tag{3.38}$$

over S^1. Since the last factor in (3.38) is a square, it has a canonical homotopy class of trivializations over S^1. Thus, the trivialization ψ_0' of the first factor on the right-hand side of (3.38) determines a homotopy class of trivializations of $(2\mathcal{O}_{\mathbb{P}^1}(1+2a), \widetilde{\eta}_{1,1}^{(1+2a)})$ over S^1 and thus an orientation on $\det D_a^b$. The next statement is the analogue of Corollary 3.12; it is deduced from Corollary 3.14 in the same way as Corollary 3.12 is obtained from Proposition 3.5.

Corollary 3.15. *Suppose $a \in \mathbb{Z}^{\geq 0}$. The orientation on $\det D_a^b$ induced by the trivialization ψ_0' as in the proof of [5, Lemma 2.5] agrees with the complex orientation.*

Suppose

$$0 \longrightarrow (V,\varphi) \longrightarrow (V_\bullet,\varphi_\bullet) \oplus (V_c,\varphi_c) \longrightarrow (\mathcal{L},\widetilde{\phi}) \longrightarrow 0 \tag{3.39}$$

is an exact sequence of real bundle pairs over $(\mathbb{P}^1,\eta)$ such that

$$(V_c,\varphi_c) = \bigoplus_{i=1}^{m} (2\mathcal{O}_{\mathbb{P}^1}(1+2a_i), \widetilde{\eta}_{1,1}^{(1+2a_i)}) \quad \text{and}$$
$$(\mathcal{L},\widetilde{\phi}) = \bigoplus_{i=1}^{m} (2\mathcal{O}_{\mathbb{P}^1}(1+2a_i'), \widetilde{\eta}_{1,1}^{(1+2a_i')})$$

for some $a_i, a_i' \in \mathbb{Z}^{\geq 0}$. Since the homotopy classes of trivializations of $(V_c,\varphi_c)|_{S^1}$ correspond to the homotopy classes of trivializations of $(\mathcal{L},\widetilde{\phi})|_{S^1}$,

a homotopy class of trivializations of $(V_\bullet,\varphi_\bullet)|_{S^1}$ determines a homotopy class of trivializations of $(V,\varphi)|_{S^1}$ via the exact sequence (3.39). Via the isomorphism (3.18) with Σ^b being the unit disk around $0\in\mathbb{C}$, a trivialization of $(V_\bullet,\varphi_\bullet)|_{S^1}$ thus determines an orientation of the determinant of a real CR-operator D_V on the real bundle pair (V,φ). It also determines an orientation of the determinant of a real CR-operator $D_{V_\bullet}$ on the real bundle pair $(V_\bullet,\varphi_\bullet)$. A short exact sequence (3.32) of real CR-operators on the real bundle pairs in (3.39) gives rise to an isomorphism as in (3.33).

Corollary 3.16. *The isomorphism (3.33) is orientation-preserving with respect to*

- *the orientations on* $\det(D_V)$ *and* $\det(D_{V_\bullet})$ *induced by a trivialization of* $(V_\bullet,\varphi_\bullet)$ *over* S^1 *and*
- *the complex orientations on* $\det(D_{\mathcal{L}})$ *and* $\det(D_{V_c})$.

Proof. By Corollary 3.15, the complex orientations on $\det(D_{\mathcal{L}})$ and $\det(D_{V_c})$ are induced by the trivializations $m\psi_0'$ of $(\mathcal{L},\widetilde{\phi})$ and (V_c,φ_c) over S^1. The short exact sequence (3.39) determines a homotopy class of isomorphisms (3.34) over $(\mathbb{P}^1,\eta)$. Thus, the orientations on (3.35) specified in the statement of this corollary are induced by homotopy classes of trivializations of the real bundle pairs

$$(V,\varphi)\oplus(\mathcal{L},\widetilde{\phi}),(V_\bullet,\varphi_\bullet)\oplus(V_c,\varphi_c)\longrightarrow\big(S^1,\eta|_{S^1}\big)$$

that are identified under the isomorphism (3.34) restricted to S^1. The isomorphism (3.33) is orientation-preserving with respect to these orientations. □

4. The compatibility of the canonical orientations

In this section, we establish Theorem 1.2. In order to do so, we study how each step in the construction of the orientation on $\mathfrak{M}_{g,l}(X,B;J)^\phi$ in [12, Section 5] extends across the strata consisting of maps from symmetric surfaces with a pair of conjugate nodes. The argument is similar to [12, Section 6], which studies the extendability of the orientation on $\mathfrak{M}_{g,l}(X,B;J)^\phi$ induced by a real orientation on (X,ω,ϕ) across the codimension-one strata. We also compare the resulting extensions with the corresponding objects over the normalizations.

4.1. Two-nodal symmetric surfaces

We begin by establishing Proposition 2.2 for symmetric surfaces with one pair of conjugate nodes. If (Σ,σ) is a symmetric surface, possibly nodal and disconnected, and G is a Lie group with a natural conjugation, such as $\mathbb{C}^*$, $\mathrm{SL}_n\mathbb{C}$, or $\mathrm{GL}_n\mathbb{C}$, denote by $\mathcal{C}(\Sigma,\sigma;G)$ the topological group of continuous maps $f\colon\Sigma\longrightarrow G$ such that $f(\sigma(z))=\overline{f(z)}$ for all $z\in\Sigma$. The restrictions of such functions to the fixed locus $\Sigma^\sigma\subset\Sigma$ take values in the real locus of G, i.e. $\mathbb{R}^*$, $\mathrm{SL}_n\mathbb{R}$, and $\mathrm{GL}_n\mathbb{R}$, in the three examples.

Lemma 4.1. *Suppose (Σ,σ) is a symmetric surface, possibly nodal and disconnected, $x\in\Sigma-\Sigma^\sigma$, and G is a connected Lie group with a natural conjugation. For every $f\in\mathcal{C}(\Sigma,\sigma;G)$ and an open neighborhood $U\subset\Sigma$ of x, there exists a path $f_t\in\mathcal{C}(\Sigma,\sigma;G)$ such that $f_0=f$, $f_1(x)=\mathrm{Id}$, and $f_t=f$ on $\Sigma-U\cup\sigma(U)$.*

Proof. By shrinking U, we can assume that $U\cap\sigma(U)=\emptyset$. Let $\rho\colon\Sigma\longrightarrow[0,1]$ be a smooth σ-invariant function so that $\rho(x)=1$ and $\rho=0$ on $\Sigma-U\cup\sigma(U)$. Choose a path $g_t\in G$ such that $g_0=\mathrm{Id}$ and $g_1=f(x)$. The path $f_t\in\mathcal{C}(\Sigma,\sigma;G)$ given by

$$f_t(z)=\begin{cases} g_{\rho(z)t}^{-1}f(z), & \text{if } z\in U;\\ \overline{g_{\rho(z)t}}^{-1}f(z), & \text{if } z\in\sigma(U);\\ f(z), & \text{if } z\notin U\cup\sigma(U);\end{cases}$$

has the desired properties. □

We will denote the nodes of a connected symmetric surface (Σ,σ) with one pair of conjugate nodes by $x_{12}^{\pm}$. A normalization of such $(\Sigma,x_{12}^{\pm},\sigma)$ is a smooth, possibly disconnected, symmetric surface $(\widetilde{\Sigma},\widetilde{\sigma})$ with two distinguished pairs of conjugate points, (x_1^+,x_1^-) and (x_2^+,x_2^-); the normalization map takes x_i^+ to x_{12}^+ and x_i^- to x_{12}^-.

Lemma 4.2. *Suppose (Σ,σ) is a connected symmetric surface with one pair of conjugate nodes, $n\in\mathbb{Z}^+$, and $f\in\mathcal{C}(\Sigma,\sigma;\mathrm{SL}_n\mathbb{C})$. If*

$$f|_{\Sigma^\sigma}\colon\Sigma^\sigma\longrightarrow\mathrm{SL}_n\mathbb{R}$$

is homotopic to a constant map, then f is homotopic to the constant map Id *through maps $f_t\in\mathcal{C}(\Sigma,\sigma;\mathrm{SL}_n\mathbb{C})$.*

Proof. By Lemma 4.1, we can assume that $f(x_{12}^+)=\mathrm{Id}$. Let $\widetilde{f}\in\mathcal{C}(\widetilde{\Sigma},\widetilde{\sigma};\mathrm{SL}_n\mathbb{C})$ be the function corresponding to $f\in\mathcal{C}(\Sigma,\sigma;\mathrm{SL}_n\mathbb{C})$. In particular,

$$\widetilde{f}(x_1^{\pm}),\widetilde{f}(x_2^{\pm})=\mathrm{Id}.$$

We proceed as in the proof of [12, Lemma 5.4], which contains a picture illustrating a similar argument. Choose a symmetric half-surface $\widetilde{\Sigma}^b\subset\widetilde{\Sigma}$ and a neighborhood $U\subset\widetilde{\Sigma}^b$ of $\partial\widetilde{\Sigma}^b$ so that either $x_1^+,x_2^+\in\widetilde{\Sigma}^b-U$ or $x_1^+,x_2^-\in\widetilde{\Sigma}^b-U$. Let $x_2=x_2^+$ in the first case, $x_2=x_2^-$ in the second case, and $x_1=x_1^+$ in both cases. Take the cutting paths C_i so that $x_1,x_2\notin C_i$ and the extensions of the homotopies of $\widetilde{f}$ from C_i to $\widetilde{\Sigma}^b$ so that they do not change $\widetilde{f}$ at x_1 or x_2. The surface D obtained by cutting $\widetilde{\Sigma}^b$ along these paths is either a disk D^2 or two disjoint copies of D^2. Choose disjoint embedded paths γ_1 and γ_2 in D as in the last paragraph of the proof of [12, Lemma 5.4] from ∂D to x_1 and x_2, respectively. Since $\widetilde{f}(x_i)=\mathrm{Id}$ in this case, we can homotope $\widetilde{f}$ to Id over γ_i while keeping it fixed at the endpoints. Similarly to the second paragraph in the proof of this lemma, this homotopy extends over D without changing $\widetilde{f}$ over ∂D or γ_{3-i} and thus descends to $\widetilde{\Sigma}^b$. We then cut D along γ_1 and γ_2 into another disk or a pair of disks and proceed as in the second half of the last paragraph in the proof of [12, Lemma 5.4]. The doubled homotopy in the proof of this lemma then satisfies $\widetilde{f}_t(x_1^{\pm})=\widetilde{f}_t(x_2^{\pm})$ and so descends to Σ. □

Corollary 4.3. *Let (Σ,σ) be a connected symmetric surface with one pair of conjugate nodes and*

$$\Phi,\Psi\colon(V,\varphi)\longrightarrow\big(\Sigma\times\mathbb{C}^n,\sigma\times\mathfrak{c}\big)$$

be isomorphisms of real bundle pairs over (Σ,σ). If the isomorphisms

$$\begin{aligned}\Phi|_{V^\varphi},\Psi|_{V^\varphi}\colon V^\varphi&\longrightarrow\Sigma\times\mathbb{R}^n,\\ \Lambda^{\mathrm{top}}_{\mathbb{C}}\Phi,\Lambda^n_{\mathbb{C}}\Psi\colon\Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\varphi)&\longrightarrow\Lambda^{\mathrm{top}}_{\mathbb{C}}\big(\Sigma\times\mathbb{C}^n,\sigma\times\mathfrak{c}\big)=\big(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c}\big)\end{aligned}$$

are homotopic, then so are the isomorphisms Φ and Ψ.

Proof. The first paragraph of the proof of [12, Corollary 5.5] applies without any changes. The second paragraph applies with [12, Lemma 5.4] replaced by Lemma 4.2 above. □

Lemma 4.4. *Proposition 2.2 holds for connected symmetric surfaces with one pair of conjugate nodes.*

Proof. Let $\widetilde{V},\widetilde{L}\longrightarrow\widetilde{\Sigma}$ be complex vector bundles and

$$\psi_1\colon \widetilde{V}\big|_{x_1^\pm}\longrightarrow\widetilde{V}\big|_{x_2^\pm}\quad\text{and}\quad \psi_2\colon \widetilde{L}\big|_{x_1^\pm}\longrightarrow\widetilde{L}\big|_{x_2^\pm}$$

be isomorphisms of complex vector spaces such that

$$V=\widetilde{V}/\!\sim,\ v\sim\psi_1(v)\ \forall\, v\in\widetilde{V}\big|_{x_1^\pm},\quad\text{and}\quad L=\widetilde{L}/\!\sim,\ v\sim\psi_2(v)\ \forall\, v\in\widetilde{L}\big|_{x_1^\pm}.$$

Denote by $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ the lift of φ to $\widetilde{V}$ and the lift of $\widetilde{\phi}$ to $\widetilde{L}$, respectively. Define

$$\big(\widetilde{W},\widetilde{\varphi}_{12}\big)=\big(\widetilde{V}\oplus 2\widetilde{L}^*,\widetilde{\varphi}_1\oplus 2\widetilde{\varphi}_2^*\big),\quad \psi_{12}=\psi_1\oplus 2(\psi_2^{-1})^*\colon \widetilde{W}\big|_{x_1^\pm}\longrightarrow\widetilde{W}\big|_{x_2^\pm}.$$

Thus, $(\widetilde{V},\widetilde{\varphi}_1)$ and $(\widetilde{L},\widetilde{\varphi}_2)$ are real bundle pairs over $(\widetilde{\Sigma},\widetilde{\sigma})$ that descend to the real bundle pairs (V,φ) and $(L,\widetilde{\phi})$ over (Σ,σ). Furthermore,

$$\psi_{12}\circ\widetilde{\varphi}_{12}=\widetilde{\varphi}_{12}\circ\psi_{12}\,. \tag{4.1}$$

For any $f\in\mathcal{C}(\widetilde{\Sigma},\widetilde{\sigma};\mathrm{GL}_{n+2}\mathbb{C})$, let

$$\widetilde{\Psi}_f\colon\big(\widetilde{\Sigma}\times\mathbb{C}^{n+2},\widetilde{\sigma}\times\mathfrak{c}\big)\longrightarrow\big(\widetilde{\Sigma}\times\mathbb{C}^{n+2},\widetilde{\sigma}\times\mathfrak{c}\big),\quad \widetilde{\Psi}_f(z,v)=\big(z,f(z)v\big).$$

Let $\widetilde{\sigma}'(x_i^\pm)=x_{3-i}^\pm$ for $i=1,2$.

The choices (RO2) and (RO3) in Definition 1.1 for (Σ,σ) lift to $(\widetilde{\Sigma},\widetilde{\sigma})$. By [12, Proposition 5.2], there thus exists an isomorphism

$$\widetilde{\Phi}\colon(\widetilde{W},\widetilde{\varphi}_{12})\longrightarrow\big(\widetilde{\Sigma}\times\mathbb{C}^{n+2},\widetilde{\sigma}\times\mathfrak{c}\big)$$

of real bundle pairs over $(\widetilde{\Sigma},\widetilde{\sigma})$ that lies in the homotopy class determined by the lifted real orientation. It satisfies the spin structure requirement of Proposition 2.2. By the proof of [12, Proposition 5.2], $\widetilde{\Phi}$ can be chosen so that it induces the isomorphism in (2.9) over $(\widetilde{\Sigma},\widetilde{\sigma})$ determined by the lift of a given isomorphism in (1.1) over (Σ,σ). This implies that

$$\begin{aligned}\big\{\widetilde{\sigma}'\times\mathrm{id}\big\}\circ\big\{\Lambda^{\mathrm{top}}_{\mathbb{C}}\widetilde{\Phi}\big\}&=\big\{\Lambda^{\mathrm{top}}_{\mathbb{C}}\widetilde{\Phi}\big\}\circ\big\{\Lambda^{\mathrm{top}}_{\mathbb{C}}\psi_{12}\big\}\colon\Lambda^{\mathrm{top}}_{\mathbb{C}}\widetilde{W}\big|_{x_1^\pm}\\&\longrightarrow\{x_2^\pm\}\times\Lambda^{\mathrm{top}}_{\mathbb{C}}\mathbb{C}^{n+2}=\{x_2^\pm\}\times\mathbb{C}.\end{aligned} \tag{4.2}$$

In the next paragraph, we homotope $\widetilde{\Phi}$ near $x_1^\pm$ so that it descends to an isomorphism Ψ over Σ; the latter satisfies the two properties in the last sentence of Proposition 2.2. By Corollary 4.3, any two such isomorphisms Ψ are homotopic.

Define $\psi^{\pm}\in\mathrm{GL}_{n+2}\mathbb{C}$ by

$$\mathrm{id}\times\psi^{\pm}=\big\{\widetilde{\sigma}'\times\mathrm{Id}\big\}\circ\widetilde{\Phi}\circ\psi_{12}\circ\widetilde{\Phi}^{-1}\colon\{x_1^{\pm}\}\times\mathbb{C}^{n+2}\longrightarrow\{x_1^{\pm}\}\times\mathbb{C}^{n+2}\,. \tag{4.3}$$

By (4.2), $\det_{\mathbb{C}}\psi^{\pm}=1$, i.e. $\psi\in\mathrm{SL}_{n+2}\mathbb{C}$. By (4.1), $\overline{\psi^{+}}=\psi^{-}$. Since $\mathrm{SL}_{n+2}\mathbb{C}$ is connected, there exist $f\in\mathcal{C}(\widetilde{\Sigma},\widetilde{\sigma};\mathrm{SL}_{n+2}\mathbb{C})$ and a neighborhood U of x_1^{+} in $\widetilde{\Sigma}$ such that

$$f(z)=\begin{cases}\psi^{\pm}, & \text{if } z=x_1^{\pm};\\ \mathrm{Id}, & \text{if } z\notin U\cup\widetilde{\sigma}(U);\end{cases}\qquad x_2^{\pm}\notin U,\quad U\cap\widetilde{\sigma}(U)=\emptyset. \tag{4.4}$$

By (4.3) and (4.4),

$$\big\{\widetilde{\sigma}'\times\mathrm{Id}\big\}\circ\widetilde{\Psi}_f\circ\widetilde{\Phi}=\widetilde{\Psi}_f\circ\widetilde{\Phi}\circ\psi_{12}\colon\widetilde{W}|_{x_1^{\pm}}\longrightarrow\{x_2^{\pm}\}\times\mathbb{C}^{n+2}\,.$$

Thus, $\widetilde{\Psi}_f\circ\widetilde{\Phi}$ descends to an isomorphism Ψ in (2.8) of real bundle pairs over (Σ,σ) that induces the isomorphism in (2.9) determined by a given isomorphism in (1.1). □

Suppose $(\Sigma,x_{12}^{\pm},\sigma)$ and $(\widetilde{\Sigma},x_1^{\pm},x_2^{\pm},\widetilde{\sigma})$ are as above. A rank n real bundle pair (V,φ) over (Σ,σ) lifts to a rank n real bundle pair $(\widetilde{V},\widetilde{\varphi})$ over $(\widetilde{\Sigma},\widetilde{\sigma})$. A real orientation on (V,φ) lifts to a real orientation on $(\widetilde{V},\widetilde{\varphi})$. A real CR-operator D on (V,φ) lifts to a real CR-operator $\widetilde{D}$ on $(\widetilde{V},\widetilde{\varphi})$. There is a short exact sequence of Fredholm operators

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\Sigma;V)^{\varphi} & \longrightarrow & \Gamma(\widetilde{\Sigma};\widetilde{V})^{\widetilde{\varphi}} & \xrightarrow{\mathrm{ev}_{x_{12}^{+}}} & V_{x_{12}^{+}} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \widetilde{D}} & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{\mathrm{j}}^{0,1}(\Sigma;V)^{\varphi} & \longrightarrow & \Gamma_{\mathrm{j}}^{0,1}(\widetilde{\Sigma};\widetilde{V})^{\widetilde{\varphi}} & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{4.5}$$

with the last homomorphism in the top row given by

$$\mathrm{ev}_{x_{12}^{+}}(\xi)=\xi(x_1^{+})-\xi(x_2^{+})\in V_{x_{12}^{+}}=\widetilde{V}_{x_1^{+}}=\widetilde{V}_{x_2^{+}}\,.$$

Thus, there is a canonical isomorphism

$$\det\widetilde{D}\approx\det D\otimes\Lambda^{2n}_{\mathbb{R}}V_{x_{12}^{+}} \tag{4.6}$$

of real lines.

If Ψ is an isomorphism as in (2.8) and $\widetilde{\Psi}$ is its lift to $(\widetilde{\Sigma},\widetilde{\sigma})$, then the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma\big(\Sigma;V\oplus 2L^*\big) & \xrightarrow{\varphi\oplus 2\widetilde{\phi}^*} & \Gamma\big(\widetilde{\Sigma};\widetilde{V}\oplus 2\widetilde{L}^*\big) & \xrightarrow{\widetilde{\varphi}\oplus 2\widetilde{\phi}^*} & V_{x_{12}^+}\oplus 2L^*_{x_{12}^+} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \widetilde{\Psi}} & & \downarrow{\scriptstyle \Psi} & & \\
0 & \longrightarrow & \mathcal{C}(\Sigma,\sigma;\mathbb{C}^{n+2}) & \longrightarrow & \mathcal{C}(\widetilde{\Sigma},\widetilde{\sigma};\mathbb{C}^{n+2}) & \xrightarrow{\mathrm{ev}_{x_{12}^+}} & \mathbb{C}^{n+2} & \longrightarrow & 0
\end{array}$$

commutes. If Ψ is an isomorphism as in (2.8) in the homotopy class determined by a real orientation on (V,φ), then the lift $\widetilde{\Psi}$ of Ψ to $(\widetilde{V},\widetilde{\varphi})$ lies in the homotopy class of isomorphisms determined by the induced real orientation on $(\widetilde{V},\widetilde{\varphi})$. These two observations yield the following comparison of the orientations on the relative determinants provided by Corollary 2.3.

Corollary 4.5. *Let (Σ,σ), $(\widetilde{\Sigma},\widetilde{\sigma})$, (V,φ), and $(\widetilde{V},\widetilde{\varphi})$ be as above. The isomorphism*

$$\widehat{\det}\,\widetilde{D}\approx\big(\widehat{\det}\,D\big)\otimes\Lambda^{2n}_{\mathbb{R}}V_{x_{12}^+}\otimes\Lambda^{2n}_{\mathbb{R}}\mathbb{C}^n \tag{4.7}$$

induced by the isomorphisms (4.6) for (V,φ) and $(\Sigma\times\mathbb{C}^n,\sigma\times\mathfrak{c})$ is orientation-preserving with respect to the orientation on $\widehat{\det}\,D$ determined by a real orientation on (V,φ), the orientation on $\widehat{\det}\,\widetilde{D}$ determined by the lifted real orientation on $(\widetilde{V},\widetilde{\varphi})$, and the complex orientations of $V_{x_{12}^+}$ and $\mathbb{C}^n$.

4.2. Smoothings of two-nodal symmetric surfaces

For a disk $\Delta\subset\mathbb{C}$ centered at the origin, let

$$\Delta^*=\Delta-\{0\},\quad \Delta^2_{\mathbb{R}}=\big\{(t,\bar{t})\colon t\in\Delta\big\},\quad \Delta^{*2}_{\mathbb{R}}=\Delta^{*2}\cap\Delta^2_{\mathbb{R}},$$
$$\tau_\Delta\colon\Delta^2\longrightarrow\Delta^2,\quad \tau_\Delta\big(t^+,t^-\big)=\big(\overline{t^-},\overline{t^+}\big).$$

Thus, $\Delta^2_{\mathbb{R}}$ is the fixed locus of the anti-complex involution τ_Δ on Δ^2.

Let $\mathcal{C}\equiv(\Sigma,z_1,\ldots,z_l)$ be a marked Riemann surface with two nodes and $\pi\colon\mathcal{U}\longrightarrow\Delta^2$ be a holomorphic map from a complex manifold with sections $s_1,\ldots,s_l\colon\Delta^2\longrightarrow\mathcal{U}$. We will call the tuple $(\pi,s_1,\ldots,s_l)$ a smoothing of $\mathcal{C}$ if

- $\Sigma_{\mathbf{t}}\equiv\pi^{-1}(\mathbf{t})$ is a smooth compact Riemann surface for all $\mathbf{t}\in\Delta^{*2}$;
- $s_i(\mathbf{t})\neq s_j(\mathbf{t})$ for all $\mathbf{t}\in\Delta^2$ and $i\neq j$;
- $(\Sigma_0,s_1(0),\ldots,s_l(0))=\mathcal{C}$.

Suppose $\mathcal{C}\equiv(\Sigma,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-))$ is a marked symmetric Riemann surface with involution σ and a pair of conjugate nodes, $(\pi,s_1,\ldots,s_l)$ is as above, and $\widetilde{\tau}_\Delta:\mathcal{U}\longrightarrow\mathcal{U}$ is an anti-holomorphic involution lifting the involution τ_Δ. We will call the tuple $(\pi,\widetilde{\tau}_\Delta,s_1,\ldots,s_l)$ a smoothing of $\mathcal{C}$ if $(\pi,s_1,\widetilde{\tau}_\Delta\circ s_1,\ldots,s_l,\widetilde{\tau}_\Delta\circ s_l)$ is a smoothing of $\mathcal{C}$ and $\widetilde{\tau}_\Delta|_{\Sigma_0}=\sigma$. In such a case, let $\sigma_{\mathbf{t}}=\widetilde{\tau}_\Delta|_{\Sigma_{\mathbf{t}}}$ for each $\mathbf{t}\in\Delta^2_{\mathbb{R}}$.

With $(\pi,\widetilde{\tau}_\Delta,s_1,\ldots,s_l)$ as above, denote by $x_{12}^\pm\in\Sigma$ and $\widetilde{\Sigma}\longrightarrow\Sigma$ the nodes and the normalization of Σ, respectively, and set $\Sigma^*=\Sigma-\{x_{12}^\pm\}$. Let

$$\mathcal{U}_0^+\equiv\big\{(t^+,t^-,z_1^+,z_2^+)\in\Delta^2\times\mathbb{C}^2\colon\ |z_1^+|,|z_2^+|<1,\ z_1^+z_2^+=t^+\big\},$$
$$\mathcal{U}_0^-\equiv\big\{(t^+,t^-,z_1^-,z_2^-)\in\Delta^2\times\mathbb{C}^2\colon\ |z_1^-|,|z_2^-|<1,\ z_1^-z_2^-=t^-\big\}.$$

As fibrations over Δ,

$$\mathcal{U}\approx\big(\mathcal{U}_0^+\sqcup\mathcal{U}_0^-\sqcup\mathcal{U}'\big)/\sim,\quad(\mathbf{t},z_1^\pm,z_2^\pm)\sim\begin{cases}(\mathbf{t},z_1^\pm),&\text{if }|z_1^\pm|>|z_2^\pm|;\\(\mathbf{t},z_2^\pm),&\text{if }|z_1^\pm|<|z_2^\pm|;\end{cases}\tag{4.8}$$

for some family $\mathcal{U}'$ of deformations of Σ^* over Δ^2, a choice of coordinates $z_i^\pm$ on $\widetilde{\Sigma}$ centered at $x_i^\pm$, and their extensions to $\mathcal{U}$. The local coordinates $z_i^\pm$ and the family $\mathcal{U}'$ in (4.8) can be chosen so that $\mathcal{U}'$ is preserved by $\widetilde{\tau}_\Delta$ and the identification in (4.8) intertwines $\widetilde{\tau}_\Delta$ with the involution

$$\mathcal{U}_0^\pm\longrightarrow\mathcal{U}_0^\mp,\quad\big(t^+,t^-,z_1^\pm,z_2^\pm\big)\longrightarrow\big(\overline{t^-},\overline{t^+},\overline{z_1^\pm},\overline{z_2^\pm}\big).\tag{4.9}$$

In particular, $\mathcal{U}$ retracts onto Σ_0 respecting the involution $\widetilde{\tau}_\Delta$.

Suppose $\pi:\mathcal{U}\longrightarrow\Delta^2$ and $\widetilde{\tau}_\Delta$ are as above, $(V,\varphi)\longrightarrow(\mathcal{U},\widetilde{\tau}_\Delta)$ is a real bundle pair, and ∇ and A are a connection and a 0-th order deformation term on (V,φ) as in Section 2.2. The restriction of ∇ and A to $(V,\varphi)|_{(\Sigma_{\mathbf{t}},\sigma_{\mathbf{t}})}$ with $\mathbf{t}\in\Delta^2_{\mathbb{R}}$ determines a real CR-operator $D_{\mathbf{t}}$. By [16, Appendix D.4] and [4, Section 3.2], the determinant lines of these operators form a line bundle

$$\det D_{(V,\varphi)}\longrightarrow\Delta^2_{\mathbb{R}}\,.\tag{4.10}$$

We denote by $\det\bar\partial_{\mathbb{C}}\longrightarrow\Delta^2_{\mathbb{R}}$ the determinant line bundle associated with the standard holomorphic structure on $(\mathcal{U}\times\mathbb{C},\widetilde{\tau}_\Delta\times\mathfrak{c})$. The proof of the next statement is essentially identical to the proof of [12, Corollary 6.7], with Lemma 4.4 replacing the use of [12, Proposition 6.2].

Corollary 4.6. *Let $(\pi,\widetilde{\tau}_\Delta)$, (V,φ), and (∇,A) be as above. Then a real orientation on (V,φ) as in Definition 1.1 induces an orientation on the line*

bundle

$$\widehat{\det} D_{(V,\varphi)} \equiv \big(\det D_{(V,\varphi)}\big) \otimes (\det \bar\partial_{\mathbb{C}})^{\otimes n} \longrightarrow \Delta^2_{\mathbb{R}}, \tag{4.11}$$

where $n=\mathrm{rk}_{\mathbb{C}}V$. *The restriction of this orientation to the fiber over each* $\mathbf{t}\in\Delta^{*2}_{\mathbb{R}}$ *is the orientation on* $\widehat{\det} D_{\mathbf{t}}$ *induced by the restriction of the real orientation to* $(V,\varphi)|_{(\Sigma_{\mathbf{t}},\sigma_{\mathbf{t}})}$ *as in Corollary 2.3.*

Let (Σ,σ) be a smooth symmetric surface and $(L,\widetilde\phi)$ be a rank 1 real bundle pair over (Σ,σ). For a pair $\mathbf{x}\equiv(x^+,x^-)$ of conjugate points of (Σ,σ), define

$$\begin{aligned} L(\mathbf{x}) = L\big(x^+ + x^-\big), \quad L^{\otimes 2}(\mathbf{x}) &= L\otimes_{\mathbb{C}} L(\mathbf{x}),\\ \big\{L(\mathbf{x})^{\otimes 2}\big\}^0_{\mathbf{x}} &= \big(L(\mathbf{x})^{\otimes 2}|_{x^+} \oplus L(\mathbf{x})^{\otimes 2}|_{x^-}\big)^{\widetilde\phi^{\otimes 2}},\\ \big\{L(\mathbf{x})^{\otimes 2}\big\}^1_{\mathbf{x}} &= \big\{L(\mathbf{x})^{\otimes 2}\big\}^0_{\mathbf{x}} \oplus \big(L^{\otimes 2}(\mathbf{x})|_{x^+} \oplus L^{\otimes 2}(\mathbf{x})|_{x^-}\big)^{\widetilde\phi^{\otimes 2}}. \end{aligned}$$

The projection

$$\big\{L(\mathbf{x})^{\otimes 2}\big\}^1_{\mathbf{x}} \longrightarrow \big\{L(\mathbf{x})^{\otimes 2}\big\}^1_{x^+} \equiv L(\mathbf{x})^{\otimes 2}|_{x^+} \oplus L^{\otimes 2}(\mathbf{x})|_{x^+}$$

is an isomorphism of real vector spaces and thus induces an orientation on its domain from the complex orientation of its target. This induced orientation is invariant under the interchange of x^+ and x^-; we will call it the canonical orientation of $\{L(\mathbf{x})^{\otimes 2}\}^1_{\mathbf{x}}$.

For a real CR-operator $D_{L(\mathbf{x})^{\otimes 2}}$ on $(L(\mathbf{x}),\widetilde\phi)^{\otimes 2}$, there is a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma\big(\Sigma; L^{\otimes 2}(\mathbf{x})\big)^{\widetilde\phi} & \longrightarrow & \Gamma\big(\Sigma; L(\mathbf{x})^{\otimes 2}\big)^{\widetilde\phi} & \longrightarrow & \big\{L(\mathbf{x})^{\otimes 2}\big\}^0_{\mathbf{x}} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle D_{L^{\otimes 2}(\mathbf{x})}} & & \downarrow{\scriptstyle D_{L(\mathbf{x})^{\otimes 2}}} & & \downarrow & & \\ 0 & \longrightarrow & \Gamma^{0,1}_{\mathrm{j}}\big(\Sigma; L^{\otimes 2}(\mathbf{x})\big)^{\widetilde\phi} & \longrightarrow & \Gamma^{0,1}_{\mathrm{j}}\big(\Sigma; L(\mathbf{x})^{\otimes 2}\big)^{\widetilde\phi} & \longrightarrow & 0 & & \end{array}$$

of Fredholm operators. By (2.4), it induces a canonical isomorphism

$$\det D_{L(\mathbf{x})^{\otimes 2}} \approx \big(\det D_{L^{\otimes 2}(\mathbf{x})}\big) \otimes \Lambda^2_{\mathbb{R}}\big(\big\{L(\mathbf{x})^{\otimes 2}\big\}^0_{\mathbf{x}}\big). \tag{4.12}$$

The analogous exact sequence for an operator $D_{L^{\otimes 2}(\mathbf{x})}$ on $(L^{\otimes 2}(\mathbf{x}),\widetilde\phi^{\otimes 2})$ yields an isomorphism

$$\det D_{L^{\otimes 2}(\mathbf{x})} \approx \big(\det D_{L^{\otimes 2}}\big) \otimes \Lambda^2_{\mathbb{R}}\big(\big(L^{\otimes 2}(\mathbf{x})|_{x^+} \oplus L^{\otimes 2}(\mathbf{x})|_{x^-}\big)^{\widetilde\phi^2}\big). \tag{4.13}$$

Combining these two isomorphisms with the identity isomorphism on $\det \bar\partial_{\Sigma;\mathbb{C}}$, we obtain an isomorphism

$$\widehat{\det}\, D_{L(\mathbf{x})^{\otimes 2}} \approx \big(\widehat{\det}\, D_{L^{\otimes 2}}\big) \otimes \Lambda^4_{\mathbb{R}}\big(\big\{L(\mathbf{x})^{\otimes 2}\big\}^1_{\mathbf{x}}\big). \tag{4.14}$$

Corollary 4.7. *With notation as above, suppose the real vector bundle $L^{\widetilde\phi} \longrightarrow \Sigma^\sigma$ is orientable. The isomorphism (4.14) is orientation-preserving with respect to the orientations induced by Corollaries 2.3 and 2.4 on $\widehat{\det}\, D_{L(\mathbf{x})^{\otimes 2}}$ and $\widehat{\det}\, D_{L^{\otimes 2}}$ and the canonical orientation on $\{L(\mathbf{x})^{\otimes 2}\}^1_{\mathbf{x}}$.*

Proof. Let $(\widehat\Sigma, \widehat\sigma)$ be the two-nodal symmetric surface consisting of (Σ,σ) with a 0-doublet $\mathbb{P}^1_+ \sqcup \mathbb{P}^1_-$ attached at x^+ and x^-; see (1.7). Let $\widehat{\mathbf{x}} \equiv (\widehat{x}^+, \widehat{x}^-)$ be a pair of conjugate points on $\widehat\Sigma - \Sigma$, with $\widehat{x}^+ \in \mathbb{P}^1_+$, and $(\widehat{L}, \widehat\phi)$ be the rank 1 real bundle pair over $(\widehat\Sigma, \widehat\sigma)$ such that

$$\big(\widehat{L}, \widehat\phi\big)\big|_\Sigma = \big(L, \widetilde\phi\big), \quad \big(\widehat{L}, \widehat\phi\big)\big|_{\mathbb{P}^1_+ \sqcup \mathbb{P}^1_-} = \big(\mathcal{O}_{\mathbb{P}^1_+} \sqcup \mathcal{O}_{\mathbb{P}^1_-}, \widehat\sigma|_{\mathbb{P}^1_+ \sqcup \mathbb{P}^1_-} \times \mathfrak{c}\big).$$

Choose a smoothing

$$\pi\colon \mathcal{U} \longrightarrow \Delta^2, \quad \widetilde\tau_\Delta\colon \mathcal{U} \longrightarrow \mathcal{U}, \quad s\colon \Delta^2 \longrightarrow \mathcal{U}$$

of $(\widehat\Sigma, (\widehat{x}^+, \widehat{x}^-), \widehat\sigma)$. For $\mathbf{t} \in \Delta^{*2}_{\mathbb{R}}$, $(\Sigma_{\mathbf{t}}, \sigma_{\mathbf{t}}) \approx (\Sigma, \sigma)$.

Let (V, φ) be a real bundle pair over $(\mathcal{U}, \widetilde\tau_\Delta)$ that restricts to $(\widehat{L}, \widehat\phi)$ over $\widehat\Sigma$ and

$$V(\mathbf{s}) = V\big(s + \widetilde\tau_\Delta \circ s\big).$$

For $\mathbf{t} \in \Delta^{*2}_{\mathbb{R}}$, the restrictions of the real bundle pairs (V,φ) and $(V(\mathbf{s}),\varphi)$ to $(\Sigma_{\mathbf{t}}, \sigma_{\mathbf{t}})$ are isomorphic to $(L, \widetilde\phi)$ and $(L(\mathbf{x}), \widetilde\phi)$, respectively. The canonical real orientations on $(\widehat{L}, \widehat\phi)^{\otimes 2}$ and $(\widehat{L}(\widehat{\mathbf{x}}), \widehat\phi)^{\otimes 2}$ provided by Corollary 2.4 (like all other real orientations) extend to real orientations on $(V,\varphi)^{\otimes 2}$ and $(V(\mathbf{s}),\varphi)^{\otimes 2}$, respectively. The restrictions of the latter to

- $(\Sigma_{\mathbf{t}}, \sigma_{\mathbf{t}})$ with $\mathbf{t} \in \Delta^{*2}_{\mathbb{R}}$ are the canonical real orientations on $(L, \widetilde\phi)^{\otimes 2}$ and $(L(\mathbf{x}), \widetilde\phi)^{\otimes 2}$, respectively,
- $\Sigma \subset \widehat\Sigma$ are the canonical real orientation on $(L, \widetilde\phi)^{\otimes 2}$,
- $\mathbb{P}^1_+ \sqcup \mathbb{P}^1_-$ are the canonical real orientations on

  $$\big(\mathcal{O}_{\mathbb{P}^1_+} \sqcup \mathcal{O}_{\mathbb{P}^1_-}, \widehat\sigma|_{\mathbb{P}^1_+ \sqcup \mathbb{P}^1_-} \times \mathfrak{c}\big)^{\otimes 2} \quad \text{and} \quad \big(\mathcal{O}_{\mathbb{P}^1_+}(\widehat{x}_+) \sqcup \mathcal{O}_{\mathbb{P}^1_-}(\widehat{x}_-), \widehat\sigma|_{\mathbb{P}^1_+ \sqcup \mathbb{P}^1_-} \times \mathfrak{c}\big)^{\otimes 2},$$

  respectively.

Let $D_{(V,\varphi)^{\otimes 2};\mathbf{t}}$ be a family of real CR-operators on $(V,\varphi)^{\otimes 2}$ as above Corollary 4.6; we can assume that it restricts to the standard $\bar\partial$-operator over the 0-doublet. It induces a family $D_{(V(\mathbf{s}),\varphi)^{\otimes 2};\mathbf{t}}$ of real CR-operators on $(V(\mathbf{s}),\varphi)^{\otimes 2}$. Let

$$D_{\widehat{L}^{\otimes 2}} = D_{(V,\varphi)^{\otimes 2};\mathbf{0}} \quad \text{and} \quad D_{\widehat{L}(\widehat{\mathbf{x}})^{\otimes 2}} = D_{(V(\mathbf{s}),\varphi)^{\otimes 2};\mathbf{0}}$$

be the restrictions of these operators to $(\widehat{\Sigma},\widehat{\sigma})$. Similarly to (4.12),

$$\widehat{\det} D_{(V(\mathbf{s}),\varphi)^{\otimes 2}} \approx \big(\widehat{\det} D_{(V,\varphi)^{\otimes 2}}\big) \otimes \Lambda^4_{\mathbb{R}}\big(\big\{V(\mathbf{s})^{\otimes 2}\big\}^1_{\mathbf{s}}\big)$$

as real line bundles over $\mathbf{t}\in\Delta^2_{\mathbb{R}}$. By the first bullet point above and Corollary 4.6, it is thus sufficient to show that the isomorphism

$$\widehat{\det} D_{\widehat{L}(\widehat{\mathbf{x}})^{\otimes 2}} \approx \big(\widehat{\det} D_{\widehat{L}^{\otimes 2}}\big) \otimes \Lambda^4_{\mathbb{R}}\big(\big\{\widehat{L}(\widehat{\mathbf{x}})^{\otimes 2}\big\}^1_{\widehat{\mathbf{x}}}\big) \tag{4.15}$$

is orientation-preserving with respect to the orientations on $\widehat{\det} D_{\widehat{L}(\widehat{\mathbf{x}})^{\otimes 2}}$ and $\widehat{\det} D_{\widehat{L}^{\otimes 2}}$ induced by the canonical real orientations on $(\widehat{L}(\widehat{\mathbf{x}}),\widehat{\phi})^{\otimes 2}$ and $(\widehat{L},\widehat{\phi})^{\otimes 2}$, respectively, and the complex orientation on $\{\widehat{L}(\widehat{\mathbf{x}})^{\otimes 2}\}^1_{\widehat{\mathbf{x}}}$.

Let $(\widetilde{L},\widetilde{\varphi})$ be the lift of $(\widehat{L},\widehat{\phi})$ to the normalization $(\widetilde{\Sigma},\widetilde{\sigma})$ of $(\widehat{\Sigma},\widehat{\sigma})$; the latter consists of (Σ,σ) and the 0-doublet $\mathbb{P}^1_+\sqcup\mathbb{P}^1_-$. The isomorphisms (4.7) induce a commutative diagram

$$\begin{array}{ccc}
\widehat{\det} D_{\widetilde{L}(\widehat{\mathbf{x}})^{\otimes 2}} & \longrightarrow & \widehat{\det} D_{\widehat{L}(\widehat{\mathbf{x}})^{\otimes 2}} \otimes \Lambda^2_{\mathbb{R}} L_{x^+} \otimes \Lambda^2_{\mathbb{R}}\mathbb{C} \\
\downarrow & & \downarrow \\
\big(\widehat{\det} D_{\widetilde{L}^{\otimes 2}}\big) \otimes \Lambda^4_{\mathbb{R}}\big(\big\{\widetilde{L}(\widehat{\mathbf{x}})^{\otimes 2}\big\}^1_{\widehat{\mathbf{x}}}\big) & \longrightarrow & \big(\widehat{\det} D_{\widehat{L}^{\otimes 2}}\big) \otimes \Lambda^2_{\mathbb{R}} L_{x^+} \otimes \Lambda^2_{\mathbb{R}}\mathbb{C} \otimes \Lambda^4_{\mathbb{R}}\big(\big\{\widehat{L}(\widehat{\mathbf{x}})^{\otimes 2}\big\}^1_{\widehat{\mathbf{x}}}\big).
\end{array}$$

By the second and third bullet points above and Corollary 4.5, the horizontal isomorphisms in this diagram are orientation-preserving with respect to the orientations on the relative determinants induced by Corollaries 2.3 and 2.4 and with respect to the complex orientations on the remaining lines.

The left vertical isomorphism in the diagram is the tensor product of the isomorphisms

$$\begin{aligned}
\widehat{\det} D_{\widetilde{L}\left(\widehat{\mathbf{x}})^{\otimes 2}|_{\Sigma}\right.} &\approx \widehat{\det} D_{\widetilde{L}^{\otimes 2}|_{\Sigma}} \quad \text{and} \\
\widehat{\det} D_{\widetilde{L}\left(\widehat{\mathbf{x}})^{\otimes 2}|_{\mathbb{P}^1_+\sqcup\mathbb{P}^1_-}\right.} &\approx \big(\widehat{\det} D_{\widetilde{L}^{\otimes 2}|_{\mathbb{P}^1_+\sqcup\mathbb{P}^1_-}}\big) \otimes \Lambda^4_{\mathbb{R}}\big(\big\{\widetilde{L}(\widehat{\mathbf{x}})^{\otimes 2}\big\}^1_{\widehat{\mathbf{x}}}\big).
\end{aligned} \tag{4.16}$$

The first of these isomorphisms is orientation-preserving with respect to the canonical real orientations because $\widetilde{L}(\widehat{\mathbf{x}})|_{\Sigma}=\widetilde{L}|_{\Sigma}$. Under the restrictions

to $\mathbb{P}^1_+$ as in (3.2), the second isomorphism in (4.16) corresponds to an isomorphism induced by two short exact sequences of $\mathbb{C}$-linear homomorphisms. Thus, it is orientation-preserving with respect to the complex orientations on $\widehat{\det} D_{\widetilde{L}(\widetilde{\mathbf{x}})^{\otimes 2}|_{\mathbb{P}^1_+\sqcup\mathbb{P}^1_-}}$ and $\widehat{\det} D_{\widetilde{L}^{\otimes 2}|_{\mathbb{P}^1_+\sqcup\mathbb{P}^1_-}}$ as in Section 3.1. By Lemma 3.1, these complex orientations are the same as the orientations induced by any real orientations on the squares. Thus, the second isomorphism in (4.16) and the left vertical isomorphism in the commutative diagram are orientation-preserving with respect to the orientations on the relative determinants induced by Corollaries 2.3 and 2.4. Along with the last sentence of the previous paragraph, this implies that the right vertical isomorphism is also orientation-preserving with respect to these orientations. □

Let $\widetilde{\Sigma}$ be a smooth Riemann surface and $x\in\widetilde{\Sigma}$. A holomorphic vector field ξ on a neighborhood of x in $\widetilde{\Sigma}$ with $\xi(x)=0$ determines an element

$$\nabla\xi\big|_x \in T^*_x\widetilde{\Sigma}\otimes_{\mathbb{C}} T_x\widetilde{\Sigma}=\mathbb{C}\,.$$

Similarly, a meromorphic one-form η on a neighborhood of x in $\widetilde{\Sigma}$ has a well-defined residue at x, which we denote by $\mathfrak{R}_x\eta$. For a holomorphic line bundle $L\longrightarrow\widetilde{\Sigma}$, we denote by $\Omega(L)$ the sheaf of holomorphic sections of L.

If (Σ,σ) is a symmetric Riemann surface with a pair of conjugate nodes $x^\pm_{12}\in\Sigma$ and $x^\pm_1,x^\pm_2\in\widetilde{\Sigma}$ are the preimages of the nodes in its normalization, let

$$\begin{aligned} T\widetilde{\Sigma}(-\mathbf{x}) &= T\widetilde{\Sigma}\big(-x_1^+-x_1^--x_2^+-x_2^-\big),\\ T^*\widetilde{\Sigma}(\mathbf{x}) &= T^*\widetilde{\Sigma}\big(x_1^++x_1^-+x_2^++x_2^-\big).\end{aligned}$$

The next statement is the analogue of [12, Lemma 6.8] in the present situation.

Lemma 4.8. *Suppose $(\pi:\mathcal{U}\longrightarrow\Delta^2,\widetilde{\tau}_\Delta)$ is a smoothing of (Σ,σ) as above. There exist holomorphic line bundles $\mathcal{T},\widehat{\mathcal{T}}\longrightarrow\mathcal{U}$ with involutions $\varphi,\widehat{\varphi}$ lifting $\widetilde{\tau}_\Delta$ such that*

$$\begin{aligned}(\mathcal{T},\varphi)\big|_{\Sigma_{\mathbf{t}}}=\big(T\Sigma_{\mathbf{t}},\mathrm{d}\widetilde{\tau}_\Delta|_{T\Sigma_{\mathbf{t}}}\big),\quad (\widehat{\mathcal{T}},\widehat{\varphi})\big|_{\Sigma_{\mathbf{t}}}=\big(T^*\Sigma_{\mathbf{t}},(\mathrm{d}\widetilde{\tau}_\Delta|_{T\Sigma_{\mathbf{t}}})^*\big)\quad\forall\,\mathbf{t}\in\Delta^{*2},\\ \Omega\big(\mathcal{T}|_{\Sigma_0}\big)=\big\{\xi\in\Omega\big(T\widetilde{\Sigma}(-\mathbf{x})\big):\ \nabla\xi|_{x_1^\pm}+\nabla\xi|_{x_2^\pm}=0\big\},\\ \Omega\big(\widehat{\mathcal{T}}|_{\Sigma_0}\big)=\big\{\eta\in\Omega\big(T^*\widetilde{\Sigma}(\mathbf{x})\big):\ \mathfrak{R}_{x_1^\pm}\eta+\mathfrak{R}_{x_2^\pm}\eta=0\big\}.\end{aligned}$$

Furthermore, $(\widehat{\mathcal{T}},\widehat{\varphi})\approx(\mathcal{T},\varphi)^*$.

Proof. We continue with the notation as in (4.8) and (4.9). Denote by $T^{\mathrm{vrt}}\mathcal{U}' \longrightarrow \mathcal{U}'$ the vertical tangent bundle. Let

$$\begin{aligned}
\mathcal{T} &= \big(\mathcal{U}_0^+ \times \mathbb{C} \sqcup \mathcal{U}_0^- \times \mathbb{C} \sqcup T^{\mathrm{vrt}}\mathcal{U}'\big)/\sim,\\
\widehat{\mathcal{T}} &= \big(\mathcal{U}_0^+ \times \mathbb{C} \sqcup \mathcal{U}_0^- \times \mathbb{C} \sqcup (T^{\mathrm{vrt}}\mathcal{U}')^*\big)/\sim,\\
(\mathbf{t}, z_1^\pm, z_2^\pm, c) &\sim \begin{cases} c\, z_1^\pm \frac{\partial}{\partial z_1^\pm}\big|_{(\mathbf{t}, z_1^\pm)}, & \text{if } |z_1^\pm| > |z_2^\pm|;\\ -c\, z_2^\pm \frac{\partial}{\partial z_2^\pm}\big|_{(\mathbf{t}, z_2^\pm)}, & \text{if } |z_1^\pm| < |z_2^\pm|; \end{cases}\\
(\mathbf{t}, z_1^\pm, z_2^\pm, c) &\sim \begin{cases} c\, \frac{\mathrm{d}_{(\mathbf{t}, z_1^\pm)} z_1^\pm}{z_1^\pm}, & \text{if } |z_1^\pm| > |z_2^\pm|;\\ -c\, \frac{\mathrm{d}_{(\mathbf{t}, z_2^\pm)} z_2^\pm}{z_2^\pm}, & \text{if } |z_1^\pm| < |z_2^\pm|. \end{cases}
\end{aligned}$$

Under the identifications (4.8), the vector field and one-form on a neighborhood of the node in $\mathcal{U}$ associated with $(\mathbf{t}, z_1^\pm, z_2^\pm, c) \in \mathcal{U}_0^\pm \times \mathbb{C}$ correspond to the vector field and one-form on $\mathcal{U}_0^\pm$ given by

$$c\left(z_1^\pm \frac{\partial}{\partial z_1^\pm} - z_2^\pm \frac{\partial}{\partial z_2^\pm}\right) \quad \text{and} \quad c\, \frac{\mathrm{d} z_1^\pm|_{\Sigma_{\mathbf{t}}}}{z_1^\pm} = -c\, \frac{\mathrm{d} z_2^\pm|_{\Sigma_{\mathbf{t}}}}{z_2^\pm},$$

respectively (the above equality of one-forms holds for $t^\pm \neq 0$). Thus, $\mathcal{T}$ and $\widehat{\mathcal{T}}$ have the desired restriction properties. The identifications in the construction of $\mathcal{T}$ and $\widehat{\mathcal{T}}$ above intertwine the trivial lift of (4.9) to a conjugation on $(\mathcal{U}_0^+ \sqcup \mathcal{U}_0^-) \times \mathbb{C}$ with the conjugations on $T^{\mathrm{vrt}}\mathcal{U}'$ and $(T^{\mathrm{vrt}}\mathcal{U}')^*$ induced by $\mathrm{d}\widetilde{\tau}_\Delta$. Thus, they induce conjugations φ and $\widehat{\varphi}$ on $\mathcal{T}$ and $\widehat{\mathcal{T}}$. By the same reasoning as in the proof of [12, Lemma 6.8], $(\widehat{\mathcal{T}}, \widehat{\varphi})$ and $(\mathcal{T}, \varphi)^*$ are isomorphic as real bundle pairs over $(\mathcal{U}, \widetilde{\tau}_\Delta)$. □

Corollary 4.9. *Let (Σ, σ), $(\pi, \widetilde{\tau}_\Delta)$, and $\mathcal{T}, \widehat{\mathcal{T}} \longrightarrow \mathcal{U}$ be as in Lemma 4.8. The orientation on the restriction of the real line bundle*

$$\widehat{\det}\, \bar{\partial}_{(\widehat{\mathcal{T}}, \widehat{\varphi})^{\otimes 2}} \equiv \big(\det \bar{\partial}_{(\widehat{\mathcal{T}}, \widehat{\varphi})^{\otimes 2}}\big) \otimes \big(\det \bar{\partial}_{\mathbb{C}}\big) \longrightarrow \Delta^2_{\mathbb{R}} \tag{4.17}$$

to $\Delta^{*2}_{\mathbb{R}}$ determined by the canonical isomorphisms of Corollaries 2.3 and 2.4 extends across $\mathbf{t}=0$ as the orientation determined by the canonical isomorphism of Corollaries 2.3 and 2.4 for the nodal symmetric surface (Σ, σ).

Proof. By Corollaries 2.3 and 2.4, the restriction of the real bundle pair

$$\big(\widehat{\mathcal{T}}^{\otimes 2} \oplus 2\mathcal{T}, \widehat{\varphi}^{\otimes 2} \oplus 2\varphi\big) \longrightarrow (\mathcal{U}, \widetilde{\tau}_\Delta) \tag{4.18}$$

to the central fiber (Σ,σ) has a canonical real orientation. Since $\mathcal{U}$ retracts onto Σ respecting the involution $\widetilde{\tau}_\Delta$, this real orientation extends to a real orientation on (4.18) which restricts to the canonical real orientation over each fiber $(\Sigma_{\mathbf{t}},\sigma_{\mathbf{t}})$ with $\mathbf{t}\in\Delta^{*2}_{\mathbb{R}}$. By Corollary 4.6, the extended real orientation induces an orientation on the real line bundle (4.17) over $\Delta^2_{\mathbb{R}}$. The restriction of this orientation to the fiber over each $\mathbf{t}\in\Delta^{*2}_{\mathbb{R}}$ is the orientation induced by the restriction of the extended real orientation to the fiber of (4.18) as in Corollary 2.3, i.e. the canonical orientation on each fiber of (4.17). □

The next two statements are the analogues of [12, Lemmas 6.9,6.10] for smoothings of two-nodal Riemann surfaces and hold for the same reasons.

Lemma 4.10 (Dolbeault Isomorphism). *Suppose (Σ,σ) and $(\pi,\widetilde{\tau}_\Delta)$ are as in Lemma 4.8 and $(L,\widetilde{\phi})\longrightarrow(\mathcal{U},\widetilde{\tau}_\Delta)$ is a holomorphic line bundle so that $\deg L|_\Sigma<0$ and $\deg L|_{\Sigma'}\le 0$ for each irreducible component $\Sigma'\subset\Sigma$. The families of vector spaces $H^1_{\bar\partial}(\Sigma_{\mathbf{t}};L)$ and $\check{H}^1(\Sigma_{\mathbf{t}};L)$ then form vector bundles $R^1_{\bar\partial}\pi_*L$ and $\check{R}^1\pi_*L$ over Δ^2 with conjugations lifting τ_Δ which are canonically isomorphic as real bundle pairs over (Δ^2,τ_Δ).*

Lemma 4.11 (Serre Duality). *Suppose (Σ,σ), $(\pi,\widetilde{\tau}_\Delta)$, and $(\widehat{\mathcal{T}},\widehat{\varphi})$ are as in Lemma 4.8 and $(L,\widetilde{\phi})\longrightarrow(\mathcal{U},\widetilde{\tau}_\Delta)$ is a holomorphic line bundle so that $\deg L|_\Sigma>2g_a(\Sigma)-2$ and $\deg L|_{\Sigma'}\ge 2g_a(\Sigma')-2$ for each irreducible component $\Sigma'\subset\Sigma$. The family of vector spaces $H^0_{\bar\partial}(\Sigma_{\mathbf{t}};L)$ then forms a vector bundle $R^0_{\bar\partial}\pi_*L$ over Δ with a conjugation lifting τ_Δ and there is a canonical isomorphism*

$$R^1_{\bar\partial}\pi_*\big(L^*\otimes\widehat{\mathcal{T}}\big)\approx\big(R^0_{\bar\partial}\pi_*L\big)^* \tag{4.19}$$

of real bundle pairs over (Δ^2,τ_Δ).

4.3. Canonical isomorphisms and canonical orientations

Let $(\Sigma,x_{12}^{\pm},\sigma)$ be a symmetric surface with a pair of conjugate nodes. We will next compare the orientations of isomorphisms of determinant lines associated with (Σ,σ) which are induced via its smoothings $(\Sigma_{\mathbf{t}},\sigma_{\mathbf{t}})$ as in Section 4.2 and via its normalization $(\widetilde{\Sigma},\widetilde{\sigma})$. We continue with the notation introduced in Section 4.2.

Let $\mathbb{C}_{\mathbf{x}}, S_{\mathbf{x}} \longrightarrow \widetilde{\Sigma}$ denote the skyscraper sheaves with fibers $\mathbb{C}$ at the preimages $x_i^{\pm}$ of the nodes of Σ and fibers $T^*_{x_i^{\pm}}\widetilde{\Sigma}$, respectively. The projections

$$\begin{aligned} H^0\big(\widetilde{\Sigma};\mathbb{C}_{\mathbf{x}}\big)^{\widetilde{\sigma}} &\longrightarrow \mathbb{C}^2 = H^0\big(\widetilde{\Sigma};\mathbb{C}_{x_1^+}\big)\oplus H^0\big(\widetilde{\Sigma};\mathbb{C}_{x_2^+}\big), \\ H^0\big(\widetilde{\Sigma};S_{\mathbf{x}}\big)^{\widetilde{\sigma}} &\longrightarrow T^*_{x_1^+}\widetilde{\Sigma}\oplus T^*_{x_2^+}\widetilde{\Sigma} = H^0\big(\widetilde{\Sigma};S_{x_1^+}\big)\oplus H^0\big(\widetilde{\Sigma};S_{x_2^+}\big) \end{aligned} \tag{4.20}$$

to the values at x_1^+ and x_2^+ are isomorphisms. We use the first isomorphism to orient $H^0(\widetilde{\Sigma};\mathbb{C}_{\mathbf{x}})^{\widetilde{\sigma}}$ from the standard orientation on $\mathbb{C}$. We use the second isomorphism to orient $H^0(\widetilde{\Sigma};S_{\mathbf{x}})^{\widetilde{\sigma}}$ from the orientations on $T^*_{x_1^+}\widetilde{\Sigma}$ and $T^*_{x_2^+}\widetilde{\Sigma}$ induced from the complex orientations on $T_{x_1^+}\widetilde{\Sigma}$ and $T_{x_2^+}\widetilde{\Sigma}$, respectively, as in the proof of Lemma 3.2. As indicated at the beginning of Section 3.1, the orientation on each $T^*_{x_i^+}\widetilde{\Sigma}$ induced from the complex orientation of $T_{x_i^+}\widetilde{\Sigma}$ is the opposite of the complex orientation of $T^*_{x_i^+}\widetilde{\Sigma}$. Thus, the induced orientation on $T^*_{x_1^+}\widetilde{\Sigma}\oplus T^*_{x_2^+}\widetilde{\Sigma}$ agrees with the complex orientation.

The residues of meromorphic one-forms on $\widetilde{\Sigma}$ provide canonical identifications

$$T^*\widetilde{\Sigma}(\mathbf{x})\big|_{x_i^+} \approx \mathbb{C}\,.$$

With the notation as in Corollary 4.7, we thus have

$$\begin{aligned} \big\{T^*\widetilde{\Sigma}(\mathbf{x})^{\otimes 2}\big\}^1_{\mathbf{x}} &\equiv \big\{T^*\widetilde{\Sigma}(\mathbf{x})^{\otimes 2}\big\}^1_{(x_1^+,x_1^-)}\oplus\big\{T^*\widetilde{\Sigma}^{\otimes 2}(\mathbf{x})\big\}^1_{(x_2^+,x_2^-)} \\ &= H^0\big(\widetilde{\Sigma};\mathbb{C}_{\mathbf{x}}\big)^{\widetilde{\sigma}}\oplus H^0\big(\widetilde{\Sigma};S_{\mathbf{x}}\big)^{\widetilde{\sigma}}. \end{aligned}$$

With $L=T^*\widetilde{\Sigma}$, (4.14) becomes

$$\begin{aligned} \widehat{\det}\bar\partial_{(T^*\widetilde{\Sigma}(\mathbf{x}),(\mathrm{d}\widetilde{\sigma})^*)^{\otimes 2}} \approx \big(\widehat{\det}\bar\partial_{(T^*\widetilde{\Sigma},(\mathrm{d}\widetilde{\sigma})^*)^{\otimes 2}}\big) \\ \otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\otimes\Lambda^4_{\mathbb{R}}\big(H^0\big(\widetilde{\Sigma};S_{\mathbf{x}}\big)^{\widetilde{\sigma}}\big). \end{aligned} \tag{4.21}$$

Let $\mathbb{C}_{x_{12}^{\pm}} \longrightarrow \Sigma$ be the skyscraper sheaf over $x_{12}^{\pm}$. By Lemma 4.8, there is an exact sequence

$$0\longrightarrow\mathcal{O}(\widehat{\mathcal{T}}^{\otimes 2})\longrightarrow\mathcal{O}\big(T^*\widetilde{\Sigma}(\mathbf{x})^{\otimes 2}\big)\longrightarrow\mathbb{C}_{x_{12}^+}\oplus\mathbb{C}_{x_{12}^-}\longrightarrow 0$$

of sheaves over Σ. Thus, (4.6) applied with $(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_{\Sigma}$ and $(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c})$ determines an isomorphism

$$\big(\widehat{\det}\,\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_{\Sigma}}\big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\approx\widehat{\det}\,\bar\partial_{(T^*\widetilde{\Sigma}(\mathbf{x}),(\mathrm{d}\widetilde{\sigma})^*)^{\otimes 2}}\,. \tag{4.22}$$

Combining this isomorphism with (4.21), we obtain an isomorphism

$$\begin{aligned}(4.23)\qquad &\big(\widehat{\det}\,\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_\Sigma}\big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\\ &\approx\big(\widehat{\det}\,\bar\partial_{(T^*\widetilde\Sigma,(\mathrm{d}\widetilde\sigma)^*)^{\otimes 2}}\big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\otimes\Lambda^4_{\mathbb{R}}\big(H^0(\widetilde\Sigma;S_{\mathbf{x}})^{\widetilde\sigma}\big).\end{aligned}$$

Corollary 4.12. *Suppose (Σ,σ), $(\widetilde\Sigma,\widetilde\sigma)$, $(\pi,\widetilde\tau_\Delta)$, and $\mathcal{T},\widehat{\mathcal{T}}\longrightarrow\mathcal{U}$ are as in Lemma 4.8. The isomorphism (4.23) is orientation-preserving with respect to*

- *the canonical orientation of Corollary 4.9 on $\widehat{\det}\,\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_\Sigma}$,*
- *the canonical orientation of Corollaries 2.3 and 2.4 on $\widehat{\det}\,\bar\partial_{(T^*\widetilde\Sigma,(\mathrm{d}\widetilde\sigma)^*)^{\otimes 2}}$,*
- *the orientation on $H^0(\widetilde\Sigma;S_{\mathbf{x}})^{\widetilde\sigma}$ described above and the complex orientation on $\mathbb{C}$.*

Proof. The canonical orientation of Corollary 4.9 on $\widehat{\det}\,\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_\Sigma}$ is the orientation induced by the canonical real orientation on the restriction of $(\widehat{\mathcal{T}},\widehat{\varphi})$ to Σ. The latter lifts to the canonical real orientation on the real bundle pair

$$(4.24)\qquad \big(T^*\widetilde\Sigma(\mathbf{x}),(\mathrm{d}\widetilde\sigma)^*\big)^{\otimes 2}\longrightarrow(\widetilde\Sigma,\widetilde\sigma).$$

By Corollary 4.5, the isomorphism (4.22) is thus orientation-preserving with respect to the orientation in the first bullet item above, the complex orientation on $\mathbb{C}$, and the orientation on $\widehat{\det}\,\bar\partial_{(T^*\widetilde\Sigma(\mathbf{x}),(\mathrm{d}\widetilde\sigma)^*)^{\otimes 2}}$ induced by the canonical real orientation on (4.24). By Corollary 4.7, the isomorphism (4.21) is orientation-preserving with respect to the latter and the orientations in the second and third bullet items above. The last two statements together imply the claim. □

Let $(\pi,\widetilde\tau_\Delta,s_1,\ldots,s_l)$ be a smoothing of a marked symmetric Riemann surface

$$(4.25)\qquad \mathcal{C}\equiv\big(\Sigma,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-)\big)$$

with a pair of conjugate nodes, $\mathcal{T},\widehat{\mathcal{T}}\longrightarrow\mathcal{U}$ be the holomorphic line bundles with involutions $\varphi,\widehat\varphi$ as in Lemma 4.8, and

$$\begin{aligned}\mathcal{T}\mathcal{C}&=\mathcal{T}\big(-s_1-\widetilde\tau_\Delta\circ s_1-\cdots-s_l-\widetilde\tau_\Delta\circ s_l\big),\\ \widehat{\mathcal{T}}\mathcal{C}&=\widehat{\mathcal{T}}\big(s_1+\widetilde\tau_\Delta\circ s_1+\cdots+s_l+\widetilde\tau_\Delta\circ s_l\big).\end{aligned}$$

By the last statement of Lemma 4.8, $\mathcal{TC}^*=\widehat{\mathcal{T}}\mathcal{C}$. Let

$$\begin{aligned}\widetilde{\mathcal{C}}&=\big(\widetilde{\Sigma},(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-),(x_1^+,x_1^-),(x_2^+,x_2^-)\big),\\ T\widetilde{\mathcal{C}}&=T\widetilde{\Sigma}\big(-z_1^+-z_1^--\cdots-z_l^+-z_l^--x_1^+-x_1^--x_2^+-x_2^-\big),\\ T^*\widetilde{\mathcal{C}}&=T^*\widetilde{\Sigma}\big(z_1^++z_1^-+\cdots+z_l^++z_l^-+x_1^++x_1^-+x_2^++x_2^-\big).\end{aligned}\tag{4.26}$$

Let $S\mathcal{C}\longrightarrow\Sigma$ and $S\widetilde{\mathcal{C}}\longrightarrow\widetilde{\Sigma}$ be the skyscraper sheaves of the cotangent bundles at the marked points as in the proof of Lemma 3.2. We also denote by $S\mathcal{C}\subset S\widetilde{\mathcal{C}}$ the lift of $S\mathcal{C}$ to $\widetilde{\Sigma}$, i.e. the natural complement of the subsheaf $S_{\mathbf{x}}$ of $S\widetilde{\mathcal{C}}$.

By Lemma 4.8, taking the (second order) residues of sections of $\widehat{\mathcal{TC}}\otimes\widehat{\mathcal{T}}$ at $x_1^+\in\widetilde{\Sigma}$ induces an isomorphism

$$\det\bar\partial_{(\widehat{\mathcal{TC}}\otimes\widehat{\mathcal{T}},\widehat{\varphi}^{\otimes 2})}|_\Sigma\approx\det\bar\partial_{(T^*\widetilde{\mathcal{C}}\otimes T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma)^{*\otimes 2})}\otimes\Lambda^2_{\mathbb{R}}\mathbb{C};\tag{4.27}$$

it corresponds to the isomorphism (2.4) for the short exact sequence of Fredholm operators represented by the middle column in Figure 1. Combining (4.27) with the isomorphism (4.6) for the trivial rank 1 real bundle pair (V,φ), we obtain an isomorphism

$$\big(\widehat{\det}\,\bar\partial_{(\widehat{\mathcal{TC}}\otimes\widehat{\mathcal{T}},\widehat{\varphi}^{\otimes 2})|_\Sigma}\big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\approx\big(\widehat{\det}\,\bar\partial_{(T^*\widetilde{\mathcal{C}}\otimes T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma^*)^{\otimes 2})}\big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}.\tag{4.28}$$

The exact sequence represented by the middle row in Figure 1 and its analogue for $\widetilde{\mathcal{C}}$ determine isomorphisms

$$\begin{aligned}\det\bar\partial_{(\widehat{\mathcal{TC}}\otimes\widehat{\mathcal{T}},\widehat{\varphi}^{\otimes 2})}|_\Sigma&\approx\big(\det\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_\Sigma}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma;S\mathcal{C})^\sigma\big),\\ \det\bar\partial_{(T^*\widetilde{\mathcal{C}}\otimes T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma^*)^{\otimes 2})}&\approx\big(\det\bar\partial_{(T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma^*))^{\otimes 2}}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\widetilde{\Sigma};S\widetilde{\mathcal{C}})^{\tilde\sigma}\big).\end{aligned}\tag{4.29}$$

The isomorphisms (4.29) induce orientations on the first factors on the two sides of (4.28) from

(1) the orientations of $H^0(\Sigma;S\mathcal{C})^\sigma$ and $H^0(\widetilde{\Sigma};S\widetilde{\mathcal{C}})^{\tilde\sigma}$ described in the proof of Lemma 3.2, and

(2) the canonical orientations on

$$\begin{aligned}\widehat{\det}\,\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_\Sigma}&\equiv\big(\det\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_\Sigma}\big)\otimes\big(\det\bar\partial_{\mathbb{C}}|_\Sigma\big)\quad\text{and}\\ \widehat{\det}\,\bar\partial_{(T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma)^*)^{\otimes 2}}&\equiv\big(\det\bar\partial_{(T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma)^*)^{\otimes 2}}\big)\otimes\big(\det\bar\partial_{\mathbb{C}}|_{\widetilde{\Sigma}}\big)\end{aligned}\tag{4.30}$$

provided by Corollaries 2.3 and 2.4.

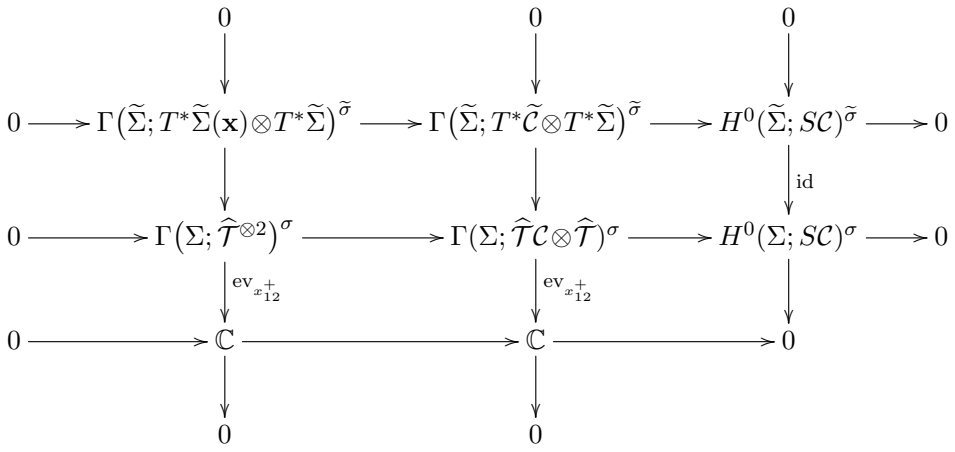

Figure 1: Commutative diagram for the proof of Corollary 4.13

Corollary 4.13. *The isomorphism (4.28) is orientation-preserving with respect to the two orientations described above and the complex orientation on* $\mathbb{C}$.

Proof. The exact sequence represented by the first row in Figure 1 and the $l=0$ case of the second isomorphism in (4.29) determine isomorphisms

$$\begin{aligned} \det\bar\partial_{(T^*\widetilde{\mathcal{C}}\otimes T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma^*)^{\otimes 2})} &\approx \big(\det\bar\partial_{(T^*\widetilde{\Sigma}(\mathbf{x})\otimes T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma^*)^{\otimes 2})}\big) \\ &\qquad \otimes \Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\widetilde{\Sigma}; S\mathcal{C})^{\tilde\sigma}\big), \\ \det\bar\partial_{(T^*\widetilde{\Sigma}(\mathbf{x})\otimes T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma^*)^{\otimes 2})} &\approx \big(\det\bar\partial_{(T^*\widetilde{\Sigma},\mathrm{d}\tilde\sigma^*)^{\otimes 2}}\big)\otimes\Lambda^4_{\mathbb{R}}\big(H^0(\widetilde{\Sigma}; S_{\mathbf{x}})^{\tilde\sigma}\big). \end{aligned} \tag{4.31}$$

The second isomorphism in (4.29) is the composition of the first isomorphism in (4.31) and the second one tensored with the identity on $\Lambda^{\mathrm{top}}_{\mathbb{R}}(H^0(\widetilde{\Sigma}; S\mathcal{C})^{\tilde\sigma})$.

Combining the analogue of (4.27) for $l=0$ (i.e. the isomorphism induced by the left column in Figure 1) with the isomorphism (4.6) for the trivial rank 1 real bundle pair (V,φ), we obtain an isomorphism

$$\big(\widehat{\det}\,\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_\Sigma}\big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C} \approx \big(\widehat{\det}\,\bar\partial_{(T^*\widetilde{\Sigma}(\mathbf{x})\otimes T^*\widetilde{\Sigma},(\mathrm{d}\tilde\sigma^*)^{\otimes 2})}\big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}. \tag{4.32}$$

The canonical orientation on the second line in (4.30) and the second isomorphism in (4.31) induce an orientation on the first factor on the right-hand

side of (4.32). By the commutativity of the squares in Figure 1, it is sufficient to show that the isomorphism (4.32) is orientation-preserving with respect to the canonical orientation on $\widehat{\det}\,\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}|_\Sigma}$, the above orientation on $\widehat{\det}\,\bar\partial_{(T^*\widetilde\Sigma(\mathbf{x})\otimes T^*\widetilde\Sigma,(\mathrm{d}\widetilde\sigma^*)^{\otimes 2})}$, and the complex orientation on $\mathbb{C}$.

The composition of the isomorphism (4.32) tensored with the identity on $\Lambda^2_{\mathbb{R}}\mathbb{C}$ and the second isomorphism in (4.31) tensored with the identities on $\det\bar\partial_{\widetilde\Sigma;\mathbb{C}}$ and two copies of $\Lambda^2_{\mathbb{R}}\mathbb{C}$ is homotopic to the isomorphism (4.23). By the previous paragraph, the claim is thus equivalent to the isomorphism (4.23) being orientation-preserving with respect to the canonical orientations on the first factors on the two sides, the complex orientation on $\mathbb{C}$, and the orientation on $\Lambda^4_{\mathbb{R}}(H^0(\widetilde\Sigma;S_{\mathbf{x}})^{\widetilde\sigma})$ induced as in the paragraph containing (4.20). This is indeed the case by Corollary 4.12. □

The next two statements are obtained from Lemmas 4.10 and 4.11 in the same way as [12, Corollary 6.12] is obtained from [12, Lemmas 6.9,6.10].

Corollary 4.14. *If the marked curve (4.25) is stable, the orientation on the restriction of the real line bundle*

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big((\check R^1\pi_*\mathcal{TC})^\sigma\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\big(R^1_{\bar\partial}\pi_*\mathcal{TC}\big)^\sigma\big)\longrightarrow\Delta^2_{\mathbb{R}}$$

to $\Delta^{*2}_{\mathbb{R}}$ induced by Dolbeault Isomorphism extends across $\mathbf{t}=0$ as the orientation induced by Dolbeault Isomorphism for the nodal symmetric surface (Σ,σ).

Corollary 4.15. *If the marked curve (4.25) is stable, the orientation on the restriction of the real line bundle*

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\big(R^1_{\bar\partial}\pi_*\mathcal{TC}\big)^\sigma\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\big((R^0_{\bar\partial}\pi_*(\widehat{\mathcal{TC}}\otimes\widehat{\mathcal{T}}))^\sigma\big)^*\big)\longrightarrow\Delta^2_{\mathbb{R}}$$

to $\Delta^{*2}_{\mathbb{R}}$ induced by Serre Duality as in the proof of [12, Proposition 5.9] extends across $\mathbf{t}=0$ as the orientation induced by Serre Duality for the nodal symmetric surface (Σ,σ).

We continue with the setup for (4.26). By Lemma 4.8, there is an exact sequence

$$0\longrightarrow\mathcal{O}(\mathcal{TC}|_\Sigma)\longrightarrow\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\longrightarrow\mathbb{C}_{x_{12}^+}\oplus\mathbb{C}_{x_{12}^-}\longrightarrow 0 \tag{4.33}$$

of sheaves with lifts of the involution σ over Σ. The projection of

$$\check H^0\big(\Sigma;\mathbb{C}_{x_{12}^+}\oplus\mathbb{C}_{x_{12}^-}\big)^\sigma\subset\check H^0\big(\Sigma;\mathbb{C}_{x_{12}^+}\big)\oplus\check H^0\big(\Sigma;\mathbb{C}_{x_{12}^-}\big)=\mathbb{C}\oplus\mathbb{C} \tag{4.34}$$

to the first component induces an isomorphism of real vector spaces.

If $\mathcal{C}$ is stable, the real part of the cohomology sequence induced by (4.33), its analogue in Dolbeault cohomology, and Dolbeault Isomorphism induce a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \check{H}^1\big(\Sigma;\mathcal{O}(\mathcal{TC}|_\Sigma)\big)^\sigma & \longrightarrow & \check{H}^1\big(\widetilde{\Sigma};\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\big)^\sigma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{DI}} & & \downarrow{\scriptstyle \mathrm{DI}} & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & H^1\big(\Sigma;\mathcal{TC}\big)^\sigma & \longrightarrow & H^1\big(\widetilde{\Sigma};T\widetilde{\mathcal{C}}\big)^\sigma & \longrightarrow & 0 \end{array} \tag{4.35}$$

of exact sequences. In particular, there are canonical isomorphisms

$$\begin{aligned} \Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\check{H}^1\big(\Sigma;\mathcal{O}(\mathcal{TC}|_\Sigma)\big)^\sigma\big) &\approx \Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\check{H}^1\big(\widetilde{\Sigma};\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\big)^\sigma\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\mathbb{C}\,, \\ \Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^1\big(\Sigma;\mathcal{TC}\big)^\sigma\big) &\approx \Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^1\big(\widetilde{\Sigma};T\widetilde{\mathcal{C}}\big)^\sigma\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\mathbb{C}\,. \end{aligned} \tag{4.36}$$

Combining them together, we obtain an isomorphism

$$\begin{aligned} &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\check{H}^1\big(\Sigma;\mathcal{O}(\mathcal{TC}|_\Sigma)\big)^\sigma\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^1(\Sigma;\mathcal{TC})^\sigma\big) \\ &\approx \Big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\check{H}^1\big(\widetilde{\Sigma};\mathcal{O}(T\widetilde{\mathcal{C}})\big)^\sigma\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^1(\widetilde{\Sigma};T\widetilde{\mathcal{C}})^\sigma\big)\Big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}. \end{aligned} \tag{4.37}$$

Corollary 4.16. *The isomorphism (4.37) is orientation-preserving with respect to the canonical orientation of Corollary 4.14 on the left-hand side, the orientation on the first tensor product on the right-hand side induced by Dolbeault Isomorphism, and the canonical orientation on the last tensor product.*

Proof. By the commutativity of the diagram (4.35), the isomorphism (4.37) is orientation-preserving with respect to the orientations on the left-hand side and on the first tensor product on the right-hand side induced by Dolbeault Isomorphism. The former is the orientation of Corollary 4.14. □

Combining the dual of (4.27) with the second isomorphism in (4.36), we obtain an isomorphism

$$\begin{aligned} &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^1(\Sigma;\mathcal{TC})^\sigma\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\big(H^0(\Sigma;\widehat{\mathcal{T}}\mathcal{C}\otimes\widehat{\mathcal{T}})^\sigma\big)^*\big) \\ &\approx \Big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^1(\widetilde{\Sigma};T\widetilde{\mathcal{C}})^\sigma\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\big(H^0(\widetilde{\Sigma};T^*\widetilde{\mathcal{C}}\otimes T^*\widetilde{\Sigma})^\sigma\big)^*\big)\Big) \\ &\quad\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\otimes\Lambda^2_{\mathbb{R}}(\mathbb{C}^\vee)\,, \end{aligned} \tag{4.38}$$

where $\mathbb{C}^\vee = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C})$. The complex orientation on $\mathbb{C}$ induces an orientation on $\mathbb{C}^\vee$ under the isomorphism (3.1). The latter is the opposite of the complex orientation of $\mathbb{C}^\vee$.

Corollary 4.17. *The isomorphism (4.38) is orientation-preserving with respect to the canonical orientation of Corollary 4.15 on the left-hand side, the orientation on the first tensor product on the right-hand side induced by Serre Duality, and the complex orientations on $\mathbb{C}$ and $\mathbb{C}^\vee$.*

Proof. Since the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C} & \longrightarrow & H^1\big(\Sigma;\mathcal{TC}\big)^\sigma & \longrightarrow & H^1\big(\widetilde{\Sigma};T\widetilde{\mathcal{C}}\big)^\sigma & \longrightarrow & 0 \\
 & & \otimes & & \otimes & & \otimes & & \\
0 & \longleftarrow & \mathbb{C} & \longleftarrow & H^0\big(\Sigma;\widehat{\mathcal{T}}\mathcal{C}\otimes\widehat{\mathcal{T}}\big)^\sigma & \longleftarrow & H^0\big(\widetilde{\Sigma};T^*\widetilde{\mathcal{C}}\otimes T^*\widetilde{\Sigma}\big)^\sigma & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbb{R} & = & \mathbb{R} & = & \mathbb{R} & &
\end{array}$$

induced by the imaginary parts of the Serre Duality pairings commutes, the isomorphism (4.38) is orientation-preserving with respect to the orientations on the left-hand side and on the first tensor product on the right-hand side induced by Serre Duality and the complex orientations on $\mathbb{C}$ and $\mathbb{C}^\vee$. The former is the orientation of Corollary 4.15. The first pairing in the above diagram is the real part of a $\mathbb{C}$-linear pairing and thus identifies the oriented real vector space $\mathbb{C}$ in the first row with the complex dual $\mathbb{C}^\vee$ of the vector space $\mathbb{C}$ in the second row. □

4.4. Comparison of the canonical orientations

Before establishing Theorem 1.2 at the end of this section, we obtain its analogue for the real Deligne-Mumford moduli spaces; see Proposition 4.18 below. This is done after comparing the behavior of the Kodaira-Spencer (KS) map under the smoothings and normalization of a symmetric surface (Σ,σ) with a pair of conjugate nodes; see Lemma 4.19.

Let $g\in\mathbb{Z}$ and $l\in\mathbb{Z}^{\geq 0}$ be such that $g+l\geq 2$. The identification of the last two pairs of conjugate marked points induces an immersion

$$\iota\colon \mathbb{R}\overline{\mathcal{M}}^\bullet_{g-2,l+2} \longrightarrow \mathbb{R}\overline{\mathcal{M}}^\bullet_{g,l}\,; \tag{4.39}$$

the image $\mathbb{R}\mathcal{N}^\bullet_{g,l}$ of $\mathbb{R}\mathcal{M}^\bullet_{g-2,l+2}$ under this immersion consists of symmetric surfaces with one pair of conjugate nodes. There is a canonical isomorphism

$$\mathcal{N}\iota \equiv \frac{\iota^* T\mathbb{R}\overline{\mathcal{M}}^\bullet_{g,l}}{T\mathbb{R}\overline{\mathcal{M}}^\bullet_{g-2,l+2}} \approx \mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$$

of the normal bundle of ι with the tensor product of the universal tangent line bundles for the first points in the last two conjugate pairs. Thus, there is a canonical isomorphism

$$\iota^*\big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathbb{R}\overline{\mathcal{M}}^\bullet_{g,l}\big)\big) \approx \Lambda^{\mathrm{top}}_{\mathbb{R}}(T\mathbb{R}\overline{\mathcal{M}}^\bullet_{g-2,l+2})\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\big) \tag{4.40}$$

of real line bundles over $\mathbb{R}\overline{\mathcal{M}}^\bullet_{g-2,l+2}$.

Combining the isomorphism (4.40) with the isomorphism (4.6) for the trivial rank 1 real bundle pair (V,φ), we obtain an isomorphism

$$\begin{aligned}&\big(\Lambda^{\mathrm{top}}_{\mathbb{R}}(T\mathbb{R}\mathcal{M}^\bullet_{g-2,l+2})\otimes(\det\bar\partial_{\mathbb{C}})\big)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\big)\\ &\approx\iota^*\big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathbb{R}\overline{\mathcal{M}}^\bullet_{g,l}\big)\otimes(\det\bar\partial_{\mathbb{C}})\big)\otimes\big(\Lambda^2_{\mathbb{R}}\mathbb{C}\big)\end{aligned} \tag{4.41}$$

of real line bundles over $\mathbb{R}\overline{\mathcal{M}}^\bullet_{g-2,l+2}$. Since the complement of $\mathbb{R}\mathcal{N}^\bullet_{g,l}$ in a small neighborhood in $\mathbb{R}\overline{\mathcal{M}}^\bullet_{g,l}$ is connected and consists of smooth curves, the canonical orientation on the real line bundle (2.12) provided by [12, Proposition 5.9] extends across $\mathbb{R}\mathcal{N}^\bullet_{g,l}$ and thus induces an orientation on the first tensor product on the right-hand side of (4.41).

Proposition 4.18. *Let $g\in\mathbb{Z}$ and $l\in\mathbb{Z}^{\ge0}$ be such that $g+l\ge2$. The isomorphism (4.41) is orientation-reversing with respect to the orientations on the real line bundles (2.12) provided by [12, Proposition 5.9] and the complex orientations of $\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$ and $\mathbb{C}$.*

Suppose $\mathcal{C}$, $\widetilde{\mathcal{C}}$, $(\pi,\widetilde\tau_\Delta,s_1,\ldots,s_l)$, $(\mathcal{T},\varphi)$, and $(\widehat{\mathcal{T}},\widehat\varphi)$ are as in (4.25) and (4.26) with $\mathcal{U}|_{\Delta^2_{\mathbb{R}}}\longrightarrow\Delta^2_{\mathbb{R}}$ embedded inside of the universal curve fibration over $\mathbb{R}\overline{\mathcal{M}}^\bullet_{g,l}$. Combining the first isomorphism in (4.36) and (4.40), we obtain an isomorphism

$$\begin{aligned}&\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\mathcal{C}]}\mathbb{R}\overline{\mathcal{M}}^\bullet_{g,l}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\check H^1\big(\Sigma;\mathcal{O}(\mathcal{TC}|_\Sigma)\big)^\sigma\big)\\ &\approx\Big(\Lambda^{\mathrm{top}}_{\mathbb{R}}(T_{[\widetilde{\mathcal{C}}]}\mathbb{R}\overline{\mathcal{M}}^\bullet_{g-2,l+2})\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\check H^1\big(\widetilde\Sigma;\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\big)^\sigma\big)\Big)\\ &\quad\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\,.\end{aligned} \tag{4.42}$$

The KS map induces an orientation on the left-hand side of (4.42) whenever $\mathcal{C}$ is a smooth curve. Since the complement of $\mathbb{R}\mathcal{N}^\bullet_{g,l}$ in a small neighborhood

in $\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g,l}$ is connected and consists of smooth curves, this orientation extends over $\mathbb{R}\mathcal{N}^{\bullet}_{g,l}$.

Lemma 4.19. *The isomorphism (4.42) is orientation-preserving with respect to the orientations on the left-hand side and the first tensor product on the right-hand side determined by the KS map and the canonical orientations of $\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$ and $\mathbb{C}$.*

Proof. The proof is similar to that of [12, Lemma 6.16]. The complex parameter t^+ in Section 4.2 can be viewed as an element of the complex line bundle $\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$ and parametrizes the smoothings of the marked symmetric surface $\mathcal{C}$ as in (4.25). In the notation of Section 4.2, they are described by $\mathbf{t}=(t^+,t^-)$ with $t^-=\overline{t^+}$. Denote by $\mathcal{TC}\longrightarrow\widetilde{\mathcal{U}}_{g-2,l+2}$ the twisted down vertical tangent bundle over the universal curve $\pi\colon\widetilde{\mathcal{U}}_{g-2,l+2}\longrightarrow\mathbb{R}\mathcal{N}^{\bullet}_{g,l}$.

As in the proof of [12, Lemma 6.16], the vector bundles

$$T\mathbb{R}\mathcal{N}^{\bullet}_{g,l},\big(\check{R}^1\pi_*(\mathcal{TC})\big)^{\sigma}\longrightarrow\mathbb{R}\mathcal{N}^{\bullet}_{g,l}$$

extend over a neighborhood of $\mathbb{R}\mathcal{N}^{\bullet}_{g,l}$ in $\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g,l}$ as a subbundle of $T\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g,l}$ and a quotient bundle of $(\check{R}^1\pi_*\mathcal{TC})^{\sigma}$. The KS map induces an isomorphism between these two extensions and gives rise to a diagram

$$\begin{array}{ccccc}
T_{\widetilde{\mathcal{C}}}\mathbb{R}\mathcal{M}^{\bullet}_{g-2,l+2} & \longrightarrow & T_{\mathcal{C}_{\mathbf{t}}}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g,l} & \longrightarrow & \mathcal{L}_1\otimes_{\mathbb{C}}\mathcal{L}_2|_{\widetilde{\mathcal{C}}}\\
\mathrm{KS}\big\downarrow\approx & & \mathrm{KS}\big\downarrow\approx & & \mathrm{KS}\big\downarrow\approx\\
\check{H}^1\big(\widetilde{\Sigma};\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\big)^{\sigma} & \longleftarrow & \check{H}^1\big(\Sigma_{\mathbf{t}};\mathcal{O}(\mathcal{TC}|_{\Sigma_{\mathbf{t}}})\big)^{\sigma_{\mathbf{t}}} & \longleftarrow & \mathbb{C}
\end{array}$$

commuting up to homotopy of the isomorphisms given by the vertical arrows. The crucial point is that the KS map sends the deformation parameter $t^+\in\mathcal{L}_1\otimes_{\mathbb{C}}\mathcal{L}_2$ to the $\mathbb{C}$-factor in (4.42) in an orientation-preserving fashion. This is shown in the next paragraph.

Similarly to the last part of the proof of [12, Lemma 6.16], we cover a neighborhood of $\Sigma_{\mathbf{t}}$ in $\mathcal{U}$ by the open sets

$$\begin{aligned}
\mathcal{U}_1^{\pm}&=\big\{(t^+,t^-,z_1^{\pm},z_2^{\pm})\in\mathcal{U}_0^{\pm}\colon\ 2|z_2^{\pm}|<1\big\}\quad\text{and}\\
\mathcal{U}_2^{\pm}&=\big\{(t^+,t^-,z_1^{\pm},z_2^{\pm})\in\mathcal{U}_0^{\pm}\colon\ 2|z_1^{\pm}|<1\big\},
\end{aligned}$$

along with coordinate charts each of which intersects at most one of $\mathcal{U}_1^{\pm}$ and $\mathcal{U}_2^{\pm}$. By the same computation as before, the Čech 1-cocycle corresponding to the radial vector field [12, (6.25)] for the smoothing parameter $t=t^+$

is given by

$$\widehat{\theta}^{\pm}_{0;12} \equiv z_1^{\pm}\frac{\partial}{\partial z_1^{\pm}} - z_2^{\pm}\frac{\partial}{\partial z_2^{\pm}}, \quad \widehat{\theta}^{\pm}_{0;21} \equiv -z_1^{\pm}\frac{\partial}{\partial z_1^{\pm}} + z_2^{\pm}\frac{\partial}{\partial z_2^{\pm}} \tag{4.43}$$

on $\mathcal{U}_1^{\pm}\cap\mathcal{U}_2^{\pm}$ after re-scaling by $|t|^{-1}$ and vanishes on all remaining overlaps. In order to determine the image of the angular vector field, we replace t with $\mathrm{e}^{\mathrm{i}\theta}t$ in the computation in the proof of [12, Lemma 6.16] and differentiate the resulting overlap maps $f_{12}^{\pm}$ and $f_{21}^{\pm}$ with respect to θ at $\theta=0$. Over $\mathcal{U}_0^{\pm}$, we then obtain the right-hand sides of the two expressions in (4.43) multiplied by $\pm\mathrm{i}$. Thus, the KS map sends $t^+\in\mathcal{L}_1\otimes_{\mathbb{C}}\mathcal{L}_2$ to the $\mathbb{C}$-factor in (4.42) in an orientation-preserving fashion. □

Proof of Proposition 4.18. Let $(\widetilde{\mathcal{C}},\widetilde{\sigma})$ be an element of $\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g-2,l+2}$. Its image under ι is a marked symmetric curve $(\mathcal{C},\sigma)$ with a pair of conjugate nodes. We continue with the notation and setup in the proof of Lemma 4.19.

The isomorphisms (4.40) and (4.27) induce an isomorphism

$$\begin{aligned}\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\mathcal{C}]}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g,l}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\big(H^0(\Sigma;\widehat{\mathcal{T}}\mathcal{C}\otimes\widehat{\mathcal{T}})^{\sigma}\big)^*\big)&\\ \approx\Big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\widetilde{\mathcal{C}}]}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(\big(H^0(\widetilde{\Sigma};T^*\widetilde{\mathcal{C}}\otimes T^*\widetilde{\Sigma})^{\sigma}\big)^*\big)\Big)&\\ \otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\big)\otimes\Lambda^2_{\mathbb{R}}(\mathbb{C}^{\vee}).&\end{aligned} \tag{4.44}$$

Orientations on the left-hand side of (4.44) and the first tensor product on the right-hand side are obtained by tensoring the orientations on the corresponding terms

(1) in (4.42) determined by the KS map,

(2) in (4.37) determined by Dolbeault Isomorphism and Corollary 4.14,

(3) in (4.38) determined by Serre Duality and Corollary 4.15.

By Lemma 4.19 and Corollaries 4.16 and 4.17, the isomorphism (4.44) is orientation-preserving with respect to these two orientations and the complex orientations on $\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$ and $\mathbb{C}^{\vee}$.

The orientations on

$$\begin{aligned}&\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\mathcal{C}]}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g,l}\big)\otimes\big(\det\bar{\partial}_{\mathbb{C}}|_{[\mathcal{C}]}\big)\quad\text{and}\\ &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\widetilde{\mathcal{C}}]}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g-2,l+2}\big)\otimes\big(\det\bar{\partial}_{\mathbb{C}}|_{[\widetilde{\mathcal{C}}]}\big)\end{aligned}$$

provided by [12, Proposition 5.17] are the tensor products of the orientations on

(1) the left-hand side of (4.44) and the first tensor product on the right-hand side described above and

(2) the first tensor products on the two sides of (4.28) described below (4.29).

The isomorphism (3.1) with $V=\mathbb{C}$ induces a homotopy class of identifications of $(\Lambda^2_{\mathbb{R}}\mathbb{C})^*\otimes\Lambda^2_{\mathbb{R}}(\mathbb{C}^\vee)$ with $\mathbb{R}$. By the previous paragraph and Corollary 4.13, the isomorphisms

$$\begin{aligned}&\Big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\mathcal{C}]}\mathbb{R}\overline{\mathcal{M}}^\bullet_{g,l}\big)\otimes\big(\det\bar\partial_{\mathcal{C};\mathbb{C}}\big)^*\Big)\otimes\big(\Lambda^2_{\mathbb{R}}\mathbb{C}\big)^*\\&\approx\Big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{[\widetilde{\mathcal{C}}]}\mathbb{R}\overline{\mathcal{M}}^\bullet_{g-2,l+2}\big)\otimes\big(\det\bar\partial_{\widetilde{\mathcal{C}};\mathbb{C}}\big)^*\Big)\\&\quad\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\big)\otimes\big(\Lambda^2_{\mathbb{R}}\mathbb{C}\big)^*\otimes\Lambda^2_{\mathbb{R}}(\mathbb{C}^\vee)\end{aligned}$$

induced by the isomorphism (4.40), the isomorphism (4.6) for the trivial rank 1 real bundle pair (V,φ), and trivializations of $(\Lambda^2_{\mathbb{R}}\mathbb{C})^*\otimes\Lambda^2_{\mathbb{R}}(\mathbb{C}^\vee)$ in the above homotopy class are orientation-preserving with respect to the orientations of Proposition 4.18 and the complex orientations of $\mathbb{C}$ and $\mathbb{C}^\vee$. Since the isomorphism (3.1) with $V=\mathbb{C}$ is orientation-reversing with respect to the complex orientations of $\mathbb{C}$ and $\mathbb{C}^\vee$, the isomorphism (4.41) is also orientation-reversing with respect to the orientations of Proposition 4.18. □

Proof of Theorem 1.2. Throughout this argument, we omit $(X,B,J)^\phi$ from the notation for the moduli spaces of maps and let

$$\mathcal{L}=\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\,.$$

By the construction of the orientations in the proofs of Corollary 5.10 and Theorem 1.3 in [12], it is sufficient to verify the claim over an element $[\widetilde{u}]\in\overline{\mathfrak{M}}'^\bullet_{g-2,l+2}$ with a smooth stable domain. Let u be the induced real map from the corresponding nodal symmetric surface. We denote the marked domains of $\widetilde{u}$ and u by $\widetilde{\mathcal{C}}$ and $\mathcal{C}$, respectively, and let $q=\mathrm{ev}_{l+1}(\widetilde{u})$.

The forgetful morphisms (2.2) induce the short exact sequences represented by the left and middle columns in the two diagrams of Figure 2. The top row in the first diagram is the exact sequence on the indices of Fredholm operators determined by the exact sequence (4.5) with $(V,\varphi)=u^*(TX,\mathrm{d}\phi)$; the middle row is the exact sequence above (1.3). The middle and bottom rows in the second diagram are the exact sequences associated with the normal bundles $\mathcal{N}\iota$ above (1.5) and (4.40), respectively.

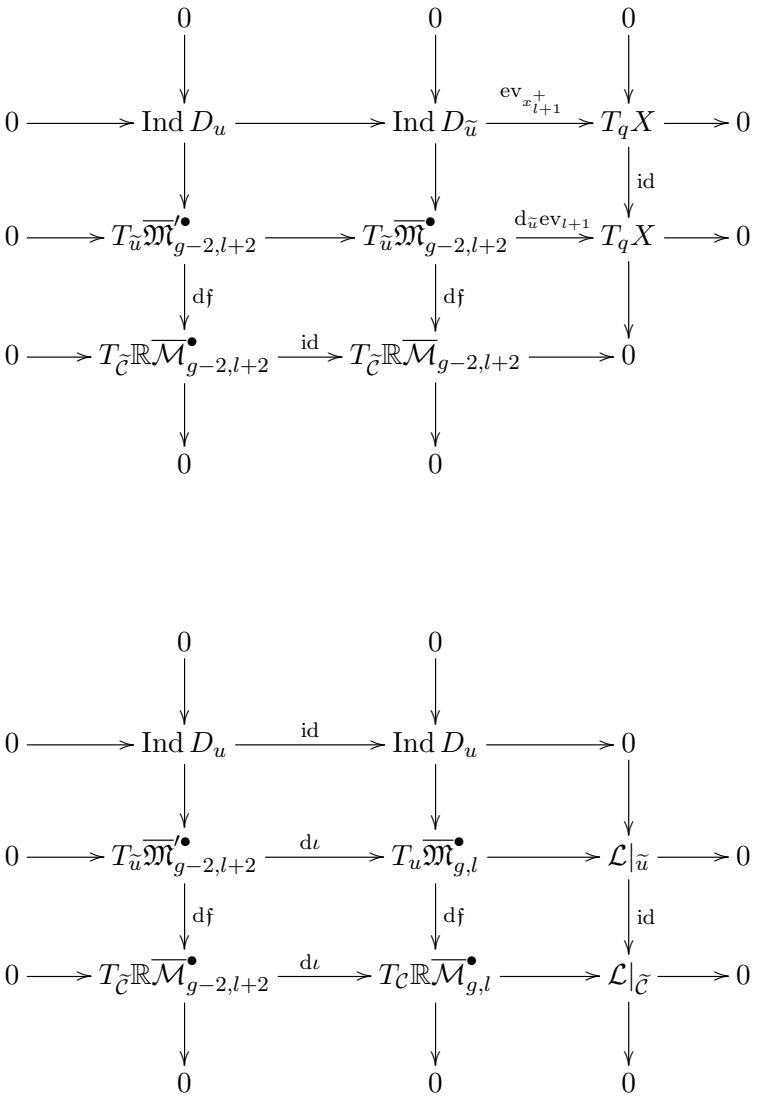

Figure 2: Commutative diagrams for the proof of Theorem 1.2

The middle row and column in the first diagram in Figure 2 determine isomorphisms

$$\begin{aligned}(4.45)\qquad &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{u}}\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\\ &\approx\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{u}}\overline{\mathfrak{M}}^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\\ &\approx\big(\det D_{\widetilde{u}}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{\mathcal{C}}}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\,.\end{aligned}$$

By the commutativity of the squares in this diagram, the composition of the two isomorphisms in (4.45) equals to the composition of the isomorphism

$$\begin{aligned}(4.46)\qquad &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{u}}\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\\ &\approx\big(\det D_u\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{\mathcal{C}}}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\end{aligned}$$

induced by the first column and the isomorphism (4.6) with (V,φ) replaced by $u^*(TX,\mathrm{d}\phi)$; the latter is induced by the first row.

The middle row and column in the second diagram in Figure 2 determine isomorphisms

$$\begin{aligned}(4.47)\qquad &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{u}}\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\\ &\approx\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_u\overline{\mathfrak{M}}^{\bullet}_{g,l}\big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\\ &\approx\big(\det D_u\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\mathcal{C}}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g,l}\big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\,.\end{aligned}$$

By the commutativity of the squares in this diagram, the composition of the two isomorphisms in (4.47) equals to the composition of the isomorphisms (4.46) and (4.40); the latter is induced by the bottom row. Thus, the isomorphism

$$\begin{aligned}(4.48)\qquad &\big(\det D_{\widetilde{u}}\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{\mathcal{C}}}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\\ &\approx\big(\det D_u\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\mathcal{C}}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g,l}\big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\end{aligned}$$

induced by (4.45) and (4.47) is the tensor product of the isomorphism (4.6) with $(V,\varphi)=u^*(TX,\mathrm{d}\phi)$ and the isomorphism (4.40).

The isomorphism (4.45) induces an isomorphism

$$\begin{aligned}(4.49)\qquad &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{u}}\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\otimes\big(\det\bar\partial_{\widetilde{\Sigma};\mathbb{C}}\big)^{\otimes(n+1)}\\ &\approx\Big(\big(\det D_{\widetilde{u}}\big)\otimes\big(\det\bar\partial_{\widetilde{\Sigma};\mathbb{C}}\big)^{\otimes n}\Big)\\ &\quad\otimes\Big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{\mathcal{C}}}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g-2,l+2}\big)\otimes\big(\det\bar\partial_{\widetilde{\Sigma};\mathbb{C}}\big)\Big)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\,.\end{aligned}$$

The isomorphism (4.47) and the isomorphisms (4.6) with

$$(V,\varphi)=u^*\big(TX,\mathrm{d}\phi\big),\big(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c}\big)$$

induce an isomorphism

$$\begin{aligned}(4.50)\qquad &\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\widetilde{u}}\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}\big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\otimes\Lambda^2_{\mathbb{R}}\big(\mathcal{L}|_{\widetilde{\mathcal{C}}}\big)\otimes\big(\det\bar\partial_{\widetilde{\Sigma};\mathbb{C}}\big)^{\otimes(n+1)}\\ &\approx\Big(\big(\det D_u\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes n}\Big)\otimes\Lambda^{2n}_{\mathbb{R}}(T_qX)\otimes\Lambda^{2n}_{\mathbb{R}}\mathbb{C}^n\\ &\quad\otimes\Big(\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T_{\mathcal{C}}\mathbb{R}\overline{\mathcal{M}}^{\bullet}_{g,l}\big)\otimes\big(\det\bar\partial_{\mathbb{C};\Sigma}\big)\Big)\otimes\Lambda^2_{\mathbb{R}}\mathbb{C}\,.\end{aligned}$$

A real orientation on (X,ω,φ) induces orientations on

$$\begin{aligned}(4.51)\qquad &\widehat{\det}\,D_{\widetilde{u}}\equiv\big(\det D_{\widetilde{u}}\big)\otimes\big(\det\bar\partial_{\widetilde{\Sigma};\mathbb{C}}\big)^{\otimes n}\quad\text{and}\\ &\widehat{\det}\,D_u\equiv\big(\det D_u\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes n}.\end{aligned}$$

The isomorphisms (4.49) and (4.50) induce orientations on their common domain from the orientations in (4.51), the orientation of (2.12) provided by [12, Proposition 5.9], and the canonical orientations on $\mathcal{L}$, TX, and $\mathbb{C}$. The substance of Theorem 1.2 is that the two induced orientations are different.

The two induced orientations are different if the composition of the inverse of the isomorphism in (4.49) with the isomorphism in (4.50) is orientation-reversing. By the sentence containing (4.48), this composition is the tensor product of

(1) the isomorphism (4.7) with $(V,\varphi)=u^*(TX,\mathrm{d}\phi)$ and

(2) the isomorphism (4.41).

By Corollaries 4.5 and 4.6, the first isomorphism is orientation-preserving. By Proposition 4.18, the second isomorphism is orientation-reversing. □

Remark 4.20. A real orientation on a $2n$-dimensional manifold X determines orientations on the moduli spaces of real spaces if n is odd. If n is even, a real orientation on X determines orientations on the tangent bundles of the moduli spaces of real maps twisted by the tangent bundles of the moduli spaces of real curves; the real spaces in this case are generally not orientable. If $n\in2\mathbb{Z}$ and $2g+l\ge3$, the comparison of Theorem 1.2 should thus be made with the tangent bundles of the moduli spaces of maps twisted by the tangent bundles of the moduli spaces of curves as in [12, (1.3)]. The isomorphism (1.6) is then replaced by its tensor product with the inverse of (4.40). The proof of Theorem 1.2 implies that this isomorphism is orientation-preserving, since the orientation-reversing isomorphism (4.41)

now enters twice. This $n \in 2\mathbb{Z}$ analogue of Theorem 1.2 is also invariant under the reordering of the nodes, since it now preserves the orientation of TX and $\mathcal{L}_{l+1} \otimes_{\mathbb{C}} \mathcal{L}_{l+2}$ appears twice.

References

[1] I. Biswas, J. Huisman, and J. Hurtubise, *The moduli space of stable vector bundles over a real algebraic curve*, Math. Ann. **347** (2010), no. 1, 201–233.

[2] C.-H. Cho, *Counting real J-holomorphic discs and spheres in dimension four and six*, J. Korean Math. Soc. **45** (2008), no. 5, 1427–1442.

[3] B. Dubrovin and Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, `arXiv: math/0108160`.

[4] T. Ekholm, J. Etnyre, and M. Sullivan, *Orientations in Legendrian contact homology and exact Lagrangian immersions*, Internat. J. Math. **16** (2005), no. 5, 453–532.

[5] M. Farajzadeh Tehrani, *Counting genus zero real curves in symplectic manifolds*, Geom. Top. **20** (2016), no. 2, 629–695.

[6] K. Fukaya and K. Ono, *Arnold Conjecture and Gromov-Witten invariant*, Topology **38** (1999), no. 5, 933–1048.

[7] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, Lagrangian Intersection Theory: Anomaly and Obstruction, AMS Studies in Advanced Mathematics **46** (2009).

[8] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Anti-symplectic involution and Floer cohomology*, Geom. Topol. **21** (2017), no. 1, 1–106.

[9] P. Georgieva, *Open Gromov-Witten invariants in the presence of an anti-symplectic involution*, Adv. Math. **301** (2016), 116–160.

[10] P. Georgieva and A. Zinger, *The moduli space of maps with crosscaps: Fredholm theory and orientability*, Comm. Anal. Geom. **23** (2015), no. 3, 81–140. Geom. Topol. **17** (2013), no. 4, 2485–2512.

[11] P. Georgieva and A. Zinger, *Enumeration of real curves in $\mathbb{C}P^{2n-1}$ and a WDVV relation for real Gromov-Witten invariants*, Duke Math. **166** (2017), no. 17, 3291–3347.

[12] P. Georgieva and A. Zinger, *Real Gromov-Witten theory in all genera and real enumerative geometry: construction*, Ann. Math. **188** (2018), no. 3, 685–752.

[13] P. Georgieva and A. Zinger, *Real Gromov-Witten theory in all genera and real enumerative geometry: computation*, `arXiv:math/1510.07568`.

[14] P. Georgieva and A. Zinger, *On the topology of real bundle pairs over nodal symmetric surfaces*, Topology Appl. **214** (2016), 109–126.

[15] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347.

[16] Y.-Z. Huang, Two-Dimensional Conformal Geometry and Vertex Operator Algebras, Progress in Math. **148**, Birkhäuser (1997).

[17] S. Katz and C.-C. Liu, *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc*, Adv. Theor. Math. Phys. **5** (2001), no. 1, 1–49.

[18] M. Kontsevich and Y. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), no. 3, 525–562.

[19] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in Symplectic 4-Manifolds, pp. 47-83, First Int. Press Lect. Ser., I, Internat. Press, (1998).

[20] C.-C. Liu, *Moduli of J-holomorphic curves with Lagrangian boundary condition and open Gromov-Witten invariants for an S^1-pair*, `arXiv:math/0210257v2`.

[21] D. McDuff and D. Salamon, *J*-Holomorphic Curves and Quantum Cohomology, University Lecture Series **6**, AMS (1994).

[22] D. McDuff and D. Salamon, J-holomorphic Curves and Symplectic Topology, Colloquium Publications **52**, AMS (2012).

[23] J. Niu and A. Zinger, *Lower bounds for the enumerative geometry of positive-genus real curves*, Adv. Math. **339** (2018), 191–247.

[24] R. Pandharipande, J. Solomon, and J. Walcher, *Disk enumeration on the quintic* 3*-fold*, J. Amer. Math. Soc. **21** (2008), no. 4, 1169–1209.

[25] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995), no. 2, 259–367.

[26] J. Solomon, *Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions*, arXiv:math/0606429.

[27] J. Walcher, *Evidence for tadpole cancellation in the topological string*, Comm. Number Theory Phys. **3** (2009), no. 1, 111–172.

[28] J.-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. **162** (2005), no. 1, 195–234.

[29] J.-Y. Welschinger, *Spinor states of real rational curves in real algebraic convex* 3*-manifolds and enumerative invariants*, Duke Math. J. **127** (2005), no. 1, 89–121.

[30] A. Zinger, *The determinant line bundle for Fredholm operators: construction, properties, and classification*, Math. Scand. **118** (2016), no. 2, 203–268.

Institut de Mathématiques de Jussieu - Paris Rive Gauche
Sorbonne Université, Campus Pierre et Marie Curie
4 Place Jussieu, 75252 Paris Cedex 5, France
E-mail address: penka.georgieva@imj-prg.fr

Department of Mathematics, SUNY Stony Brook
Stony Brook, NY 11794-3651, USA
E-mail address: azinger@math.stonybrook.edu

Received April 27, 2017
Accepted November 29, 2017