Thin compactifications and relative fundamental classes

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We define a notion of relative fundamental class that applies to moduli spaces in gauge theory and in symplectic Gromov–Witten theory. For universal moduli spaces over a parameter space, the relative fundamental class specifies an element of the Čech homology of the compactification of each fiber; it is defined if the compactification is "thin" in the sense that the boundary of the generic fiber has homological codimension at least two.

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The moduli spaces that occur in gauge theories and in symplectic Gromov–Witten theory are often orbifolds that can be compactified by adding "boundary strata" of lower dimension. Often, it is straightforward to prove that each stratum is a manifold, but more difficult to prove "collar theorems" that describe how these strata fit together. The lack of collar theorems is an impediment to applying singular homology to the compactified moduli space, and in particular to defining its fundamental homology class. The purpose of this paper is to show that collar theorems are not needed to define a (relative) fundamental class as an element of Čech homology for families of appropriately compactified manifolds.

One can distinguish two types of homology theories. Type I theories, exemplified by singular homology, are based on finite chains and are functorial under continuous maps. Type II theories, exemplified by Borel-Moore singular homology, are based on locally finite (possibly infinite) chains, and are functorial under *proper* continuous maps. We will use two theories of the second type: (type II) Steenrod homology ${}^s\mathrm{H}_*$ and (type II) Čech homology $\check{\mathrm{H}}_*$. These have two features that make them especially well-suited for applications to compactified moduli spaces:

(1) For any closed subset A of a locally compact Hausdorff space X, the relative group ${}^s\mathrm{H}_p(X, A)$ is identified with ${}^s\mathrm{H}_p(X \setminus A)$. As Massey notes [Ma2, p. vii]:

> . . . one does not need to consider the relative homology or cohomology groups of a pair (X, A); the homology or cohomology groups of the complementary space $X - A$ serve that function. In many ways these "single space" theories are simpler than the usual theories involving relative homology groups of pairs. The analog of the excision property becomes a tautology, and never needs to be considered. It makes possible an intuitive and straightforward discussion of the homology and cohomology of a manifold in the top dimension, without any assumption of differentiability, triangulability, compactness, or even paracompactness!

(2) Čech homology satisfies a "continuity property" ((1.10) below) that allows one to define relative fundamental classes by a limit process.

We briefly review Steenrod and Čech homology in Section 1. Then, in Section 2, we apply Property (1) to manifolds M that admit compactifications $\overline{M}$ whose "boundary" $\overline{M} \setminus M$ is "thin" in the sense that it has homological codimension at least 2. There may be many such compactifications. If M is oriented and d-dimensional, every thin compactification carries a fundamental class

$$[\overline{M}] \in {}^s\mathrm{H}_d(\overline{M}; \mathbb{Z})$$

in Steenrod homology. This class pushes forward under maps $M \to Y$ that extend continuously over $\overline{M}$, and many properties of fundamental classes of manifolds continue to hold.

We next enlarge the setting by considering thinly compactified *families*. We consider a proper continuous map

$$(0.1) \qquad\qquad \begin{array}{c} \overline{\mathcal{M}} \\ \downarrow \overline{\pi} \\ \mathcal{P} \end{array}$$

from a Hausdorff space to a locally path-connected Baire metric space whose generic fiber is a thin compactification in the sense of Section 2. More precisely, as in Definition 3.1, we call (0.1) a "relatively thin family" if there is a Baire second category subset $\mathcal{P}^*$ of $\mathcal{P}$ such that (i) the fiber $\overline{\mathcal{M}}_p$ over each $p \in \mathcal{P}^*$ is a thin compactification of a d-dimensional oriented manifold, and (ii) a similar condition holds for a dense set of paths in $\mathcal{P}$. Then the fiber over each $p \in \mathcal{P}^*$ has a fundamental class, *which we now regard as an element of Čech homology* (see Lemma 1.1). Because $\mathcal{P}^*$ is dense, a limiting process using Property (2) then yields a class — now called a relative fundamental class — in the Čech homology of *every* fiber of $\overline{\pi}$. This important fact, stated as Extension Lemma 3.4, is used repeatedly in subsequent arguments. We then give a precise definition of a relative fundamental class (Definition 4.1) and prove:

Theorem 4.2. *Every thinly compactified family $\pi : \overline{\mathcal{M}} \to \mathcal{P}$ admits a unique relative fundamental class.*

The end of Section 4 explains how a relative fundamental class yields numerical invariants associated to the family.

Section 5 describes how relatively thin families arise from Fredholm maps. Suppose that $\pi : \mathcal{M} \to \mathcal{P}$ is a Fredholm map between Banach manifolds with index d. A "Fredholm-stratified thin compactification" is an extension of π to a proper map $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ such that the boundary $\mathcal{S} = \overline{\mathcal{M}} \setminus \mathcal{M}$

is stratified by Banach manifolds $\mathcal{S}_\alpha$ so that, for each α, $\overline{\pi}$ restricts to a Fredholm map $\mathcal{S}_\alpha \to \mathcal{P}$ of index at most $d-2$ (see Definition 5.2). The Sard-Smale theorem implies that such compactifications fit into the context of Section 4:

Lemma 5.3. *A Fredholm-stratified thin family is a relatively thin family.*

Section 6 describes how a relative fundamental class on one thinly compactified family extends or restricts to relative fundamental classes on related families.

The remaining sections give examples. In each example, we show that the relevant moduli space admits a Fredholm-stratified thin compactification. Lemma 5.3 and Theorem 4.2 then immediately imply the existence of a relative fundamental class.

Sections 7 and 8 apply these ideas to Donaldson theory. Given an oriented Riemannian four-manifold (X, g), one constructs moduli spaces $\mathcal{M}_k(g)$ of g-anti-self-dual $U(2)$-connections. Donaldson's polynomial invariants are defined by evaluating certain natural cohomology classes on $\mathcal{M}_k(g)$ for a generic g. We show that results already present in Donaldson's work imply the existence of relative fundamental classes for the Uhlenbeck compactification $\overline{\mathcal{M}}_k(g)$ for *any* metric.

Sections 9 and 10 give applications to Gromov–Witten theory. Here the central object is the moduli space of stable maps into a closed symplectic manifold (X, ω), viewed as a family

$$(0.2) \qquad\qquad \mathcal{M}_{A,g,n}(X) \to \mathcal{JV}$$

over the space of Ruan-Tian perturbations, as described in Section 9. Again, the theme is that many results in the literature can be viewed as giving conditions under which there exist thin compactifications of the Gromov–Witten moduli spaces (0.2) over $\mathcal{JV}$, or over some subset of $\mathcal{JV}$. In these situations, the results of Sections 2–6 produce a relative fundamental class over a subset of $\mathcal{JV}$. Section 10 presents two examples: the moduli space of somewhere-injective J-holomorphic maps, and the moduli space of domain-fine (J, ν)-holomorphic maps.

We note that John Pardon, building on the work of McDuff and Wehrheim [MW], has constructed a virtual fundamental class on the space of stable maps for any genus and any closed symplectic manifold [Pd]. While Pardon's approach is different from the one presented here, both produce classes in the dual of Čech cohomology, and we expect that they are equal whenever both are defined.

1. Steenrod and Čech homologies

Expositions of Steenrod homology are surprisingly hard to find in the literature. We will use the type II version of Steenrod homology that is based on "infinite chains", as presented in Chapter 4 of W. Massey's book [Ma2]. We call this simply "Steenrod homology" and denote it by ${}^s\mathrm{H}_*$ (Massey's notation is H_* in Chapters 4-9 and H_*^∞ in Chapters 10 and 11). For background, see also [Ma1], [Mil], and the introduction to [Ma2].

Steenrod homology with abelian coefficient group G assigns, for each integer p, an abelian group ${}^s\mathrm{H}_p(X) = {}^s\mathrm{H}_p(X;G)$ to each locally compact Hausdorff space X, and a homomorphism $f_* : {}^s\mathrm{H}_p(X) \to {}^s\mathrm{H}_p(Y)$ to each *proper* continuous map. The axioms for this homology theory [Ma2, p. 86] include:

- For each open subset $U \subseteq X$ and each p, there is a natural "restriction" map

(1.1) $$\rho_{X,U} : {}^s\mathrm{H}_p(X) \to {}^s\mathrm{H}_p(U).$$

- For each closed set $\iota : A \hookrightarrow X$, there is a natural long exact sequence

(1.2) $$\cdots \longrightarrow {}^s\mathrm{H}_p(A) \xrightarrow{i_*} {}^s\mathrm{H}_p(X) \xrightarrow{\rho} {}^s\mathrm{H}_p(X - A) \xrightarrow{\partial} {}^s\mathrm{H}_{p-1}(A) \longrightarrow \cdots$$

- If X is the union of disjoint open subsets $\{X_\alpha\}$, then the inclusions $\iota_\alpha : X_\alpha \to X$ induce monomorphisms in homology, and ${}^s\mathrm{H}_p(X)$ is the cartesian product

(1.3) $$ {}^s\mathrm{H}_p(X) = \prod_\alpha (\iota_\alpha)_* {}^s\mathrm{H}_p(X_\alpha).$$

- For any inverse system $\{\cdots \to Y_3 \to Y_2 \to Y_1\}$ of compact metric spaces with limit Y, the maps $Y \to Y_\alpha$ induce a natural exact sequence [Mil, Theorem 4]

(1.4) $$0 \longrightarrow \lim{}^1 [{}^s\mathrm{H}_{p+1}(Y_\alpha;G)] \longrightarrow {}^s\mathrm{H}_p(Y;G) \longrightarrow \varprojlim {}^s\mathrm{H}_p(Y_\alpha;G) \longrightarrow 0.$$

The corresponding cohomology theory is Alexander-Spanier cohomology with compact support. For compact Haudorff spaces, this is isomorphic to

both Alexander-Spanier and Čech cohomology $\check{H}^*$ [Sp, p. 334], and there is a universal coefficient theorem [Ma2, Cor. 4.18],

$$0 \longrightarrow \operatorname{Ext}(\check{H}^{p+1}(\overline{M};\mathbb{Z}),G) \longrightarrow {}^s\!H_p(\overline{M};G) \longrightarrow \operatorname{Hom}(\check{H}^p(\overline{M},\mathbb{Z}),G) \longrightarrow 0.$$

In Sections 1–4, the term "manifold", or "topological manifold" for emphasis, of dimension d, means a Hausdorff space in which each point has an open neighborhood homeomorphic to $\mathbb{R}^d$. Any further assumptions (such as compactness, connectedness, or second countability) will be explicitly specified as needed. Orientations can be defined as in [Ma2, §3.6]. One has the following facts for any oriented d-dimensional topological manifold and any abelian coefficient group G:

- For all $p > d$,

$$(1.5) \qquad\qquad\qquad\qquad {}^s\!H_p(M) = 0.$$

- For each topological d-ball B in a connected component M_α of M,

$$(1.6) \qquad\qquad\qquad\qquad {}^s\!H_d(B;G) \cong G$$

  and

$$(1.7) \qquad \rho_{BM_\alpha} : {}^s\!H_d(M_\alpha) \to {}^s\!H_d(B) \qquad \text{is an isomorphism.}$$

- The orientation determines a fundamental class $[M] \in {}^s\!H_d(M;\mathbb{Z})$ such that for each open ball $B \subseteq M$, regarded as a manifold with the induced orientation,

$$\rho_{BM}[M] = [B].$$

  More generally, if N is an open subset of M with the induced orientation, then

$$(1.8) \qquad\qquad\qquad\qquad \rho_{NM}[M] = [N].$$

- If M has components $\{M_\alpha\}$, the fundamental class is given under the isomorphism (1.3) by

$$(1.9) \qquad\qquad\qquad\qquad [M] = \prod_\alpha [M_\alpha].$$

For proofs, see [Ma2], Theorems 2.13 and 3.21a, page 112, and Lemma 11.6.

Note that (1.6) shows a key difference between type I and type II homology theories: in a type II homology, a ball $B \subseteq \mathbb{R}^d$ has a fundamental class. This, as well as the existence of the restriction map (1.1), stem from the fact that type II homology is constructed using chains that are dual to *compactly supported* cochains. For the same reason, type II homology is invariant only under *proper* homotopies.

In Section 2, we work exclusively with Steenrod homology. In Section 3, where we consider families of spaces, we pass instead to Čech homology, because it satisfies the following

Continuity Property. For every inverse system of compact Hausdorff spaces as in (1.4), the maps $Y \to Y_\alpha$ induce a natural isomorphism

$$(1.10) \qquad \check{\mathrm{H}}_*(Y; G) \xrightarrow{\cong} \varprojlim \check{\mathrm{H}}_*(Y_\alpha; G)$$

[ES, pages 260-261].

In general, Steenrod homology does not satisfy the continuity property (it satisfies (1.4) instead), and Čech homology does not satisfy the exactness axiom. However, for every compact Hausdorff space X, abelian group G, and commutative ring R, there are natural maps

$$(1.11) \quad {}^s\mathrm{H}_p(X; G) \longrightarrow \check{\mathrm{H}}_p(X; G) \qquad \text{and} \qquad \check{\mathrm{H}}_p(X; R) \longrightarrow \check{\mathrm{H}}^p(X; R)^\vee,$$

where $\check{\mathrm{H}}^p(X; R)^\vee = \mathrm{Hom}(\check{\mathrm{H}}^p(X; R), R)$ is the dual to Čech cohomology (cf. Remark 5.0.2 in [Pd]). Furthermore, when restricted to compact metric spaces and rational coefficients, both arrows in (1.11) are isomorphisms (the first by Milnor's uniqueness theorem [Mil]), giving a theory that is both exact and continuous (cf. [ES, p. 233]).

Lemma 1.1. *Let $\check{\mathcal{H}}_*(X)$ denote one of the three possibilities:*

$$(1.12) \qquad \check{\mathcal{H}}_*(X) \;=\; \begin{cases} \check{\mathrm{H}}_*(X; \mathbb{Z}) & \text{Čech homology, or} \\ \check{\mathrm{H}}^*(X; \mathbb{Z})^\vee & \text{Dual Čech cohomology, or} \\ \check{\mathrm{H}}_*(X; \mathbb{Q}) & \text{Rational Čech homology.} \end{cases}$$

Then there is a natural transformation ${}^s\mathrm{H}_*(X; \mathbb{Z}) \to \check{\mathcal{H}}_*(X)$ defined on the category of compact Hausdorff spaces, and $\check{\mathcal{H}}_*$ satisfies the Continuity Property (i.e. (1.10) holds with $\check{\mathrm{H}}_*$ replaced by $\check{\mathcal{H}}_*$).

Proof. For any abelian group G, Čech homology satisfies (1.10) while, with the same notation, Čech cohomology satisfies

$$(1.13) \qquad \check{\mathrm{H}}^p(Y, \mathbb{Z}) = \varinjlim_{\alpha} \check{\mathrm{H}}^p(Y_\alpha; \mathbb{Z})$$

[ES, pages 260-261]. Hence by Proposition 5.26 in [Ro],

$$\check{\mathrm{H}}^p(Y; \mathbb{Z})^\vee = \mathrm{Hom}(\varinjlim_{\alpha} \check{\mathrm{H}}^p(Y_\alpha; \mathbb{Z}), \mathbb{Z}))$$
$$= \varprojlim_{\alpha} \mathrm{Hom}(\check{\mathrm{H}}^p(Y_\alpha; \mathbb{Z}), \mathbb{Z}) = \varprojlim_{\alpha} \check{\mathrm{H}}^p(Y_\alpha; \mathbb{Z})^\vee.$$

$\square$

Each of the possibilities in Lemma 1.1 pairs with Čech cohomology; there is no longer any need for Alexander-Spanier cohomology. Čech cohomology, of course, is different from singular cohomology but, for any G and any paracompact Hausdorff space X, there is a natural map

$$(1.14) \qquad \check{\mathrm{H}}^p(X; G) \to H^p_{sing}(X; G)$$

that is an isomorphism if X is a manifold, or more generally if X is locally contractible [Sp, Corollaries 6.8.8 and 6.9.5].

2. Thin compactifications

In Steenrod homology with integer coefficients, oriented open manifolds M have a fundamental class, but this class is of limited use because it does not push forward under general continuous maps. This deficiency can be rectified by considering maps that extend continuously over a compactification $\overline{M} = M \cup S$ of M, and showing that the fundamental class $[M] \in {}^s\mathrm{H}_*(M)$ extends canonically to a class $[\overline{M}]$ in ${}^s\mathrm{H}_*(\overline{M})$. Many such compactifications are possible; making S larger allows more maps to extend continuously to $\overline{M}$, but making S too large interferes with the fundamental class. Definition 2.1 identifies a class of compactifications — "thin compactifications" — that is appropriate for working with fundamental classes. These have the form

$$\overline{M} = M \cup S$$

where S is a space of "homological codimension 2". There are no assumptions about differentiability or about how M and S fit together, other than the requirement that $\overline{M}$ is a compact Hausdorff space.

Definition 2.1. *Let M be an oriented d-dimensional topological manifold. A* thin compactification *of M is a compact Hausdorff space $\overline{M}$ containing M such that the complement $S = \overline{M} \setminus M$ (the "singular locus") is a closed subset of codimension 2 in the sense that*

$$(2.1) \qquad\qquad {}^s\mathrm{H}_p(S) = 0 \quad \forall\, p > d - 2.$$

Every compact manifold is a thin compactification of itself (with S empty), and for each oriented manifold of finite dimension $d \geq 2$, the 1-point compactification is a thin compactification. Further examples arise from stratified spaces of the following type (as was communicated to us by both J. Morgan and J. Pardon).

Lemma 2.2 (Stratified thin compactification). *Suppose that an oriented d-dimensional topological manifold M is a subset of a compact Hausdorff space $\overline{M}$ that, as a set, is a disjoint union*

$$(2.2) \qquad\qquad \overline{M} = M \cup \bigcup_{k \geq 2} S_k,$$

where for each $k \geq 2$, S_k is a manifold of dimension at most $d - k$, and $T_k := \bigcup_{i \geq k} S_i$ is closed. Then $\overline{M}$ is a thin compactification of M.

Proof. By induction on k, we will show that ${}^s\mathrm{H}_p(T_k) = 0$ for all $p > d - k$, which implies that the singular set $S = T_2$ satisfies (2.1). The induction starts with $k = d + 1$ (T_{d+1} is empty) and descends. For $p > d - (k-1)$, we have $\dim S_{k-1} \leq d - (k-1) < p + 1$, so ${}^s\mathrm{H}_{p+1}(S_{k-1}) = 0$. The long exact sequence

$$\to {}^s\mathrm{H}_{p+1}(S_{k-1}) \to {}^s\mathrm{H}_p(T_{k-1}) \to {}^s\mathrm{H}_q(T_k) \to$$

and the induction assumption then imply that ${}^s\mathrm{H}_p(T_{k-1}) = 0$, as required. □

In practice, singular strata are usually unions of a large number of strata S_α. One must form the S_k of (2.2) as unions of the S_α and verify that $S_k \setminus S_{k-1}$ are manifolds. One way of doing this is described in Lemma 5.4.

Example 2.3. (a) The closure $\overline{V}$ of a smooth quasi-projective variety $V \subset \mathbb{P}^N$ is a thin compactification.

(b) For a nodal complex curve C, the regular part $M = C^{reg}$ can be thinly compactified in three ways: by its 1-point compactification, by C, and by its normalization $\tilde{C}$, which may be disconnected.

(c) Define an infinite chain of 2-spheres as follows. For each $n = 1, 2, \ldots$, let p_n be the point $(\frac{1}{n}, 0, 0)$ in $\mathbb{R}^3$. Let S_n be the sphere with center $q_n = \frac{1}{2}(p_n + p_{n+1})$ and radius $R_n = |p_n - q_n|$ with the two points p_n and p_{n+1} removed. Then $M = \bigcup S_n$ is an embedded 2-manifold in $\mathbb{R}^3$, and $\overline{M} = M \cup S$ is a thin compactification with a singular set $S = \bigcup p_n \cup (0, 0, 0)$ of dimension zero.

(d) In contrast, $M = \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \right\} \subset \mathbb{R}$ is a 0-manifold, but its compactification $M \cup \{0\}$ is not thin.

We now come to the key point of these definitions: in Steenrod homology, the fundamental class of an oriented manifold M extends to every thin compactification.

Theorem 2.4. *Let M be an oriented d-dimensional manifold with fundamental class $[M]$. Every thin compactification $\overline{M}$ of M has a fundamental class*

$$[\overline{M}] \in {}^s\mathrm{H}_d(\overline{M}; \mathbb{Z})$$

uniquely characterized by the requirement that

(2.3) $$\rho_M([\overline{M}]) = [M],$$

where $\rho_M : {}^s\mathrm{H}_d(\overline{M}; \mathbb{Z}) \to {}^s\mathrm{H}_d(M; \mathbb{Z})$ is the map (1.1).

Proof. The exact sequence (1.2) for the closed subset $A = S$ of $\overline{M}$, together with (2.1), implies that the map

(2.4) $$\rho_M : {}^s\mathrm{H}_\ell(\overline{M}) \xrightarrow{\cong} {}^s\mathrm{H}_\ell(M)$$

is an isomorphism for all $\ell \geq d$. Taking $\ell = d$ shows that there is a unique class $[\overline{M}]$ satisfying (2.3). $\qquad\square$

In general, a manifold M has many thin compactifications, each with a fundamental class related to $[M]$ by (2.3). If $\overline{M}$ is one such thin compactification with singular locus S, and $Z \subset M$ is a closed subset such that $Z \cup S$ has homological codimension 2, then $\overline{M}$ is also a thin compactification of $M \setminus Z$, and $[\overline{M \setminus Z}] = [\overline{M}]$. In this sense, one can ignore sets of codimension 2 in computations with fundamental classes.

Example 2.5. For two thin compactifications $\overline{M}_1$ and $\overline{M}_2$ of the same d-dimensional manifold M, there are isomorphisms $\rho_i : {}^s\mathrm{H}_d(\overline{M}_i) \to {}^s\mathrm{H}_d(M)$, as in (2.4), and the composition

$$\rho_2^{-1} \circ \rho_1 : {}^s\mathrm{H}_d(\overline{M}_1) \to {}^s\mathrm{H}_d(\overline{M}_2)$$

takes $[\overline{M}_1]$ to $[\overline{M}_2]$. This is true even when there is no continuous map from $\overline{M}_1$ to $\overline{M}_2$. If there is a map $f : \overline{M}_1 \to \overline{M}_2$, then $f_*[\overline{M}_1] = [\overline{M}_2]$ by the naturality of ρ. In particular:

(a) Let $\pi : M_Z \to M$ be the blowup of a closed complex manifold M along a complex submanifold Z. Then M and M_Z are two different thin compactifications of $M \setminus Z$, and $\pi_*[M_Z] = [M]$.

(b) More generally, a birational map $X \dashrightarrow Y$ between complex projective varieties induces an identification of $[X]$ with $[Y]$.

(c) If $\dim M \geq 2$, every thin compactification $\overline{M}$ has a map p to the 1-point compactification M^+, and $p_*[\overline{M}] = [M^+]$.

The fundamental class of a manifold M need not push forward under a general continuous map $f : M \to X$. However, if f extends to a continuous map $\overline{f} : \overline{M} \to X$ from *some* thin compactification $\overline{M}$ of M, then $\overline{f}$ is proper, so induces a map $\overline{f}_*$ in Steenrod homology:

$$
\begin{array}{ccc}
{}^s\mathrm{H}_d(\overline{M}) & & \\
\rho \downarrow \cong & \searrow^{\overline{f}_*} & \\
{}^s\mathrm{H}_d(M) & \dashrightarrow & {}^s\mathrm{H}_d(X).
\end{array}
$$

In this situation, $[M]$ corresponds to $[\overline{M}]$ by (2.3), and the class $\overline{f}_*([\overline{M}]) \in {}^s\mathrm{H}_d(X)$ serves as a surrogate for $f_*[M]$. Alternatively, one can take a Čech class $\alpha \in \check{\mathrm{H}}^d(X)$ and evaluate $\overline{f}^*\alpha$ on the image of $[\overline{M}]$ under (1.11).

2.1. Covering maps

The isomorphism (2.4) implies several statements about how fundamental classes behave under covering maps.

Lemma 2.6. *Suppose that $\overline{f} : \overline{M} \to \overline{N}$ is a continuous map between thinly compactified oriented manifolds. If f restricts to a degree ℓ oriented covering*

$f : M \to N$, *then*

(2.5) $$\overline{f}_*[\overline{M}] = \ell\,[\overline{N}].$$

More generally, if N has components $\{N_\alpha\}$ and f restricts to a degree ℓ_α cover over some nonempty open ball U_α in each N_α then, in the notation of (1.3) and (1.9),

(2.6) $$\overline{f}_*[\overline{M}] = \prod_\alpha \ell_\alpha\,[\overline{N}_\alpha].$$

Here $\ell_\alpha = 0$ if $f^{-1}(U_\alpha)$ is empty.

Proof. First assume that M and N are both connected. Fix an open ball $U \subseteq N$ so that $f^{-1}(U)$ is the disjoint union of ℓ open balls $V_1, \ldots, V_\ell$. In this situation, there is an isomorphism $\rho_U : {}^s\mathrm{H}_d(N) \to {}^s\mathrm{H}_d(U)$ as in (1.7), and similar isomorphisms $\rho_i : {}^s\mathrm{H}_d(M) \to {}^s\mathrm{H}_d(V_i)$ for each i. These fit into a commutative diagram

$$
\begin{array}{ccccccc}
{}^s\mathrm{H}_d(\overline{M}) & \xrightarrow{\;\cong\;{}_{\rho_M}\;} & {}^s\mathrm{H}_d(M) & \xrightarrow{\;(\rho_1,\ldots,\rho_\ell)\;} & \bigoplus_i {}^s\mathrm{H}_d(V_i) & \xrightarrow{\;\cong\;} & \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \\
\overline{f}_* \downarrow & & f_* \downarrow & & f_* \downarrow & & \varphi \downarrow \\
{}^s\mathrm{H}_d(\overline{N}) & \xrightarrow{\;\cong\;{}_{\rho_N}\;} & {}^s\mathrm{H}_d(N) & \xrightarrow[\;\rho_U\;]{\;\cong\;} & {}^s\mathrm{H}_d(U) & \xrightarrow{\;\cong\;} & \mathbb{Z}
\end{array}
$$

where $\varphi(a_1, \ldots, a_\ell) = \sum a_i$, where ρ_M and ρ_N are isomorphisms by (2.4), and where the first two squares commute by the naturality of ρ. Restricting the diagram to generators gives (2.5).

In general, for each component N_α of N, $f^{-1}(U_\alpha)$ is the disjoint union of components $V_{\alpha\beta}$, and (2.5) applies to each restriction $f_{\alpha\beta} = f|_{V_{\alpha\beta}}$, and the homologies of $\overline{M}$ and $\overline{N}$ are cartesian products as in (1.3). This implies (2.6) with $\ell_\alpha = \sum_\beta \deg f_{\alpha\beta}$, and (2.5) if all ℓ_α are equal to ℓ. $\qquad\square$

Example 2.7. Lemma 2.6 applies to branched covers of complex analytic varieties.

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2.2. Components

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Suppose that an oriented manifold M has finitely many connected components M_α, and that $\overline{M}$ is a thin compactification of M with singular locus S. We then have:

Lemma 2.8. *For each α, $\overline{M}_\alpha = M_\alpha \cup S$ is a thin compactification of M_α, and*

$$(2.7) \qquad [\overline{M}] = \sum_\alpha [\overline{M}_\alpha].$$

Proof. The first statement holds because $\overline{M}_\alpha = M_\alpha \cup S$ is a closed, hence compact, subset of $\overline{M}$ and S satisfies (2.1). The disjoint union $\bigsqcup \overline{M}_\alpha$ is therefore another thin compactification of M, and $[\bigsqcup \overline{M}_\alpha] = \sum_\alpha [\overline{M}_\alpha]$. Moreover, the identity $M \to M$ extends to a continuous map $\iota : \bigsqcup \overline{M}_\alpha \to \overline{M}$. Lemma 2.6 then gives $\iota_*[\bigsqcup \overline{M}_\alpha] = [\overline{M}]$, and hence (2.7). $\qquad \square$

2.3. Thin compactifications with boundary

It is useful to extend the notion of thin compactifications to manifolds M with boundary ∂M.

Definition 2.9. *A* thin compactification *of $(M, \partial M)$ is a compact Hausdorff pair $(\overline{M}, \overline{\partial M})$ containing $(M, \partial M)$ such that*

  (i) $S = \overline{M} \setminus M$ is a closed subset of $\overline{M}$ of codimension 2,

  (ii) $S' = \overline{\partial M} \setminus \partial M$ is a closed subset of $\overline{\partial M}$ of codimension 2, and

 (iii) $S' \subseteq S$.

Note that (ii) implies that $\overline{\partial M}$ is a thin compactification of ∂M, while (iii) implies that the interior $M^0 = M \setminus \partial M$ is a subset of $\overline{M} \setminus \overline{\partial M}$ and that $\partial M = M \cap \overline{\partial M}$. The exact sequence (1.2) of such a pair $(\overline{\partial M}, \overline{M})$ is, in part,

$$(2.8) \qquad {}^s\mathrm{H}_d(\overline{M}) \xrightarrow{\rho} {}^s\mathrm{H}_d(\overline{M} \setminus \overline{\partial M}) \xrightarrow{\partial} {}^s\mathrm{H}_{d-1}(\overline{\partial M}) \xrightarrow{\iota_*} {}^s\mathrm{H}_{d-1}(\overline{M}).$$

When M is oriented, there is an induced orientation on ∂M, and the interior M^0 carries a fundamental class $[M^0] = [M \setminus \partial M] \in H_d(M^0)$. This is related to the fundamental class $[\partial M]$ of ∂M by

$$(2.9) \qquad \partial[M^0] = [\partial M] \in {}^s\mathrm{H}_{d-1}(\partial M),$$

where ∂ is the boundary operator in the sequence (1.2) for the pair $(M, \partial M)$ (see [Ma2, Theorem 11.8], being mindful of orientations and noting the change of notation $H_p \mapsto H_p^\infty$ on page 302).

Lemma 2.10. *A thin compactification* $(\overline{M}, \overline{\partial M})$ *of an oriented d-dimensional manifold with boundary* $(M, \partial M)$ *has a natural fundamental class* $[\overline{M}] \in {}^s\mathrm{H}_d(\overline{M} \setminus \overline{\partial M})$ *such that, for the maps in* (2.8),

(2.10)　　　　　　$a)\ \partial[\overline{M}] = [\overline{\partial M}]$　　　*and*　　$b)\ \bar{\iota}_*[\overline{\partial M}] = 0.$

Furthermore, $\rho'[\overline{M}] = [M^0]$ *under the restriction to* $M^0 \subseteq \overline{M} \setminus \overline{\partial M}$.

Proof. Combining (2.8) with the similar sequence for the pair $(M, \partial M)$ gives the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & {}^s\mathrm{H}_d(\overline{M}) & \xrightarrow{\ \rho\ } & {}^s\mathrm{H}_d(\overline{M} \setminus \overline{\partial M}) & \xrightarrow{\ \partial\ } & {}^s\mathrm{H}_{d-1}(\overline{\partial M}) & \xrightarrow{\ \bar{\iota}_*\ } & {}^s\mathrm{H}_{d-1}(\overline{M}) \\
& & \downarrow{\scriptstyle \rho_{\overline{M},M}} & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho_{\partial M}} & & \downarrow{\scriptstyle \rho} \\
0 & \longrightarrow & {}^s\mathrm{H}_d(M) & \xrightarrow{\ \rho\ } & {}^s\mathrm{H}_d(M \setminus \partial M) & \xrightarrow{\ \partial\ } & {}^s\mathrm{H}_{d-1}(\partial M) & \xrightarrow{\ \iota_*\ } & {}^s\mathrm{H}_{d-1}(M)
\end{array}
$$

where the rows are exact and the vertical maps are restriction maps to open subsets. Using properties 3b, 4b, and 4c listed on page 86 of [Ma2], one sees that the three squares are commutative. The first and third vertical arrows are isomorphisms by parts (i) and (ii) of Definition 2.9, and the exact sequence (1.2) for the pair $(\overline{M}, S)$ shows that ρ is an injection. The Five Lemma then implies that ρ' is an isomorphism.

We can define $[\overline{M}] \in {}^s\mathrm{H}_d(\overline{M} \setminus \overline{\partial M})$ uniquely by the requirement that

$$\rho'[\overline{M}] = [M^0].$$

Then (2.10a) follows from (2.9) and the uniqueness of (2.3), while (2.10b) follows from exactness of the top row of the diagram.　　　□

Example 2.11. (a) If $\overline{X}$ is a thin compactification of a manifold X of dimension $d \geq 1$, then the cone $C\overline{X}$ on $\overline{X}$ is a thin compactification of the cone on X minus its vertex.

(b) In the picture, $\overline{M}$ is the union of a cone on S^2 and a cylinder $S^2 \times [0,1]$, intersecting at one point p. Then the complement of the cone point p is a manifold with boundary, and $\overline{M}$ satisfies the conditions of Definition 2.9 with $S = S' = \{p\}$.
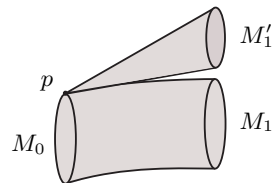
2.4. Cobordisms

Lemma 2.10 can be applied to cobordisms. A *thin compactified cobordism* between $\overline{M}_0$ and $\overline{M}_1$ is a compact Hausdorff pair $(\overline{W}, S)$ such that

(i) $W = \overline{W} \setminus S$ is an oriented cobordism between two manifolds M_0 and M_1.

(ii) $\overline{M}_i \subset \overline{W}$ is a thin compactification of M_i for $i = 0, 1$, and $\overline{M}_0$ is disjoint from $\overline{M}_1$.

(iii) ${}^s\mathrm{H}_{k+1}(S) = 0$ for all $k \geq d - 2$.

Corollary 2.12. *Suppose that W is an oriented topological cobordism between d-dimensional manifolds M_0 and M_1. If W admits a thin compactification $\overline{W}$, then the fundamental classes of $\overline{M}_0$ and $\overline{M}_1$ represent the same class in $\overline{W}$:*

$$(2.11) \qquad (\iota_0)_*[\overline{M}_0] = (\iota_1)_*[\overline{M}_1] \qquad in \; {}^s\mathrm{H}_d(\overline{W}),$$

where ι_0, ι_1 are the inclusions of $\overline{M}_0$ and $\overline{M}_1$ into $\overline{W}$.

Proof. The hypothesis means that W is an oriented topological manifold with boundary $\partial W = M_1 \sqcup -M_0$ and that $(\overline{W}, \overline{\partial W})$ is a thin compactification of $(W, \partial W)$, where $\overline{\partial W} = \overline{M}_1 \cup \overline{M}_0$. Then Lemmas 2.8 and 2.10 apply, and (2.10b) becomes (2.11). $\qquad \square$

3. Relatively thin families and the Extension Lemma

The notion of thin compactification has a relative version. Consider a continuous map $\pi : \mathcal{M} \to \mathcal{P}$ between Hausdorff spaces, which we regard as a family of spaces (the fibers of π) parameterized by $\mathcal{P}$. A compactification of this family is a Hausdorff space $\overline{\mathcal{M}}$ with maps

$$(3.1) \qquad \begin{array}{ccc} \mathcal{M} & \lhook\joinrel\dashrightarrow & \overline{\mathcal{M}} \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \overline{\pi}} & \\ \mathcal{P} & & \end{array}$$

where the horizontal arrow is an inclusion of $\mathcal{M}$ as an open subset, and $\overline{\pi}$ is continuous and proper. The fibers of $\mathcal{M}$ and $\overline{\mathcal{M}}$ over a point $p \in \mathcal{P}$ are denoted $\mathcal{M}_p$ and $\overline{\mathcal{M}}_p$ respectively; these may be empty because we are not assuming that $\overline{\pi}$ is surjective.

To extend the notion of a thin compactification to families, one might require that the fiber $\overline{\mathcal{M}}_p$ be a thin compactification of $\mathcal{M}_p$ for every $p \in \mathcal{P}$. The aim of this section is to show that it is enough to use a weaker notion, in which the fiber is required to be thin only for generic points $p \in \mathcal{P}$.

In the following definition, the term "second category subset" means a countable intersection of open dense subsets. We will assume that $\mathcal{P}$ has two properties:

(a) $\mathcal{P}$ is a locally path-connected metric space, and

(b) $\mathcal{P}$ is a Baire space, i.e. every second category subset of $\mathcal{P}$ is dense in $\mathcal{P}$.

By the Baire Category Theorem, both (a) and (b) hold if $\mathcal{P}$ is a metrizable separable Banach manifold.

The space of paths in $\mathcal{P}$ is the set of continuous maps $\gamma : [0, 1] \to \mathcal{P}$ with the C^0 topology. For each such γ, the pullback of $\overline{\mathcal{M}}$ by γ is a space

$$\overline{\mathcal{M}}_\gamma = \big\{ (x, y) \in [0, 1] \times \overline{\mathcal{M}} \,\big|\, \gamma(x) = \overline{\pi}(y) \big\}.$$

There is an associated pullback diagram

(3.2)
$$\begin{array}{ccc} \overline{\mathcal{M}}_\gamma & \xrightarrow{\hat{\gamma}} & \overline{\mathcal{M}} \\ {\scriptstyle \overline{\pi}_\gamma} \downarrow & & \downarrow {\scriptstyle \overline{\pi}} \\ [0, 1] & \xrightarrow{\gamma} & \mathcal{P}, \end{array}$$

with natural embeddings $\iota_0 : \overline{\mathcal{M}}_p \to \overline{\mathcal{M}}_\gamma$, $\iota_1 : \overline{\mathcal{M}}_q \to \overline{\mathcal{M}}_\gamma$ of the fibers over the endpoints.

Definition 3.1. *A relatively thin family of relative dimension d is a proper continuous map*

(3.3)
$$\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$$

from a Hausdorff space $\overline{\mathcal{M}}$ to a space $\mathcal{P}$ satisfying (a) and (b) above, such that there exists a second category subset $\mathcal{P}^* \subseteq \mathcal{P}$ satisfying:

(I) *for each $p \in \mathcal{P}^*$, the fiber $\overline{\mathcal{M}}_p$ over p is a thin compactification of a d-dimensional oriented topological manifold $\mathcal{M}_p$.*

(II) *for each $p, q \in \mathcal{P}^*$, there is a second category subset of paths from p to q such that, for each γ in this subset, $(\overline{\mathcal{M}}_\gamma, \overline{\mathcal{M}}_p \sqcup \overline{\mathcal{M}}_q)$ is a thin compactification of an oriented cobordism from $\mathcal{M}_p$ to $\mathcal{M}_q$.*

The assumptions on $\mathcal{P}$ ensure that $\mathcal{P}^*$ is dense in $\mathcal{P}$. Relatively thin families often appear as compactifications:

Definition 3.2. *A thin compactification of a family* $\pi : \mathcal{M} \to \mathcal{P}$ *is a relatively thin family* (3.3) *together with an embedding as in Diagram* (3.1).

The lemmas below use elementary topological arguments to show that assumptions (I) and (II) imply the existence and uniqueness of a consistent relative fundamental class. In subsequent sections, we will use the Sard-Smale theorem to obtain (I) and (II).

By Lemma 1.1, Assumption (I) implies that for each $p \in \mathcal{P}^*$ there is an associated fundamental class

$$(3.4) \qquad [\overline{\mathcal{M}}_p] \in {}^s\mathrm{H}_d(\overline{\mathcal{M}}_p, \mathbb{Z})$$

in integral Steenrod homology. Corollary 2.12 and Assumption (II) imply that this association has the consistency property

$$(3.5) \qquad (\iota_0)_*[\overline{M}_p] = (\iota_1)_*[\overline{M}_q] \qquad \text{in } {}^s\mathrm{H}_d(\overline{\mathcal{M}}_\gamma)$$

along a dense set of paths γ from p to q.

We now pass from Steenrod to Čech homology using the natural transformation in Lemma 1.1. The fundamental class (3.4) in Steenrod homology determines a fundamental class, still denoted $[\overline{\mathcal{M}}_p]$, in Čech homology. Thus there is an association

$$(3.6) \qquad p \mapsto [\overline{\mathcal{M}}_p] \in \check{\mathcal{H}}_d(\overline{\mathcal{M}}_p)$$

for each $p \in \mathcal{P}^*$ that satisfies the consistency property (3.5) in $\check{\mathcal{H}}_d(\overline{\mathcal{M}}_\gamma)$. To proceed, it is helpful to temporarily move to a general context that does not involve fundamental classes (as done in Definition 3.3 and Lemma 3.4). We will return to (3.6) in Section 4.

Definition 3.3. *Let* $\check{\mathcal{H}}_*$ *be as in Lemma 1.1. We say an association*

$$p \mapsto \alpha_p \in \check{\mathcal{H}}_*(\overline{\mathcal{M}}_p)$$

is consistent *along a path* γ *from* p *to* q *if the images of* α_p *and* α_q *become equal in the homology of* $\overline{\mathcal{M}}_\gamma$:

$$(3.7) \qquad (\iota_0)_*\alpha_p = (\iota_1)_*\alpha_q \quad in \quad \check{\mathcal{H}}_*(\overline{\mathcal{M}}_\gamma).$$

We can now apply the continuity property (1.10) to extend any such consistent association to all $p \in \mathcal{P}$:

Extension Lemma 3.4. *Let $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ be a proper continuous map from a Hausdorff space to a locally path-connected metric space $\mathcal{P}$. Suppose that there is a dense subset $\mathcal{P}^*$ of $\mathcal{P}$ and an assignment*

$$(3.8) \qquad\qquad p \mapsto \alpha_p \in \check{\mathcal{H}}_*(\overline{\mathcal{M}}_p)$$

defined for $p \in \mathcal{P}^*$ and consistent along paths in a dense subset of the space of paths in $\mathcal{P}$ from p to q for each $p, q \in \mathcal{P}^*$. Then (3.8) extends to all $p \in \mathcal{P}$ so that (3.7) holds for all paths γ in $\mathcal{P}$, and this extension is unique.

Proof. Fix a point $p \in \mathcal{P}$, and let B_k be the ball of radius $1/k$ centered at p. Using the definition of locally path-connected, one can inductively choose a sequence of path-connected open neighborhoods U_k of p with $U_k \subset B_k$ and $U_{k+1} \subset U_k$, for all $k \geq 1$. Then each U_k contains a dense set of values $q \in \mathcal{P}^* \cap U_k$ for which (3.8) is defined. Moreover, any two values in $\mathcal{P}^* \cap U_k$ can be joined by a path in U_k which, by assumption, can be perturbed, keeping the endpoints fixed, to a path in U_k for which (3.7) holds.

Choose any sequence $p_k \in U_k \cap \mathcal{P}^*$ (so p_k converge to p) and paths $\gamma_k : [0,1] \to \mathcal{P}$ from p_k to p_{k+1} satisfying (3.7) and whose image is in U_k. For each $m \geq 1$, set $K_m = [0, \frac{1}{m}]$, and define a "segmented" path $\varphi_m : K_m \to \mathcal{P}$ by $\varphi_m(0) = p$ and

$$\varphi_m(t) = \gamma_k \left(\tfrac{1}{t} - k \right) \quad \text{for } t \in \left[\tfrac{1}{k+1}, \tfrac{1}{k} \right], \quad k \geq m.$$

Then each φ_m is a proper continuous map whose image is a path through the points $p_k = \varphi_m(1/k)$ for $k \geq m$. The pullback spaces $\overline{\mathcal{M}}_{\varphi_m}$ (defined as in (3.2)) form a nested sequence of compact Hausdorff spaces whose intersection is the compact space $\overline{\mathcal{M}}_p$. There are also natural inclusions $\iota_{km} : \overline{\mathcal{M}}_{p_k} \to \overline{\mathcal{M}}_{\varphi_m}$ for each $k \geq m$. Applying the consistency condition (3.7) inductively, one sees that the class

$$(\iota_{km})_* \alpha_{p_k} \ \in \ \check{\mathcal{H}}_d(\overline{\mathcal{M}}_{\varphi_m}).$$

is independent of k for $k \geq m$. These homology classes are consistently related by the inclusions $\overline{\mathcal{M}}_{\varphi_{m_1}} \hookrightarrow \overline{\mathcal{M}}_{\varphi_{m_2}}$ for $m_1 \geq m_2$, so define an element

of the inverse limit

$$(3.9) \qquad \varprojlim (\iota_{km})_* \alpha_{p_k} \in \varprojlim_m \check{\mathcal{H}}_d(\overline{\mathcal{M}}_{\varphi_m}).$$

By the continuity property (1.10), this determines a unique Čech homology class

$$(3.10) \qquad \alpha_p \in \check{\mathcal{H}}_d(\overline{\mathcal{M}}_p)$$

which, at this point, depends on the choices of the p_k and the γ_k.

Next, fix an arbitrary continuous path $\gamma : [0, 1] \to \mathcal{P}$ from $p \in \mathcal{P}$ to $p' \in \mathcal{P}$. Choose segmented paths φ_m and $\varphi'_m : K_m \to \mathcal{P}$ as above that limit to $p = \varphi_m(0)$ and $p' = \varphi'_m(0)$, respectively. Then, for each k, choose a path σ_k from p_k to p'_k that lies in the $1/k$ neighborhood of γ, and for which (3.7) holds (specifically, $U_k \cup \gamma \cup U'_k$ is path-connected, so contains a path from $p_k \in \mathcal{P}^*$ to $p'_k \in \mathcal{P}^*$ which, by assumption, can be perturbed to the desired path σ_k). For each m, let $L_m \subset \mathbb{R}^2$ denote the "ladder" consisting of the union of the segments:

$$I_m = \{(0, y) \mid 0 \le y \le 1/m\} \quad I_0 = \{(x, 0) \mid 0 \le x \le 1\}$$
$$I'_m = \{(1, y) \mid 0 \le y \le 1/m\} \quad J_k = \{(x, \tfrac{1}{k}) \mid 0 \le x \le 1\}, \quad k \ge m.$$

Now let $\Phi_m : L_m \to \mathcal{P}$ be the continuous map whose restriction (i) to I_0 is γ (after identifying I_0 with $[0, 1]$), and whose restrictions

(ii) to I_m is φ_m,

(iii) to I'_m is φ'_m, and

(iv) to each J_k is σ_k.

Ladder L_m
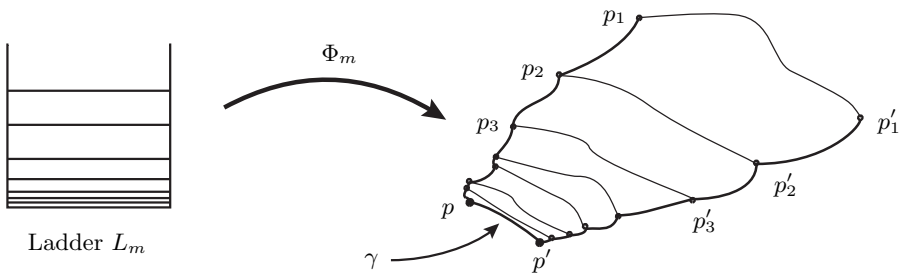
Each L_m is compact, so Φ_m is proper, and the pullback spaces $\overline{\mathcal{M}}_{\Phi_m}$ are a nested sequence of compacta whose intersection is $\overline{\mathcal{M}}_\gamma$. Again the consistency condition (3.7) implies that, for $k \geq m$, the classes

$$(\iota_{km})_* \alpha_{p_k}, (\iota'_{km})_* \alpha_{p'_k} \in \check{\mathcal{H}}_d(\overline{\mathcal{M}}_{\Phi_m})$$

are equal and independent of k, and hence form an inverse system that defines an element

(3.11) $\alpha_\gamma = \varprojlim (\iota_{km})_* \alpha_{p_k} = \varprojlim (\iota'_{km})_* \alpha_{p'_k} \in \check{\mathcal{H}}_d(\overline{\mathcal{M}}_\gamma).$

Recall that the class (3.10) depends on the choice of the points p_k and the connecting paths γ_k. But given another choice $\{p'_k, \gamma'_k\}$, we can construct ladder maps Φ_m for the constant path $\gamma(t) \equiv p$. For constant paths, $\overline{\mathcal{M}}_\gamma$ is equal to $\overline{\mathcal{M}}_p \times [0,1]$, so there is a projection $\rho : \overline{\mathcal{M}}_\gamma \to \overline{\mathcal{M}}_p$. Applying ρ_* to (3.11) then shows the class (3.10) constructed from the two choices are equal.

With this understood, the consistency condition (3.7) along γ follows simply by comparing (3.10) and (3.11).

Finally, to check uniqueness, assume α' is another extension which agrees with α on $\mathcal{P}^*$ and satisfies (3.7) for all paths γ in $\mathcal{P}$. Pick any point $p \in \mathcal{P}$ and segmented paths $\varphi_m : K_m \to \mathcal{P}$ as above. Then for any $k \geq m$, the inclusions induce equalities

$$\alpha'_p = (\iota_{km})_* \alpha'_{p_k} = (\iota'_{km})_* \alpha_{p_k} = \alpha_p$$

in $\check{\mathcal{H}}_d(\overline{\mathcal{M}}_{\varphi_m})$. Therefore, again by continuity, we have

$$\alpha'_p = \varprojlim_m (\iota'_{km})_* \alpha'_p = \varprojlim_m (\iota_{km})_* \alpha_{p_k} = \alpha_p$$

as elements of $\varprojlim \check{\mathcal{H}}_d(\overline{\mathcal{M}}_{\varphi_m}) = \check{\mathcal{H}}_d(\overline{\mathcal{M}}_p)$. Thus the extension is unique.   □

4. Relative fundamental classes

We now return to the homology theory (1.12) and define relative fundamental classes for relatively thin families. The definition is axiomatic, and we prove both existence and uniqueness.

Definition 4.1. *A* relative fundamental class *for the relatively thin family (3.3) of relative dimension d associates to each $p \in \mathcal{P}$ an element*

$$[\overline{\mathcal{M}}_p]^{rel} \in \check{\mathcal{H}}_d(\overline{\mathcal{M}}_p) \tag{4.1}$$

such that, for some choice of $\mathcal{P}^*$ as in Definition 3.1,

A1. *(Normalization) For each $p \in \mathcal{P}^*$, $[\overline{\mathcal{M}}_p]^{rel}$ is the fundamental class $[\overline{\mathcal{M}}_p]$.*

A2. *(Consistency) For every path γ in $\mathcal{P}$ from p to q,*

$$(\iota_0)_*[\overline{\mathcal{M}}_p]^{rel} = (\iota_1)_*[\overline{\mathcal{M}}_q]^{rel} \quad in \quad \check{\mathcal{H}}_*(\overline{\mathcal{M}}_\gamma). \tag{4.2}$$

Note that a relative fundamental class is *not a single class*, but rather is a consistent *collection* of classes. It assigns a d-dimensional class (4.1) to *every* fiber $\overline{\mathcal{M}}_p$, including those that are not thinly compactified manifolds, and those whose dimension is not d. Similarly, the consistency condition (4.2) is a *collection* of equalities, one for each path in $\mathcal{P}$. The proof of Theorem 4.2 below shows how $[\overline{\mathcal{M}}_p]^{rel}$ is defined at each p as a limit of the fundamental classes of the fibers $\overline{\mathcal{M}}_p$ for p in the dense set $\mathcal{P}^*$.

Of course, the relative fundamental class depends on the relatively thin family (3.3), and in particular on its relative dimension d. A priori, it also depends on the second category set $\mathcal{P}^*$, but we show next that it does not.

Using the terminology of Definitions 3.1 and 4.1, our main result can be stated simply:

Theorem 4.2. *A relatively thin family $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ admits a unique relative fundamental class. This class satisfies satisfies **A1** and **A2** in Definition 4.1 for each choice of the second category set $\mathcal{P}^*$ in Definition 3.1, and is independent of the choice of $\mathcal{P}^*$.*

Proof. For each $p \in \mathcal{P}^*$, the fiber $\overline{\mathcal{M}}_p$ is a thin compactification of an oriented d-manifold, and we define $[\overline{\mathcal{M}}_p]^{rel}$ to be its fundamental class. As in (3.6), properties I and II of Definition 3.1 imply that the association

$$p \mapsto [\overline{\mathcal{M}}_p]^{rel} \tag{4.3}$$

has the consistency property (3.5). Thus the Extension Lemma 3.4 applies, giving a unique extension of (4.3) to all $p \in \mathcal{P}$ that satisfies the consistency condition Axiom A2.

To show independence of $\mathcal{P}^*$, suppose that a relatively thin family satisfies conditions I and II of Definition 3.1 for two second category sets $\mathcal{Q}^*$ and $\mathcal{Q}^{**}$. Then it also satisfies these conditions for the second category set $\mathcal{P}^* = \mathcal{Q}^* \cap \mathcal{Q}^{**}$. The sets $\mathcal{P}^*, \mathcal{Q}^*$ and $\mathcal{Q}^{**}$ each define a relative fundamental class, and these three classes are equal for all p in dense set $\mathcal{P}^*$. By the uniqueness in the Extension Lemma 3.4, they must agree for all $p \in \mathcal{P}$. $\square$

A relative fundamental class can be used to define numerical invariants. For each $p \in \mathcal{P}$, there is map

$$(4.4) \qquad\qquad I_p : \check{\mathrm{H}}^d(\overline{\mathcal{M}}, \mathbb{Z}) \to \mathbb{Z}$$

defined on a Čech cohomology class $\alpha \in \check{\mathrm{H}}^*(\overline{\mathcal{M}})$ by

$$(4.5) \qquad\qquad I_p(\alpha) = \left\langle \alpha, \ [\overline{\mathcal{M}}_p]^{rel} \right\rangle.$$

Here we are implicitly restricting α to the fiber $\overline{\mathcal{M}}_p$, and the pairing is well defined because $\overline{\mathcal{M}}_p$ is compact.

Corollary 4.3. *For a relatively thin family $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ the map (4.4) is independent of p on each path component of $\mathcal{P}$.*

Proof. Given points p and q in the same path component, fix a path $\gamma : [0, 1] \to \mathcal{P}$ from p to q. Pushing the consistency condition (4.2) forward by the homology map induced by the proper map $\hat{\gamma}$ in diagram (3.2) shows that $[\overline{\mathcal{M}}_p]^{rel}$ is homologous to $[\overline{\mathcal{M}}_q]^{rel}$ in $\check{\mathrm{H}}_d(\overline{\mathcal{M}})$. Hence $I_p(\alpha)$ is equal to $I_q(\alpha)$ for all cohomology classes α. $\square$

5. Fredholm families

In many gauge theories, the universal moduli space admits a compactification that is stratified by Banach manifolds in the manner described in Definition 5.2 below. If so, and more generally if such a stratification exists over an open dense subset of the parameter space, one can obtain a relative fundamental class using the Sard-Smale theorem and Theorem 4.2.

In this and later sections, the term "Banach manifold" means a metrizable separable C^l Banach manifold, finite or infinite dimensional. Such manifolds are second countable and paracompact. We say that a property holds "for generic p" if it holds for all p in some second category subset of $\mathcal{P}$. We

will consider Fredholm maps

(5.1)
$$\begin{array}{c} \mathcal{M} \\ \downarrow \pi \\ \mathcal{P} \end{array}$$

between Banach manifolds, which we again regard as a family parameterized by $\mathcal{P}$ and, to emphasize this viewpoint, call it a "Fredholm family". Such a map has an associated Fredholm index d, and we assume that

(5.2)
$$l > \max(d + 1, 0).$$

The Sard-Smale theorem shows that the generic fibers of π are manifolds of dimension d. It also yields a similar statement about generic paths in the Banach manifolds $\Omega(p, q)$ of C^1 paths $\gamma : [0, 1] \to \mathcal{P}$ from $p = \gamma(0)$ to $q = \gamma(1)$. The precise statements are as follows.

Theorem 5.1. *For a Fredholm map* (5.1) *of index d that satisfies* (5.2),

(a) *The set $\mathcal{P}_0^{reg}$ of regular values of π is a second category subset of $\mathcal{P}$, and for each $p \in \mathcal{P}_0^{reg}$, the fiber $\mathcal{M}_p = \pi^{-1}(p)$ is a manifold of dimension d, and is empty if $d < 0$.*

(b) *For each $p, q \in \mathcal{P}_0^{reg}$, there is a second category subset of $\Omega(p, q)$ consisting of paths γ for which the pullback space $\mathcal{M}_\gamma$ (cf.* (3.2)*) is manifold of dimension $d + 1$.*

Proof. Statement (a) is the Sard-Smale theorem; see Section 1 of [S]. For (b), set $\Omega = \Omega(p, q)$ and let $\varepsilon : [0, 1] \times \Omega \to \mathcal{P}$ be the evaluation map $\varepsilon(t, \gamma) = \gamma(t)$. The pullback of (5.1) by ε is a map $\varepsilon^*\mathcal{M} \to [0, 1] \times \Omega$. Composing with the projection to Ω yields a Fredholm map $\varepsilon^*\mathcal{M} \to \Omega$ whose fiber over $\gamma \in \Omega$ is $\mathcal{M}_\gamma$. Part (b) follows by applying part (a) to this map, as explained, for example, in Sections 4.3.1 and 4.3.2 of [DK]. □

The data (5.1) also determines a real line bundle $\det(d\pi)$ over $\mathcal{M}$ — the determinant line bundle of the Fredholm map π — whose restriction to each regular fiber $\mathcal{M}_p$, $p \in \mathcal{P}_0^{reg}$, is the orientation bundle $\Lambda^d T\mathcal{M}_p$. We will always assume that (5.1) has a *relative orientation* specified by a nowhere zero section of $\det(d\pi)$. We will use the term *oriented Fredholm family* to mean a Fredholm map (5.1) together with a choice of a relative orientation.

Given an oriented Fredholm family, we can consider compactifications as in Section 3 which are stratified by Fredholm families. In fact, in the

applications given in Sections 7-10 below, the relevant compactifications will have the following structure.

Definition 5.2. *A* Fredholm-stratified thin family *of index d is proper continuous map* $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ *from a Hausdorff space* $\overline{\mathcal{M}}$ *which, as a set, is a disjoint union*

$$\overline{\mathcal{M}} = \mathcal{M} \cup \bigcup_{k=2}^{\infty} \mathcal{S}_k$$

such that

(a) *The restriction of* $\overline{\pi}$ *to* $\mathcal{M}$ *is an index d oriented Fredholm family* $\pi : \mathcal{M} \to \mathcal{P}$.

(b) *For each* $k \geq 2$, *the restriction of* $\overline{\pi}$ *to* $\mathcal{S}_k$ *is an index* $d - k$ *Fredholm family* $\pi_k : \mathcal{S}_k \to \mathcal{P}$.

(c) $\mathcal{T}_k = \bigcup_{i \geq k} \mathcal{S}_i$ *is closed in* $\overline{\mathcal{M}}$ *for each k.*

We then say that $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ *is a* Fredholm-stratified thin compactification *of the Fredholm family* π *with top stratum* $\mathcal{M}$ *and strata* $\mathcal{S}_k$.

The first key observation is that Fredholm-stratified thin families fit into the context of the previous section: the Sard-Smale theorem implies that they are relatively thin families in the sense of Definition 3.1.

Lemma 5.3. *A Fredholm-stratified thin family is a relatively thin family with* $\mathcal{P}^*$ *equal to the set of regular values defined in (5.3) below.*

Proof. By assumption, $\mathcal{P}$ is a Banach manifold, so is locally path-connected. Apply the Sard-Smale Theorem to (5.1) and to each map $\pi_k : \mathcal{S}_k \to \mathcal{P}$, and intersect the corresponding second category sets of regular values. The result is a single second category subset

$$(5.3) \qquad\qquad\qquad \mathcal{P}^{reg} \subseteq \mathcal{P}$$

whose points are simultaneous regular values of π and all π_k; we call these *regular values of* $\overline{\pi}$.

For each regular $p \in \mathcal{P}^{reg}$, the fiber $\overline{\mathcal{M}}_p$ of $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ is stratified as in (2.2), so is a thin compactification of $\mathcal{M}_p$ by Lemma 2.2. Thus Assumption I of Definition 3.1 holds.

Similarly, for any $p, q \in \mathcal{P}^{reg}$, the Sard-Smale theorem shows that there is a second category subset of the space of paths γ in $\mathcal{P}$ from p to q for

which γ is transverse to π_k for all k, and hence the pullback $(\mathcal{S}_k)_\gamma$ of π_k over γ is a manifold (with boundary) of dimension $d - k + 1$. Then $\overline{\mathcal{M}}_\gamma$ is the union of $\mathcal{M}_\gamma$ and the manifolds $(\mathcal{S}_k)_\gamma$, so Assumption II of Definition 3.1 also holds. $\qquad\square$

The following simple lemma provides a useful way of verifying that a given family satisfies the conditions of Definition 5.2.

Lemma 5.4. *Consider an index d oriented Fredholm family (5.1) of C^l manifolds with l satisfying (5.2). Suppose that there exists a Hausdorff space $\overline{\mathcal{M}}$ containing $\mathcal{M}$ as an open set and an extension of π to a proper continuous map $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ such that*

(a) $\overline{\mathcal{M}}$ can be written as a disjoint union of sets $\{\mathcal{S}_\alpha | \alpha \in \mathcal{A}\}$ indexed by a finite set $\mathcal{A}$ with $0 \in \mathcal{A}$ and $\mathcal{S}_0 = \mathcal{M}$.

(b) Each $\mathcal{S}_\alpha$ is a manifold, and $\pi_\alpha = \overline{\pi}|_{\mathcal{S}_\alpha}$ is a C^l Fredholm map $\mathcal{S}_\alpha \to \mathcal{P}$ of index d_α.

(c) $d_\alpha \leq d - 2$ for all $\alpha \neq 0$, and

$$\overline{\mathcal{S}}_\alpha \setminus \mathcal{S}_\alpha \subseteq \bigcup_{\{\beta \,|\, d_\beta < d_\alpha\}} \mathcal{S}_\beta.$$

Then $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ is a Fredholm-stratified thin compactification of the family $\pi : \mathcal{M} \to \mathcal{P}$.

Proof. Condition (c) implies that the accumulation points of $\mathcal{S}_\alpha$ that are not in $\mathcal{S}_\alpha$ lie in strata of strictly smaller index. Hence for each k, the union of strata of index $d - k$

$$X_k = \bigcup_{d_\alpha = d - k} \mathcal{S}_\alpha$$

is topologically a disjoint union of manifolds. This means that each X_k is a manifold, and that the restriction of $\overline{\pi}$ to X_k is a Fredholm map of index $d - k$. It also means that $\bigcup_{i \geq k} X_i$ is closed for each k. Definition 5.2 then applies, showing that

$$\overline{\mathcal{M}} = \mathcal{M} \cup \bigcup_{\alpha \neq 0} \mathcal{S}_\alpha = \mathcal{M} \cup \bigcup_{k \geq 2} X_k$$

is a Fredholm-stratified thin compactification of $\mathcal{M} \to \mathcal{P}$ with strata X_k. $\quad\square$

We conclude this section with two finite-dimensional examples, both of which come from algebraic geometry. The first shows that the relative fundamental class can be different from the actual fundamental class even when the fiber is a manifold.

Example 5.5 (Elliptic surfaces). An elliptic surface is a compact complex algebraic surface S with a holomorphic projection $\pi : X \to C$ to an algebraic curve C whose fiber is an elliptic curve except over a finite number of points $p_i \in C$. The singular fibers F_{p_i} are unions of rational curves, each possibly with singularities and multiplicities, and elliptic curves with multiplicity. The restriction of π to the union of the smooth fibers is a Fredholm map $X^* \to C$ of index 2, and $\pi : X \to C$ is a thin compactification of X^* regarded as a family over C. Thus by Theorem 4.2, every fiber F_p carries a relative fundamental class

$$[F_p]^{rel} \in \check{\mathrm{H}}_2(F_p, \mathbb{Z})$$

whose image $\iota_*[F_p]^{rel}$ in $\check{\mathrm{H}}_2(X, \mathbb{Q})$ is the homology class of the generic fiber.

In particular, if F_p is a smooth elliptic fiber with multiplicity $m > 1$, then F_p has a fundamental class $[F_p]$, but the relative fundamental class is

(5.4) $$[F_p]^{rel} = m[F_p].$$

Example 5.6 (Lefschetz pencils and fibrations). Consider a complex projective manifold X with a complete linear system $|D|$ of divisors of complex dimension at least 3. Lefschetz showed that a generic 2-dimensional linear system $[D]$ determines a holomorphic map $\pi : X \setminus B \to \mathbb{P}^1$, where B is the base locus of $[D]$. The generic fiber of π is smooth and the other fibers have only quadratic singularities. This map π is therefore Fredholm, and its index is the real dimension $d = 2(\dim_{\mathbb{C}} X - 1)$ of the generic fiber. While π does not extend continuously to X, it does extend continuously over the blowup X_B of X along B, and $\tilde{\pi} : X_B \to \mathbb{P}^1$ is a thin compactification of $X \setminus B \to \mathbb{P}^1$. Theorem 4.2 therefore defines a relative fundamental class

$$[F_p]^{rel} \in \check{\mathrm{H}}_d(F_p, \mathbb{Z})$$

on the fiber $F_p = \tilde{\pi}^{-1}(p)$ over each $p \in \mathbb{P}^1$.

6. Enlarging the parameter space

In gauge theories, one starts with a parameterized family of elliptic PDEs, and considers the moduli space of solutions as a family over the space of

parameters. After completing in appropriate Sobolev norms, this yields a map $\pi : \mathcal{M} \to \mathcal{P}$ to a separable Banach space $\mathcal{P}$ of parameters. Often, there is a natural compactification $\overline{\mathcal{M}}$ as in diagram (3.1).

One can then hope to obtain a relative fundamental class by applying Theorem 4.2. This involves defining a stratification of $\mathcal{S} = \overline{\mathcal{M}} \setminus \mathcal{M}$, and proving lemmas of two types:

(i) Formal dimension counts for all strata.

(ii) Transversality results showing that $\mathcal{M}$ and each stratum $\mathcal{M}_\alpha$ of $\mathcal{S}$ is a manifold of the expected dimension.

In general, (ii) can be done only if the space of parameters $\mathcal{P}$ is sufficiently large. Thus it may be necessary to enlarge the space of parameters in order to define relative fundamental classes. Enlarged spaces of parameters may also be needed to show independence of added geometric structure, such as the choice of a Riemannian metric used to define Donaldson polynomials (see Section 7 and 8), and the choice of an almost complex structure used to define Gromov–Witten invariants (Sections 9 and 10).

When enlarging the parameter space, some care is needed because the relative fundamental classes depend on the choice of $\mathcal{P}$ and of the thin compactification. Thus enlarging the space of parameters may change the problem that one is trying to solve. Lemma 6.2 below gives a stability result that ensures that a base expansion yields a compatible relative fundamental class.

Definition 6.1. *A* base expansion *of a relatively thin compactification (3.3) is a relatively thin compactification of $\pi' : \mathcal{M}' \to \mathcal{P}'$ with a commutative diagram of continuous maps*

(6.1)
$$
\begin{array}{ccc}
\overline{\mathcal{M}} & \xrightarrow{\ F\ } & \overline{\mathcal{M}'} \\
\Big\downarrow{\overline{\pi}} & & \Big\downarrow{\overline{\pi}'} \\
\mathcal{P} & \xrightarrow{\ f\ } & \mathcal{P}'
\end{array}
$$

where there exist a second category subset $\mathcal{P}^*$ of $\mathcal{P}$ that satisfies the conditions of Definition 3.1 for $\overline{\pi}$, and a similar subset $(\mathcal{P}')^*$ of $\mathcal{P}'$ for $\overline{\pi}'$, such that:

(a) $f(\mathcal{P}^*) \subseteq (\mathcal{P}')^*$.

(b) *for each $p \in \mathcal{P}^*$, F restricts to a degree 1 map $\mathcal{M}_p \to \mathcal{M}'_{f(p)}$ between oriented topological manifolds.*

Note that these conditions imply that $\bar{\pi}$ and $\bar{\pi}'$ have the same relative dimension.

Lemma 6.2. *For a base expansion* (6.1), *the relative fundamental classes of π and π' agree over $\mathcal{P}$, i.e. for all $p \in \mathcal{P}$ we have*

$$(6.2) \qquad (F_p)_*[\overline{\mathcal{M}}_p]^{rel} = [\overline{\mathcal{M}}'_{f(p)}]^{rel}$$

in $\check{\mathcal{H}}_*(\overline{\mathcal{M}}'_{f(p)})$, where $F_p : \overline{\mathcal{M}}_p \to \overline{\mathcal{M}}'_{f(p)}$ denotes the restriction of F.

Proof. For each p in the set $\mathcal{P}^*$ of Definition 6.1, both $\mathcal{M}_p$ and $\mathcal{M}_{f(p)}$ are oriented topological manifolds. By Definition 3.1(I), $\overline{\mathcal{M}}_p$ and $\overline{\mathcal{M}}'_{f(p)}$ are thin compactifications of $\mathcal{M}_p = \mathcal{M}'_{f(p)}$, respectively. Each carries a fundamental class by Theorem 2.4, and these are equal to the corresponding relative fundamental class by Axiom A1 of Definition 4.1. Therefore, for each $p \in \mathcal{P}^*$,

$$(F_p)_*[\overline{\mathcal{M}}_p]^{rel} = (F_p)_*[\overline{\mathcal{M}}_p] = [\overline{\mathcal{M}}'_{f(p)}] = [\overline{\mathcal{M}}'_{f(p)}]^{rel},$$

where the middle equality holds by Lemma 2.6 and Definition 6.1(b). This then implies (6.2) for all $p \in \mathcal{P}$, as follows.

As in the proof of the Extension Lemma 3.4, pick nested open sets $V_k \subseteq \mathcal{P}$ with $\bigcap V_k = \{p\}$, and $V'_k \subseteq \mathcal{P}'$ with $\bigcap V'_k = \{f(p)\}$, and set $U_k = V_k \cap f^{-1}(V'_k)$. Next, choose a sequence $p_k \to p$ with $p_k \in U_k \cap \mathcal{P}^*$, and segmented paths φ_m in $\mathcal{P}$ converging to p. Then, as in (3.9),

$$[\overline{\mathcal{M}}_p]^{rel} = \varprojlim \, (\iota_{km})_*[\overline{\mathcal{M}}_{p_k}]^{rel},$$

and therefore by the naturality of (1.10)

$$(F_p)_*[\overline{\mathcal{M}}_p]^{rel} = (F_p)_* \varprojlim \, (\iota_{km})_*[\overline{\mathcal{M}}_{p_k}]^{rel} = \varprojlim \, (F_p \circ \iota_{km})_*[\overline{\mathcal{M}}_{p_k}]^{rel}$$

On the other hand, the images $F \circ \varphi_m$ converge to $f(p)$ in $\mathcal{P}'$, and therefore

$$[\overline{\mathcal{M}}'_{f(p)}]^{rel} = \varprojlim \, (j_{km})_*[\overline{\mathcal{M}}'_{f(p_k)}]^{rel},$$

where $j_{km} = F \circ \iota_{km}$ is the inclusion of $f(p_k)$ into V'_m. Combining the last three displays give (6.2) for all $p \in \mathcal{P}$. □

Example 6.3. (a) If both vertical arrows in (6.1) are Fredholm-stratified families, and p is a regular value of $\bar{\pi}$, then the inclusion of $\overline{\mathcal{M}}_p \to \{p\}$ into $\bar{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ is a base expansion. Equation (6.2) becomes $[\overline{\mathcal{M}}_p] = [\overline{\mathcal{M}}_p]^{rel}$, which is Axiom A1 of Definition 4.1.

(b) Example 5.5 shows the importance of condition (a) in Definition 6.1. Let F_p be a smooth elliptic fiber in an elliptic surface with multiplicity $m > 1$. Then $F_p \to \{p\}$ is a thinly compactified family with $[F_p]^{rel} = [F_p]$, and the inclusion of $F_p \to \{p\}$ into $X \to C$ satisfies all of the conditions of Definition 6.1 except (a). But, as in (5.4), the relative fundamental class induced by the extended family $X \to C$ is $m[F_p]$ rather than $[F_p]$.

(c) Similarly, in Example 2.5(a), the family $\pi_Z : \pi^{-1}(Z) \to Z$ embeds into $\pi : M_Z \to M$. In this case, the dimensions of the generic fibers and the indices are different, so this embedding is not a base expansion, and the two relative fundamental classes lie in different dimensions.

Examples (b) and (c) above are instances where the relative fundamental class $[\overline{\mathcal{M}}]^{rel}$ *depends on the choice of the parameter space $\mathcal{P}$.* Thus it does not make sense to speak of "the" relative fundamental class of a single fiber $\mathcal{M}_p$: relative fundamental classes are, by their nature, associated with relatively thin families over parameter spaces.

Example 6.4. For moduli spaces of solutions to an elliptic differential equation, one obtains base expansions by lowering the regularity of the parameters, for example, by including a space $\mathcal{P}^l$ of C^l parameters into the corresponding C^{l-1} space. Often, elliptic theory implies that, for sufficiently large l, all conditions in Definition 6.1 are satisfied, and hence the relative fundamental class is unchanged in the sense of Lemma 6.2. In particular, for each smooth parameter $p \in \mathcal{P}^\infty = \bigcap \mathcal{P}^l$, the moduli space $\overline{\mathcal{M}}_p$ of solutions is canonically identified with the fibers $\overline{\mathcal{M}}_p^l$ over p in $\mathcal{P}^l$ for each large l, and the relative fundamental classes $[\overline{\mathcal{M}}_p^l]^{rel}$ consistently induce a relative fundamental class on $\overline{\mathcal{M}}_p$.

In some applications, one has a family $\overline{\mathcal{M}} \to \mathcal{P}$ which is not itself Fredholm-stratified, but whose restriction to an open dense subset $\mathcal{P}^o$ of $\mathcal{P}$ is Fredholm-stratified. The next result, which will be used in Section 8, gives conditions under which this is sufficient to make $\overline{\mathcal{M}} \to \mathcal{P}$ a relatively thin family.

Lemma 6.5. *Let $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ be a proper continuous map from a Hausdorff space to a Banach manifold. Suppose that there is an open, dense subset $\mathcal{P}^o$ of $\mathcal{P}$ such that*

(i) *Every path in $\mathcal{P}$ is a limit of paths in $\mathcal{P}^o$.*

(ii) *The restriction $\overline{\pi}^o : \overline{\mathcal{M}}^o \to \mathcal{P}^o$ of $\overline{\pi}$ over $\mathcal{P}^o$ is a Fredholm-stratified thin family of index d.*

Then $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ is a relatively thin family of relative dimension d with $\mathcal{P}^*$ defined by (6.3), and therefore admits a unique relative fundamental class $[\overline{\mathcal{M}}_p]^{rel} \in \check{\mathcal{H}}_d(\overline{\mathcal{M}}_p)$ for all $p \in \mathcal{P}$.

Proof. By Lemma 5.3, the set

$$(6.3) \qquad\qquad \mathcal{P}^* = \mathcal{P}^{reg}$$

of regular values of $\overline{\pi}^o$ is a second category subset of $\mathcal{P}^o$, i.e. is a countable intersection of open dense subsets. But open dense subsets of $\mathcal{P}^o$ are open and dense in $\mathcal{P}$ (because $\mathcal{P}^o$ is open and dense in $\mathcal{P}$), so $\mathcal{P}^*$ is also a second category subset of $\mathcal{P}$.

Next observe that any path γ in $\mathcal{P}$ whose endpoints p, q are in $\mathcal{P}^* \subseteq \mathcal{P}^o$ is a limit of paths in $\mathcal{P}^o$ with the same endpoints p, q as follows. By assumption (i), γ is the limit of a sequence of paths γ_k in $\mathcal{P}^o$ with endpoints p_k, q_k, where $p_k \to p$ and $q_k \to q$. Because $p, q \in \mathcal{P}^o$ and $\mathcal{P}^o$ is open subset of a Banach manifold, for sufficiently large k we can find paths σ_k in $\mathcal{P}^o$ from p to p_k converging to the constant path at p, and similarly paths τ_k in $\mathcal{P}^o$ from q_k to q converging to the constant path at q. The concatenation of these paths is a sequence $\{\sigma_k \# \gamma_k \# \tau_k\}$ of paths in $\mathcal{P}^o$, each with endpoints p, q, which limit to the path γ.

With these observations, one sees that Definition 3.1 applies to $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ with this $\mathcal{P}^*$:

(i) Condition I holds as in the proof of Lemma 5.3.

(ii) Condition II holds because, again as in the proof of Lemma 5.3, it holds for a dense set of paths in $\mathcal{P}^o$ from p to q described above, and this set of paths is dense in the space of paths in $\mathcal{P}$ from p to q.

The lemma then follows by Theorem 4.2. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

7. Donaldson theory

Let X be a smooth, closed, oriented 4-manifold that satisfies the Betti number condition $b_2^+(X) > 1$. Donaldson theory uses moduli spaces of connections to construct invariants of the smooth structure of X. This section and the next describe how Donaldson's polynomial invariants fit into the context

of the previous sections. We follow Donaldson's exposition in Sections 5.6 and 6.3 of [D2].

Let $E \to X$ be a $U(2)$ vector bundle with first Chern class $c_1 = c_1(E)$ and instanton number $k = (c_2(E) - \frac{1}{4}c_1^2(E))[X]$. Fix a connection ∇^0 on $\Lambda^2 E$. After completing in appropriate Sobolev norms (see, for example, Section 4.2 of [DK]), we obtain three separable Banach manifolds: a space $\mathcal{A} = \mathcal{A}_E(\nabla^0)$ of connections on E that induce ∇^0 on $\Lambda^2 E$, a space $\mathcal{R}$ of Riemannian metrics on X, and the group $\mathcal{G}$ of gauge transformations of E with determinant 1. Furthermore, $\mathcal{G}$ acts smoothly on $\mathcal{A}$, the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ is metrizable, and the subset $\mathcal{B}^{irred} \subset \mathcal{B}$ of irreducible connections is also a separable Banach manifold.

A pair (A, g) in $\mathcal{A} \times \mathcal{R}$ is called an *instanton* if its curvature F^A satisfies $*ad(F^A) = -ad(F^A)$, where $*$ is the Hodge star operator on 2-forms for the metric g. The universal moduli space $\mathcal{M}_E \subset \mathcal{B} \times \mathcal{R}$ is the set of all $\mathcal{G}$-equivalence classes $([A], g)$ of instantons for $A \in \mathcal{A}_E$. Up to isomorphism, $\mathcal{M}_E$ depends on the bundle E only though the pair (k, c_1), and is independent of the connection ∇^0 (see page 146 of [D2]).

Now fix c_1 and consider the sequence of moduli spaces $\mathcal{M}_k$ associated with bundles E with instanton number k and this fixed c_1. Projection onto the second factor is a map

$$(7.1) \qquad \begin{array}{c} \mathcal{M}_k \\ \downarrow {\scriptstyle \pi} \\ \mathcal{R} \end{array}$$

whose restriction to $\mathcal{M}_k^{irred} = \mathcal{M}_k \cap (\mathcal{B}^{irred} \times \mathcal{R})$ is a smooth Fredholm map of index $2d_k$, where d_k is given in terms of the Betti numbers $b_1(X)$ and $b_2^+(X)$ of X by

$$(7.2) \qquad d_k = 4k - \tfrac{3}{2}\big(1 - b_1(X) + b_2^+(X)\big).$$

This Fredholm family is oriented by the choice of a homology orientation for X [DK, 7.1.39].

Let $\mathcal{M}_k(g)$ denote the fiber of $\mathcal{M}_k$ over a metric $g \in \mathcal{R}$. We say that c_1 is *odd* if it represents a class in $H^2(X; \mathbb{Z})/\text{Torsion}$ that is not divisible by 2.

Lemma 7.1. *Suppose that $b_2^+(X) > 1$ and c_1 is odd. Then there is an open dense subset $\mathcal{R}^o$ of $\mathcal{R}$ such that*

(i) *For each $g \in \mathcal{R}^o$ and each $0 \leq j \leq k$, the fiber $\mathcal{M}_j(g)$ contains no reducible connection.*

(ii) *Every path in $\mathcal{R}$ is the limit of paths in $\mathcal{R}^o$.*

Proof. This follows directly from the discussion on page 147 of [D2] and Corollary 4.3.15 of [DK]. Note that the assumption that c_1 is odd implies that the space $\mathcal{A}_E$ contains no flat connections [D2, Section 5.6]. $\qquad\square$

Lemma 7.2. *Under the hypotheses of Lemma 7.1, the map (7.1) extends to a proper continuous map $\overline{\pi} : \overline{\mathcal{M}}_k \to \mathcal{R}$ whose restriction over $\mathcal{R}^o$ is a Fredholm-stratified thin compactification of $\pi^o : \mathcal{M}_k|_{\mathcal{R}^\circ} \to \mathcal{R}^o$.*

Proof. We follow the notation and discussion in Section 4.4 of [DK]. Using the topology of weak convergence (as defined by Condition 4.4.2 in [DK]), one defines the Uhlenbeck compactification $\overline{\mathcal{M}}_k$ by setting

$$(7.3) \qquad\qquad \overline{\mathcal{M}}_k = \mathcal{M}_k \cup \mathcal{S},$$

where $\mathcal{S}$ is the union of the strata $S_{jk} = \mathcal{M}_{k-j} \times \operatorname{Sym}^j(X)$ for $0 < j < k$ (noting that $\mathcal{M}_0$ is empty because there are no flat connections). Then $\overline{\mathcal{M}}_k$ is paracompact and metrizable [DK, Section 4.4], and π extends to a map $\overline{\pi} : \overline{\mathcal{M}}_k \to \mathcal{R}$ whose restriction to each stratum is Fredholm.

The proof is completed by applying Lemma 5.4. For this, it suffices to define a stratification on $\overline{\mathcal{M}}_k$, different from the one in (7.3), whose restriction $\overline{\mathcal{M}}_k^o = \overline{\mathcal{M}}_k|_{\mathcal{R}^\circ}$ satisfies the hypotheses of Lemma 5.4.

The new strata are labeled by partitions. A partition is a non-increasing sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers; its length $\ell(\alpha) = \ell$ and its degree $|\alpha| = \sum \alpha_i$ satisfy $\ell(\alpha) \leq |\alpha|$. We also consider (0) to be a partition with $\ell(0) = |(0)| = 0$. Let $\mathcal{P}_k$ be the set of all partitions α with $|\alpha| \leq k$. Define the *level* of α to be

$$(7.4) \qquad\qquad \Lambda(\alpha) = 2|\alpha| - \ell(\alpha),$$

and note that $\Lambda(\alpha) \geq 0$ with equality if and only if $\alpha = (0)$.

Given a four-manifold X and an integer $k \geq 0$, regard $\operatorname{Sym}^k X$ as formal positive sums $\sum \alpha_i x_i$ of distinct points of X associated with some partition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $|\alpha| = k$. Let Δ_α be the set of all such sums associated with a given α. Then Δ_α is a manifold of dimension $4\ell(\alpha)$, and $\operatorname{Sym}^k X$ is the disjoint union of the sets Δ_α over all α with $|\alpha| = k$.

With these preliminaries understood, we re-stratify the compactification (7.3) by writing

$$(7.5) \qquad \overline{\mathcal{M}}_k = \mathcal{M}_k \ \cup \ \bigcup_{\alpha \in \mathcal{P}_k} \mathcal{S}_\alpha,$$

where $\mathcal{S}_0 = \mathcal{M}_k$ and

$$\mathcal{S}_\alpha = \mathcal{M}_{k-|\alpha|} \times \Delta_\alpha.$$

By the choice of $\mathcal{R}^o$, the restriction $\mathcal{S}_\alpha^o$ of $\mathcal{S}_\alpha$ over $\mathcal{R}^o$ is, for each α, a Banach manifold with a Fredholm projection $\pi_\alpha : \mathcal{S}_\alpha^o \to \mathcal{R}^o$ of index

$$(7.6) \qquad \iota_\alpha = 2d(k - |\alpha|) + 4\ell(\alpha) = 2d_k - 4\Lambda(\alpha),$$

where d_k is the index (7.2).

One then sees that conditions (a) and (b) of Lemma 5.4 hold for the restriction of (7.5) over $\mathcal{R}^o$. To verify (c), suppose that a sequence $(A_n, \sum \alpha_i(x_n)_i)$ converges in the weak topology. Then $\{A_n\}$ converges to a formal instanton $(B, \sum \beta_j y_j)$ with $B \in \mathcal{M}_{k-|\alpha|-|\beta|}^o$, and $\sum \alpha_i(x_n)_i$ converges to $\sum \gamma_m z_m$ with $\ell(\gamma) \le \ell(\alpha)$ and $|\gamma| = |\alpha|$. Thus the limit is

$$\left(B, \sum \beta_j y_j + \sum \gamma_m z_m \right) \in \mathcal{M}_{k-|\delta|}^o \times \Delta_\delta,$$

with $\ell(\delta) \le \ell(\beta) + \ell(\gamma) \le \ell(\alpha) + \ell(\beta)$ and $|\delta| = |\beta| + |\gamma| = |\alpha| + |\beta|$. The level (7.4) of this limit stratum is therefore

$$\Lambda(\delta) = 2|\delta| - \ell(\delta) \ \ge \ \Lambda(\alpha) + \Lambda(\beta) \ge \Lambda(\alpha),$$

with equality if and only if $\beta = (0)$ and $\gamma = \alpha$. This, together with (7.6), implies property (c) of Lemma 5.4. Thus Lemma 5.4 applies. $\qquad \square$

8. Relative fundamental classes and Donaldson polynomials

As in Section 7, the universal moduli space (7.1) of anti-self-dual instantons on a 4-manifold X admits a compactification, the Uhlenbeck compactification $\overline{\pi} : \overline{\mathcal{M}}_k \to \mathcal{R}$. Under the assumptions of Lemmas 7.1 and 7.2, there is

an open dense subset $\mathcal{R}^o$ of $\mathcal{R}$ and a diagram

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_k^o & \longrightarrow & \overline{\mathcal{M}}_k \\
\downarrow{\scriptstyle \overline{\pi}^o} & & \downarrow{\scriptstyle \overline{\pi}} \\
\mathcal{R}^o & \hookrightarrow & \mathcal{R}
\end{array}
$$

where $\overline{\mathcal{M}}_k^o$ is the restriction of $\overline{\mathcal{M}}_k$ over $\mathcal{R}^o$, and

(i) $\overline{\pi}^o : \overline{\mathcal{M}}_k^o \to \mathcal{R}^o$ is a Fredholm-stratified thin family.

(ii) Every path in $\mathcal{R}$ is the limit of paths in $\mathcal{R}^o$.

Let $\mathcal{R}^{reg}$ be the set of regular values of the family (i). By the Sard-Smale theorem, $\mathcal{R}^{reg}$ is dense in $\mathcal{R}^o$, and hence is dense in $\mathcal{R}$.

With this setup, Lemma 5.3 and Theorem 4.2 produce a relative fundamental class for $\overline{\mathcal{M}}_k^o \to \mathcal{R}^o$. In fact, Lemma 6.5 gives a stronger conclusion: it shows that the Uhlenbeck compactification is a relatively thin family over the entire space of metrics. Thus we obtain a relative fundamental class for Donaldson theory:

Proposition 8.1. *Let X be a closed, oriented 4-manifold with $b_2^+(X) > 1$, and let $E \to X$ a $U(2)$ vector bundle with instanton number k and $c_1(E)$ odd. Then*

(a) *The Uhlenbeck compactification is a relatively thin family with index $2d_k$ with $\mathcal{R}^*$ equal to $\mathcal{R}^{reg}$ and d_k given by (7.2).*

(b) *A homology orientation for X determines a relative fundamental class*

(8.1) $$\left[\overline{\mathcal{M}}_k(g)\right]^{rel} \in \check{\mathcal{H}}_{2d_k}\left(\overline{\mathcal{M}}_k(g)\right),$$

where $\overline{\mathcal{M}}_k(g)$ is the fiber of $\overline{\mathcal{M}}_k$ over $g \in \mathcal{R}$.

To obtain invariants, one would like, as in (4.5), to consider pairings

$$
\left\langle \alpha, \ [\overline{\mathcal{M}}_k(g)]^{rel} \right\rangle
$$

where α is the restriction to $\overline{\mathcal{M}}_k(g)$ of a Čech cohomology class defined on $\mathcal{B}_k$. Unfortunately, this is not as straightforward as one might hope, and one must work harder.

Following Donaldson, the natural cohomology classes to consider are those in the image of the μ-map

$$\mu : H_2(X; \mathbb{Q}) \to \check{\mathrm{H}}^2(\mathcal{B}_k^{irred}; \mathbb{Q})$$

(cf. Chapter 5 of [DK]). For each choice of classes $A_1, \ldots, A_{d_k} \in H_2(X; \mathbb{Q})$, the product $\mu(A_1) \cup \cdots \cup \mu(A_{d_k})$ restricts to a class

$$\mu = \mu(A_1, \ldots, A_{d_k}) \in \check{\mathrm{H}}^{2d_k}(\mathcal{M}_k^{irred}; \mathbb{Q})$$

whose dependence on the A_i is multilinear and symmetric. For each $g \in \mathcal{R}$, this further restricts under the inclusion $\iota_g : \mathcal{M}_k^{irred}(g) \hookrightarrow \mathcal{M}_k^{irred}$ of the fiber over g to a class

(8.2) $$\iota_g^* \mu \in \check{\mathrm{H}}^{2d_k}(\mathcal{M}_k^{irred}(g); \mathbb{Q}).$$

But these are not classes in the cohomology of $\overline{\mathcal{M}}_k(g)$, so cannot be directly paired with the relative fundamental class. Thus we proceed more indirectly.

The key observation is that, for each *regular* metric g, the classes (8.2) extend over the compatification $\overline{\mathcal{M}}_k(g)$ in a way that is consistent along paths. (Here "regular" means $g \in \mathcal{R}^{reg}$, which is equivalent to conditions 9.2.4 and implies 9.2.13 in [DK].) One can then apply Extension Lemma 3.4 to obtain a relative fundamental class in 0-dimensional Čech homology, which yields invariants. The remainder of this section gives the details.

Lemma 8.2. *For each $A_1, \ldots, A_{d_k} \in H_2(X; \mathbb{Z})$,*

(a) *For each $g \in \mathcal{R}^{reg}$, the class (8.2), which depends on $A_1, \ldots, A_{d_k}$, extends uniquely to an element μ_g of $\check{\mathrm{H}}^{2d_k}(\overline{\mathcal{M}}_k(g); \mathbb{Q})$.*

(b) *There is a unique association*

$$g \mapsto \alpha_g \in \check{\mathrm{H}}_0(\overline{\mathcal{M}}_k(g); \mathbb{Q})$$

such that
(i) for each $g \in \mathcal{R}^{reg}$, α_g is the cap product with the fundamental class (8.1):

(8.3) $$\alpha_g = [\overline{\mathcal{M}}_k(g)]^{rel} \cap \mu_g.$$

(ii) the consistency condition (8.5) below holds for every path γ in $\mathcal{R}$.

Proof. (a) Donaldson and Kronheimer showed [DK, Subsection 9.2.3] that for each regular g, $\iota_g^* \mu$ has a Čech representative with compact support in $\mathcal{M}_k^{irred}(g)$, which is equal to $\mathcal{M}_k(g)$ by Lemma 7.1(i). Because $\overline{\mathcal{M}}_k(g)$ is a Fredholm-stratified thin compactification of $\mathcal{M}_k(g)$, the long exact sequence in Čech cohomology, used as in the proof of Lemma 2.2, shows that $\iota_g^* \mu$ extends uniquely to a Čech class in the compactification

$$\mu_g \in \check{\mathrm{H}}^{2d_k}(\overline{\mathcal{M}}_k(g); \mathbb{Q}).$$

Furthermore, for each regular path γ in $\mathcal{R}^o$ with endpoints g, g', the pullback $\overline{\mathcal{M}}_k(\gamma)$ over γ of the compactified moduli space contains no reducible connections and is a thin compactified cobordism as defined in Section 2.4 above. Again as in [DK], the class $\iota_\gamma^* \mu$ has a representative compactly supported in $\mathcal{M}_k(\gamma)$, so extends uniquely to a class μ_γ on $\overline{\mathcal{M}}_k(\gamma)$. The uniqueness of these extensions implies that

$$(8.4) \quad \mu_g = \iota_g^* \mu^\gamma \ \text{ in } \ \check{\mathrm{H}}^{2d_k}(\overline{\mathcal{M}}_k(g); \mathbb{Q}) \quad \text{and} \quad \mu_{g'} = \iota_{g'}^* \mu^\gamma \ \text{ in } \ \check{\mathrm{H}}^{2d_k}(\overline{\mathcal{M}}_k(g'); \mathbb{Q}).$$

(b) For each regular g, define α_g to be the cap product (8.3). By the naturality of cap products, (8.4) implies a consistency condition for α_g of the form (3.7), namely

$$(8.5) \qquad\qquad (\iota_0)_* \alpha_g = (\iota_1)_* \alpha_{g'} \ \text{ in } \ \check{\mathrm{H}}_0(\overline{\mathcal{M}}_k(\gamma); \mathbb{Q})$$

for every regular path γ. Lemma 7.1(ii), together with the middle paragraph of the proof of Lemma 6.5, shows that each path γ in $\mathcal{R}$ with endpoints $g, g' \in \mathcal{R}^{reg}$ is a limit of paths γ_k in $\mathcal{R}^o$ with the same endpoints. But each γ_k is a limit of regular paths in $\mathcal{R}^o$ with the same endpoints (cf. the proof of Lemma 5.3), which means that the regular paths are dense in the space of all paths in $\mathcal{R}$ from g to g'. The hypotheses of Lemma 3.4 then hold for $g \mapsto \alpha_g$, with $\mathcal{P}^*$ taken to be $\mathcal{R}^{reg}$, and the conclusion of Lemma 3.4 gives (b). $\qquad \square$

Remark 8.3. Alternatively, one could work with the index 0 universal "cutdown" moduli spaces defined by [DK, (9.2.8)], and regard the class α_g in (8.3) as the relative fundamental class of the cutdown moduli space.

We can now use the class α_g of Lemma 8.2, which depends on $A_1, \ldots, A_{d_k}$, to define numerical invariants. For each $g \in \mathcal{R}$ there is a map

$$q_k(g) : \mathrm{Sym}^{d_k} H_2(X; \mathbb{Q}) \to \mathbb{Q}$$

defined by evaluating α_g on the class $1 \in \check{H}^0(\overline{\mathcal{M}}_k(g); \mathbb{Q})$:

$$(8.6) \qquad\qquad\qquad q_k(g) = \langle 1, \alpha_g \rangle.$$

Proposition 8.4. *The map $q_k(g)$ is independent of $g \in \mathcal{R}$, and is equal to Donaldson's polynomial invariants.*

Proof. First note that the space $\mathcal{R}$ of Riemannian metrics is path-connected; in fact, it is contractible. The consistency condition (8.5) then shows that $q_k(g)$ is independent of g, exactly as in the proof of Corollary 4.3. For regular g, we can use (8.3) to rewrite (8.6) as

$$q_k(A_1, \ldots, A_{d_k})(g) = \left\langle \mu_g, \ [\overline{\mathcal{M}}_k(g)]^{rel} \right\rangle = \left\langle \iota_g^* \mu(A_1, \ldots, A_{d_k}), \ [\mathcal{M}_k^{irred}(g)] \right\rangle,$$

where the last term is a pairing between a compactly supported cohomology class and the fundamental class of a non-compact manifold. This agrees with Donaldson's definition of q_k: see Section 9.2 of [DK], especially (9.2.18) and the top of page 360. □

Proposition 8.4 is a re-casting of Donaldson's theorem [D1] in the form presented in [D2]: it implies that the Donaldson polynomials are invariants of the smooth structure of the manifold X, depending on the class $c_1(E)$, the orientation, and the homology orientation. In fact, changes in $c_1(E)$ and the homology orientation change the Donaldson polynomial in a specific way [MM]. In the literature, the story is completed by removing the assumption that c_1 is odd by using the stabilizing trick of Morgan and Mrowka; see [MM] or [D2, Section 6.3].

This viewpoint makes clear that the invariance of the Donaldson polynomials follows directly from two core facts: (i) the Uhlenbeck compactification is a Fredholm-stratified thin family over an open, dense, path-connected subset $\mathcal{R}^o$ of the space of metrics, and (ii) $2d_k$-dimensional products of classes $\mu(A_i)$ extend to the compactification of regular fibers. Both appear explicitly in the work of Donaldson. As we have seen, these same two facts imply the existence of a relative fundamental class $[\overline{\mathcal{M}}_k(g)]^{rel}$ defined for every metric g and every k.

9. Gromov–Witten theory

In the remaining two sections, we consider thin compactifications in Gromov–Witten theory. This section summarizes the well-known setup; details can

be found in [MS], [RT1], [RT2], and [IP]. Throughout, we work in the stable range $2g - 2 + n > 0$.

The Deligne-Mumford spaces $\overline{\mathcal{M}}_{g,n}$ are at the foundation of Gromov–Witten theory. Points in $\overline{\mathcal{M}}_{g,n}$ represent equivalence classes $[C]$ of stable, connected nodal complex curves C of arithmetic genus g with n marked points $x_1, \ldots, x_n$; those without nodes form the principal stratum $\mathcal{M}_{g,n}$. There is a universal curve

$$
(9.1) \qquad\qquad
\begin{array}{c}
\overline{\mathcal{U}}_{g,n} \\
\downarrow{\scriptstyle \pi} \\
\overline{\mathcal{M}}_{g,n}
\end{array}
$$

with the property that for each stable curve C as above there is a map $C \to \overline{\mathcal{U}}_{g,n}$ whose image is a fiber of (9.1) that is biholomorphic (as a marked curve) to $C/\mathrm{Aut}(C)$. More generally, for any connected, n-marked genus g nodal curve C, there is a map

$$
(9.2) \qquad\qquad \varphi : C \to \overline{\mathcal{U}}_{g,n}
$$

defined as the composition $C \to C^{st} \to \overline{\mathcal{U}}_{g,n}$ where C^{st} is the stable curve (the stable model of C) obtained by collapsing all unstable irreducible components of C, and the second map is as above.

Now fix a closed symplectic manifold (X, ω), a large integer l and a number $r > 2$. As in Section 3.1 of [MS], let $\mathcal{J}$ be the smooth separable Banach manifold of all C^l ω-tame almost complex structures J on X. We consider maps $f : C \to X$ whose domain is an n-marked connected nodal curve C with complex structure j. Such a map is called J-holomorphic if

$$
\overline{\partial}_J f = \tfrac{1}{2}(df + J df j) = 0,
$$

and two such maps are regarded as equivalent if they differ by reparametrization. Let $\mathcal{M}_{A,g,n}(X)$ denote the moduli space of all equivalence classes $([f], J)$ of pairs (f, J), where $J \in \mathcal{J}$ and f is a J-holomorphic map of Sobolev class $W^{l,r}$ with smooth stable domain that represents $A = [f(C)] \in H_2(X; \mathbb{Z})$. One then has a continuous projection π and a continuous

stabilization-evaluation map se

(9.3)
$$\mathcal{M}_{A,g,n}(X) \xrightarrow{\;se\;} \overline{\mathcal{M}}_{g,n} \times X^n$$
$$\downarrow{\scriptstyle\pi}$$
$$\mathcal{J}$$

defined by $\pi(f,J) = J$ and $se(f,J) = ([C], f(x_1), \ldots, f(x_n))$.

More generally, each map $f : C \to X$ from a connected nodal curve has an associated graph map

(9.4)
$$F = F_f : C \to \overline{\mathcal{U}}_{g,n} \times X$$

defined by $F(x) = (\varphi(x), f(x))$; this is an embedding if $\mathrm{Aut}(C) = 1$. Following Ruan and Tian [RT2], we can use F to expand the base of (9.3), as follows.

The universal curve $\overline{\mathcal{U}}_{g,n}$ is projective; fix a holomorphic embedding $\overline{\mathcal{U}}_{g,n} \hookrightarrow \mathbb{P}^M$. For each fixed almost complex structure J, consider sections ν of the bundle $\mathrm{Hom}(\pi_1^* T\mathbb{P}^M, \pi_2^* TX)$ over $\mathbb{P}^M \times X$ that satisfy $J \circ \nu + \nu \circ j = 0$, where this j is the complex structure on $\mathbb{P}^M$. The space of all C^l pairs (J, ν) of this form is also a smooth separable Banach manifold, which we denote by $\mathcal{J}\mathcal{V}$. Each $(J, \nu) \in \mathcal{J}\mathcal{V}$ defines a deformation J_ν of the product almost complex structure on $\mathbb{P}^M \times X$, and therefore on $\overline{\mathcal{U}}_{g,n} \times X$, by writing

(9.5)
$$J_\nu = \begin{pmatrix} j & 0 \\ -\nu \circ j & J \end{pmatrix}.$$

We identify such J_ν with the pair (J, ν) and call it a *Ruan-Tian perturbation*.

A map $f : C \to X$ is (J, ν)-holomorphic if its graph satisfies $\overline{\partial}_{J_\nu} F = 0$, or equivalently if f satisfies

(9.6)
$$\overline{\partial}_J f(x) = \nu(\varphi(x), f(x)).$$

Such a map is called *stable* if, for each irreducible component C_i of C, either C_i is stable or $f(C_i)$ is not a single point.

The map $J \mapsto (J, 0)$ induces a smooth inclusion $\mathcal{J} \hookrightarrow \mathcal{J}\mathcal{V}$ of Banach manifolds. Furthermore, the maps π and se extend continuously over the universal moduli space $\overline{\mathcal{M}}_{A,g,n}(X)$ of all triples (f, J, ν) where f is a stable

(J, ν)-holomorphic map, giving continuous maps

(9.7)
$$\overline{\mathcal{M}}_{A,g,n}(X) \xrightarrow{\;se\;} \overline{\mathcal{M}}_{g,n} \times X^n$$
$$\downarrow{\scriptstyle\pi}$$
$$\mathcal{J}\mathcal{V}.$$

The analysis of these maps is standard; see, for example, Chapter 3 of [MS], Section 3 of [RT2], and Sections 4 and 5.1 of [IP]. Let $\mathcal{E}^{m,r}$ (resp. $\mathcal{F}^{m,r}$) denote the space of $W^{m,r}$ sections of the bundle f^*TX (resp. $T^{0,1}C \otimes_{\mathbb{C}} f^*TX$) over C. The space of first order deformations of the complex structure on C is the finite-dimensional vector space $H^{0,1}(TC)$. The linearization of the (J, ν)-holomorphic map equation (9.6) at (f, J, ν) is a bounded linear operator

$$\mathbf{D}_{f, J_\nu} : \mathcal{E}^{m,r} \times H^{0,1}(TC) \times T_{J_\nu}\mathcal{J}\mathcal{V} \longrightarrow \mathcal{F}^{m-1,r}$$

given by formula [RT2, (3.10)]; see also [MS, Prop 3.1.1]. The elliptic theory of this operator leads to two important regularity properties:

Reg 1. If $\mathbf{D}_{f, J_\nu}$ is surjective and $\mathrm{Aut}(f) = 1$, the universal moduli space $\pi : \mathcal{M}_{A,g,n}(X) \to \mathcal{J}\mathcal{V}$ in (9.3) is a manifold near (f, J, ν) with a natural relative orientation (see the proofs of [RT2, Theorem 3.2] or [MS, Theorem 3.1.5]).

Reg 2. If **Reg 1** holds, then at each regular value (J, ν) of π, the fiber $\mathcal{M}_{A,g,n}^{J,\nu}(X)$ is a manifold whose dimension is the index of D_{f, J_ν}, which is

(9.8) $$\iota(A, g, n) = 2[c_1(A) + (N - 3)(1 - g) + n]$$

where $\dim X = 2N$.

The construction of Gromov–Witten invariants now hinges on a single issue: can one find a thin compactification of (9.3) so that the map se extends over the compactification to give diagram (9.7)? Doing so, even over a portion of $\mathcal{J}\mathcal{V}$, allows us to define the Gromov–Witten numbers

(9.9) $$GW_{A,g,n}(\alpha) = \left\langle (se)^*\alpha, \; [\overline{\mathcal{M}}_{A,g,n}^J]^{rel} \right\rangle \text{ for all } \alpha \in \check{\mathrm{H}}^*(\overline{\mathcal{M}}_{g,n} \times X^n; \mathbb{Q}).$$

Note that $\overline{\mathcal{M}}_{g,n} \times X^n$ is locally contractible, so by (1.14) α can also be regarded as an element of rational singular cohomology.

More specifically, assuming the existence of a thin compactification, we can apply the results of Sections 1-6, with the following payoffs:

(a) A thin compactification for the fiber over a single regular $J \in \mathcal{J}$ yields a relative fundamental class $[\overline{\mathcal{M}}^J_{A,g,n}]^{rel}$. (Note, however, that the numbers (9.9) may not be invariant under changes in J.)

(b) A thin compactification over a connected neighborhood $\mathcal{P}$ of J gives a relative fundamental class at each $J \in \mathcal{P}$, and by Corollary 4.3 the numbers (9.9) are independent of J in $\mathcal{P}$.

(c) A thin compactification over all of $\mathcal{J}$ or $\mathcal{J}\mathcal{V}$ gives numbers (9.9) that depend only on the symplectic structure of (X, ω).

(d) A thin compactification over the larger space $\mathcal{J}_{symp}$ of all tame pairs (ω, J), completed in an appropriate Sobolev norm, implies that the numbers (9.9) are invariants of the isotopy class of the symplectic structure on X.

We will give examples of this procedure in the next section. Before proceeding, here are some simple examples that illustrate the ideas of this section.

Example 9.1 (Rational ghost maps). For each $J \in \mathcal{J}$, every J-holomorphic map $f : S^2 \to X$ representing the trivial class $A = 0$ is a constant map. It follows that $D_{f,J}$ is the $\overline{\partial}$ operator on the trivial holomorphic bundle f^*TX, and f is regular because the sheaf cohomology group $H^1(S^2, f^*TX)$ vanishes. Hence for $n \geq 3$ the fibers of the moduli space $\overline{\mathcal{M}}^J_{0,0,n}(X) \to \mathcal{J}$ are all regular and canonically identified with $\overline{\mathcal{M}}_{0,n} \times X$. The relative fundamental class $[\overline{\mathcal{M}}^J(X)]^{rel}$ is therefore equal to the actual fundamental class $[\overline{\mathcal{M}}_{0,n} \times X]$ and the GW invariants (9.9) are independent of $J \in \mathcal{J}$.

Example 9.2 (K3 surfaces). Let X be a K3 surface, and consider the moduli space $\mathcal{M}(X) \to \mathcal{J}_{alg}$ of smooth rational holomorphic maps (f, J) for algebraic $J \in \mathcal{J}$. By a theorem of Mumford and Mori (see [MMu]), every algebraic K3 contains a non-trivial rational curve, so for each algebraic J the fiber $\mathcal{M}^J_{A,0,0}(X)$ is non-empty for some $A \neq 0$. But by (9.8) the index $\iota(A, 0, 0) = -2$ is negative. Thus $\mathcal{M}_{A,0,0}(X) \to \mathcal{J}_{alg}$ does not satisfy condition **Reg 1** for any algebraic J.

Now expand the base by considering $\pi : \mathcal{M}(X) \to \mathcal{J}_{cx}$ over the space of all integrable almost complex structures. Each $J \in \mathcal{J}_{cx}$ determines a 20-dimensional subspace $H^{1,1}(X; \mathbb{R})$ of $H^2(X; \mathbb{R}) \cong \mathbb{R}^{22}$, and the resulting map $\mathcal{J}_{cx} \to \mathrm{Gr}(20, 22)$ to the Grassmannian is a submersion. But $A \in H_2(X; \mathbb{Z})$ can be represented by a J-holomorphic curve only if the Poincaré dual of A is an integral $(1, 1)$ class. It follows that $\mathcal{M}^J_{A,g,n}(X)$ is empty for all J

in a subset $\mathcal{P} \subset \mathcal{J}_{cx}$ whose complement is a locally finite countable union of codimension 2 submanifolds. Since empty fibers are regular, a relative fundamental class exists over $\mathcal{P}$ and is equal to 0. Lemma 6.5 then applies, showing that

$$[\overline{\mathcal{M}}_{A,g,n}^{J}(X)]^{rel} = 0$$

for all $A \neq 0$, g and n, and all $J \in \mathcal{J}_{cx}$, including the algebraic J.

Example 9.3 (Convex manifolds). A complex algebraic manifold (X, ω, J) is called *convex* if $H^1(C, f^*TX) = 0$ for all stable J-holomorphic maps $f : S^2 \to X$. Examples include projective spaces, Grassmannians, and Flag manifolds. Convexity implies that all J-holomorphic maps with smooth domain are regular, so $\mathcal{M}_{A,0,n}^{J}(X)$ is smooth and complex. It is also a quasi-projective variety (cf. [FP]), so its closure is a thin compactification. Hence by Lemma 2.2 there is a relative fundamental class $[\overline{\mathcal{M}}_{A,0,n}^{J}(X)]^{rel}$ for the given J; more work is needed to determine if the associated GW numbers (9.9) are symplectic invariants.

10. Moduli spaces of stable maps

The space of stable maps is the most commonly-used compactification of the moduli space (9.3) of smooth pseudo-holomorphic maps. Indeed, it is often regarded as the central object of Gromov–Witten theory. This section uses existing results to show that, in certain rather special circumstances, the space of stable maps is a thin compactification over parts of $\mathcal{J}$ or $\mathcal{J}\mathcal{V}$. In these cases, the space of stable maps carries a relative fundamental class.

Each stable map $f : C \to X$ has an associated dual graph $\tau(f)$, whose vertices correspond to the irreducible components C_i of C and whose edges correspond to the nodes of C. Each vertex of the graph is labeled by the homology class $A_i = [f(C_i)] \in H_2(X; \mathbb{Z})$, by the genus g_i of C_i, and by the number n_i of marked points on C_i. Every such graph τ defines a stratum $\mathcal{S}_\tau$ consisting of all stable maps f with $\tau(f) = \tau$. The trivial graph, which consists of a single vertex and no edges, corresponds to the moduli space $\mathcal{M}_{A,g,n}$ in (9.3). The universal moduli space of all stable maps is then the disjoint union

$$(10.1) \qquad \overline{\mathcal{M}}_{A,g,n} = \mathcal{M}_{A,g,n} \cup \bigcup \mathcal{S}_\tau,$$

where the last union is over all non-trivial graphs τ with $\sum A_i = A$, $\sum n_i = n$, and with $\sum g_i$ plus the first Betti number of the graph equal to g.

The Gromov Compactness Theorem (cf. [IS]) implies that the projection $\overline{\pi} : \overline{\mathcal{M}}_{A,g,n} \to \mathcal{J}$ is proper.

To check whether (10.1) is a thin compactification one must, as always, compute the index of the restriction $\pi_\tau : \mathcal{S}_\tau \to \mathcal{JV}$ of $\overline{\pi}$ to each stratum $\mathcal{S}_\tau$, and prove transversality results that show that $\mathcal{S}_\tau$ is a manifold over $\mathcal{J}$. In this case, the index calculations have been done many times in the literature (for example, see Theorem 6.2.6(i) in [MS] or Section 4 in [RT1] for the $g = 0$ case, and Section 3 in [RT2] in general). These calculations show that, for each τ,

$$(10.2) \qquad \qquad \text{index } \pi_\tau \; = \; \iota(A, g, n) - 2k$$

where $\iota(A, g, n)$ is the index (9.8) of the principal stratum $\pi : \mathcal{M}_{A,g,n} \to \mathcal{J}$, and k is the number of nodes of the domain. Lemma 5.4 then shows that (10.1) is a Fredholm-stratified thin compactification of the principal stratum *provided all strata satisfy the transversality condition* **Reg 1** *in Section 9.*

Unfortunately, transversality can only be shown for certain classes of stable maps. In the remainder of this section, we examine two such classes of maps.

10.1. Moduli spaces of somewhere injective maps

A stable map $f : C \to X$ is called *somewhere injective* (si) if each irreducible component C_i of C contains a non-special point p_i such that

$$(df)_{p_i} \neq 0 \qquad \text{and} \qquad f^{-1}(f(p_i)) = \{p_i\}.$$

(cf. [MS, Section 2.5]). In the literature, it is usual to consider the universal moduli space of stable maps $\overline{\mathcal{M}} \to \mathcal{J}$, and to show that the subset $\overline{\mathcal{M}}^*$ consisting of the somewhere injective maps has good properties. We will shift perspective: *instead of restricting to a subset of $\overline{\mathcal{M}}$, we restrict to the subset of $\mathcal{J}$ consisting of those "nice" J for which the entire fiber $\overline{\mathcal{M}}^J$ consists of somewhere injective maps.*

Thus we fix (A, g, n) and define the (possibly empty) subset $\mathcal{J}_{si} = \mathcal{J}_{si}(A, g, n)$ of $\mathcal{J}$ by

$$\mathcal{J}_{si} = \left\{ J \in \mathcal{J} \;\middle|\; \text{all } (f, J) \in \overline{\mathcal{M}}^J_{A,g,n} \text{ are si} \right\}.$$

We then consider the map

$$\overline{\mathcal{M}}'_{A,g,n}(X)$$

(10.3) $\downarrow \overline{\pi}'$

$$\mathcal{J}_{si}$$

obtained by restricting the space of stable maps (9.7) over $\mathcal{J}_{si}$, with the stratification

$$\overline{\mathcal{M}}'_{A,g,n} = \mathcal{M}'_{A,g,n} \cup \bigcup \mathcal{S}'_\tau,$$

obtained by restricting (10.1) over $\mathcal{J}_{si}$. First note that:

Lemma 10.1. $\mathcal{J}_{si}(A, g, n)$ *is an open subset of $\mathcal{J}$, so is a Banach manifold.*

The proof of Lemma 10.1 is given at the end of this subsection. Assuming it, one obtains a relative fundamental class, in any one of the homology theories (1.12), for the space of stable maps over $\mathcal{J}_{si}$:

Proposition 10.2. *The family* (10.3) *is a Fredholm-stratified thin compactification whose index $d = \iota(A, g, n)$ is given by* (9.8). *It therefore admits a unique relative fundamental class which, in particular, assigns an element*

$$[\overline{\mathcal{M}}^J_{A,g,n}(X)]^{rel} \in \check{\mathcal{H}}_d \left(\overline{\mathcal{M}}^J_{A,g,n}(X) \right)$$

to each $J \in \mathcal{J}_{si}(A, g, n)$.

By Corollary 4.3, the corresponding GW numbers (9.9) are constant on each path-component of $\mathcal{J}_{si}(A, g, n)$.

Proof. Following the discussion in Section 9, it suffices to verify the assumptions of **Reg 1**. First, observe that somewhere injective maps have no non-trivial automorphisms. Next, standard arguments show that for each somewhere injective f, one can use the variation in the parameter $J \in \mathcal{J}$ to show that the linearization of the equation $\overline{\partial}_J f = 0$ (with fixed domain and map f) is onto. Specifically, for the $g = 0$ case, Proposition 6.2.7 and Theorem 6.3.1 in [MS] imply that each stratum $\mathcal{S}'_\tau$ of (10.3) is a Banach manifold and $\pi'_\tau : \mathcal{S}'_\tau \to \mathcal{J}_{si}$ has index given by (10.2). As mentioned before (9.8), the principal stratum is relatively oriented. Therefore (10.3) is a Fredholm-stratified thin compactification when $g = 0$.

The same proofs (Propositions 6.2.7 and 6.2.8 and the proof of Theorem 6.3.1) in [MS] also apply for $g > 0$: they show that the linearization is

surjective using variations that fix the complex structure on the domain, which implies, *a fortiori*, surjectivity as the domain is allowed to vary. □

While Proposition 10.2 implies that the Gromov–Witten numbers are invariant under small deformations of J, it does not imply that they are symplectic invariants unless one can show that $\mathcal{J}_{si}$ is equal to $\mathcal{J}$, or at least is path-connected, and open and dense in $\mathcal{J}$. The following examples give two simple cases where this occurs.

Example 10.3. For $X = \mathbb{C}P^N$, the universal space $\overline{\mathcal{M}}_{L,0,0}(X)$ of stable rational maps representing the class of a line is smooth and equal to $\mathcal{M}_{L,0,0}(X)$, and $\mathcal{J}_{si}(L,0,0)$ is all of $\mathcal{J}$.

Example 10.4. Assume X is a Calabi-Yau 3-fold. As in (10.3), consider the universal moduli space $\overline{\mathcal{M}}'_{A,0,0}(X) \to \mathcal{J}_{si}$ of unmarked stable rational curves representing a primitive homology class $A \in H_2(X;\mathbb{Z})$. In this case, $\overline{\pi}$ has index 0 and, we claim, $\mathcal{J}_{si} = \mathcal{J}_{si}(A,0,0)$ is not only open, but is also dense and path-connected. Hence Proposition 10.2 gives a relative fundamental class

$$[\overline{\mathcal{M}}^J_{A,0,0}(X)]^{rel} \in \check{\mathcal{H}}_0(\overline{\mathcal{M}}^J_{A,0,0}(X))$$

defined for all $J \in \mathcal{J}_{si}$, and therefore for all $J \in \mathcal{J}$ by the Extension Lemma 3.4. Evaluating on $1 \in \check{\mathrm{H}}^0(\overline{\mathcal{M}}^J)$ then gives a well-defined numerical GW invariant.

To prove the claim, note that, by Corollaries 1.4 and 6.6 of [IP], there is a path-connected dense subset $\mathcal{J}^E_{isol}$ of $\mathcal{J}$ (with $E = \omega(A)$) such that, for each $J \in \mathcal{J}^E_{isol}$, all somewhere injective J-holomorphic maps with energy at most E are embeddings, and their images are disjoint. Fix $J \in \mathcal{J}^E_{isol}$. Then by Lemma 1.5(a) of [IP] any J-holomorphic map $f \in \overline{\mathcal{M}}^J_{A,0,0}(X)$ factors as a composition $f = g \circ \varphi$ of a holomorphic map $\varphi : C \to C_{red}$ of (connected) complex curves and a J-holomorphic embedding $g : C_{red} \to X$. But A is primitive so the degree of φ is 1, and C is an unmarked rational curve, so φ cannot have any constant components. Therefore f is an embedding of a smooth curve; in particular, f is somewhere injective. Thus $\mathcal{J}^E_{isol} \subseteq \mathcal{J}_{si}$. But this means that $\mathcal{J}_{si}$ is an open subset of the manifold $\mathcal{J}$ that contains a dense path-connected set. It follows that $\mathcal{J}_{si}$ itself is dense and path-connected, as claimed.

We conclude this subsection by supplying the deferred proof.

Proof of Lemma 10.1. From the discussion in [MS, Section 2.5], one sees that the complement of $\mathcal{J}_{si}$ in $\mathcal{J}$ is the set of all J such that there exists a

J-holomorphic map $f : C \to X$ in $\overline{\mathcal{M}}_{A,g,n}(X)$ and an irreducible component C_i of C with either

(i) $f(C_i) = p$ is a single point,

(ii) the restriction $f|_{C_i}$ is a multiple cover of its image, or

(iii) there is another component C_j of C with $f(C_i) = f(C_j)$.

We will show that each of these is a closed condition on J, so the complement of $\mathcal{J}_{si}$ is the union of three closed sets.

Suppose that a sequence $\{J_k\}$ converges to $J \in \mathcal{J}$ and that there are stable J_k-holomorphic maps $f_k : C_k \to X$ and components $C_k' \subset C_k$ with $f_k(C_k') = p_k$ as in (i). By Gromov compactness, after passing to a subsequence and then a diagonal subsequence, $\{f_k\}$ and $\{f_k|_{C_k'}\}$ converge to J_0-holomorphic maps $f : C \to X$ and $f' : C' \to X$, respectively, for some nodal curve C and subcurve C' with $f' = f|_{C'}$. But then f' is a constant map. Thus (i) is a closed condition on J.

If each $\{f_k|_{C_k'}\}$ is multiply covered then, by the proof of [MS, Proposition 2.5.1], there exist curves B_k and holomorphic maps $\varphi_k : C_k' \to B_k$ of degree > 1 such that $f_k|_{C_k'}$ is the composition $g_k \circ \varphi_k$ for some J_k-holomorphic map $g_k : B_k \to X$. Again by Gromov compactness, we may assume that, after restricting to C_k', these converge to maps f', g and φ with $f' = g \circ \varphi$ and $\deg \varphi > 1$. Then $f' = f|_{C'}$ satisfies (ii), so (ii) is a closed condition on J.

The proof for (iii) is similar after using [MS, Corollary 2.5.3] to write $f_k|_{C_i}$ as the composition of $\varphi_k : C_k^i \to C_k^j$ and $g_k : C_k^j \to X$. $\qquad\square$

10.2. Moduli spaces of domain-fine maps

The somewhere injective condition is too restrictive for most applications. In the genus 0 case, the needed transversality results hold for the slightly larger class of maps ("simple maps") that are somewhere injective on the complement of ghost components; see [MS, Example 6.2.5]. But it is more effective to expand the base space $\mathcal{J}$ to the space $\mathcal{J}\mathcal{V}$ of Ruan-Tian perturbations and work with the universal moduli space (9.7) of (J, ν)-holomorphic maps. Here one has results analogous to those of the previous section for a different class of maps:

Definition 10.5. *A (J, ν)-holomorphic map $f : C \to X$ is called* domain-fine *if* $\mathrm{Aut}\, C = 1$.

Note that any domain-fine map $f : C \to X$ is a stable map. Furthermore, the map $C \mapsto \varphi(C)$ defined by (9.2) is an embedding, so the graph map (9.4) is also an embedding, and hence is somewhere injective. While the proofs in the previous subsection do not automatically apply (because the set of almost complex structures on $\overline{\mathcal{U}}_{g,n} \times X$ is restricted to be of the form (9.5)), their conclusions hold, as we show next.

Again, we fix (A, g, n), set

$$\mathcal{JV}_{df} = \mathcal{JV}_{df}(A, g, n) = \left\{ J \in \mathcal{JV} \ \middle| \ \text{all } (f, J) \in \overline{\mathcal{M}}^J_{A,g,n} \text{ are domain-fine} \right\},$$

and consider the map

(10.4)
$$\begin{array}{c} \overline{\mathcal{M}}''_{A,g,n}(X) \\ \downarrow {\scriptstyle \pi''} \\ \mathcal{JV}_{df} \end{array}$$

obtained by restricting the space of stable maps (9.7) over $\mathcal{J}_{df}$. Then (10.1) restricts to a stratification

$$\overline{\mathcal{M}}''_{A,g,n} = \mathcal{M}''_{A,g,n} \ \cup \ \bigcup \mathcal{S}''_\tau.$$

Corresponding to Lemma 10.1, we have:

Lemma 10.6. *$\mathcal{JV}_{df}$ is an open subset of $\mathcal{JV}$, so is a Banach manifold.*

Proof. Under Gromov convergence, the order of the automorphism group of the domain is upper semi-continuous, and limits of unstable domain components are unstable. Thus each domain-fine map f has a neighborhood with the same property. For $(J, \nu) \in \mathcal{JV}_{df}$, these open sets cover the moduli space $\overline{\mathcal{M}}^{J,\nu}_{A,g,n}(X)$, and hence by compactness cover the moduli spaces $\pi^{-1}(U)$ for some open neighborhood U of (J, ν). $\square$

Lemma 10.6 enables us to rephrase a result of Ruan and Tian in [RT2] to show that the moduli space (10.4) over $\mathcal{JV}_{df}$ admits a relative fundamental class in the homology theories (1.12).

Proposition 10.7. *Fix (A, g, n) and $\mathcal{JV}_{df}$ as above. Then the restriction (10.4) of the universal moduli space of stable maps over $\mathcal{JV}_{df}$ is a Fredholm-stratified thin compactification $\overline{\mathcal{M}}_{A,g,n}(X) \to \mathcal{JV}_{df}$ of index $d = \iota(A, g, n)$.*

Therefore it admits a unique relative fundamental class

$$[\overline{\mathcal{M}}^J_{A,g,n}(X)]^{rel} \in \check{\mathcal{H}}_* \left(\overline{\mathcal{M}}^J_{A,g,n}(X) \right).$$

Again, by Corollary 4.3, the corresponding GW numbers (9.9) are invariant under small deformations of (J, ν), and are constant on each path-component of $\mathcal{JV}_{df}(A, g, n)$.

Proof. For domain-fine maps f, we have $\text{Aut}(f) = 1$ and the graph map F is an embedding. Hence one can use the variation in ν to show that the linearization of the equation $\overline{\partial}_J f = \nu$ is onto, as in the proof of [RT2, Proposition 3.2]. The proof is completed exactly as the proof of Proposition 10.2. □

Example 10.8. Let $\overline{\mathcal{M}}_{0,0,n}(X) \to \mathcal{JV}$ be the moduli space of stable (J, ν)-holomorphic rational maps representing the class $0 \in H_2(X)$ and with $n \geq 3$ marked points. Because stable rational curves have no non-trivial automorphisms, maps of this type are domain-fine for all (J, ν). Thus in this case $\mathcal{J}_{df}(0, 0, n)$ is all of $\mathcal{JV}$.

In general, $\mathcal{JV}_{df}$ may not be all of $\mathcal{JV}$, and may even be empty. Proposition 10.7 is then insufficient to define symplectic invariants. This is a manifestation of a well-known problem in symplectic Gromov–Witten theory originally identified by Ruan and Tian: the space of stable maps may not be a relatively thin family because of the presence of multiply-covered unstable domain components. On such components, the perturbation ν vanishes and cannot be used to verify condition **Reg 1** of Section 9. In subsequent papers, we will extend and apply the constructions in Sections 1-6 with the aim of moving past this obstacle.

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