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Quaternionic toric manifolds

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In the present paper we introduce and study a new notion of toric manifold in the quaternionic setting. We develop a construction with which, starting from appropriate m-dimensional Delzant polytopes, we obtain manifolds of real dimension $4m$, acted on by m copies of the group $\mathrm{Sp}(1)$ of unit quaternions. These manifolds, are *quaternionic regular* in the sense of [11] and can be endowed with a 4-plectic structure and a generalized moment map. Convexity properties of the image of the moment map are studied. Quaternionic toric manifolds appear to be a large enough class of examples where one can test and study new results in quaternionic geometry.

1. Introduction

Toric varieties are geometric objects that can be defined by combinatorial information encoded in convex polyhedra. They provide a large and interesting class of examples in algebraic geometry and many notions in this field such as singularities, birational maps, cycles, homology, intersection theory

can be interpreted in terms of properties of the convex polyedra on which these varieties are modeled. An exahustive introduction to this topic can be found in the book [7] by Cox, Little and Schenck.

The study of toric manifolds (i.e., smooth toric varieties) has many different motivations and a wide spectrum of applications. In particular, in symplectic geometry, toric manifolds provide examples of extremely symmetric and completely integrable Hamiltonian spaces. Properties of symplectic toric manifolds are extensively studied, and are in the mainstream of current mathematical research.

The term moment map was introduced by Souriau, [18], under the French name *application moment*, to indicate one of the main tools used to study problems in geometry and topology when there is a suitable symmetry, as illustrated in the book by Gelfand, Kapranov and Zelevinsky [10]. The role of the moment map is fundamental in the symplectic setting: in fact the geometry encoded in its image, the so-called moment polytope, identifies the symplectic toric manifold.

Recent developments in the theory of regular functions over the quaternions encourage to go through the already existing approaches and to elaborate new tools to study quaternionic toric manifolds. This class of manifolds seems suitable to become large enough to give interesting examples, where to test and develop new results of quaternionic differential (and 4-plectic) geometry.

The purpose of the present paper is to introduce the notion of quaternionic toric manifolds. The starting point is the definition of 4-plectic manifold, originally introduced by Foth in [9], as a natural generalization of symplectic manifold. In the symplectic case, whenever a compact Lie group acts on the manifold in a Hamiltonian fashion it is possible to define a moment map which takes values in the dual of the Lie algebra of the acting group; in particular when the group is a torus T^m and the action is effective the moment map takes values in $\mathbb{R}^m$. These manifolds are called symplectic toric manifolds. We consider 4-plectic manifolds, i.e. $4m$-dimensional real manifolds endowed with a non degenerate closed 4-form, acted on by the group $\mathrm{Sp}(1)^m$ in a *generalized Hamiltonian* fashion so that it is possible to define a *tri-moment map* which takes values in $(\Lambda^3\mathfrak{sp}(1)^{*m}) \cong \mathbb{R}^m$. Inspired by the symplectic setting we give the following

Definition 1.1. Let M be a connected, compact $4m$-dimensional 4-plectic manifold on which $\mathrm{Sp}(1)^m$ acts effectively in a generalized Hamiltonian fashion with discrete principal isotropy. Then M is called a *quaternionic toric manifold.*

The celebrated Atiyah's convexity Theorem, [2], establishes the convexity of the image of the moment map for symplectic toric manifolds.

In some cases we are able to prove that the image of the tri-moment map is a convex polytope. More generally, when a 4-plectic manifold (M, ψ), acted on in a generalized Hamiltonian fashion by $Sp(1)^m$ with tri-moment map σ, is equipped with a *partitioned strongly non degenerate* form ψ, we prove that $\sigma(M)$ is contained in the convex envelope of a finite set of points, see Theorem 3.2.

In the other direction, in the symplectic setting the Theorem of Delzant proves that there is a one-to-one correspondence between symplectic toric manifolds and a special class of polytopes, the Delzant polytopes; in particular in its well known paper [8] the author provides a procedure to recover the symplectic manifold starting from a Delzant polytope. In the quaternionic setting the idea of defining toric manifolds starting from polytopes can already be found in [17] where the author begins the study of a new class of topological spaces analogous to real and complex toric varieties, but with the underlying structure provided by the skew field of quaternions. Starting from a m-dimensional convex polytope P and a *characteristic function* he defines a *quaternionic toric variety* to be a certain topological quotient of $P \times (S^3)^m$, where S^3 is the unit sphere of the quaternionic space $\mathbb{H}$. The author emphasizes that these are not algebraic varieties, and moreover he observes that the notion of "quaternionic variety" is unclear, because quaternions are non-commutative and general polynomials are not well behaved. The author studies the topology and the homology Betti numbers of the resulting objects.

In the present paper we introduce a procedure that, starting from a m-dimensional Delzant polytope with appropriate hypotheses, leads to obtain a compact manifold acted on effectively and with trivial principal isotropy by $Sp(1)^m$. The advantage of our construction is that it suggests a way, in the spirit of the symplectic cut [15], to equip the resulting manifold with a non-degenerate 4-form.

Indeed we define the 4-plectic cut as follows. Let (M, ψ) be a 4-plectic manifold equipped with a generalized Hamiltonian $Sp(1)^m$-action. Consider the restricted $Sp(1)$-action, and let $h : M \to \mathbb{R}$ be the corresponding tri-moment map. Let ε be a regular value of h. For simplicity we assume that the $Sp(1)$-action on $h^{-1}(\varepsilon)$ is free. We denote by $M_{h>\varepsilon}, M_{h\geq\varepsilon}$ the pre-images of (ε, ∞) and $[\varepsilon, \infty)$ under $h : M \to \mathbb{R}$, and denote by $\overline{M_{h\geq\varepsilon}}$ *the* 4*-plectic cut,* i.e. the manifold which is obtained by collapsing the boundary $h^{-1}(\varepsilon)$ of $M_{h\geq\varepsilon}$ along the orbits of the $Sp(1)$-action.

Let ψ_0 be the standard 4-plectic form on $\mathbb{H}$. With the above notations we prove

Theorem 1.2. *Let (M, ψ) be a 4-plectic manifold. Whenever the induced form $\psi \oplus \psi_0$ on $M \times \mathbb{H}$ is* horizontal *along $(h - \frac{1}{4}|q|^4)^{-1}(\varepsilon)$, there is a natural 4-plectic structure Ψ_ε on $\overline{M_{h\geq\varepsilon}}$ such that the restriction of Ψ_ε to $M_{h>\varepsilon} \subseteq \overline{M_{h\geq\varepsilon}}$ equals ψ.*

As an application we find a correspondence between a special class of Delzant polytopes and some quaternionic toric manifolds. In these cases we are also able to show that the involved manifolds admit an action of $(\mathbb{H}^*)^m$ with an open dense orbit, in analogy with what happens in the complex setting. We observe that all these examples, together with quaternionic tori, [4], and Hopf surfaces, [1], enrich the class of *quaternionic regular* manifolds in the sense of [11].

The paper is organized as follows. In the second section we give the basic definitions and notions of 4-plectic manifolds and generalized Hamiltonian actions. In Section 3 we present a sub-convexity result, Theorem 3.2, and in Section 4 we describe the above mentioned procedure, providing necessity and sufficiency conditions under which it can be applied. Section 5 is devoted to study the 4-plectic reduction and the 4-plectic cut. In the last section we collect the obtained examples, we give the explicit form of the tri-moment map and consequently we obtain a convexity theorem for this class of examples. We finally make some remarks on the $\mathbb{H}^*$ action and on the manifold $G_2/SO(4)$ which deserves further investigation.

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2. The 4-plectic viewpoint

We begin this section by introducing a possible counterpart of symplectic forms and structures on $4m$-dimensional real manifolds.

Definition 2.1. Let M be real manifold of dimension $4m$. A 4-form ψ on M is said to be 4-*plectic* if

1) ψ is closed, i.e. $d\psi = 0$;
2) ψ is *non-degenerate*, i.e. the map $v \mapsto \iota_v\psi$ that contracts ψ along a tangent vector field v has trivial kernel.

A 4-plectic form defines a 4-*plectic structure* on M, and M equipped with such a form is called a 4-*plectic manifold*.

A natural class of examples of 4-plectic manifolds is given by $4m$-dimensional symplectic manifolds. Indeed starting from a manifold with symplectic form ω we obtain a 4-plectic manifold by endowing it with the 4-form $\omega \wedge \omega$. Quaternion Kähler manifolds equipped with the Kraines form, [13], give a class of examples of non-symplectic 4-plectic manifolds; a particularly large class of 4-dimensional 4-plectic manifolds is given by the Kulkarni four-folds, [14].

An interesting basic example that we will use in the sequel is the quaternionic space $\mathbb{H}^m$, naturally identified with $\mathbb{R}^{4m}$, endowed with the 4-plectic form ψ_0 defined by

$$\psi_0 = \sum_{i=1}^{m} dx_{4i-3} \wedge dx_{4i-2} \wedge dx_{4i-1} \wedge dx_{4i}$$

where $x_1, \dots, x_{4m}$ are the coordinates on $\mathbb{R}^{4m}$. Note that $(\mathbb{H}^m, \psi_0)$ is not symplectic for $m > 1$; indeed the form ψ_0 cannot be obtained as the square of a symplectic form.

The notion of Hamiltonian action in the symplectic setting is very useful and powerful. Indeed, whenever a Lie group G acts on a symplectic manifold in a Hamiltonian fashion, it is possible to define a map $\mu : M \to \mathfrak{g}^*$, commonly known as moment map, which encodes many geometric information on the manifold and on the action. Whenever the action of the group $G = Sp(1)^k$ on a 4-plectic $4m$-dimensional manifold M is *generalized Hamiltonian*, our aim is to define an analog of the moment map also in the 4-plectic setting, following the path indicated by Foth in [9].

Whenever a Lie group G acts on a manifold M, it is possible to define a canonical map $\mathfrak{g} \to \Gamma(M, TM)$ which sends the vector $X \in \mathfrak{g}$ to the *fundamental vector field* $\widehat{X}$ in M, such that at a point $p \in M$,

$$\widehat{X}_p = \frac{d}{dt}_{|_{t=0}} \exp tX \cdot p.$$

Now, if M is equipped with a 4-plectic form ψ, there is also a natural map $\mathfrak{g} \to A^3(M)$ which sends the generic vector $X \in \mathfrak{g}$ to the contraction of ψ along $\widehat{X}$, i.e. to a 3-form on M. Given a tangent vector field $Y \in \Gamma(M, TM)$, if the 3-form $\iota_Y \psi$ is closed we say that Y is a *locally Hamiltonian vector field*; if moreover $\iota_Y \psi$ is exact we say that Y is a *Hamiltonian vector field*.

From now on we assume M is a $4m$-dimensional real manifold.

Definition 2.2. Let (M, ψ) be a 4-plectic manifold on which the group $Sp(1)^k$ acts preserving ψ. We say that this action is *generalized Hamiltonian* if for any $X \in \mathfrak{sp}(1)^k$ the fundamental vector field $\widehat{X}$ is Hamiltonian.

The standard basis of $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ is given by

$$H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

which represent respectively the quaternion imaginary units i, j, k.

The space of 3-vectors $\Lambda^3(\mathfrak{sp}(1))$ can be identified with $\mathbb{R}$ by the isomorphism that sends $X \wedge Y \wedge H \mapsto 1$.

Any

$$\delta = (\delta_1, \delta_2, \ldots, \delta_k) = (U_1 \wedge V_1 \wedge W_1, \ldots, U_k \wedge V_k \wedge W_k) \in (\Lambda^3 \mathfrak{sp}(1))^k$$

induces a k-tuple of 3-vector fields on M

$$\widetilde{\delta} = (\widetilde{\delta}_1, \widetilde{\delta}_2, \ldots, \widetilde{\delta}_k) = (\widehat{U}_1 \wedge \widehat{V}_1 \wedge \widehat{W}_1, \ldots, \widehat{U}_k \wedge \widehat{V}_k \wedge \widehat{W}_k) \in (\Lambda^3(TM))^k.$$

Definition 2.3. Let $Sp(1)^k$ act on a 4-plectic manifold (M, ψ) in a generalized Hamiltonian fashion. A tri-moment map σ is a map

$$\sigma : M \to ((\Lambda^3 \mathfrak{sp}(1))^k)^* \cong ((\Lambda^3 \mathfrak{sp}(1))^*)^k \cong \mathbb{R}^k$$

satisfying the following conditions:

1) σ is $Sp(1)^k$-invariant, i.e. $\sigma(g \cdot p) = \sigma(p)$ for all $g \in Sp(1)^k$, $p \in M$;

2) for any $\delta = (\delta_1, \delta_2, \dots, \delta_k)$ in $(\Lambda^3\mathfrak{sp}(1))^k$, $p \in M$ and $v \in T_pM$ we have

$$d\sigma_p(v)(\delta) = \sum_{i=1}^{k} \iota_{\widetilde{\delta}_{i_p}} \psi(v) =: \iota_{\widetilde{\delta}_p} \psi(v)$$

where $\widetilde{\delta_i}$ is the tri-vector field induced by δ_i.

Since the coadjoint action of $Sp(1)$ on $\mathfrak{sp}(1)^*$ induces the trivial action on $(\Lambda^3\mathfrak{sp}(1))^*$, the action is indeed equivariant.

A further property of the tri-moment map is that for any $p \in M$ such that $\sigma(p)$ is regular, setting $V = (\Lambda^3(T_p(Sp(1)^k \cdot p)))^k$,

$$\ker\ d\sigma_p = V^{\perp_{\psi_p}}.$$

In fact

$$\ker d\sigma_p = \{v \in T_pM \mid 0 = d\sigma_p(v)(\delta) = \iota_{\widetilde{\delta}_p} \psi(v) \text{ for any } \delta \in (\Lambda^3\mathfrak{sp}(1))^k\}$$

equals

$$V^{\perp_{\psi_p}} = \{v \in T_pM \mid \iota_{\widetilde{\delta}_p} \psi(v) = 0 \text{ for any } \delta \in (\Lambda^3(T_p(Sp(1)^k \cdot p))^k\},$$

since $(\Lambda^3\mathfrak{sp}(1))^k$ and $(\Lambda^3(T_p(Sp(1)^k \cdot p)))^k$ are isomorphic.
Notice that the tri-moment map σ is defined up to a constant $C \in \mathbb{R}^k$; the $Sp(1)^k$-invariance of σ implies that there is no further hypothesis on C.

We are now ready to prove

Proposition 2.4. *Let (M, ψ) be a 4-plectic manifold acted on by $Sp(1)^k$ in a generalized Hamiltonian fashion. The tri-moment map exists when $b_1(M) = 0$.*

Proof. It is enough to prove the statement for a component of the tri-moment map. Consider the standard basis X, Y, H of the Lie algebra $\mathfrak{sp}(1)$ defined above. Recall that

$$[X, Y] = 2H, \quad [Y, H] = 2X \text{ and } \quad [H, X] = 2Y.$$

The element

$$\delta_i = (0, 0, \dots, 0, \overbrace{X \wedge Y \wedge H}^{i-\text{th component}}, 0, \dots, 0) \in (\Lambda^3(\mathfrak{sp}(1)))^k$$

is identified with the usual canonical basis vector $e_i \in \mathbb{R}^k$. For each i-th component of the tri-moment map, and any tangent vector Z we have

$$< d\sigma(Z), \delta_i >= d\sigma_i(Z) = \psi(\widehat{X}, \widehat{Y}, \widehat{H}, Z) = \iota_{\widehat{X}}\iota_{\widehat{Y}}\iota_{\widehat{H}}\psi(Z).$$

So a necessary and sufficient condition for the existence of a map satisfying condition (2) of Definition 2.3 is that the 1-form $\iota_{\widehat{X}}\iota_{\widehat{Y}}\iota_{\widehat{H}}\psi$ is closed. We prove the closedness of this form by applying the well known relations involving the Lie derivative $\mathcal{L}$ and valid for any tangent vector fields U, W

$$\mathcal{L}_U = d\iota_U + \iota_U d \tag{2.1}$$

and

$$[\mathcal{L}_U, \iota_W] = \iota_{[U,W]}. \tag{2.2}$$

We compute

$$\begin{aligned} d(\iota_{\widehat{X}}\iota_{\widehat{Y}}\iota_{\widehat{H}}\psi) &= -\iota_{\widehat{X}}d\iota_{\widehat{Y}}\iota_{\widehat{H}}\psi + \mathcal{L}_{\widehat{X}}\iota_{\widehat{Y}}\iota_{\widehat{H}}\psi \\ &= -\iota_{\widehat{X}}d\iota_{\widehat{Y}}\iota_{\widehat{H}}\psi + \iota_{\widehat{Y}}\mathcal{L}_{\widehat{X}}\iota_{\widehat{H}}\psi + \iota_{[\widehat{X},\widehat{Y}]}\iota_{\widehat{H}}\psi. \end{aligned}$$

Now the last term of the equality is zero since $[X, Y] = 2H$. Applying again equation (2.2) the equality above becomes

$$-\iota_{\widehat{X}}d\iota_{\widehat{Y}}\iota_{\widehat{H}}\psi + \iota_{\widehat{Y}}\mathcal{L}_{\widehat{X}}\iota_{\widehat{H}}\psi = -\iota_{\widehat{X}}d\iota_{\widehat{Y}}\iota_{\widehat{H}}\psi + \iota_{\widehat{Y}}\iota_{\widehat{H}}\mathcal{L}_{\widehat{X}}\psi + \iota_{\widehat{Y}}\iota_{[\widehat{X},\widehat{H}]}\psi.$$

Both the last two terms are zero: $\mathcal{L}_{\widehat{X}}\psi = 0$ since the action is via 4-plectomorphisms, and $[H, X] = 2Y$. Applying twice equation (2.1) and once equation (2.2) and using the closeness of ψ we get that $-\iota_{\widehat{X}}d\iota_{\widehat{Y}}\iota_{\widehat{H}}\psi$ equals

$$\begin{aligned} \iota_{\widehat{X}}\iota_{\widehat{Y}}d\iota_{\widehat{H}}\psi - \iota_{\widehat{X}}\mathcal{L}_{\widehat{Y}}\iota_{\widehat{H}}\psi = \iota_{\widehat{X}}\iota_{\widehat{Y}}\mathcal{L}_{\widehat{H}}\psi - \iota_{\widehat{X}}\iota_{\widehat{Y}}\iota_{\widehat{H}}d\psi \\ - \iota_{\widehat{X}}\iota_{\widehat{H}}\mathcal{L}_{\widehat{Y}}\psi - \iota_{\widehat{X}}\iota_{[\widehat{Y},\widehat{H}]}\psi = 0. \end{aligned}$$

Now by averaging over the group, which is compact, we get an invariant tri-moment map. □

Proposition 2.5. *The multiplicative action of $Sp(1)^m$ on $(\mathbb{H}^m, \psi_0)$ given by*

$$(\lambda_1, \lambda_2, \ldots, \lambda_m)(q_1, q_2, \ldots, q_m) := (\lambda_1 \cdot q_1, \lambda_2 \cdot q_2, \ldots, \lambda_m \cdot q_m)$$

is generalized Hamiltonian, and the tri-moment map is given by

$$\sigma(q_1, q_2, \ldots, q_m) = -\frac{1}{4}(|q_1|^4, |q_2|^4, \ldots, |q_m|^4) + C.$$

Proof. It is sufficient to compute the tri-moment map for $m = 1$ since each factor of $Sp(1)^m$ acts on each copy of $\mathbb{H}$ separately. We firstly determine the fundamental vector fields, starting from H, X, Y previously defined, at $q_1 = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$:

$$\begin{aligned}
\widehat{H}_{q_1} &= -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4};\\
\widehat{X}_{q_1} &= -x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} - x_2\frac{\partial}{\partial x_4};\\
\widehat{Y}_{q_1} &= -x_4\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4}.
\end{aligned}$$

Denoting by $e_{ijk} = \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$ the 3-vector $\widehat{H} \wedge \widehat{X} \wedge \widehat{Y}$ at q_1 is therefore given by

$$(-x_4 e_{123} + x_3 e_{124} + x_1 e_{234} - x_2 e_{134}) \cdot \sum_{i=1}^{4} (x_i)^2.$$

Thus, the contraction of ψ_0 along $\widehat{H}_{q_1} \wedge \widehat{X}_{q_1} \wedge \widehat{Y}_{q_1}$, is

$$\iota_{\widehat{H}_{q_1} \wedge \widehat{X}_{q_1} \wedge \widehat{Y}_{q_1}} \psi_0 = -|q_1|^2 \left(\sum_{i=1}^{4} x_i dx_i \right).$$

By the definition of the tri-moment map we get that

$$\iota_{\widehat{H}_{q_1} \wedge \widehat{X}_{q_1} \wedge \widehat{Y}_{q_1}} \psi_0(\cdot) = d\sigma_{q_1}(\cdot)(H \wedge X \wedge Y) = \sum_{i=1}^{4} \frac{\partial \sigma}{\partial x_i} dx_i$$

thus we conclude that the first component of the moment map is

$$\sigma(q_1) = -\frac{1}{4}|q_1|^4 + C_1$$

and so we get the claim. □

Note that also $(\lambda, q) \mapsto q \cdot \lambda^{-1}$ defines a multiplicative action of $Sp(1)$ on $\mathbb{H}$ which is still generalized Hamiltonian with the same tri-moment map $\sigma(q) = -|q|^4/4 + C$. Hence all the actions of the type

$$\begin{aligned}
&(\lambda_1, \lambda_2, \ldots, \lambda_m)(q_1, q_2, \ldots, q_m)\\
&:= (\lambda_1 \cdot q_1, \ldots, \lambda_k \cdot q_k, q_{k+1} \cdot \lambda_{k+1}^{-1}, \ldots, q_m \cdot \lambda_m^{-1})
\end{aligned}$$

are generalized Hamiltonian as well.

Inspired by the definition of toric symplectic manifolds we give the following

Definition 2.6. Let M be a connected, compact, $4m$-dimensional, 4-plectic manifold on which $\mathrm{Sp}(1)^m$ acts effectively in a generalized Hamiltonian fashion with discrete principal isotropy. Then M is called a *quaternionic toric manifold.*

3. Towards a convexity theorem

In this section we prove a theorem on the *sub-convexity* of the image of the tri-moment map. This can be done under some additional hypotheses on the 4-plectic form. The Darboux Theorem, that allows a local canonical expression for symplectic forms, does not hold in general for 4-plectic forms. Our assumptions in Theorem 3.2 compensate for this lack. A general convexity result can be proven for quaternionic Flag manifolds, [9], and in particular for quaternionic projective spaces, quaternionic Grassmannians and moreover for the Blow-ups of $\mathbb{HP}^m$ (see Remark 6.1). We do not know if a convexity result holds for every 4-plectic manifold acted on by $Sp(1)^k$. Recall that in Atiyah's proof of the convexity of the image of the moment map for a toric manifold, [2], a key ingredient is the fact that the moment map is a Morse-Bott function, i.e. a function with non-degenerate Hessian at the critical points. This does not hold in general for the tri-moment map; indeed for example the tri-moment map $q \mapsto -|q|^4/4$ for the action of $Sp(1)$ on $(\mathbb{H}, \psi_0)$ has degenerate Hessian at the critical point. Moreover observe that even if the components of the tri-moment map are *minimally degenerate* in the sense of Kirwan, [12], we cannot conclude that the image is convex. To see this consider the usual moment map of the complex projective space $\mathbb{CP}^2$, whose image is the standard simplex $\Delta_2 \subseteq \mathbb{R}^2$; taking the square of the components of this map (which are Morse-Bott) we get minimally degenerate functions (see p.290 in [12]), whose image is not convex, but still contained in the convex envelope of the three vertices of the simplex Δ_2.

Take (M, ψ) a 4-plectic $4m$-dimensional manifold acted on in a generalized Hamiltonian fashion by $Sp(1)^m$ with tri-moment map σ. Denote by σ_i, for $i = 1, \ldots, m$, its components. The critical set of each σ_i will be denoted by $C_i = \mathrm{Crit}\ \sigma_i$. The function σ_i is constant on each connected component of $\mathrm{Crit}\ \sigma_i$.

Consider, for each point $p \in M$, the set of linearly independent tangent vectors $\{\widehat{H}^1_p, \widehat{X}^1_p, \widehat{Y}^1_p, \ldots, \widehat{H}^m_p, \widehat{X}^m_p, \widehat{Y}^m_p\}$ where for each H^i, X^i, Y^i in the i-th term of the Lie algebra $\oplus_{i=1}^m \mathfrak{sp}(1)$ we have defined the fundamental vector

fields $\widehat{H}^i, \widehat{X}^i, \widehat{Y}^i$. We assume that it is possible to decompose the tangent space at each point $p \in M$ as the direct sum of m 4-dimensional subspaces $\{V_i\}_{i=1}^m$ with $\widehat{H}^i_p, \widehat{X}^i_p, \widehat{Y}^i_p \subseteq V_i$ in such a way that the restriction of the form ψ to V_i is non degenerate. In this case we call ψ *strongly non degenerate*. Under this assumption it is possible to define an isomorphism $L_\psi : \Lambda^3(V_i) \to V_i$ that allows to construct a basis of T_pM given by $\{\widehat{H}^i_p, \widehat{X}^i_p, \widehat{Y}^i_p, \widetilde{\delta}^i_p\}_{i=1}^m = \{v^i_1, v^i_2, v^i_3, v^i_4\}_{i=1}^m$ where $\widetilde{\delta}^i_p = L_\psi(\widehat{H}^i_p \wedge \widehat{X}^i_p, \wedge \widehat{Y}^i_p)$ for all i.

The critical points of each component of the tri-moment map σ_i can be easily found using the defining properties of σ. Indeed a critical point q of σ_i is such that

$$0 = d\sigma_i(q)(v) = \iota_{\widetilde{\delta}^i}\psi_q(v) = \psi_q(v, \widehat{H}^i_q, \widehat{X}^i_q, \widehat{Y}^i_q)$$

for all $v \in T_qM$. Therefore q is a fixed point of at least one of the 1-parameter subgroups generated by H^i, X^i or Y^i. Hence a point q in M is critical for each component σ_i if and only if the isotropy of q has maximal rank (i.e. contains a maximal torus $T \subseteq \mathrm{Sp}(1)^m$). The previous fact implies that $\cap_{i=1}^m C_i$ is a closed submanifold of the compact manifold M, and therefore it has a finite number of connected components. Since σ is constant on the connected components of $\cap_{i=1}^m C_i$, then $A = \sigma(\cap_{i=1}^m C_i)$ is a finite set. Our aim is, now, to show that the image of M via σ is contained in the convex envelope of A, $\mathrm{Conv}(A)$. The key ingredient, inspired by [12], is the following

Lemma 3.1. *Let $F = (f_1, \ldots, f_m)$ with f_i real-valued differentiable functions on a compact manifold M such that their gradient vector fields commute. Assume that* Crit f_i *is a submanifold for $i = 1, \ldots, m$. Setting $B = F(\cap_{i=1}^m \mathrm{Crit}\ f_i)$, then we have $F(M) \subseteq \mathrm{Conv(B)}$.*

Proof. In view of the Hyperplane Separation Theorem, it is sufficient to prove that, for a generic $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$, the restriction of the linear functional $y \mapsto \sum_{i=1}^m \lambda_i y_i$, to $F(M)$ takes its maximum value at a point of B. Equivalently, it is enough to prove that, for a generic $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, the maximum of the differentiable function $\varphi = \sum_{i=1}^m \lambda_i f_i$ defined on M is attained in $Z = \cap_{i=1}^m \mathrm{Crit}\ f_i$. The set Z of common critical points of $f_1, \ldots, f_m$ is also the fixed-point set of the torus T generated by the vector fields $\mathrm{grad} f_1, \ldots, \mathrm{grad} f_m$. Moreover if $\sum_{i=1}^m \lambda_i f_i$ is a generic linear combination, so that the corresponding gradient vector field generates T, then its critical set is precisely Z, and in particular φ takes its maximum on Z. □

Let us assume, as a further hypothesis, that the form ψ is zero whenever computed on at least two vectors $v^i \in V_i$, and $v^j \in V_j$ belonging to two

different 4-dimensional subspaces of the tangent space T_pM. Then we can define a diagonal metric g in terms of ψ in the following way:

$$g_p(v_j^i, L_\psi(v_t^k \wedge v_s^k \wedge v_r^k)) = \psi_p(v_j^i, v_t^k, v_s^k, v_r^k)$$

for any $i, k = 1, \dots, m$ and $j, t, s, r = 1, \dots, 4$. Hence

$$g_p(v_j^i, L_\psi(v_t^k \wedge v_s^k \wedge v_r^k)) = 0 \ \text{ for } i \neq k \ \text{ or } \ j = t, s, r.$$

Observe that the metric g is non degenerate since ψ is strongly non degenerate. Whenever the form ψ satisfies all the previous assumptions we call it *partitioned strongly non degenerate*.

With the previous notations and with an additional algebraic assumption, we can prove

Theorem 3.2. *Let (M, ψ) be a compact, connected, $4m$-dimensional manifold equipped with a partitioned strongly non degenerate 4-plectic form, acted on in generalized Hamiltonian fashion by $Sp(1)^m$ with tri-moment map $\sigma = (\sigma_1, \dots, \sigma_m)$. Suppose that $v_4^j(g_p(v_4^k, v_4^k)) = 0$ for all $j, k = 1, \dots, m$. Then $\sigma(M)$ is contained in the convex envelope of the points of $A = \sigma\,(\cap_{i=1}^m \mathrm{Crit}\ \sigma_i)$.*

Proof. In order to prove the theorem we show that $[\mathrm{grad}\sigma_i, \mathrm{grad}\sigma_j] = 0$ for all i, j. We fix a basis in $\mathbb{R}^m \cong (\Lambda^3 \mathfrak{sp}(1))^m$ given by $\delta^1, \delta^2, \dots, \delta^m$ where $\delta^i = H^i \wedge X^i \wedge Y^i$ for all i. We have that the differential of the i-th component of the tri-moment map is such that

$$\begin{aligned}(d\sigma_i)_p(v) = d\sigma_p(v)(\delta^i) &= \iota_{v_1^i}\iota_{v_2^i}\iota_{v_3^i}\psi_p(v) \\ &= \psi_p(v, v_1^i, v_2^i, v_3^i) = g_p(v, v_4^i) = g_p(v, \widetilde{\delta^i}_p)\end{aligned}$$

so that $(\mathrm{grad}\sigma_i)_p = \widetilde{\delta^i}_p$ for all $i = 1, \dots, m$, and

$$[\mathrm{grad}\sigma_i, \mathrm{grad}\sigma_j] = [\widetilde{\delta^i}, \widetilde{\delta^j}].$$

Let us now prove that $[\widetilde{\delta^i}, \widetilde{\delta^j}] = 0$, i.e. that

$$\iota_{[\widetilde{\delta^i}, \widetilde{\delta^j}]}\psi_p(a, b, c) = 0$$

for any three vectors $a, b, c \in T_pM$. First observe that if a, b, c do not belong to the same V_k in the decomposition of the tangent space or if $k \neq i, j$ then the equality is trivially true. Then, assuming that $k = i$ there are two possibilities: without loss of generality, either $a = \widetilde{\delta^i}$ or $(a, b, c) = (v_1^i, v_2^i, v_3^i)$.

The fact that g is diagonal combined with compatibility of the metric with the Levi Civita connection, gives in the first case $\iota_{[\widetilde{\delta}^i,\widetilde{\delta}^j]}\psi_p(a,b,c)=0$, while in the second $\iota_{[\widetilde{\delta}^i,\widetilde{\delta}^j]}\psi_p(a,b,c)=-\widetilde{\delta}^j g_p(\widetilde{\delta}^i,\widetilde{\delta}^i)=v_4^j(g_p(v_4^i,v_4^i))$ which vanishes by the hypothesis. We can now apply Lemma 3.1 and get the claim. □

In the particular case of a toric quaternionic manifold, we can find a bound on the cardinality of the set A. To prove this we give the following definition

Definition 3.3. Let G be a compact Lie group acting on a manifold M with principal isotropy G_{princ}. We call the *homogeneity rank* the numerical invariant of the action

$$hrk(M,G) := \operatorname{rank}(G) - \operatorname{rank}(G_{princ}) - \dim M + \dim G - \dim(G_{princ})$$

where $\operatorname{rank}(G)$ is the dimension of a maximal torus T in G.

Theorem 3.4 ([5]). *If M^T is not empty and $hrk(M,G)\leq 0$ then M^T is finite, and moreover its cardinality is equal to the Euler Characteristic of M.*

Note that for quaternionic toric manifolds the homogeneity rank is 0, thus the fixed point set M^T is finite and its cardinality $\#M^T=\chi(M)$. Using this property, the isometric actions of compact Lie groups on quaternion Kähler projective spaces with vanishing homogeneity rank have been classified in [3].

Remark 3.5. The Maximal Tori Theorem states that, in a compact Lie group, all maximal tori are conjugate; so the cardinality of the set M^T, when finite, does not depend on the chosen torus T.

In particular for quaternionic toric manifolds, endowed with a strongly non degenerate 4-plectic form, the cardinality of $A=\sigma\left(\cap_{i=1}^m \mathrm{Crit}\ \sigma_i\right)$ is bounded above an below respectively by the cardinality of the set M^T of points in M fixed by a maximal torus $T\subset Sp(1)^m$, and the cardinality of the set $M^{\mathrm{Sp}(1)^m}$ of points fixed by $\mathrm{Sp}(1)^m$.

We have therefore proved the following

Proposition 3.6. *If (M,ψ) is a quaternionic toric manifold, and ψ is strongly non degenerate, the cardinality $\#A$ of A satisfies the inequalities*

$$\#M^{\mathrm{Sp}(1)^m}\leq \#A\leq \#M^T.$$

4. From polytopes to manifolds

The introduction of the tri-moment map for the multiplicative action of $Sp(1)^n$ on $\mathbb{H}^n$ can be used to construct, starting from appropriate Delzant polytopes in $\mathbb{R}^{m*}$, real manifolds of dimension $4m$ acted on by $Sp(1)^m$ with trivial principal isotropy.

Our procedure is inspired by the Delzant construction that associates a symplectic manifold with a polytope. We recall the following, [8]

Definition 4.1. A Delzant polytope P in $\mathbb{R}^m$ is a convex polytope such that

1) P is *simple*, i.e., there are n edges meeting at each vertex;
2) P is *rational*, i.e., the edges meeting at a vertex p are rational in the sense that each edge is of the form $p + tu_i, t \geq 0$, where $u_i \in \mathbb{Z}^m$;
3) P is *smooth*, i.e., for each vertex, the corresponding $u_1, \ldots, u_m$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^m$.

In what follows, we will always consider polytopes in the dual space $\mathbb{R}^{m*}$. Let $P \subset \mathbb{R}^{m*}$ be a Delzant polytope with d *facets*, i.e. $(m-1)$-dimensional faces. Let $v_i \in \mathbb{Z}^m$ with $i = 1, \ldots, d$ be the primitive outward-pointing normal vectors to the facets. For some $\lambda_i \in \mathbb{R}$ we can write

$$P = \{x \in \mathbb{R}^{m*} | < x, v_i > \leq \lambda_i,\ i = 1, \ldots, d\}.$$

Let $\{e_1, \ldots, e_d\}$ be the standard basis of $\mathbb{R}^d$. Consider the map $\pi : \mathbb{R}^d \to \mathbb{R}^m$ defined by $e_i \mapsto v_i$. By Lemma 2.5.1 in [6] we know that the map π is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^m$. Therefore π induces a surjective map, still called π, between tori

$$\pi : \mathrm{T}^d \to \mathrm{T}^m. \tag{4.1}$$

The kernel N of π is a $(d-m)$-dimensional Lie subgroup of T^d, with inclusion map $i : N \to \mathrm{T}^d$. The exact sequence of tori

$$1 \to N \xrightarrow{i} \mathrm{T}^d \xrightarrow{\pi} \mathrm{T}^m \to 1$$

induces an exact sequence at the Lie algebra level

$$0 \to \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^m \to 0$$

with dual exact sequence

$$0 \to (\mathbb{R}^m)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \to 0.$$

Now consider $\mathbb{H}^d$ with the 4-plectic form ψ_0 and the standard generalized Hamiltonian action of $Sp(1)^d$. In Proposition 2.5 we have computed the tri-moment map σ for this action. The subtorus $N = (S^1)^{d-m}$ acts on $\mathbb{H}^d$. Our procedure works whenever the action of N on $\mathbb{H}^d$ extends to $\widehat{N} = Sp(1)^{d-m} \cong (S^3)^{d-m}$.

4.1. Extendibility of the action of N

Due to the non-commutativity of quaternions, it is not always possible to extend the action of the subtorus N. In fact the only way to define a multiplicative action of $Sp(1)^{d-m}$ on $\mathbb{H}^d$ is the following

$$(h_1, \dots, h_{d-m})(q_1, \dots, q_d) = (h_{k_1}^{\alpha_1} q_1 h_{j_1}^{\beta_1}, \dots, h_{k_d}^{\alpha_d} q_d h_{j_d}^{\beta_d})$$

with $k_\ell, j_\ell \in \{1, \dots, d-m\}$ and $\alpha_\ell \in \{0, 1\},\ \beta_\ell \in \{0, -1\}$.

We want to collect some necessary and sufficient conditions under which the action of N can be extended to an action of $\widehat{N}$ on $\mathbb{H}^{d-m}$. We state these conditions in terms of the basis of the Lie algebra $\mathfrak{n}$ of N.

Proposition 4.2. *A necessary condition for the action of N to be extendable to $\widehat{N}$ is that there exists a basis $\mathcal{B} = \{b_1, \dots, b_{d-m}\}$ of $\mathfrak{n} \subset \mathbb{R}^d$ such that the $d \times (d-m)$ matrix $A_{\mathcal{B}} = (b_1 | \dots | b_{d-m})$, whose columns are $b_1, \dots, b_{d-m}$, has the following properties:*

1) its only possible entries are $1, -1$ *and* 0*;*

2) in each of its rows at most two entries are not zero.

In this case we say that $\mathfrak{n}$ admits a reduced *basis.*

Proof. Suppose that $d - m \geq 3$ and that there exists no basis of $\mathfrak{n}$ satisfying (2). Then for any $\mathcal{B}$ there exists at least one row of $A_{\mathcal{B}}$ that has (at least) 3 entries which are non-vanishing. This means that one of the defining equations of $\mathfrak{n}$ is of the form $x_i = \alpha x_j + \beta x_k + \gamma x_l$. In terms of $\widehat{N}$ this becomes a multiplicative relation involving 4 unitary quaternions $h_i = h_j^\alpha h_k^\beta h_l^\gamma$, and this equation does not allow to define a multiplicative action of $Sp(1)^{d-m}$ on $\mathbb{H}^d$ since at least two (not commuting) factors have to be placed on the same side of q_i.

Suppose now that for any basis satisfying (2) condition (1) is not fulfilled. This means that one of the defining equations of $\mathfrak{n}$ is of the form $x_i = \alpha x_j + \beta x_k$ with α or β different from $0, 1, -1$. As before we have a relation between elements of $\widehat{N}$ of the form $h_i = h_j^{\alpha} h_k^{\beta}$ which does not define a multiplicative action. □

A basis satisfying condition (1) exists if and only if the starting polytope facets form angles which are multiple of $\frac{\pi}{4}$, otherwise entries different from 1 or -1 necessarily occur.

The necessary condition given in the previous proposition is not sufficient, indeed

Example 4.3. Assume that $m = 2$, $d = 4$ and consider the polytope whose normals are $\{(-1, 0), (0, -1), (1, 1,), (-1, 1)\}$. The matrix $A_{\mathcal{B}}$ is given by a 4×2 matrix whose columns are $(1, 1, 1, 0)$ and $(-1, 1, 0, 1)$ and correspondingly $(h_1, h_2)(q_1, q_2, \cdot, \cdot) = (h_1 q_1 h_2, h_1 q_2 h_2^{-1}, \cdot, \cdot)$ which does not define an action of $Sp(1)^2$ on $\mathbb{H}^4$.

The previous fact is general, indeed by direct computation we can prove

Theorem 4.4. *Let P be a Delzant polytope in $\mathbb{R}^{m*}$ with d facets, let $\pi : T^d \to T^m$ be defined as in (4.1) and let $N = \ker \pi$. The action of N can be extended to an action of $\widehat{N}$ on $\mathbb{H}^d$ if and only if one can find a reduced basis $\mathcal{B}$ of the Lie algebra $\mathfrak{n}$ of N in such a way that the matrix $A_{\mathcal{B}}$ does not contain a sub matrix of the form*

$$\mathcal{M} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Theorem 4.4 applies in particular to polytopes obtained by cutting the standard simplex by means of hyperplanes parallel to the facets of the simplex, but these do not cover all the polytopes for which the action of N can be extended, indeed

Example 4.5. Consider the polytope in $\mathbb{R}^{3*}$ whose primitive outward-pointing normals to the facets are

$$\{(0, -1, 0), (0, 0, -1), (0, 1, 0), (-1, 0, 0), (1, 1, 0), (1, 1, 1)\}.$$

In this case, the vector $(1, 1, 0)$ is not orthogonal to any facet of the simplex, but a reduced basis for N given by

$$\mathcal{B} = \{(1, 0, 1, 0, 0, 0), (0, 1, 0, 0, -1, 1), (1, 0, 0, 1, 1, 0)\}$$

is such that $A_{\mathcal{B}}$ does not contain the matrix $\mathcal{M}$. Hence, thanks to Theorem 4.4, the action of N can be extended. For any $n = (h_1, h_2, h_3) \in \widehat{N}$ and $q = (q_1, \dots, q_6)$

$$n \cdot q := (h_1 q_1 h_3^{-1}, q_2 h_2^{-1}, h_1 q_3, q_4 h_3^{-1}, h_2 q_5 h_3^{-1}, q_6 h_2^{-1}). \tag{4.2}$$

4.2. Towards the construction of the manifold

Let P be the Delzant polytope defined by

$$P = \{x \in \mathbb{R}^{m*} | < x, v_i > \leq \lambda_i,\ i = 1, \dots, d\}.$$

From now on we will assume that the action of N can be extended to $\widehat{N}$. The tri-moment map for the action of $\widehat{N} \cong Sp(1)^{(d-m)}$ on $\mathbb{H}^d$ is given by $i^* \circ \sigma$ where $\sigma(q_1, q_2, \dots, q_d) = -\frac{1}{4}(|q_1|^4, |q_2|^4, \dots, |q_d|^4) + C$, for some $C = (C_1, \dots, C_d)$, is the tri-moment map for the standard action of $\mathrm{Sp}(1)^d$ on $\mathbb{H}^d$ and

$$i^* : \mathbb{R}^{d^*} \cong ((\Lambda^3 \mathfrak{sp}(1))^d)^* \to ((\Lambda^3 \mathfrak{sp}(1))^{d-m})^* \cong \mathbb{R}^{(d-m)^*}$$

is the dual of the inclusion map. Now we choose the constant C to be $(\lambda_1, \dots, \lambda_d)$.

Lemma 4.6. *Let $Z = (i^* \circ \sigma)^{-1}(0)$. Then Z is compact and, if we assume that the action of N extends to $\widehat{N}$, the action of $\widehat{N}$ on Z is free.*

The proof of the first part of the statement is the same as in the symplectic case (see, e.g., [6]). We present it here for the convenience of the reader.

Proof. It is sufficient to show that Z is bounded, since it is clearly closed. Let $P' = \pi^*(P)$. We want to show that $\sigma(Z) = P'$. Observe that if $y \in \mathbb{R}^{d^*}$ then $y \in P'$ if and only if y is in the image of Z via σ. Indeed, we have that $y \in \sigma(Z)$ if and only if

- y is in the image of σ;
- $i^*(y) = 0$.

Now we use the expression of σ and the fact that $\mathrm{Im}\, \pi^* = \ker i^*$, so we have that these two conditions are equivalent to

1) $< y, e_\ell >= y_\ell = -\frac{1}{4}|q_\ell|^4 + \lambda_\ell \leq \lambda_\ell$ for all $\ell = 1, \dots, d$;
2) $y = \pi^*(x)$ for some $x \in \mathbb{R}^{m*}$.

From this we have, for all ℓ,

$$< y, e_\ell >\leq \lambda_\ell \iff < \pi^*(x), e_\ell >\leq \lambda_\ell \iff$$
$$< x, \pi(e_\ell) >\leq \lambda_\ell \iff < x, v_\ell >\leq \lambda_\ell \iff x \in P$$

thus $y \in \sigma(Z) \iff y \in \pi^*(P)$. Thanks to the properness of σ and the compactness of P' we get that Z is bounded, and therefore compact.

A key ingredient to prove the freeness of the action of $\widehat{N}$ on Z is the fact that N acts freely on Z, as proven in Proposition 29.1 in [6]. Consider a vertex $p \in P$ and let $I = \{i_1, \ldots, i_m\}$ be the set of indices for the m facets meeting at p. Pick $q \in Z$ such that $\sigma(q) = \pi^*(p)$. Then p is characterized by m equations $< p, v_\ell >= \lambda_\ell$ with $\ell \in I$. Now

$$< p, v_\ell >= \lambda_\ell \iff < p, \pi(e_\ell) >= \lambda_\ell \iff < \pi^*(p), e_\ell >= \lambda_\ell$$
$$\iff < \sigma(q), e_\ell >= \lambda_\ell \iff -\frac{1}{4}|q_\ell|^4 + \lambda_\ell = \lambda_\ell \iff q_\ell = 0.$$

Hence those $q \in Z$ wich are sent to vertices in P are points whose coordinates in the set I are zero. Without loss of generality, we can assume that $I = \{1, \ldots, m\}$ so that the stabilizer of q is $(Sp(1)^d)_q = \{(h_1, \ldots, h_m, 1, \ldots, 1) \in Sp(1)^d\} \cong Sp(1)^m$.

The group $\widehat{N} \cong Sp(1)^{d-m}$ acts on Z and its action on non-zero coordinates is of the form $h_{k_\ell}^{\alpha_\ell} q_\ell h_{j_\ell}^{\beta_\ell}$ where $\alpha_\ell = 0$ or 1 and $\beta_\ell = 0$ or -1. Each h_{k_ℓ}, h_{j_ℓ} for $k_\ell, j_\ell \in \{1, \ldots, m\}$ acts on at least a non-zero coordinate of q, since otherwise N would not act freely on Z.

If the action on the ℓ-th coordinate of q is $q_\ell \mapsto h_{k_\ell} q_\ell h_{j_\ell}^{-1}$ we say that (k_ℓ, j_ℓ) form a *couple*. We use all the couples to construct a graph Γ with m distinct vertices labeled $1, \ldots, m$ in the standard way. We want to show that each connected component of Γ contains a vertex s such that h_s acts as a simple multiplication on a non-zero coordinate of q. On the contrary suppose that there is a connected component of Γ whose vertices are $V_1, \ldots, V_t$ such that $h_{V_1}, \ldots, h_{V_t}$ do not act as simple multiplication on any coordinate of q. This implies that setting $h_{V_1} = \cdots = h_{V_t} = -1$ identifies a non-trivial element in the stabilizer of q in N, contradicting the freeness of the action of N on Z. This allows us to conclude that for each connected component of Γ at least a vertex s is such that h_s acts as a simple multiplication on a non-zero coordinate of q. Therefore $h_s = 1$ and the same holds true for all h_{V_i} in the same connected component. Note that for all other $q' \in Z$, which are not sent to vertices, the stabilizer is necessarly smaller and therefore trivial. □

Thanks to the previous lemma we can consider the orbit space

$$M_P = Z/\widehat{N}$$

which is a compact manifold of (real) dimension

$$\dim Z - 3(d-m) = 4d - (d-m) - 3(d-m) = 4m.$$

In the complex setting the group N is normal in $(S^1)^d$ so that one can directly define the group G that acts on Z/N as the quotient group $(S^1)^d/N$. In the present setting we have to define appropriately the group $\widehat{G}$ as a $3m$-dimensional group, isomorphic to $Sp(1)^m$, acting on $Z/\widehat{N}$. And this is not always possible. In particular, this can be done when $\widehat{N}$ acts by simple multiplication (either on the right or on the left) on at least m coordinates in $\mathbb{H}^d$. We now state some conditions under which the action of $\widehat{G}$ is effective and has trivial isotropy.

Proposition 4.7. *If a polytope in $\mathbb{R}^{m*}$ is obtained by cutting the standard simplex by means of hyperplanes parallel to the facets of the simplex, then the group $\widehat{G}$ can be defined in such a way that its action on $Z/\widehat{N}$ is effective and with trivial isotropy.*

Proof. Since we start from the simplex in $\mathbb{R}^{m*}$, the $m+1$ oriented vectors orthogonal to the facets are given by $\{-e_1, -e_2, \ldots, -e_m, e_1 + e_2 + \cdots + e_m\}$. If we cut our simplex with $m+1$ hyperplanes parallel to its facets, we can get a polytope with $d = 2m+2$ facets. The matrix associated with π is an $m \times (2m+2)$ matrix given by $(-Id_m, C_{m+1}, -C_{m+1}, Id_m)$ where C_{m+1} is the column vector $e_1 + e_2 + \cdots + e_m$ and Id_m is the identity matrix of order m. The kernel $\mathfrak{n} \subseteq \mathbb{R}^d$ of π is defined by the linear system of m equations

$$\begin{cases} x_1 = x_{m+1} - x_{m+2} + x_{m+3} \\ x_2 = x_{m+1} - x_{m+2} + x_{m+4} \\ \cdots \\ x_m = x_{m+1} - x_{m+2} + x_{2m+2} \end{cases} \tag{4.3}$$

We now choose the new parameters $x'_{m+1} = x_{m+2} - x_{m+1}, x_{m+2}, \ldots, x_{2m+2}$ instead of $x_{m+1}, x_{m+2}, \ldots, x_{2m+2}$ and we get a basis $\mathcal{B}$ for $\mathfrak{n}$, given by the $m+2$ vectors

$$\{-f_1 - f_2 - \cdots - f_m + f_{m+1}, f_{m+2}, f_1 + f_{m+3}, f_2 + f_{m+4}, \ldots, f_m + f_{2m+2}\}$$

where f_i are the elements of the standard basis in $\mathbb{R}^{2m+2}$. This basis is reduced and the corresponding $A_{\mathcal{B}}$ does not contain the sub matrix $\mathcal{M}$. Thanks to Proposition 4.4 we can extend the action of N to $\widehat{N}$ on $\mathbb{H}^{2m+2}$. For a generic element $n=(h_{m+1}, h_{m+2}, \ldots, h_{2m+2})$ in $\widehat{N}$ and $q=(q_1, q_2, \ldots, q_{2m+2})$ in $\mathbb{H}^{2m+2}$ we have

$$\begin{aligned} n \cdot q := (&h_{m+3} q_1 h_{m+1}^{-1}, \ldots, h_{2m+2} q_m h_{m+1}^{-1}, \\ &h_{m+2} q_{m+1} h_{m+1}^{-1}, h_{m+2} q_{m+2}, \ldots, h_{2m+2} q_{2m+2}); \end{aligned}$$

Then we say that a generic element $g = (g_1, g_2, \ldots, g_m)$ of $\widehat{G} \subseteq Sp(1)^d$ acts on the orbit $(\widehat{N} \cdot q)$ as

$$\begin{aligned} g \cdot (\widehat{N} \cdot q) := (&h_{m+3} q_1 h_{m+1}^{-1}, \ldots, h_{2m+2} q_m h_{m+1}^{-1}, h_{m+2} q_{m+1} h_{m+1}^{-1}, \\ &h_{m+2} q_{m+2} g_1^{-1}, \ldots, h_{2m+1} q_{2m+2} g_m^{-1}, h_{2m+2} q_{2m+2}). \end{aligned}$$

It is clear that $\widehat{G}$ takes $\widehat{N}$-orbits to $\widehat{N}$-orbits. Moreover its action is effective and has trivial principal isotropy. Indeed suppose the group $\widehat{G}$ fixes an orbit through a point $\widetilde{q}$, whose coordinates are all non-vanishing, thus from the relation $h_{2m+2}\widetilde{q}_{2m+2} = \widetilde{q}_{2m+2}$ we deduce that $h_{2m+2} = 1$ and consequently that $h_{m+1} = h_{m+2} = \cdots = h_{2m+2} = 1$. To conclude note that the previous argument (slightly modified) can be applied also if we cut by means of less than $m+1$ hyperplanes. □

Note that Example 4.5 shows that the assumption in Proposition 4.7, on the way the simplex is cut to construct the polytope, is not necessary to define an action of $\widehat{G}$ on the quotient manifold. Indeed in this case the action of $\widehat{G}$ given by

$$g \cdot \widehat{N} q := (h_1 q_1 h_3^{-1}, g_1 q_2 h_2^{-1}, h_1 q_3 g_2^{-1}, g_3 q_4 h_3^{-1}, h_2 q_5 h_3^{-1}, q_6 h_2^{-1})$$

has trivial principal isotropy (and hence is effective).

4.3. Examples of the procedure

We now collect some useful examples of manifolds constructed starting from a given polytope.

Example 4.8. *Starting from the standard simplex Δ_m in $\mathbb{R}^{m*}$ we obtain the quaternionic projective space $\mathbb{HP}^m$. Indeed, the kernel $\mathfrak{n}$ of $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$*

is defined by the linear system

$$\begin{cases} x_1 = x_{m+1} \\ x_2 = x_{m+1} \\ \cdots \\ x_m = x_{m+1} \end{cases} \tag{4.4}$$

So the action of $\widehat{N}$ *is given by*

$$n \cdot q := (hq_1, \ldots, hq_{m+1}),$$

and the corresponding tri-moment map is

$$i^* \circ \sigma(q) := -\frac{1}{4} \sum_{i=1}^{m+1} |q_i|^4 + 1.$$

Thus $Z = (i^* \circ \sigma)^{-1}(0)$ *is diffeomorphic to the* $(4m+3)$*-dimensional sphere in* $\mathbb{R}^{4m+4}$*. Recalling the classical Hopf fibration* $\widehat{N} \cong S^3 \to S^{4m+3} \to \mathbb{HP}^m$*, we get that* $M_{\Delta_m} = \mathbb{HP}^m$*. The action of* $\widehat{G} \cong Sp(1)^m$ *can be defined on* $\widehat{N}$*-orbits as*

$$g \cdot \widehat{N} q = (hq_1 g_1^{-1}, \ldots, hq_m g_m^{-1}, hq_{m+1})$$

and it has clearly trivial isotropy (and hence it is effective).

Example 4.9. *If the polytope* P *is a square* $[0,1] \times [0,1] \in \mathbb{R}^{2^*}$ *the corresponding manifold* M_P *is* $\mathbb{HP}^1 \times \mathbb{HP}^1$*. Indeed in this case* $d = 4$ *the kernel* $\mathfrak{n}$ *of* $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ *is defined by the linear system*

$$\begin{cases} x_1 = x_3 \\ x_2 = x_4 \end{cases} \tag{4.5}$$

So the action of $\widehat{N}$ *is given by*

$$n \cdot q := (h_1 q_1, h_2 q_2, h_1 q_3, h_2 q_4),$$

and the corresponding tri-moment map is

$$i^* \circ \sigma(q) := \left(-\frac{1}{4}(|q_1|^4 + |q_3|^4) + 1, -\frac{1}{4}(|q_2|^4 + |q_4|^4) + 1 \right).$$

Thus $Z = (i^* \circ \sigma)^{-1}(0,0)$ *is diffeomorphic to the product of two* 7*-dimensional spheres in* $\mathbb{R}^8$ *on which* $\widehat{N} \cong Sp(1)^2$ *acts separately on each factor. Thus, using again the Hopf fibration* $S^3 \to S^7 \to S^4$*, we find that the quotient space is*

the above mentioned product. The action of $\widehat{G} \cong Sp(1)^2$ *which can be defined on* $\widehat{N}$*-orbits as*

$$g \cdot \widehat{N}q = (h_1 q_1 g_1^{-1}, h_2 q_2 g_2^{-1}, h_1 q_3, h_2 q_4),$$

has clearly trivial isotropy (and hence is effective).

This procedure can be naturally generalized to prove that, starting from the m*-dimensional cube* $[0,1] \times \cdots \times [0,1] \in \mathbb{R}^{m*}$*, we find the product of* m*-copies of* $\mathbb{HP}^1$ *acted on by* $\widehat{G} \cong Sp(1)^m$ *whose action is defined on* $\widehat{N} \cong Sp(1)^m$*-orbits as*

$$g \cdot \widehat{N}q = (h_1 q_1 g_1^{-1}, h_2 q_2 g_2^{-1}, \ldots, h_m q_m g_m^{-1}, h_1 q_{m+1}, \ldots, h_m q_{2m}).$$

Note that the m*-dimensional cube can be obtained by cutting the standard simplex by means of* m *hyperplanes parallel to the coordinate hyperplanes of* $\mathbb{R}^{m*}$*.*

Example 4.10. *We here start from the trapezoid* T *in* $\mathbb{R}^{2^*}$*, defined by*

$$T = \{x \in \mathbb{R}^{2^*} | < x, v_1 > \leq 0, < x, v_2 > \leq 0, < x, v_3 > \leq 1, < x, v_4 > \leq 2\}$$

with $v_1 = -e_2, v_2 = -e_1, v_3 = e_2$ *and* $v_4 = e_1 + e_2$*. The kernel* $\mathfrak{n}$ *of* $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ *is defined by the linear system*

$$\begin{cases} x_1 = x_3 + x_4 \\ x_2 = x_4 \end{cases} \tag{4.6}$$

so the action of $\widehat{N} \cong Sp(1)^2$ *is given by*

$$n \cdot q := (h_1 q_1 h_2^{-1}, q_2 h_2^{-1}, h_1 q_3, q_4 h_2^{-1})$$

and the corresponding tri-moment map is

$$i^* \circ \sigma(q) := \left(-\frac{1}{4}(|q_1|^4 + |q_3|^4) + 1, -\frac{1}{4}(|q_1|^4 + |q_2|^4 + |q_4|^4) + 2 \right).$$

Thus $Z = (i^* \circ \sigma)^{-1}(0,0)$ *is is given by*

$$\begin{cases} |q_1|^4 + |q_3|^4 = 4 \\ |q_1|^4 + |q_2|^4 + |q_4|^4 = 8. \end{cases}$$

It is not difficult to show that the orbit space $M_T = Z/\widehat{N}$ *is a (non-trivial)* $\mathbb{HP}^1$*-bundle on* $\mathbb{HP}^1$ *and hence coincides with the blow-up of* $\mathbb{HP}^2$ *at a point.*

The action of $\widehat{G} \cong Sp(1)^2$ *which can be defined on* $\widehat{N}$*-orbits as*

$$g \cdot \widehat{N}q = (h_1 q_1 h_2^{-1}, g_1 q_2 h_2^{-1}, h_1 q_3 g_2^{-1}, q_4 h_2^{-1})$$

has clearly trivial isotropy (and hence is effective).

As proved in, e.g., [11], the blow-up of $\mathbb{HP}^2$ at a point, i.e. the manifold M_T, is the connected sum $\mathbb{HP}^2 \# \overline{\mathbb{HP}^2}$ where the symbol $\overline{\mathbb{HP}^2}$ denotes the quaternionic projective space with the reversed orientation.

The trapezoid T is obtained via a cut by means of a hyperplane (a straight line) of the standard simplex Δ_2, and the corresponding manifold M_T is the blow-up at a point of $M_{\Delta_2} \cong \mathbb{HP}^2$. The fact that cutting certain polytopes corresponds to blowing-up the associated manifolds is indeed general as shown in the following section.

Remark 4.11. Note that if one considers the trapezoid whose vertices are $(0,0); (l+1,0); (0,1); (1,1)$ the action of N cannot be extended to $\widehat{N} \cong Sp(1)^2$ for $l > 1$. Indeed in this case the kernel $\mathfrak{n}$ of $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ is spanned by the vectors $\{(1,0,1,0), (1,l,0,1)\}$ and does not admit a basis $\mathcal{B}$ such that condition (1) in Proposition 4.2 is satisfied. Therefore we cannot apply the procedure starting from this Delzant polytope.

5. 4-plectic reduction and 4-plectic cut

Starting from a symplectic manifold (M, ω) acted on by a compact Lie group K in a Hamiltonian fashion with moment map μ, the Marsden-Weinstein reduction is a tool that permits to equip the manifold $\mu^{-1}(x)/K$, when $x \in \mathfrak{k}^*$ ($\mathfrak{k}$ is the Lie algebra of K) is a regular value of μ, with a symplectic form ω_{red} (see e.g. [16]).

In the 4-plectic setting the procedure holds under stronger hypotheses. Indeed, let (M^{4m}, ψ) be a 4-plectic manifold on which the group $G = Sp(1)^m$ acts with tri-moment map σ. Let x be a regular value in $\mathbb{R}^{m*}$ for the map σ. Consider the G-invariant and smooth $Z_x = \sigma^{-1}(x)$. The stabilizers of points in Z_x form a group bundle over it, which we assume to be smooth. We say that these stabilizers form a *spheroid bundle* if they are isomorphic to the product of copies of Sp(1). Then the quotient space $Y_x := Z_x/G$, usually known as *reduced space* is a smooth manifold. We say that the 4-plectic form ψ is *horizontal* on Z_x if and only if the contraction of ψ along every

fundamental vector field $\hat{\beta}$ (for each $\beta \in \mathfrak{g}$) is zero along Z_x. Formally

$$\iota_{\hat{\beta}} \psi_{|_{Z_x}} = 0 \quad \forall \beta \in \mathfrak{g}.$$

With this notation we recall

Theorem 5.1 (Theorem 3.1 [9]). *Let (M^{4m}, ψ) be a 4-plectic manifold on which the group $G = Sp(1)^m$ acts with tri-moment map σ and let x be a regular value of σ. Assume that the stabilizers of all points in Z_x form a smooth spheroid bundle over Z_x, and that ψ is horizontal. Then the* reduced space *Y_x is a smooth manifold admitting a 4-plectic form ψ_{red}, such that*

$$\pi^*(\psi_{red}) = i^*(\psi)$$

where $\pi := Z_x \to Y_x$ and $i := Z_x \to M$ denote respectively the projection and the inclusion map.

Example 5.2. Let $X = \mathbb{H}^2$ with the standard diagonal action of Sp(1) and let the 4-form ψ_h be given by

$$\psi_h = d(|q_1|^4 - |q_2|^4) \wedge d(\alpha_1 - \alpha_2) \wedge d(\beta_1 - \beta_2) \wedge d(\gamma_1 - \gamma_2),$$

where $q_r = x_r + \alpha_r i + \beta_r j + \gamma_r k$ for $r := 1, 2$. The form ψ_h is horizontal. The reduced space is isomorphic to $\mathbb{HP}^1$, with an appropriate 4-plectic form ψ_{red}. Similarly we can obtain an invariant 4-plectic form on $\mathbb{HP}^m$ that we will denote again by ψ_{red}, [9].

We would like to define a 4-plectic analog of blowing-up M at a point p. In [11] we have proved that if M is a regular quaternionic manifold of real dimension $4m$, then the blow-up of M at one point is a $4m$-dimensional regular quaternionic manifold which is diffeomorphic to $M \# \mathbb{HP}^m$. Note that topologically $M \# \mathbb{HP}^m$ can be obtained by removing a ball centered at p and then collapsing the boundary S^{4m-3} along the fibers of the Hopf fibration $S^3 \to S^{4m-3} \to \mathbb{HP}^m$. The generalization in the 4-plectic setup of the so-called symplectic cutting due to E. Lerman is a particular case of the quaternionic blow-up. Let (M^{4m}, ψ) be a 4-plectic manifold equipped with a generalized Hamiltonian $Sp(1)^m$-action. Consider the restricted diagonal $Sp(1)$-action, and let $h : M \to \mathbb{R}$ be the corresponding tri-moment map. Let ε be a regular value of h. Assume that the $Sp(1)$-action on $h^{-1}(\varepsilon)$ is free.

We introduce the following notations: we denote by $M_{h>\varepsilon}, M_{h\geq\varepsilon}$ the pre-images of (ε, ∞) and $[\varepsilon, \infty)$ under $h : M \to \mathbb{R}$, and denote by $\overline{M_{h\geq\varepsilon}}$ the

manifold which is obtained by collapsing the boundary $h^{-1}(\varepsilon)$ of $M_{h\geq\varepsilon}$ along the orbits of the $Sp(1)$-action. The manifold $\overline{M_{h\geq\varepsilon}}$ can be therefore identified with the blow-up of M at a point and is called *the* 4*-plectic cut* of M.

Theorem 5.3. *Let (M^{4m},ψ) be a 4-plectic manifold equipped with a generalized Hamiltonian $Sp(1)^m$-action. Consider the restricted action of a single $Sp(1)$, and let $h : M \to \mathbb{R}$ be the corresponding tri-moment map. Let ε be a regular value of h. We assume that the $Sp(1)$-action on $h^{-1}(\varepsilon)$ is free. Whenever the form $\psi \oplus \psi_0$ on $M \times \mathbb{H}$ is horizontal along $(h - \frac{1}{4}|q|^4)^{-1}(\varepsilon)$, there is a natural 4-plectic structure Ψ_ε on $\overline{M_{h\geq\varepsilon}}$ such that the restriction of Ψ_ε to $M_{h>\varepsilon} \subseteq \overline{M_{h\geq\varepsilon}}$ equals ψ .*

Proof. Consider the product $(M \times \mathbb{H}, \Phi = \psi \oplus \psi_0)$ and the Hamiltonian $Sp(1)$-action

$$\lambda(m,q) = (\lambda(m), \lambda q); \quad \lambda \in Sp(1),\ m \in M,\ q \in \mathbb{H}.$$

The tri-moment map is

$$F(m,q) = h(m) - \frac{1}{4}|q|^4.$$

Observe the following identification

$$F^{-1}(\varepsilon) = \{(m,q)\ |h(m) > \varepsilon, |q|^2 = 2\sqrt{(h(m)-\varepsilon)}\} \cup \{(m,0)|h(m) = \varepsilon\}$$

$$= M_{h>\varepsilon} \times S^3 \cup h^{-1}(\varepsilon).$$

So that

$$F^{-1}(\varepsilon)/Sp(1) = \left(M_{h>\varepsilon} \times S^3 \cup h^{-1}(\varepsilon)\right)/Sp(1) = \overline{M_{h\geq\varepsilon}}$$

since a fundamental set in $M_{h>\varepsilon} \times S^3$ for the action of $Sp(1)$ is given by $M_{h>\varepsilon} \times \{1\}$. The assumption on the horizontality of Φ implies that we can apply Theorem 5.1, and equip $F^{-1}(\varepsilon)/Sp(1)$ with a 4-plectic structure that equals ψ when restricted to the open submanifold $M_{h>\varepsilon}$. □

Definition 5.4. Let M be a $4m$-dimensional manifold obtained as a reduced space from the 4-plectic manifold (N,ψ_h) acted on in a generalized Hamiltonian fashion by a group $Sp(1)^k$, with tri-moment map σ, and let ψ_h be horizontal on $\sigma^{-1}(x)$ for some regular value x of σ. Then M equipped with the natural 4-plectic form ψ^x_{red} such that $\pi^*(\psi^x_{red}) = i^*(\psi_h)$, is said to be *obtained by reduction.*

Those manifolds M obtained by reduction are special: as an application of Theorem 5.3 we will see that any 4-plectic cut of such an M can be equipped with a canonical 4-plectic structure.

5.1. 4-plectic form on the blow-up

We here do explicit calculations for the case $M = \mathbb{HP}^2$, using the notation established in the proof of Theorem 5.3.

We want to prove that the form $\Phi = \psi^{\varepsilon}_{red} \oplus \psi_0$ on $\mathbb{HP}^2 \times \mathbb{H}$ is horizontal on $F^{-1}(\varepsilon)$ so that, applying Theorem 5.3, we can equip the blow-up of $\mathbb{HP}^2$ with a 4-plectic form. More precisely we show that $\iota_{\hat{\beta}}\Phi|_{F^{-1}(\varepsilon)} = 0$, for all $\beta \in \mathfrak{sp}(1)$.

First note that the contraction of the form Φ along $\hat{\beta}_q$ for $q \in S^3$ is zero. Indeed for example, using the standard basis of $\mathfrak{sp}(1)$, if $\beta = H$, then

$$\widehat{H}_q = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}$$

and the contraction of Φ along it is therefore given by

$$\begin{aligned}\iota_{\widehat{H}_q}\Phi = \iota_{\widehat{H}_q}\psi_0 = {} & x_2 dx_2 \wedge dx_3 \wedge dx_4 + x_1 dx_1 \wedge dx_3 \wedge dx_4 \\ & - x_3 dx_1 \wedge dx_2 \wedge dx_3 - x_4 dx_1 \wedge dx_2 \wedge dx_4.\end{aligned}$$

The claim follows since

$$S^3 = \{q = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H} \mid |q|^2 = |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 = c\}$$

and thus $x_1 dx_1 = -\sum_{i=2}^{4} x_i dx_i$ on it. Analogously for $\widehat{X}_q$ and $\widehat{Y}_q$ we have on S^3, $\iota_{\widehat{X}_q}\Phi = \iota_{\widehat{Y}_q}\Phi = 0$. In [9, p. 337] it is proven that $(\mathbb{HP}^2, \psi^{\varepsilon}_{red})$ can be obtained via reduction from $(\mathbb{H}^3, \psi_h)$, acted on by $Sp(1)$ with tri-moment map σ, where ψ_h is horizontal.

Note that $Sp(1)^2$ acts on $\sigma^{-1}(\varepsilon)$ and on $\mathbb{HP}^2$. Here we use the fact that the projection $\pi : \sigma^{-1}(\varepsilon) \subseteq \mathbb{H}^3 \to \sigma^{-1}(\varepsilon)/Sp(1)$ is $Sp(1)^2$-equivariant. By the equivariance, for every $m \in \sigma^{-1}(\varepsilon)$

$$\pi_*(\hat{\beta}_m) = \hat{\beta}_{\pi(m)}.$$

Take a point $p = \pi(m)$ in $\mathbb{HP}^2 = \sigma^{-1}(\varepsilon)/Sp(1)$,

$$\iota_{\hat{\beta}_p}\Phi|_{F^{-1}(\varepsilon)} = \iota_{\hat{\beta}_p}\psi^{\varepsilon}_{red}|_{\mathbb{HP}^2_{\sigma\geq\varepsilon}} = \iota_{\hat{\beta}_{\pi(m)}}\psi^{\varepsilon}_{red}|_{\mathbb{HP}^2_{\sigma\geq\varepsilon}} = \iota_{\pi_*(\hat{\beta}_m)}\psi^{\varepsilon}_{red}|_{\mathbb{HP}^2_{\sigma\geq\varepsilon}}.$$

The tangent space of $\mathbb{HP}^2_{\sigma>\varepsilon}$ at $p = \pi(m)$ corresponds via π_* to a subspace of the tangent space of $\sigma^{-1}(\varepsilon)$ at m. Now

$$\iota_{\pi_*(\hat{\beta}_m)}\psi^{\varepsilon}_{red} = \iota_{(\hat{\beta}_m)}\pi^*\psi^{\varepsilon}_{red} = \iota_{(\hat{\beta}_m)}\psi_h$$

since the contraction of ψ^{ε}_{red} along $\pi_*(\hat{\beta}_m)$ is given by the pull back

$$\pi^*(\psi^{\varepsilon}_{red}) = i^*\psi_h.$$

The fact that ψ_h is appropriately chosen horizontal on $\sigma^{-1}(\varepsilon)$ implies that $\iota_{(\hat{\beta}_m)}\psi_h = 0$. We can give the following general definition.

Definition 5.5. A 4-plectic manifold M, obtained by reduction from (N, ψ_h) with tri-moment map $\sigma : N \to \mathbb{R}^{k*}$ for the action of $G = Sp(1)^k$, is said to be obtained by an $Sp(1)^m$-*equivariant* reduction if the projection $\pi : \sigma^{-1}(x) \to \sigma^{-1}(x)/G$ is $Sp(1)^m$-equivariant.

With the above notations, with the same argument used for $M = \mathbb{HP}^2$, it is not difficult to prove

Theorem 5.6. *Let M be a 4-plectic manifold obtained via an $Sp(1)^m$-equivariant reduction from (N, ψ_h). Then the blow-up of M at a point can be endowed with a 4-plectic form reduced from the horizontal form ψ_h.*

In [9] it is proven that all quaternionic flag manifolds can be obtained by equivariant reduction. Therefore the previous theorem can be applied to this class of examples, showing that it is possible to equip their blow-up with a 4-plectic structure.

5.2. Polytopes vs quaternionic toric manifolds

Let $P \subseteq \mathbb{R}^{m*}$ be obtained by cutting the standard simplex Δ_m with $d - m - 2$ hyperplanes parallel to the original facets (in order to have $d - 1$ facets), and let M_P be the corresponding manifold. We prove here that if we cut another time P with a hyperplane parallel to one of its facets then the manifold that we get is the blow-up at a point of M_P.

With the notation of Section 4, the kernel $N_1 = \ker \pi$ of the map $\pi : (S^1)^{d-1} \to (S^1)^m$ has real dimension $d - m - 1$. Let $\widehat{N_1} \cong Sp(1)^{d-m-1}$ be

its extension and let

$$\sigma_{\widehat{N}_1} : \mathbb{H}^{d-1} \to \mathbb{R}^{d-m-1*}$$

be the tri-moment map associated with the action of $\widehat{N}_1$ on $\mathbb{H}^{d-1}$. By construction, the manifold M_P is given by

$$M_P = \sigma^{-1}_{\widehat{N}_1}(a_1, a_2, \dots, a_{d-m-1})/\widehat{N}_1 \tag{5.1}$$

where $(a_1, a_2, \dots, a_{d-m-1}) \in \mathbb{R}^{d-m-1}$ is determined by the polytope P. Cutting P with a hyperplane parallel to one of its facets to obtain a new polytope $\widetilde{P}$ with d facets, we get that the kernel of the new projection $\tilde{\pi}$ is isomorphic to $N = N_1 \times N_2$ where $N_2 \cong S^1$. Since the action of N_1 is trivial on the d-th coordinate of $\mathbb{H}^d$, it is possible to define the action of the extension $\widehat{N} = \widehat{N}_1 \times \widehat{N}_2$ where $\widehat{N}_2 \cong Sp(1)$. If we enumerate the facets of P as $j = 1, \dots, d-1$ and the cut is parallel to the j-th facet, then the tri-moment map $\sigma_{\widehat{N}} : \mathbb{H}^{d-1} \times \mathbb{H} \to \mathbb{R}^{d-m*}$ corresponding to the action of $\widehat{N}$ on $\mathbb{H}^d$ is given by

$$\sigma_{\widehat{N}}(q_1, \dots, q_{d-1}, q_d) = (\sigma_{\widehat{N}_1}(q_1, \dots, q_{d-1}), \langle \sigma(q_1, \dots, q_{d-1}), e_j \rangle - \frac{1}{4}|q_d|^4)$$

where σ is the tri-moment map associated with the standard action of $Sp(1)^{d-1}$ on $\mathbb{H}^{d-1}$ and e_j is the j-th element of the standard basis of $\mathbb{R}^{d-1}$. Therefore, given $(a_1, a_2, \dots, a_{d-m}) \in \mathbb{R}^{d-m}$, we get

$$\sigma^{-1}_{\widehat{N}}(a_1, a_2, \dots, a_{d-m}) = \begin{cases} \sigma^{-1}_{\widehat{N}_1}(a_1, a_2, \dots, a_{d-m-1}) \times \mathbb{H} \\ \quad \langle \sigma(q), e_j \rangle - \dfrac{1}{4}|q_d|^4 = a_{d-m} \end{cases} \tag{5.2}$$

where $q = (q_1, q_2, \dots, q_{d-1})$. Following the procedure, the manifold $M_{\widetilde{P}}$ is obtained as the quotient of $\sigma^{-1}_{\widehat{N}}(a_1, a_2, \dots, a_{d-m})$ via the action of $\widehat{N}_1 \times \widehat{N}_2$. Now $\widehat{N}_1$ acts only on the first $(d-1)$-coordinates. So the quotient

$$M_{\widetilde{P}} = \sigma^{-1}_{\widehat{N}}(a_1, a_2, \dots, a_{d-m})/\widehat{N}$$

is given by

$$M_{\widetilde{P}} = \{(q, q_d) \in M_P \times \mathbb{H} \mid \langle \sigma(q), e_j \rangle - \tfrac{1}{4}|q_d|^4 = a_{d-m}\}/\widehat{N}_2. \tag{5.3}$$

If we denote by $h(q) = \langle \sigma(q), e_j \rangle$, we get that $M_{\widetilde{P}} = \overline{(M_P)}_{h \geq a_{d-m}}$. This fact is relevant, indeed it implies that the manifold $M_{\widetilde{P}}$ is obtained via 4-plectic cut from M_P. Since P is obtained by cutting appropriately the standard

simplex, applying iteratively Theorem 5.6 at each cut, we can conclude that $M_{\widetilde{P}}$ admits a 4-plectic form.

Theorem 5.7. *The manifold corresponding to a polytope with $m+k+1$ facets, obtained via cutting the standard simplex Δ_m in $\mathbb{R}^{m*}$ by means of k-hyperplanes parallel to facets of Δ_m is the blow-up at k points of $\mathbb{HP}^m$. Moreover it admits a 4-plectic form.*

We observe here that this class of manifolds, thanks to Theorem 3.12 in [11], has also the property of being *quaternionic regular*.

6. Quaternionic toric manifolds

In the symplectic setting, the Delzant Theorem establishes a one-to-one correspondence between symplectic toric manifolds and Delzant polytopes (up to symplectomorphisms). In the 4-plectic case, in Theorem 3.2, we obtained a sub-convexity result on the image of the tri-moment map. For the class of 4-plectic manifolds, associated with polytopes obtained by cutting the standard simplex by means of hyperplanes parallel to its facets, the image of the tri-moment map turns out to be convex and it coincides with the starting polytope. This establishes, for this class of 4-plectic manifolds, a correspondence completely analogous to the one stated by the Delzant Theorem in the symplectic case.

We here present some significant examples in the 4-plectic setting; we explicitly describe the generalized Hamiltonian action and the corresponding tri-moment map.

Quaternionic projective spaces

We recall that if $(q_1,\dots,q_{n+1}) \in \mathbb{H}^{n+1}\setminus\{0\}$, then $[q_1,\dots,q_{n+1}]$ denotes the (right) vector line $\{(q_1\lambda,\dots,q_{n+1}\lambda) \in \mathbb{H}^{n+1} : \lambda \in \mathbb{H}\}$ of $\mathbb{H}^{n+1}$. As usual $\mathbb{HP}^n$ denotes the set of (right) vector lines in $\mathbb{H}^{n+1}$. Using the reduced form obtained in [9, p. 337] on $\mathbb{HP}^m$ acted on freely by the group $Sp(1)^m$ as

$$(\lambda_1,\lambda_2,\dots,\lambda_m)[q_1 : q_2 : \cdots : q_{m+1}] = [\lambda_1 q_1 : \lambda_2 q_2 : \cdots \lambda_m q_m : q_{m+1}]$$

the tri-moment map turns out to be

$$\sigma([q_1 : q_2 : \cdots : q_{m+1}]) = -\left(\frac{|q_1|^4}{\sum_{i=1}^{m+1}|q_i|^4}, \frac{|q_2|^4}{\sum_{i=1}^{m+1}|q_i|^4}, \dots, \frac{|q_m|^4}{\sum_{i=1}^{m+1}|q_i|^4}\right) \in \mathbb{R}^{m*}.$$

The image, which is an m-simplex, is given by the convex envelope of the images of the points fixed by the group $Sp(1)^m$ that coincide, in this case, with the common critical points of the components of the tri-moment map.

Blow up of $\mathbb{HP}^2$

In general the idea is the following: we start with the quaternionic projectve space $\mathbb{HP}^2$ acted on by the group $Sp(1)^2$

$$(\lambda_1, \lambda_2)[q_1 : q_2 : q_3] = [\lambda_1 q_1 : \lambda_2 q_2 : q_3].$$

At each blow-up corresponds an extended action. We blow-up first at the point $[0 : 1 : 0]$, then at $[1 : 0 : 0]$ and finally at $[0 : 0 : 1]$. Thus the action at the third step (after three blow-ups) is the following

$$(\lambda_1, \lambda_2)([q_1 : q_2 : q_3], [p_1 : p_2], [r_1 : r_2], [s_1 : s_2]) \tag{6.1}$$

$$= ([\lambda_1 q_1 : \lambda_2 q_2 : q_3], [\lambda_1 p_1 : p_2], [\lambda_2 r_1 : r_2], [\lambda_1 s_1 : \lambda_2 s_2]). \tag{6.2}$$

If $(\psi_1)_{red}, (\psi_2)_{red}$ denote the reduced 4-plectic structures on $\mathbb{HP}^1$ and $\mathbb{HP}^2$ respectively (see [9] for the precise expression), we can equip the exceptional divisors $E_i \cong \mathbb{HP}^1$ with the 4-plectic structures $\alpha_i(\psi_1)_{red}$ with $i = 1, 2, 3$. Thus the tri-moment map is given by (σ_1, σ_2) where

$$4\sigma_1 = -\frac{|q_1|^4}{|q_1|^4 + |q_2|^4 + |q_3|^4} - \alpha_1 \frac{|p_1|^4}{|p_1|^4 + |p_2|^4} - \alpha_3 \frac{|s_1|^4}{|s_1|^4 + |s_2|^4} \tag{6.3}$$

$$4\sigma_2 = -\frac{|q_2|^4}{|q_1|^4 + |q_2|^4 + |q_3|^4} - \alpha_2 \frac{|r_1|^4}{|r_1|^4 + |r_2|^4} - \alpha_3 \frac{|s_2|^4}{|s_1|^4 + |s_2|^4}. \tag{6.4}$$

Note that if $\alpha_i = 0$ for $i = 1, 2, 3$ we find again the tri-moment map of the quaternionic projective space, and for $\alpha_2 = \alpha_3 = 0$ we find the tri-moment map of the blow-up at $[0 : 1 : 0]$ (the first step), and analogously for $\alpha_1 = \alpha_3 = 0$ the blow-up at $[1 : 0 : 0]$ and for $\alpha_1 = \alpha_2 = 0$ at $[0 : 0 : 1]$. Moreover all the α_i must be less or equal to 1. We finally observe that, in order to obtain an image which is a polytope to which one can apply our procedure necessarily α_1 equals α_2, otherwise the slope of the edge is not a multiple of $\frac{\pi}{4}$. Looking at the action in equation (6.1), we can observe that the fixed points are more than 6. However, the image is given by the convex envelope of all the fixed points, and is an hexagon (admissible for our procedure only if $\alpha_1 = \alpha_2$).

Remark 6.1. One can observe that in all the cases considered so far i.e. when M equals $\mathbb{H}^m$, $\mathbb{HP}^{m-1}$ and their iterated blow-ups, the tri-moment map σ for the $Sp(1)^m$ action is given by the composition of the usual moment map ν for the $(S^1)^m$ action on the complex manifolds $\mathbb{C}^m$, $\mathbb{CP}^{m-1}$ and their iterated blow-ups, with the surjective map α defined on $\mathbb{H}^m$ with values in $\mathbb{C}^m$ as

$$\alpha(x_1 + y_1 I_1, \ldots, x_m + y_m I_m) = ((x_1 + y_1 i)^2, \ldots, (x_m + y_m i)^2)$$

and on $\mathbb{HP}^{m-1}$ with values in $\mathbb{CP}^{m-1}$ as

$$\alpha([x_1 + y_1 I_1 : \ldots : x_m + y_m I_m]) = [(x_1 + y_1 i)^2 : \ldots : (x_m + y_m i)^2]$$

where $x_\ell, y_\ell \in \mathbb{R}$, $y_\ell \geq 0$ and I_ℓ is a purely imaginary unit in $\mathbb{H}$ for any $\ell = 1, \ldots, m$. Therefore in all these cases the image $\sigma(M) = \nu \circ \alpha(M)$ is the same of its complex analog and thus it is a Delzant polytope.

Remark 6.2. Action of $(\mathbb{H}^*)^m$. It is easy to show that the examples considered so far admit an $(\mathbb{H}^*)^m$ action with an open dense orbit. We point out this fact since it is in complete analogy with what happens for the action of $(\mathbb{C}^*)^m$ on the corresponding complex manifolds.

In particular,

1) the action of $(\mathbb{H}^*)^m$ on $\mathbb{HP}^m$ is given by

$$(a_1, \ldots, a_m)[q_1 : \ldots : q_m : q_{m+1}] := [a_1 q_1 : \ldots : a_m q_m : q_{m+1}]$$

and it has an open dense orbit since the generic stabilizer is trivial.

2) The group $(\mathbb{H}^*)^2$ acting on the base space $\mathbb{H}^2$ as

$$(a_1, a_2)(q_1, q_2) := (a_1 q_1, a_2 q_2)$$

has an open dense orbit and lifts to the blow up $Bl_p(\mathbb{HP}^2)$, naturally: the group acts taking a direction in the exceptional fiber $\mathbb{HP}^1$ to another direction in $\mathbb{HP}^1$. Indeed the action of $(\mathbb{H}^*)^2$ on $Bl_p(\mathbb{HP}^2)$ is given by

$$(a_1, a_2)([q_1 : q_2 : q_3], [p_1 : p_2]) := ([a_1 q_1 : a_2 q_2 : q_3], [a_1 p_1 : p_2])$$

and the generic orbit is open and dense.

3) The same argument also applies for $(\mathbb{H}^*)^m$ on $\mathbb{HP}^m \# k\overline{\mathbb{HP}}^m$.

The manifold $G_2/SO(4)$

Going through the list of quaternion Kähler manifolds, endowed with the Kraines form, the only one admitting a generalized Hamiltonian action of $Sp(1)^n$ with discrete principal isotropy is the 8-dimensional quotient $M = \frac{G_2}{SO(4)}$ with $n = 2$. Thus it is a toric quaternionic manifold.

We can actually compute the number of fixed points for the action of $Sp(1)^2$ on $\frac{G_2}{SO(4)}$ proving that the fixed point set is given by a single point. Indeed the Euler characteristic of M is 3, given by the quotient of the order of the Weyl group of G_2, $|W(G_2)| = 12$, over the order of the Weyl group of $SO(4)$, $|W(SO(4))| = 4$. Since the action is polar, the Euler characteristic is equal to the number of points fixed by a maximal abelian subgroup T in $Sp(1)^2$. Let H be the normalizer of $K = SO(4)$ in G_2. The order of the fixed point set of K on M equals the order of $\frac{H}{K}$. The quotient $\frac{H}{K}$ has order 1 or 3. If the order is 3 then G_2/H would have fundamental group $\mathbb{Z}_3$ (we here use the homotopy sequence and the connectedness of SO(4)) and would be therefore orientable (since it does not admits subgroups of index two), so that its Euler characteristic should be strictly greater than 1 and therefore equal to 3. Hence $\frac{H}{K}$ would have order 1 which is a contradiction. So the image, via the moment map, is in this case contained in the convex envelope of a set of points whose cardinality runs from 1 to $\#M^T = \chi(M) = 3$.

A further investigation could clarify whether $(\mathbb{H}^*)^2$ acts on this manifold with an open dense orbit and whether the manifold is quaternionic regular (in the sense of [11]). Moreover it would be interesting to understand if the image of $\frac{G_2}{SO(4)}$ via the tri-moment map is related with the moment map image of its twistor space.

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