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The Dehn surgery characterization of the trefoil and the figure eight knot

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We give a Dehn surgery characterization of the trefoil and the figure eight knots. These results are obtained by combining surgery formulas in Heegaard Floer homology from an earlier paper with the characterization of these knots in terms of their knot Floer homology due to Ghiggini.

1. Introduction

In [4], it was shown that the unknot is characterized by its Dehn surgeries, in the sense that if Dehn surgery with some slope on a knot K in S^3 is orientation-preserving homeomorphic to Dehn surgery with the same slope on the unknot, then K is in fact unknotted. The computation uses calculational techniques for Seiberg-Witten monopole Floer homology (the surgery exact triangle), to reduce to the case of the 0-surgery, where the techniques of Gabai [2] apply.

A proof of this fact can also be given using Heegaard Floer homology [10]. Specifically, there is a Heegaard Floer homology theory for knots introduced in [9] and [14]. Surgery formulas for this invariant [12] allow one to express the Heegaard Floer homology for p/q-Dehn surgery of K in terms of the knot Floer homology of K. The hypothesis that p/q surgery on K agrees with that of the unknot forces the knot Floer homology of K to agree with that of the unknot. Combining this with the fact that knot Floer homology detects the unknot [8], the Dehn surgery characterization of the unknot follows.

In a beautiful paper, Ghiggini [3] shows that Heegaard Floer homology also detects the trefoil and the figure eight knot. Appealing to the same strategy outlined above, in the form of the surgery formulas for knot Floer homology, we obtain here a similar Dehn surgery characterization of both of these knots. Specifically, we have the following:

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Theorem 1.1. *Let T be a trefoil knot. If K is a knot with the property that there is a rational number r and an orientation-preserving diffeomorphism $S^3_r(K) \cong S^3_r(T)$, then K is in fact the trefoil T.*

In a similar vein, we have the following

Theorem 1.2. *Let S be the figure eight knot. If K is a knot with the property that there is a rational number r and an orientation-preserving diffeomorphism $S^3_r(K) \cong S^3_r(S)$, then K is in fact the figure eight knot S.*

The case where $r = 0$, the above theorems follow from results of Gabai [2, Corollary 8.23].

The condition that the diffeomorphism preserves orientations is important, here. For example, there are identifications $S^3_{+1}(T_\ell) \cong -S^3_{+1}(S)$, and also $S^3_{+5}(T_r) \cong -S^3_{+5}(O)$, where here O is the unknot.

Similarly, the condition that the surgery coefficient is fixed is also crucial; $S^3_{1/n}(K_0)$ where K_0 is a trefoil or the figure eight knot can be realized alternatively as $+1$ surgery on a suitable twist knot.

In his paper, Ghiggini proves that the trefoil is the only knot in S^3 which admits a surgery giving the Poincaré homology sphere. A consequence of Theorems 1.1 and 1.2, we obtain a similar result for the Brieskorn sphere $\Sigma(2, 3, 7)$.

Corollary 1.3. *The only surgeries on knots in S^3 which realize the Brieskorn sphere $\Sigma(2, 3, 7)$ (with either orientation) are $S^3_{-1}(T_r) \cong S^3_{+1}(S) \cong \Sigma(2, 3, 7)$ and $S^3_{+1}(T_\ell) \cong S^3_{-1}(S) \cong -\Sigma(2, 3, 7)$.*

In Section 2, we review the relevant aspects of Heegaard Floer homology which are used in the proofs of the above results. In Section 3 we give the proofs of the above two theorems and the corollary.

2. Background

2.1. Heegaard Floer homology

In its most basic form, Heegaard Floer homology is a $\mathbb{Z}/2\mathbb{Z}$-graded Abelian group associated to a three-manifold, but it comes in several variants and can be endowed with additional structure [10].

In this paper, we will consider primarily the version $HF^+(Y)$ for rational homology three-spheres Y. This group admits a splitting according to Spin^c

structures over Y

$$HF^+(Y) \cong \bigoplus_{\mathfrak{t}\in\mathrm{Spin}^c(Y)} HF^+(Y,\mathfrak{t}).$$

Moreover, $HF^+(Y,\mathfrak{t})$ is equipped with an absolute grading (defined in [11, Section 7] and studied extensively in [6]). Recall that there is a natural involution on the space of Spin^c structures over Y, denoted $\mathfrak{t} \mapsto \bar{\mathfrak{t}}$. There is a corresponding isomorphism

$$HF^+(Y,\mathfrak{t}) \cong HF^+(Y,\bar{\mathfrak{t}}), \tag{1}$$

which, in the case of rational homology spheres Y, is an isomorphism of $\mathbb{Q}$-graded Abelian groups.

The group $HF^+(Y,\mathfrak{t})$ has the following algebraic structure.

$$HF^+(Y,\mathfrak{t}) = \bigoplus_{d\in\mathbb{Q}} HF^+_d(Y,\mathfrak{t}).$$

In fact, $HF^+(Y,\mathfrak{t})$ is supported only in rational degrees d within some fixed equivalence class (depending on Y and $\mathfrak{t}$) modulo the integers. For any degree $d \in \mathbb{Q}$, $HF^+_d(Y,\mathfrak{t})$ is a finitely generated $\mathbb{Z}$-module. There is an endomorphism U of $HF^+(Y,\mathfrak{t})$ that lowers degree by 2, i.e.

$$U\colon HF^+_d(Y,\mathfrak{t}) \longrightarrow HF^+_{d-2}(Y,\mathfrak{t}).$$

Moreover, $HF^+_d(Y,\mathfrak{t}) = 0$ for all sufficiently small d. Finally, for any sufficiently large rational number d_0, if we consider the quotient module $HF^+_{\geq d_0}(Y,\mathfrak{t})$ of $HF^+(Y,\mathfrak{t})$, obtained by dividing out by all elements with degree less than d_0, then that module is isomorphic to the $\mathbb{Z}[U]$-module

$$\mathcal{T}^+ = \frac{\mathbb{Z}[U,U^{-1}]}{U\cdot\mathbb{Z}[U]}.$$

From the above properties, it is clear that there is a canonical short exact sequence

$$0 \longrightarrow \mathcal{T}^+ \longrightarrow HF^+(Y,\mathfrak{t}) \longrightarrow HF^+_{\mathrm{red}}(Y,\mathfrak{t}) \longrightarrow 0,$$

where $HF^+_{\mathrm{red}}(Y,\mathfrak{t})$ a $\mathbb{Z}[U]$ module, which is also finitely generated as a $\mathbb{Z}$-module. There is a three-manifold invariant,

$$d\colon \mathrm{Spin}^c(Y) \longrightarrow \mathbb{Q},$$

the *correction terms* of Y, where $d(Y,\mathfrak{t})$ is the minimal $\mathbb{Q}$-grading of any homogeneous element of $HF^+(Y,\mathfrak{t};\mathbb{Q})$ in the image of $\mathcal{T}^+\otimes\mathbb{Q}$. The correction terms are analogous to a gauge-theoretic invariant introduced by Frøyshov [1]; for more information on the correction terms, see [6]. Replacing $\mathbb{Q}$ by $\mathbb{Z}/p\mathbb{Z}$, we obtain analogous d-invariants $d(Y,\mathfrak{t},\mathbb{Z}/p\mathbb{Z})$. We will drop the field when it is clear from the context.

The Floer homology group $HF^+(Y,\mathfrak{t})$ also inherits a $\mathbb{Z}/2\mathbb{Z}$-grading; a non-zero element in $HF^+_d(Y,\mathfrak{t})$ has *even parity* if $d\equiv d(Y,\mathfrak{t}) \pmod{2\mathbb{Z}}$, and it has *odd parity* if $d\equiv d(Y,\mathfrak{t})+1 \pmod{2\mathbb{Z}}$.

If C is a chain complex of $\mathbb{F}[U]$-modules (where $\mathbb{F}=\mathbb{Q}$ or $\mathbb{Z}/n\mathbb{Z}$ for prime n), we can form

$$H^{\mathrm{red}}(C)=\lim_{\overleftarrow{d}}\frac{H_*(C)}{U^dH_*(C)}.$$

Then, if $CF^+(Y,\mathfrak{t})$ is the chain complex calculating $HF^+(Y,\mathfrak{t})$, then $H^{\mathrm{red}}(CF^+(Y,\mathfrak{t}))=HF^+_{\mathrm{red}}(Y,\mathfrak{t})$.

2.2. Knot Floer homology and the surgery formula

Heegaard Floer homology can be extended as in [9] and [14] to invariants for null-homologous knots K in closed three-manifolds. We restrict attention to the case where the ambient three-manifold is the three-sphere S^3. We recall now the notation of knot Floer homology, following [9]. This data can be used to calculate Heegaard Floer homology groups of Dehn fillings of S^3 along K, cf. [12]. After setting up notation for knot Floer homology, we state this surgery formula.

A $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex is a free Abelian group which splits as a direct sum $C=\bigoplus_{(i,j)\in\mathbb{Z}\oplus Z}C\{(i,j)\}$ and which is endowed with a boundary operator which carries elements in $C\{(i,j)\}$ to elements in

$$\bigoplus_{(i',j')\leq(i,j)}C\{(i',j')\},$$

writing $(i',j')\leq(i,j)$ if $i'\leq i$ and $j'\leq j$. In the present paper, we will consider $\mathbb{Z}\oplus\mathbb{Z}$-filtered $\mathbb{Z}[U]$-complexes. These come equipped with a chain map isomorphism $U\colon C\longrightarrow C$ which carries $C\{(i,j)\}$ to $C\{(i-1,j-1)\}$.

Consider a subset $X\subset\mathbb{Z}\oplus\mathbb{Z}$ with the property that if $(i,j)\in X$, then for any $(i',j')\leq(i,j)$, we also have that $(i',j')\in X$. If C is any $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex, we can form a subcomplex $C\{X\}\subset C$ generated by $\bigoplus_{(i,j)\in X}C\{(i,j)\}$.

Suppose $Y \subset \mathbb{Z} \oplus \mathbb{Z}$ a set with the property that for any $(i,j) \in Y$, if $(i',j') \geq (i,j)$, then $(i',j') \in Y$. In this case, we can endow $\bigoplus_{(i,j)\in Y} C\{(i,j)\}$ with the structure of a quotient complex, which we will also denote by $C\{Y\}$.

If $K \subset S^3$ is a knot, we obtain an associated $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex C with total homology isomorphic to $\mathbb{Z}[U, U^{-1}]$. The filtered chain homotopy type of this complex $C = \mathrm{CFK}^\infty(S^3, K)$ is a knot invariant, [9, 14].

The differential on C induces also a differential on each summand $C\{(i,j)\}$. The homology group $H_*(C\{(0,s)\})$ is called the *knot Floer homology group in filtration level* s, and it is denoted $\widehat{\mathrm{HFK}}(K,s)$.

The filtered chain homotopy type $\mathrm{CFK}^\infty(S^3, K)$ gives rise to some further algebraic structure.

Let $B^+ = C\{i \geq 0\}$. This is a model for $CF^+(S^3)$, and indeed, so is $C\{j \geq 0\}$. Since both represent S^3, we can fix a chain homotopy equivalence between them.

We have also chain complexes $A_s^+(K) = C\{\max(i, j-s) \geq 0\}$, equipped with a pair of maps

$$v_s^+ \colon A_s^+(K) \longrightarrow B^+ \quad \text{and} \quad h_s^+ \colon A_s^+(K) \longrightarrow B^+,$$

the first of which is simply the projection map (from $C\{\max(i, j-s) \geq 0\}$ to $C\{i \geq 0\}$), while the second is a composite of the projection map $C\{\max(i,j-s) \geq 0\}$ to $C\{j \geq s\}$, followed by the identification with $C\{j \geq 0\}$ (induced by multiplication by U^s), followed by a chain homotopy equivalence of this with B^+. These maps are the data necessary to calculate the Heegaard Floer homology of arbitrary Dehn fillings of S^3 along K.

Recall that the Heegaard Floer homology of Y admits a direct sum splitting indexed by the set of Spin^c structures over Y, which in turn is an affine space for $H^2(Y;\mathbb{Z})$. In particular, if $K \subset S^3$, then there is a splitting

$$HF^+(S^3_{p/q}(K)) \cong \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} HF^+(S^3_{p/q}(K), i).$$

Fix an integer i, and consider the chain complexes

$$\mathbb{A}_i^+ = \bigoplus_{s\in\mathbb{Z}} (s, A^+_{\lfloor \frac{i+ps}{q} \rfloor}(K)) \quad \text{and} \quad \mathbb{B}_i^+ = \bigoplus_{s\in\mathbb{Z}} (s, B^+),$$

where here $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x. We view the above chain homomorphisms v^+ and h^+ as maps

$$v^+ \colon (s, A^+_{\lfloor \frac{i+ps}{q} \rfloor}(K)) \longrightarrow (s, B^+) \quad \text{and} \quad h^+ \colon (s, A^+_{\lfloor \frac{i+ps}{q} \rfloor}(K)) \longrightarrow (s+1, B^+).$$

Adding these up, gives a chain map $D^+_{i,p/q}\colon \mathbb{A}^+_i \longrightarrow \mathbb{B}^+_i$, where

$$D^+_{i,p/q}\{(s,a_s)\}_{s\in\mathbb{Z}} = \{(s,b_s)\}_{s\in\mathbb{Z}},$$

is given by

$$b_s = v^+_{\lfloor \frac{i+ps}{q}\rfloor}(a_s) + h^+_{\lfloor \frac{i+p(s-1)}{q}\rfloor}(a_{s-1}).$$

Let $\mathbb{X}^+_{i,p/q}$ denote the mapping cone of $D^+_{i,p/q}$. Note that $\mathbb{X}^+_{i,p/q}$ depends on i only through its congruence class modulo p. Note also that A^+_s and B^+_s are relatively $\mathbb{Z}$-graded, and the homomorphisms v^+_s and h^+_s respect this relative grading. The mapping cone $\mathbb{X}^+_i$ can be endowed with a relative grading, with the convention that $D^+_{i,p/q}$ drops the grading by one.

The following is proved (in somewhat more generality) in [12, Theorem 1.1 and 6.1]; see also [12, Section 7.2] for the absolute gradings:

Theorem 2.1. *Let $K\subset S^3$ be a knot, and let p,q be a pair of relatively prime integers. Then, there is an identification $\sigma\colon \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{Spin}^c(S^3_{p/q}(K))$ such that for each $i\in\mathbb{Z}/p\mathbb{Z}$, there is a relatively graded isomorphism of groups*

$$\Phi_{K,i}\colon H_*(\mathbb{X}^+_{i,p/q}(K)) \xrightarrow{\cong} HF^+(S^3_{p/q}(K),\sigma(i)).$$

Indeed, there is a uniquely specified absolute grading on the subcomplex $\mathbb{B}^+_i\subset \mathbb{X}^+_{i,p/q}(K)$ (which is independent of K) for which the map $\Phi_{O,i}$ is an isomorphism (where here O is the unknot). With the corresponding induced grading on $\mathbb{X}^+_{i,p/q}(K)$, Φ_K becomes an absolutely graded isomorphism.

It is useful to note that there is a conjugation on knot Floer homology related to the conjugation invariance on closed manifolds, cf. Equation (1). In the form which we need it, this is an isomorphism

$$\Psi\colon H_*(A^+_s) \xrightarrow{\cong} H_*(A^+_{-s})$$

for all integers s. Indeed, under this isomorphism, we have a commutative diagram

$$\begin{array}{ccc} H(A^+_s) & \xrightarrow{\Psi} & H(A^+_{-s}) \\ {\scriptstyle H(v^+_s)}\downarrow & & \downarrow{\scriptstyle H(h^+_{-s})} \\ H(B^+) & \xrightarrow{=} & H(B^+) \end{array}$$

It is also useful to note that, although $\mathbb{X}^+_{i,p/q}(K)$ is a very large chain complex, if we are interested in the homology in degrees less than or equal to some fixed quantity d, then this is contained in much smaller chain complex. More precisely, suppose that $p, q > 0$. Then, since v_s^+ is an isomorphism for all sufficiently large s and h_s^+ is an isomorphism for all sufficiently small s, we can consider the subsets

$$\mathbb{A}_{i,\leq\sigma}^+ = \bigoplus_{\{s\in\mathbb{Z}||s|\leq\sigma\}} (s, A^+_{\lfloor\frac{i+ps}{q}\rfloor}(K)) \subset \mathbb{A}_i^+$$
$$\mathbb{B}_{i\leq\sigma}^+ = \bigoplus_{\{s\in\mathbb{Z}|-\sigma<s\leq\sigma\}} (s, B^+) \subset \mathbb{B}_i^+,$$

where σ is sufficiently large. The map $D_{i,p/q}^+$ induces a map from $\mathbb{A}_{i,\leq\sigma}^+$ to $\mathbb{B}_{i,\leq\sigma}^+$, whose mapping cone, denoted $\mathbb{X}_{i,p/q,\leq\sigma}^+$, is a quotient complex of $\mathbb{X}_{i,p/q}^+$. Since the subcomplex has trivial homology, the homology of the chain complex $\mathbb{X}_{i,p/q}^+(K)$ agrees with the homology of its quotient complex $\mathbb{X}_{i,p/q,\leq\sigma}^+(K)$ for some σ.

2.3. Examples

For K the unknot, C has a single generator a as a $\mathbb{Z}[U, U^{-1}]$-module, which is supported in filtration level $(0, 0)$ and grading zero. The differentials are trivial.

We let T_ℓ denote the left-handed trefoil, T_r denote the right-handed trefoil, and S denote the figure eight knot.

For $K = T_r$, the complex C has three generators as a $\mathbb{Z}[U, U^{-1}]$-module, a, b, c, in filtration levels $(-1, 0)$ $(0, 0)$, and $(0, -1)$ respectively, with the differential $Db = a + c, Da = Dc = 0$.

For $K = T_\ell$, C has three generators as a $\mathbb{Z}[U, U^{-1}]$-module, a, b, c, in filtration levels $(0, 1)$ $(0, 0)$, and $(1, 0)$ respectively, with the differential $Da = Dc = b$ and $Db = 0$.

Finally, for $K = S$, C has five generators as a $\mathbb{Z}[U, U^{-1}]$ module, a, b, c, d, and e. Here a is supported in filtration level $(1, 1)$, b in $(0, 1)$, c in $(1, 0)$, and d and e in $(0, 0)$. Differentials are given by $Da = b + c$, $Db = -Dc = d$, $Dd = De = 0$.

These answers are illustrated in Figure 1. These computations are fairly straightforward; one could find an appropriate Heegaard diagram where the computations are easy, or refer to more general results [7, 13]. These results have the following consequences for A_s^+.

Proposition 2.2. *The groups $H_*(A_s^+(K))$ and the homomorphisms on homology induced by v_s^+ (for $s \geq 0$) for $K = T_r$, T_ℓ, and S are determined*

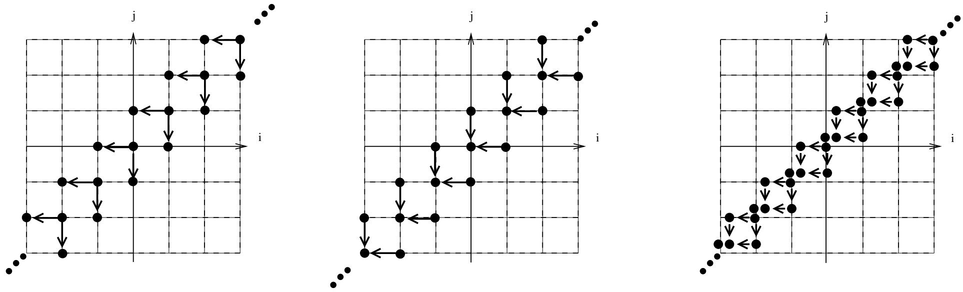

Figure 1: **Filtered complexes for T_r, T_ℓ, and S.** We have illustrated the $\mathbb{Z}\oplus\mathbb{Z}$-filtered complexes associated to the three knots listed above. Dots represent generators, and arrows represent differentials.

as follows. For all $s\in\mathbb{Z}$, $H_*(A^+_s(T_r))\cong\mathcal{T}^+$; indeed, for all $s>0$, the map induced by v_s is an isomorphism, while for $s=0$, the map

$$H(v_0)\colon H_*(A^+_0(T_r))\cong\mathcal{T}^+\longrightarrow HF^+(S^3)\cong\mathcal{T}^+$$

is modeled on multiplication by U.

For all $s>0$, $H_*(A^+_s(T_\ell))\cong\mathcal{T}^+$, while $H_*(A^+_0(T_\ell))\cong\mathcal{T}^+_{(0)}\oplus\mathbb{Z}_{(0)}$, where the extra $\mathbb{Z}$ has grading zero. Moreover, the kernel of $H(v_0)$ is one-dimensional.

For all $s>0$, $H_*(A^+_s(S))\cong\mathcal{T}^+$, with an isomorphism induced by v^+; while $H_*(A^+_0(S))\cong\mathcal{T}^+_{(0)}\oplus\mathbb{Z}_{(-1)}$, and the kernel of $H(v_0)$ is one-dimensional.

Proof. These are straightforward consequences of the chain complexes described above. □

Remark 2.3. *Recall that the surgery formula uses a chain homotopy equivalence between $C\{i\geq 0\}$ and $C\{j\geq 0\}$. Our computations will be independent of the choice of homotopy equivalence. This is unsurprising: when the ambient manifold is S^3, any automorphism of $CF^+(S^3)$ up to homotopy, is determined by its action on homology, $\mathcal{T}^+$. The automorphism group of $\mathcal{T}^+$, in turn is $\mathbb{Z}/2\mathbb{Z}$, generated by multiplication by -1. Finally, multiplying each h^+_s by -1 can be shown to give an isomorphic chain complex $\mathbb{X}^+_{i,p/q}$.*

2.4. Genus bounds and Ghiggini's theorem

In [8], it is shown that if $K\subset S^3$ is a knot with genus g, then

$$\max\{s\big|H_*(C\{(0,s)\})=\widehat{\mathrm{HFK}}(K,s)\neq 0\}=g.$$

It was conjectured that if $\widehat{\mathrm{HFK}}(K,g) \cong \mathbb{Z}$, then K is a fibered knot. Ghiggini [3] verified this conjecture for knots with genus one; see [5] for the case $g > 1$. Since the only genus one fibered knots are the figure eight and the trefoil, it follows that:

Theorem 2.4. *(Ghiggini) Let K be a knot with $\widehat{\mathrm{HFK}}(K,s) = 0$ for all $s > 1$ and $\widehat{\mathrm{HFK}}(K,1)$ is isomorphic to $\mathbb{Z}$. Then, K is a trefoil or the figure eight knot.*

3. Proof of Theorems 1.1 and 1.2

We will fix throughout some field $\mathbb{F}$ (which is $\mathbb{Q}$ or $\mathbb{Z}/n\mathbb{Z}$ where n is prime), computing Floer homology (and d invariants) with coefficients in $\mathbb{F}$. When writing $\mathcal{T}^+$, we mean $\mathcal{T}^+ \otimes \mathbb{F}$.

Let K_0 be a trefoil or the figure eight knot. From a graded isomorphism

$$HF^+(S^3_{p/q}(K)) \cong HF^+(S^3_{p/q}(K_0)),$$

we would like to use the surgery formula to conclude an isomorphism between the knot Floer homologies of K and K_0. To this end, we find it useful to identify $HF^+(S^3_{p/q}(K_0))$.

Proposition 3.1. *Let p and q be relatively prime, positive integers. We have that*

$$\mathrm{rk} HF^+_{\mathrm{red}}(S^3_{p/q}(T_r)) < q,$$

while

$$\mathrm{rk} HF^+_{\mathrm{red}}(S^3_{p/q}(T_\ell)) = \mathrm{rk} HF^+_{\mathrm{red}}(S^3_{p/q}(S)) = q.$$

In the case where $K_0 = T_\ell$, $HF^+_{\mathrm{red}}(S^3_{p/q}(T_\ell))$ is supported in even degree, while in $HF^+(S^3_{p/q}(S))$, it is supported in odd degree. Moreover, for $K_0 = T_r$, T_ℓ, and S, we have that

$$\sum_{i \in \mathbb{Z}/p\mathbb{Z}} \mathrm{rk} HF^+_{\mathrm{red}}(S^3_{p/q}(K_0), i) - \left(\frac{d(S^3_{p/q}(K_0), i) - d(S^3_{p/q}(O), i)}{2} \right) = q.$$

Proof. By Proposition 2.2, for $K_0 = T_r, T_\ell$, or S, the map $v^+_s \colon A^+_s \to B^+$ induces an isomorphism on homology for all $s > 0$; by symmetry, $h^+_{-s} \colon A^+_{-s} \to B^+$ induces isomorphisms on homology for $s > 0$. Let $k = k_i$ denote the number of times A_0 appears in the model $\mathbb{X}^+_{i,p/q}$. It follows that, when $k > 0$, then

$\mathbb{X}^+_{i,p/q}$ is quasi-isomorphic to the subcomplex consisting of k copies of A_0^+ and $k-1$ copies of B^+, which can be viewed as a mapping cone of maps obtained by adding up v_0^+ and h_0^+. Moreover, the map is surjective on homology, so we obtain a short exact sequence of the form

$$0 \longrightarrow HF^+(S^3_{p/q}(K_0), i) \longrightarrow \bigoplus^k H(A_0^+(K_0)) \xrightarrow{H(\delta)} \bigoplus^{k-1} H(B^+) \longrightarrow 0.$$

When $K_0 = S$, the results of Proposition 2.2 give a k_i-dimensional kernel to $H(\delta)$ in odd dimensions, and the portion in even degrees agrees with that for the knot, giving a summand $\mathcal{T}^+$. In fact,

$$HF^+(S^3_{p/q}(S), i) = \mathcal{T}^+_{(d)} \oplus \mathbb{F}^k_{(d-1)}, \tag{2}$$

where $d = d(S^3_{p/q}(O), i)$. Moreover for i where $k_i = 0$,

$$HF^+(S^3_{p/q}(S), i) \cong HF^+(S^3_{p/q}(O), i) \cong \mathcal{T}^+_{(d)}.$$

Adding up over all $i \in \mathbb{Z}/p\mathbb{Z}$, and noting that $\sum k_i = q$, we obtain the stated result for S.

When $K_0 = T_\ell$, the argument works similarly, except in this case, when $k_i \neq 0$, Equation (2) is replaced by

$$HF^+(S^3_{p/q}(T_\ell), i) = \mathcal{T}^+_{(d)} \oplus \mathbb{F}^k_{(d)}.$$

When $K_0 = T_r$, the short exact sequence looks like

$$0 \longrightarrow HF^+(S^3_{p/q}(T_r, i)) \longrightarrow \bigoplus^k \mathcal{T}^+_{(d-2)} \xrightarrow{H(\delta)} \oplus^{k-1}\mathcal{T}^+_{(d)} \longrightarrow 0,$$

where $d = d(S^3_{p/q}(O), i)$, and Equation (2) is replaced by

$$HF^+(S^3_{p/q}(S), i) = \mathcal{T}^+_{(d-2)} \oplus \mathbb{F}^{k-1}_{(d-2)}.$$

$\square$

Proposition 3.2. *If $K \subset S^3$ is a knot with the property that for some $s > 0$, the map $H_*(A_s^+(K)) \to HF^+(S^3)$ induced by v^+ is not an isomorphism, then for any $p/q > 0$, we have that*

$$\sum_{i=0}^{p-1} \mathrm{rk} HF^+_{\mathrm{red}}(S^3_{p/q}(K), i) - \left(\frac{d(S^3_{p/q}(K), i) - d(S^3_{p/q}(O), i)}{2} \right) \geq 2q.$$

Proof. We start by proving the following *extension property* for homology classes in the kernel of $H(v_s^+)$ for $s > 0$. Fix $i \in \{0, \ldots, p-1\}$ and choose $\ell \in \mathbb{Z}$ so that $s = \lfloor \frac{i+p\ell}{q} \rfloor$ is positive, and fix some homology class $[\xi_s] \in \mathrm{Ker}(H(v_s^+)\colon H_*(A_s^+) \to H_*(B^+))$ in with fixed grading δ. Then, there is a homology class in $H(\mathbb{X}_{i,p/q}^+)$ whose projection to $H(A^+_{\lfloor \frac{i+pj}{q} \rfloor})$ vanishes for $j < \ell$, and whose projection to $H(A^+_{\lfloor \frac{i+p\ell}{q} \rfloor})$ coincides with $[\xi_s]$. This follows from the fact that for $t \in \mathbb{Z}$, the map

$$H(v_t)\colon H(A_t^+) \to H(B^+)$$

is surjective, as follows. Fix $[\xi_s] \in \mathrm{Ker} H(v_s^+)$, and fix a cycle $\xi_s \in A_s^+$ representing $[\xi_s]$. By hypothesis, there is some $\eta_\ell \in B^+$ so that $v_\ell^+(\xi_\ell) = \partial \eta_\ell$. Since

$$v^+\colon A^+_{\lfloor \frac{i+p(\ell+1)}{q} \rfloor} \to B^+$$

induces a surjection in homology, we can find a cycle $\xi_{\ell+1} \in A^+_{\lfloor \frac{i+p(\ell+1)}{q} \rfloor}$ and a chain $\eta_{\ell+1} \in B^+$ with

$$h_\ell^+(\xi_\ell) = v^+_{\lfloor \frac{i+p(\ell+1)}{q} \rfloor}(\xi_{\ell+1}) + \partial \eta_{\ell+1}.$$

Proceeding inductively, we end up completing the initial cycle ξ_ℓ with a desired sequence of elements with $j \geq \ell$, $\xi_j \in (j, A^+_{\lfloor \frac{i+pj}{q} \rfloor})$ and $\eta_j \in (t, B^+)$ with $t \geq s$, so that

$$h_j^+(\xi_j) = v^+_{\lfloor \frac{i+p(j+1)}{q} \rfloor}(\xi_{j+1}) + \partial \eta_{j+1}.$$

Note that by degree homogeneity, this procedure terminates for sufficiently large ℓ. Thus, the sum of these elements can be viewed as a homology class in $H_*(\mathbb{X}_{i,p/q}^+)$ whose projection to $H_*(A^+_{\lfloor \frac{i+ps}{q} \rfloor})$ is $[\xi_s]$.

By symmetry, if $-s = \lfloor \frac{i+p\ell}{q} \rfloor < 0$, if $[\xi_{-s}] \in \mathrm{Ker}(H(h_{-s}^+)\colon H_*(A_{-s}^+) \to H_*(B^+))$, there is a homology class in $H(\mathbb{X}_{i,p/q}^+)$ whose projection to $H(A^+_{\lfloor \frac{i+pj}{q} \rfloor})$ vanishes for $j > \ell$, and whose projection to $H(A^+_{\lfloor \frac{i+p\ell}{q} \rfloor})$ coincides with $[\xi_{-s}]$.

With these remarks in place, we turn to the proof of the proposition, distinguishing two cases according to whether or not $H^{\mathrm{red}}(A_s^+) = 0$.

Suppose first that $H^{\mathrm{red}}(A_s^+) \neq 0$. There is a homology class $[\xi_s] \in H(A_s^+)$ in $\mathrm{Ker}(H(v^+)\colon H(A_s^+) \to H(B^+))$ so that $[\xi_s]$ represents a non-zero element element of $H^{\mathrm{red}}(A_s^+)$. The extension property verified at the beginning of the proof gives linearly independent homology classes in $H^{\mathrm{red}}(\mathbb{X}^+_{i,p/q})$, one for each times A^+_s appears in $\mathbb{X}^+_{i,p/q}$. Thus, the surgery formula gives a q-dimensional lattice in $HF^+_{\mathrm{red}}(S^3_{p/q})$. Another q-dimensional lattice is supplied by corresponding element in $H^{\mathrm{red}}(A^+_{-s})$, showing that $\mathrm{rk} HF^+_{\mathrm{red}}(S^3_{p/q}(K)) \geq 2q$.

We argue that $d(S^3_{p/q}(K), i) \leq d(S^3_{p/q}(O), i)$ for all i, as follows. In the model of the unknot O, there is some $A^+_t(O)$ summand in $\mathbb{X}^+_{i,p/q}(O)$ (the minimal positive element in $\{\frac{\lfloor i+p\ell}{q}\rfloor\}_{\ell\in\mathbb{Z}}$) so that the non-zero class $H_0(A^+_t(O))$ (where now we use the grading on A^+_t for which the map $H(v^+_t)\colon H(A^+_t) \to H(B^+) \cong \mathcal{T}^+_{(0)}$ is grading preserving) extends to give a homology class in $H_d(S^3_{p/q}(O), i)$ with $d = d(S^3_{p/q}(O), i)$. Returning to K, since the projection maps $\mathbb{X}^+_{i,p/q} \to A_t \xrightarrow{v^+_t} B^+$ induce isomorphisms in homology for all sufficiently large positive degrees, it follows that there is an element from $U^\ell H_*(\mathbb{X}^+_{p/q})$ for all $\ell \geq 0$ that restricts to a non-zero element of $H_0(A_t)$, which in turn corresponds to an element of $H_d(S^3_{p/q}(K))$. It follows at once that $d(S^3_{p/q}(K), i) \leq d(S^3_{p/q}(O))$, as stated.

Suppose next that $H^{\mathrm{red}}(A^+_s) = 0$, but the map on homology induced by v^+_s is not an isomorphism. In this case, $H_*(A^+_s) = \mathcal{T}^+_{(-2n)}$, and the map $H(v^+_s)\colon H(A^+_s) \to H(B^+)$ is modelled on the projection map $\mathcal{T}^+_{(-2n)} \to \mathcal{T}^+_{(0)}$ for some $n > 0$; and in particular its kernel has rank n.

For all $0 \leq t \leq s$, we claim that $H(A^+_t)$ has a submodule isomorphic to $\mathcal{T}^+_{(-2m)}$ for some $m \geq n$. This follows easily from the fact that $v^+\colon A^+_t \to B^+$ factors through the natural projection from A^+_t to A^+_s and also $v^+\colon A^+_t \to B^+$ is an isomorphism in sufficiently large degree. In particular, the kernel of $H(v^+_t)\colon H_*(A^+_t) \to H_*(B^+)$ for each such $0 \leq t \leq s$ has dimension at least n. Moreover, when $t = 0$, since $v^+_0\colon A^+_0 \to B^+$ and $h^+_0\colon A^+_0 \to B^+$ have the same degree shift, we obtain an at least n-dimensional kernel of $H(v^+_0) \oplus H(h^+_0)\colon H(A^+_0) \to H(B^+) \oplus H(B^+)$.

Consider a Spin^c structure $i \in \mathbb{Z}/p\mathbb{Z}$ modeled on $\mathbb{X}^+_{i,p/q}(K)$. Let $k = k_i$ denote the number of copies of A^+_t with $|t| \leq s$ which appear in this model, and suppose that $k > 0$. The extension property applied to the above remarks gives an element in $\widetilde{\xi} \in H_*(\mathbb{X}^+_{i,p/q})$ which restricts to a non-zero element in $\xi \in H_{-2n}(A^+_t)$ to an element, which is in the image of $U^\ell H_{2\ell-2n}(A^+_t)$ for all $\ell \geq 0$. Indeed, since the projection map from $H_*(\mathbb{X}^+_{i,p/q}) \to H_*(A^+_t)$ is surjective in all sufficiently large degrees, it follows that $\widetilde{\xi} \in U^\ell H_*(\mathbb{X}^+_{i,p/q})$ for all $\ell \geq 0$.

On the other hand, in the model for the unknot O, the generator of $H_d(\mathbb{X}^+_{i,p/q}(O))$ with $d = d(S^3_{p/q}(O), i)$ has some non-trivial component in $H_0(A^+_t)$ where t is chosen with minimal absolute value among all integers in the set $\{\lfloor\frac{i+p\ell}{q}\rfloor\}_{\ell\in\mathbb{Z}}$. We can thus conclude that $d(S^3_{p/q}(K), i) \leq d(S^3_{p/q}(O), i) - 2n$. Moreover, there is a remaining $(k-1)n$-dimensional subspace in $H(\mathbb{X}^+_{i,p/q})$ corresponding to generators in $H_{2j}(A^+_t)$ with $-2n < 2j < 0$ with $|t| \leq s$ that induce non-trivial elements in $HF^+_{\mathrm{red}}(S^3_{p/q}(K), i)$, showing that $\mathrm{rk}HF_{\mathrm{red}}(S^3_{p/q}(K), i) \geq n(k-1)$. It follows that

$$\mathrm{rk}HF^+_{\mathrm{red}}(S^3_{p/q}(K), i) - \left(\frac{d(S^3_{p/q}(K), i) - d(S^3_{p/q}(O), i)}{2}\right) \geq nk_i.$$

Summing over all $i = 0, \ldots, p-1$, and noting that $\sum_{i=0}^{p-1} k_i = q(2|s| + 1)$, we see that

$$\sum_{i=0}^{p-1} \mathrm{rk}HF^+_{\mathrm{red}}(S^3_{p/q}(K), i) - \left(\frac{d(S^3_{p/q}(K), i) - d(S^3_{p/q}(O), i)}{2}\right) \geq 2q,$$

completing the proposition in the case where $H^{\mathrm{red}}(A^+_s) = 0$.

□

Lemma 3.3. *Let $K \subset S^3$ be a knot with genus g. Then, there is a short exact sequence*

$$0 \longrightarrow \widehat{\mathrm{HFK}}(K, g) \longrightarrow H_*(A^+_{g-1}) \xrightarrow{H(v^+_{g-1})} H_*(B^+) \longrightarrow 0$$

Proof. There is an obvious short exact sequence

$$0 \longrightarrow C\{(-1, g-1)\} \longrightarrow A^+_{g-1} \xrightarrow{v^+_{g-1}} B^+ \longrightarrow 0$$

inducing a long exact sequence in homology. On the other hand, the map on homology v^+_{g-1} is surjective for simple algebraic reasons. (as it is an isomorphism in all sufficiently large degrees, it is U-equivariant, and the automorphism of $H_*(B^+)$ induced by U is surjective.) Finally, note that $H_*(C\{(-1, g-1)\}) \cong \widehat{\mathrm{HFK}}(K, g)$. □

Proof of Theorems 1.1 and 1.2. The case where $p = 0$ follows from [2, Corollary 8.23]. Thus, by reflecting the knot if necessary, we can assume that $p/q > 0$. Assume that K is a knot with $S^3_{p/q}(K) \cong S^3_{p/q}(K_0)$, with $K_0 \in$

$\{T_r, T_\ell, S\}$. Combining Propositions 3.1 and 3.2, we conclude that for all $s > 0$, $v_s^+ \colon A^+ \to B^+$ induces an isomorphism on homology. From Lemma 3.3, we conclude that $\widehat{\mathrm{HFK}}(K, s) = 0$ for all $s > 1$. In view of [8], this already proves that K has genus one.

Let M denote the rank of the kernel of $H(v_0^+) \colon H(A_0^+) \to H(B^+)$. Using the extension property from the proof of Proposition 3.2 (now for $\ker H(v_0^+)$, instead of $\ker H(v_s^+)$ with $s > 0$) it follows that

$$q \cdot M \leq \mathrm{rk} HF^+_{\mathrm{red}}(S^3_{p/q}(K)) - \sum_{i \in \mathbb{Z}/p\mathbb{Z}} \left(\frac{d(S^3_{p/q}(K), i) - d(S^3_{p/q}(O), i)}{2} \right).$$

From Proposition 3.1, it follows that $M = 1$. Thus, by Lemma 3.3, we see that $\widehat{\mathrm{HFK}}(K, 1)$ has rank one, computed choosing $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}/n\mathbb{Z}$ (for any prime n), so $\widehat{\mathrm{HFK}}(K, 1) \cong \mathbb{Z}$. By Ghiggini's theorem, it follows that K is either the figure eight knot or the trefoil.

Another look at the Floer homology groups $S^3_{p/q}(K_0)$ as stated in Proposition 3.1 then allows one to conclude that $K = K_0$. □

Proof of Corollary 1.3. Note that $\Sigma(2, 3, 7)$ cannot be realized as $1/n$ surgery on any knot in S^3 with $|n| > 1$. This follows from the surgery formula for Casson's invariant λ, together with the fact that $|\lambda(\Sigma(2, 3, 7))| = 1$. The corollary is now a direct application of Theorems 1.1 and 1.2. □

Remark 3.4. *Let $K_0 \in \{T_\ell, T_r, S\}$. The proof we gave shows that if the knot Floer homology of K is different from that of K_0, then for all $r \neq 0$, the Heegaard Floer homology of $S^3_r(K)$ is different from that of $S^3_r(K_0)$. In fact, this result also holds when $r = 0$. This follows by reducing to the case $r = \pm 1$, using the surgery exact sequence in gradings, as in [6, Section 8].*

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