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On the Chern numbers for pseudo-free circle actions

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Let (M, ψ) be a $(2n+1)$-dimensional oriented closed manifold with a pseudo-free S^1-action $\psi : S^1 \times M \to M$. We first define a *local data* $\mathcal{L}(M, \psi)$ of the action ψ which consists of pairs $(C, (p(C); \overrightarrow{q}(C)))$ where C is an exceptional orbit, $p(C)$ is the order of isotropy subgroup of C, and $\overrightarrow{q}(C) \in (\mathbb{Z}^{\times}_{p(C)})^n$ is a vector whose entries are the weights of the slice representation of C. In this paper, we give an explicit formula of the Chern number $\langle c_1(E)^n, [M/S^1] \rangle$ modulo $\mathbb{Z}$ in terms of the local data, where $E = M \times_{S^1} \mathbb{C}$ is the associated complex line orbibundle over M/S^1. Also, we illustrate several applications to various problems arising in equivariant symplectic topology.

1. Introduction

Let N be a $2n$-dimensional oriented closed manifold and $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle group acting on N effectively where we denote the action by $\phi : S^1 \times N \to N$. The localization theorem due to Atiyah-Bott

[AB] and Berline-Vergne [BV] is a very powerful technique for computing global (topological) invariants of N in terms of local data

$$\mathcal{L}(N, \phi) = \{(F, \nu_{S^1}(F))\}_{F \subset N^{S^1}}$$

where F is a connected component of the fixed point set N^{S^1} and $\nu_{S^1}(F)$ is an S^1-equivariant normal bundle of F in N. In particular if N admits an S^1-invariant almost complex structure, then we can compute the Chern numbers of the tangent bundle TN in terms of the local data $\mathcal{L}$.

In this paper, we attempt to find an odd dimensional analogue of the ABBV-localization theorem in the sense that if we have a $(2n+1)$-dimensional oriented closed manifold M equipped with an effective fixed-point-free S^1-action $\psi : S^1 \times M \to M$, then our aim is to find a method for computing global invariants in terms of local data. Here, local data means

$$\mathcal{L}(M, \psi) = \{(M^{\mathbb{Z}_p}, \nu_{S^1}(M^{\mathbb{Z}_p}))\}_{p \in \mathbb{N}, p > 1}$$

where $\mathbb{Z}_p$ is the cyclic subgroup of S^1 of order p, $M^{\mathbb{Z}_p}$ is a submanifold of M fixed by $\mathbb{Z}_p$, and $\nu_{S^1}(M^{\mathbb{Z}_p})$ is an S^1-equivariant normal bundle of $M^{\mathbb{Z}_p}$ in M. To do this, let us consider the following commutative diagram

$$\begin{array}{ccc} M \times \mathbb{C} & \xrightarrow[q]{/S^1} & M \times_{S^1} \mathbb{C} = E \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[q]{/S^1} & M/S^1 = B \end{array}$$

where S^1 acts on $M \times \mathbb{C}$ by

$$t \cdot (x, z) = (t \cdot x, tz)$$

for every $t \in S^1$ and $(x, z) \in M \times \mathbb{C}$.

If the action is free, then B is a smooth manifold and E becomes a complex line bundle over B with the first Chern class $c_1(E) \in H^2(B; \mathbb{Z})$. In particular, the Chern number $\langle c_1(E)^n, [B] \rangle$ is an integer where $[B] \in H_{2n}(B; \mathbb{Z})$ is the fundamental homology class of B.

On the other hand, if the action is not free, then B is an orbifold with cyclic quotient singularities and E becomes a complex line orbibundle over B. Then the first Chern class $c_1(E) \in H^2(B, \mathbb{R})$ is defined, via the Chern-Weil construction, as a cohomology class represented by a differential 2-form Θ_α on B where α is a normalized connection 1-form on M and Θ_α is a 2-form on B such that $d\alpha = q^*\Theta_\alpha$. Then the Chern number of E is given by

$$\langle c_1(E)^n, [B]\rangle = \int_B \Theta_\alpha \wedge \Theta_\alpha \wedge \cdots \wedge \Theta_\alpha = \int_M \alpha \wedge (d\alpha)^n$$

which is a rational number in general (see [W, Theorem 1]). However, the local data $\mathcal{L}(M,\psi)$ does not detect any information about free orbits by definition of $\mathcal{L}(M,\psi)$. In fact, if the S^1-action is free, then the local data $\mathcal{L}(M,\psi)$ is an empty set. Thus to make our work to be meaningful, we will construct an invariant, namely $e(M,\psi)$, of (M,ψ) which is zero if ψ is a free action, and it measures the contributions of exceptional orbits to the Chern number of the complex line orbibundle associated to (M,ψ).

Now, let us define

$$e(M,\psi) = \langle c_1(E)^n, [B]\rangle \pmod{\mathbb{Z}}.$$

This number is well-defined up to S^1-equivariant diffeomorphism. Also, we have $e(M,\psi) = 0$ if ψ is a free action. Thus the invariant $e(M,\psi)$ is a good candidate which can be computed in terms of the local data $\mathcal{L}(M,\psi)$.

Now, consider an S^1-manifold M and fix a point x in the interior $\mathring{M}$ of M. Let C be an orbit of x whose isotropy subgroup is $\mathbb{Z}_{p(C)}$ where $p(C)$ be the order of the isotropy subgroup of C. By the slice theorem (see Theorem 2.1), there exists an S^1-equivariant neighborhood $\mathcal{U}$ of C such that

$$\mathcal{U} \cong S^1 \times_{\mathbb{Z}_{p(C)}} V_x$$

where V_x is the slice representation of $\mathbb{Z}_{p(C)}$ at x.

Proposition 1.1. *Let (M,ψ) be a $(2n+1)$-dimensional fixed-point-free S^1-manifold. Suppose that $C \subset \mathring{M}$ is an orbit with the isotropy subgroup $\mathbb{Z}_p$, which is possibly trivial. Then there exists an S^1-equivariant tubular neighborhood $\mathcal{U}$ of C which is S^1-equivariantly diffeomorphic to $S^1 \times \mathbb{C}^n$ where S^1 acts on $S^1 \times \mathbb{C}^n$ by*

$$t \cdot (w, z_1, z_2, \ldots, z_n) = (t^p w, t^{q_1} z_1, t^{q_2} z_2, \ldots, t^{q_n} z_n)$$

for some integers $q_1, q_2, \ldots, q_n$. Moreover, the (unordered) integers q_j's are uniquely determined modulo p.

In other words, Proposition 1.1 says that an S^1-equivariant tubular neighborhood of the form $S^1 \times_{\mathbb{Z}_{p(C)}} V_x$ can be trivialized as a product space and the given action can be expressed as a linear action.

In this paper, we deal with the case where the action is *pseudo-free*. Recall that an S^1-action on a smooth manifold M is called *pseudo-free* if there is no fixed point and there are only finitely many exceptional orbits. Equivalently, the action on M is pseudo-free if the quotient space M/S^1 has only isolated cyclic quotient singularities. Let $\mathcal{E} = \mathcal{E}(M, \psi)$ be the set of exceptional orbits of (M, ψ). Then Proposition 1.1 implies that each $C \in \mathcal{E}$ with the stabilizer $\mathbb{Z}_{p(C)}$ assigns a vector

$$\overrightarrow{q}(C) = (q_1(C), q_2(C), \dots, q_n(C)) \in (\mathbb{Z}_{p(C)}^\times)^n,$$

where $\mathbb{Z}_p^\times$ is a multiplicative group consisting of elements in $\mathbb{Z}_p$ which are coprime to p. We call $\overrightarrow{q}(C)$ the *weight-vector*, and say that C is *of* $(p(C); \overrightarrow{q}(C))$*-type*.

Remark 1.2. Note that $\overrightarrow{q}(C)$ is unique up to ordering of $q_i(C)$'s.

Thus if the action $\psi : S^1 \times M \to M$ is pseudo-free, then the local data of (M, ψ) is given by

$$\mathcal{L}(M, \psi) = \{(C, (p(C); \overrightarrow{q}(C)))\}_{C \in \mathcal{E}}.$$

In Section 4, we give an explicit formula (Theorem 1.4) of $e(M, \psi)$ in terms of the local data $\mathcal{L}(M, \psi)$ if ψ is a pseudo-free S^1 action on M. As a first step, we prove the following.

Proposition 1.3. *Let $p > 1$ be an integer and let $\overrightarrow{q} = (q_1, \dots, q_n) \in (\mathbb{Z}_p^\times)^n$. Then there exists a $(2n+1)$-dimensional oriented closed pseudo-free S^1-manifold (M, ψ) having exactly one exceptional orbit C of $(p; \overrightarrow{q})$-type. Moreover,*

$$e(M, \psi) = \frac{q_1^{-1} q_2^{-1} \cdots q_n^{-1}}{p} \pmod{\mathbb{Z}}$$

where q_j^{-1} is the inverse of q_j in $\mathbb{Z}_p^\times$.

Using Proposition 1.3, we prove our main theorem as follows.

Theorem 1.4. *Suppose that (M, ψ) is a $(2n+1)$-dimensional oriented closed pseudo-free S^1-manifold with the set $\mathcal{E}$ of exceptional orbits. Then*

$$e(M, \psi) = \sum_{C \in \mathcal{E}} \frac{q_1(C)^{-1} q_2(C)^{-1} \cdots q_n(C)^{-1}}{p(C)} \pmod{\mathbb{Z}}$$

where $q_j(C)^{-1}$ is the inverse of $q_j(C)$ in $\mathbb{Z}_{p(C)}^\times$.

Theorem 1.4 has particularly interesting applications when we consider a pseudo-free S^1-manifold (M,ψ) such that $e(M,\psi)=0$. In this case, our theorem gives a constraint on the local data $\mathcal{L}(M,\psi)$ given by

$$\sum_{C\in\mathcal{E}} \frac{q_1^{-1}(C)q_2^{-1}(C)\cdots q_n^{-1}(C)}{p(C)} \equiv 0 \pmod{\mathbb{Z}}.$$

As immediate applications, we can obtain the following corollaries where the proof will be given in Section 5.

Corollary 1.5. *Suppose that (M,ψ) is an oriented closed pseudo-free S^1-manifold with $e(M,\psi)=0$. If the action is not free, then M contains at least two exceptional orbits. If M contains exactly two exceptional orbits, then they must have the same isotropic subgroup.*

Corollary 1.6. *Suppose that (M,ψ) is an oriented closed pseudo-free S^1-manifold with $e(M,\psi)=0$. If C is an exceptional orbit with the isotropy subgroup $\mathbb{Z}_p$ for some $p>1$, there exists an exceptional orbit $C'\neq C$ with the isotropy subgroup $\mathbb{Z}_{p'}$ for some integer p' such that* $\gcd(p,p')\neq 1$.

Now, we illustrate two types of such examples. One is a product manifold equipped with a pseudo-free S^1-action.

Proposition 1.7. *Let (M,ψ) be a $(2n+1)$-dimensional oriented closed fixed-point-free S^1-manifold. If $M=M_1\times M_2$ for some closed S^1-manifolds M_1 and M_2 with positive dimensions, then $e(M,\psi)=0$.*

By using Theorem 1.4 and Proposition 1.7, we can prove the following.

Corollary 1.8. *Let (M,J) be a closed almost complex S^1-manifold. Suppose that the action preserves J and that there are only isolated fixed points. Then,*

$$\sum_{z\in M^{S^1}} \frac{1}{\prod_{i=1}^n q_i(z)} = 0$$

where $q_1(z),\ldots,q_n(z)$ are the weights of the S^1-representation on T_zM.

Remark 1.9. Note that Corollary 1.8 also can be obtained by the ABBV-localization theorem (see Section 5 for the detail). Thus the authors expect that there would be some equivariant cohomology theory which covers both the odd-dimensional theory (Theorem 1.4) and the even-dimensional theory (ABBV-localization theorem). This work is still in progress.

The other type of examples comes from equivariant symplectic geometry as follows. Recall that for a given symplectic S^1-action ψ on a closed symplectic manifold (M,ω) where $[\omega] \in H^2(M;\mathbb{Z})$, there exists an S^1-invariant map $\mu : M \to \mathbb{R}/\mathbb{Z} \cong S^1$ called a *generalized moment map* defined by

$$\mu(x) := \int_{\gamma_x} i_{\underline{X}}\omega \pmod{\mathbb{Z}}$$

where x_0 is a base point, and γ_x is any path $\gamma_x : [0,1] \to M$ such that $\gamma_x(0) = x_0$ and $\gamma_x(1) = x$. When ψ has no fixed point, then M becomes a fiber bundle over S^1 via μ (see [CKS] for the details).

Proposition 1.10. *Let (M,ω) be a closed symplectic manifold equipped with a fixed-point-free S^1-action ψ preserving ω. Let $\mu : M \to \mathbb{R}/\mathbb{Z}$ be a generalized moment map and let $F_\theta = \mu^{-1}(\theta)$ for $\theta \in \mathbb{R}/\mathbb{Z}$. Then $e(F_\theta, \psi|_{F_\theta}) = 0$.*

Finally, here we discuss the Weinstein's theorem [W, Theorem 1] and pose some conjecture.

Theorem 1.11. *[W] Let (M,ψ) be a $(2n+1)$-dimensional closed oriented fixed-point-free S^1-manifold. Let α be a normalized connection 1-form on M. Then*

$$\ell^n \cdot \int_M \alpha \wedge (d\alpha)^n \in \mathbb{Z}$$

where ℓ is the least common multiple of the orders of the isotropy subgroups of the points in M.

Let $(M/S^1)_{\text{sing}}$ be the set of singular points in M/S^1. Our main theorem 1.4 implies that if $\dim(M/S^1)_{\text{sing}} = 0$, then we have

$$\ell \cdot \int_M \alpha \wedge (d\alpha)^n \in \mathbb{Z}.$$

We pose the following conjecture.

Conjecture 1.12. Under the same assumption of Theorem 1.11, we have

$$\ell^{k+1} \cdot \int_M \alpha \wedge (d\alpha)^n \in \mathbb{Z}.$$

where $k = \dim(M/S^1)_{\text{sing}}$.

It is obvious that Conjecture 1.12 is true when $k = 0$ by Theorem 1.4. One can verify that Conjecture 1.12 is true when M is an odd-dimensional sphere with a fixed-point-free linear S^1-action (see Proposition 3.8).

This paper is organized as follows. In Section 2, we define a *local data* for a fixed-point-free S^1-action. In Section 3, we define a Chern class of a closed fixed-point-free S^1-manifold and give the explicit computation of the Chern class of an odd-dimensional sphere equipped with a linear S^1-action. In Section 4, we give the complete proof of Proposition 1.3 and Theorem 1.4. Finally in Section 5, we discuss several applications of Theorem 1.4 and give the proofs of Corollary 1.5, 1.6, and 1.8. Also, we deal with the examples illustrated above and give the complete proof of Proposition 1.7 and 1.10.

2. Local invariants

The main purpose of this section is to define a *local invariant* for each exceptional orbit, which is invariant under S^1-equivariant diffeomorphisms. To do this, we first describe a neighborhood of each orbit.

Theorem 2.1 (Slice theorem). *[Au] Let G be a compact Lie group acting on a manifold M. Let $x \in M$ be a point whose isotropy subgroup is H. Then there exist a G-equivariant tubular neighborhood $\mathcal{U}$ of the orbit $G \cdot m$ and a G-equivariant diffeomorphism*

$$G \times_H V_x \to \mathcal{U}$$

where G acts on $G \times_H V_x$ by

$$g \cdot [g', v] = [gg', v]$$

for every $g \in G$ and $[g', v] \in G \times_H V_x$. Here V_x, called a slice *at x, is the vector space $T_xM/T_x(G \cdot x)$ with the linear H-action induced by the G-action on T_xM.*

In our case, $G = S^1$ and the isotropy subgroup H of x is isomorphic to $\mathbb{Z}_p$ for some $p \geq 1$ if x is not fixed by the S^1-action. The following lemma will be used frequently throughout this paper.

Lemma 2.2. *Let $m > 1$ be a positive integer and let $(w_0, w_1, \ldots, w_n)$ be the coordinate system of $S^1 \times \mathbb{C}^n$. Define an S^1-action on $S^1 \times \mathbb{C}^n$ given by*

$$t \cdot (w_0, w_1, \ldots, w_n) = (t^{x_0} w_0, t^{x_1} w_1, \ldots, t^{x_n} w_n)$$

for some $(x_0, x_1, \ldots, x_n) \in \mathbb{Z}^{n+1}$ with $\gcd(x_0, m) = 1$. Similarly, for $\xi = e^{\frac{2\pi i}{m}}$, define a $\mathbb{Z}_m$-action on $S^1 \times \mathbb{C}^n$ by

$$\xi \cdot (w_0, w_1, \ldots, w_n) = (\xi^{m_0} w_0, \xi^{m_1} w_1, \ldots, \xi^{m_n} w_n)$$

for some $(m_0, m_1, \ldots, m_n) \in \mathbb{Z}^{n+1}$ with $\gcd(m, m_0) = 1$. Then,

1) *the S^1-action and the $\mathbb{Z}_m$-action commutes,*
2) *the $\mathbb{Z}_m$-quotient $S^1 \times_{\mathbb{Z}_m} \mathbb{C}^n$ with the induced S^1-action is S^1-equivariantly diffeomorphic to $S^1 \times \mathbb{C}^n$ with an S^1-action given by*

$$t \cdot (z_0, z_1, \ldots, z_n) = (t^{x_0 m} z_0, t^{-x_0 a_1 + x_1} z_1, \ldots, t^{-x_0 a_n + x_n} z_n),$$

   where $a_i = m_0^{-1} m_i$ modulo m, and
3) *if $\mathbb{Z}_m$ act as a subgroup of S^1 on $S^1 \times \mathbb{C}^n$, or equivalently, if $m_i = x_i$ for every $i = 0, 1, \ldots, n$, then $S^1 \times_{\mathbb{Z}_m} \mathbb{C}^n$ with the induced $S^1/\mathbb{Z}_m$-action is equivariantly diffeomorphic to $S^1 \times \mathbb{C}^n$ with an S^1-action given by*

$$t \cdot (z_0, z_1, \ldots, z_n) = (t^{x_0} z_0, t^{s_i} z_1, \ldots, t^{s_n} z_n),$$

   where $s_i = m^{-1} x_i$ modulo x_0.

Proof. The first claim (1) is straightforward by direct computation. For (2), since m_0 is coprime to m, for each $i \geq 1$, there exist integers a_i and s_i such that

$$m_0 a_i + m s_i = m_i.$$

Then we can easily see that $a_i = m_0^{-1} m_i$ modulo m.

Now, we define a map $\Phi : S^1 \times_{\mathbb{Z}_m} \mathbb{C}^n \to S^1 \times \mathbb{C}^n$ as

$$\phi([w_0, \ldots, w_n]) = (w_0^m, w_0^{-a_1} w_1, \ldots, w_0^{-a_n} w_n).$$

Then Φ is well-defined since

$$\begin{aligned}
&\Phi([\xi^{m_0}w_0, \xi^{m_1}w_1, \ldots, \xi^{m_n}w_n]) \\
&= (\xi^{m_0 m} w_0^m, \xi^{-m_0 a_1 + m_1} w_0^{-a_1} w_1, \ldots, \xi^{-m_0 a_n + m_n} w_0^{a_n} w_n) \\
&= (w_0^m, \xi^{ms_1} w_0^{-a_1} w_1, \ldots, \xi^{ms_n} w_0^{-a_n} w_n) \\
&= (w_0^m, w_0^{-a_1} w_1, \ldots, w_0^{-a_n} w_n) = \Phi([w_0, w_1, \ldots, w_n]).
\end{aligned}$$

The surjectivity of Φ is obvious so that it is enough to show that Φ is injective. If

$$\Phi([w_0, \ldots, w_n]) = \Phi([w_0', \ldots, w_n']),$$

then

- $w_0^m = (w_0')^m$ and
- $w_0^{-a_i} w_i = (w_0')^{-a_i} w_i'$ for every $i = 1, 2, \ldots, n$.

These imply that

- $w_0' = \xi^{km_0} w_0$ for some $k \in \mathbb{Z}$ (since ξ^{m_0} is also a generator of $\mathbb{Z}_m$), and
- $w_0^{-a_i} w_i = \xi^{-km_0 a_i} w_0^{-a_i} w_i'$.

Thus we have $w_i' = \xi^{km_0 a_i} w_i$ for every $i = 1, 2, \ldots, n$. Therefore, we have

$$\begin{aligned}
[w_0, w_1, \ldots, w_n] &= [\xi^{km_0} w_0, \xi^{km_1} w_1, \ldots, \xi^{km_n} w_n] \\
&= [\xi^{km_0} w_0, \xi^{k(m_0 a_1 + ms_1)} w_1, \ldots, \xi^{k(m_0 a_n + ms_n)} w_n] \\
&= [\xi^{km_0} w_0, \xi^{km_0 a_1} w_1, \ldots, \xi^{km_0 a_n} w_n] = [w_0', w_1', \ldots, w_n'].
\end{aligned}$$

To show that Φ is S^1-equivariant, we define an S^1-action on $S^1 \times \mathbb{C}^n$ as

$$t \cdot (z_0, z_1, \ldots, z_n) = (t^{mx_0} z_0, t^{-x_0 a_1 + x_1} z_1, \ldots, t^{-x_0 a_n + x_n} z_n).$$

Then the S^1-equivariance of Φ is as following.

$$\begin{aligned}
\Phi(t \cdot [w_0, w_1, \ldots, w_n]) &= \Phi([t^{x_0} w_0, t^{x_1} w_1, \ldots, t^{x_n} w_n]) \\
&= (t^{mx_0} w_0^m, t^{-x_0 a_1 + x_1} w_0^{-a_1} w_1, \ldots, t^{-x_0 a_n + x_n} w_0^{-a_n} w_n) \\
&= t \cdot (w_0^m, w_0^{-a_1} w_1, \ldots, w_0^{-a_n} w_n) \\
&= t \cdot \Phi([w_0, w_1, \ldots, w_n]).
\end{aligned}$$

To show (3), suppose that $\mathbb{Z}_m$ acts on $S^1 \times \mathbb{C}^n$ as a subgroup of S^1, i.e., $m_i = x_i$ for every $i = 0, 1, \ldots, n$. By definition of a_i and s_i, we have

$$-x_0 a_i + x_i = -m_0 a_0 + m_i = m s_i.$$

Then $s_i = m^{-1} x_i$ modulo x_0 since x_0 is coprime to m. Thus for every $i = 1, 2, \ldots, n$, the number $-x_0 a_i + x_i$ is a multiple of m. Hence the S^1-action given as above is non-effective and it has a weight-vector $(m x_0, m s_1, \ldots, m s_n)$. Therefore, after taking a quotient by $\mathbb{Z}_m$ which acts trivially on $S^1 \times \mathbb{C}^n$, the residual $S^1/\mathbb{Z}_m$-action is given as in (3). □

Now, let us consider a $(2n+1)$-dimensional S^1-manifold (M, ψ). Then for each $x \in \mathring{M}$, Theorem 2.1 implies that $V_x \cong \mathbb{R}^{2n}$ and the orbit $S^1 \cdot x$ has an S^1-equivariant tubular neighborhood diffeomorphic to $S^1 \times_H \mathbb{R}^{2n}$ where H is the isotropy subgroup of x. The following proposition states that $S^1 \times_H \mathbb{R}^{2n}$ is in fact S^1-equivariantly diffeomorphic to the product space $S^1 \times \mathbb{C}^n$ with a certain linear S^1-action.

Proposition 2.3 (Proposition 1.1). *Let (M, ψ) be a $(2n+1)$-dimensional fixed-point-free S^1-manifold. Suppose that $C \subset \mathring{M}$ is an orbit with the isotropy subgroup $\mathbb{Z}_p$, which is possibly trivial. Then there exists an S^1-equivariant tubular neighborhood $\mathcal{U}$ which is S^1-equivariantly diffeomorphic to $S^1 \times \mathbb{C}^n$ where S^1 acts on $S^1 \times \mathbb{C}^n$ by*

$$t \cdot (z_0, z_1, z_2, \ldots, z_n) = (t^p z_0, t^{q_1} z_1, t^{q_2} z_2, \ldots, t^{q_n} z_n)$$

for some integers $q_1, q_2, \ldots, q_n$. Moreover, the (unordered) integers q_j's are uniquely determined modulo p.

Proof. Let $x \in M$ be a point in M with the isotropy subgroup $\mathbb{Z}_p \subset S^1$, and let $V_x \cong \mathbb{R}^{2n}$ be the slice at x. Recall that any orientation preserving irreducible real representation of $\mathbb{Z}_p$ is two-dimensional, and it is isomorphic to a one-dimensional complex representation of $\mathbb{Z}_p$ determined by a rotation number modulo p. Thus $V_x \cong \mathbb{C}^n$ and a $\mathbb{Z}_p$-action on $S^1 \times V_x$ is given by

$$\xi \cdot (w_0, w_1, \ldots, w_n) = (\xi w_0, \xi^{-q_1} w_1, \ldots, \xi^{-q_n} w_n)$$

for every $(w_0, w_1, \ldots, w_n) \in S^1 \times V_x$ where $\xi = e^{\frac{2\pi i}{p}}$ and q_i's are integers uniquely determined modulo p, see [Ko, p.647] for more details.

Let C be an orbit containing x. By the slice theorem 2.1, there exists an S^1-equivariant tubular neighborhood $\mathcal{U}$ of C which can be identified with

$S^1 \times_{\mathbb{Z}_p} V_x$ where the S^1-action on $S^1 \times_{\mathbb{Z}_p} V_x$ is induced from the S^1-action on $S^1 \times V_x$ given by

$$t \cdot (w_0, w_1, \dots, w_n) = (tw_0, w_1, \dots, w_n)$$

for every $t \in S^1$ and $(w_0, w_1, \dots, w_n) \in S^1 \times V_x$.

Now we apply Lemma 2.2 with $m = p$, $x_0 = m_0 = 1$ and $x_i = 0$, $m_i = -q_i$ for $i \geq 1$. Then we may choose $a_i = -q_i$ and $s_i = 0$ for $i \geq 1$ so that

$$m_0 a_i + m s_i = 1 \cdot (-q_i) + m \cdot 0 = -q_i = m_i.$$

Therefore, we obtain an S^1-equivalent diffeomorphism

$$\Phi : S^1 \times_{\mathbb{Z}_p} V_x \to S^1 \times \mathbb{C}^n,$$

where S^1-action on the target is given by

$$\begin{aligned} t \cdot (z_0, z_1, \dots, z_n) &= (t^{mx_0} z_0, t^{-x_0 a_1 + x_1} z_1, \dots, t^{-x_0 a_n + x_n} z_n) \\ &= (t^p z_0, t^{q_1} z_1, \dots, t^{q_n} z_n). \end{aligned}$$

This completes the proof. □

By Proposition 2.3, each exceptional orbit C assigns a vector

$$\overrightarrow{q}(C) = (q_1(C), q_2(C), \dots, q_n(C)) \in (\mathbb{Z}_{p(C)})^n$$

which is uniquely determined up to ordering of $q_i(C)$'s where $p(C)$ is an order of the isotropy subgroup of C. We call $\overrightarrow{q}(C)$ a *weight-vector of* C.

Now, assume that (M, ψ) is a $(2n+1)$-dimensional closed pseudo-free S^1-manifold and let $\mathcal{E}$ be the set of exceptional orbits. Then each $C \in \mathcal{E}$ is isolated so that $\gcd(p(C), q_i(C)) = 1$ for every $i = 1, 2, \dots, n$, i.e.,

$$\overrightarrow{q}(C) \in (\mathbb{Z}^\times_{p(C)})^n.$$

Definition 2.4. Let (M, ψ) be a $(2n+1)$-dimensional pseudo-free S^1-manifold with the set $\mathcal{E}$ of exceptional orbits.

1) A *local data* $\mathcal{L}(M, \psi)$ is defined by

$$\mathcal{L}(M, \psi) = \left\{ (C, (p(C); \overrightarrow{q}(C))) \;\middle|\; p(C) \in \mathbb{N}, \overrightarrow{q}(C) \in \left(\mathbb{Z}^\times_{p(C)}\right)^n \right\}_{C \in \mathcal{E}}.$$

2) We call $(p(C); \overrightarrow{q}(C))$ the *local invariant* of C, and we say that C is of $(p(C); \overrightarrow{q}(C))$*-type*.

3. Chern numbers of fixed-point-free circle actions

In this section, we give a brief review of the definition of the first Chern class of fixed-point-free S^1-manifolds. Also we give an explicit computation of the Chern number of an odd-dimensional sphere equipped with a linear action and explain how the Chern number (modulo $\mathbb{Z}$) can be computed in terms of a local data.

We first review the classical result about a principal bundle over a smooth manifold.

Definition 3.1. Let G be a compact Lie group and $\mathfrak{g}$ be the Lie algebra of G. Let M be a principal G-bundle. A *connection form* α on M is a smooth $\mathfrak{g}$-valued 1-form such that

- $\alpha(\underline{X}) = X$ for every $X \in \mathfrak{g}$, and
- α is G-invariant

where $\underline{X}$ is a vector field on M, called the *fundamental vector field* of X, defined by

$$\underline{X}_x := \left.\frac{d}{dt}\right|_{t=0} (\exp(tX) \cdot x)$$

for every $x \in M$.

For a given connection form α on M, the *curvature form* Ω_α associated to α is a $\mathfrak{g}$-valued 2-form on M defined by

$$\Omega_\alpha = d\alpha + [\alpha, \alpha].$$

In particular, if G is abelian, then the Lie bracket $[\cdot, \cdot]$ vanishes so that we have $\Omega_\alpha = d\alpha$.

Suppose that $G = S^1 \in \mathbb{C}$ be the unit circle group with the Lie algebra $\mathfrak{s}^1$. Also, let (M, ψ) be a fixed-point-free S^1-manifold with a connection form α. Then α can be viewed as an $\mathbb{R}$-valued 1-form via a linear identification map $\varepsilon : \mathfrak{s}^1 \to \mathbb{R}$. Note that ε is determined by the image $\varepsilon(X) \in \mathbb{R}$ where X is the generator of the kernel of the exponential map $\exp\colon \mathfrak{s}^1 \to S^1$. We say that α is normalized if an identification map ε is chosen to be

$$\varepsilon(X) = 1.$$

Equivalently, α is normalized if $S^1 = \mathfrak{s}^1 / \ker(\exp) \cong \mathbb{R}/\mathbb{Z}$ and $\alpha(\underline{X}) = 1$ where $X = \frac{\partial}{\partial\theta}$ and θ is a parameter of $\mathbb{R}$. In particular, if α is normalized, then we have

$$\int_F \alpha = 1$$

for any free orbit F (see also Remark 3.6).

The following proposition is well-known and the proof is given in [Au]. But we give the complete proof here to show that it can be extended to the case of a fixed-point-free action.

Proposition 3.2. *[Au] Let M be a principal S^1-bundle over a smooth manifold B and let $\alpha \in \Omega^1(M)$ be a normalized connection 1-form on M. Then,*

- *there exists a unique closed 2-form Θ_α on B such that $q^*\Theta_\alpha = d\alpha$ where $q : M \to B$ is the quotient map,*
- *$[\Theta_\alpha] \in H^2(B;\mathbb{R})$ is independent of the choice of α, and*
- *$[\Theta_\alpha]$ is equal to the first Chern class of the associated complex line bundle $M \times_{S^1} \mathbb{C}$ over B where S^1 acts on $M \times \mathbb{C}$ by*

$$t \cdot (x, z) = (t \cdot x, tz)$$

for every $t \in S^1$ and $(x, z) \in M \times \mathbb{C}$.

Proof. Recall the Cartan's formula which is given by

$$\mathcal{L}_{\underline{X}} = i_{\underline{X}} \circ d + d \circ i_{\underline{X}}.$$

By applying the Cartan's formula to α, we have

$$\mathcal{L}_{\underline{X}}\alpha = i_{\underline{X}} \circ d\alpha + d \circ i_{\underline{X}}\alpha = 0.$$

Since $i_{\underline{X}}\alpha \equiv 1$, we have $i_{\underline{X}}d\alpha = 0$, i.e. $d\alpha$ is horizontal. Also, by applying the Cartan's formula to $d\alpha$, we have

$$\mathcal{L}_{\underline{X}}d\alpha = i_{\underline{X}}d^2\alpha + di_{\underline{X}}d\alpha = 0.$$

Therefore, there exists a push-forward of $d\alpha$, namely Θ_α, on B such that $q^*\Theta_\alpha = d\alpha$. It is straightforward that such a Θ_α is unique.

To prove the second statement, let β be another connection form on M. Then it is obvious that $\alpha - \beta$ is S^1-invariant and $i_{\underline{X}}(\alpha - \beta) = 0$. Thus there exists an 1-form γ on B such that $q^*\gamma = \alpha - \beta$. In other words, $d\gamma = \Theta_\alpha - \Theta_\beta$ so that $[\Theta_\alpha] = [\Theta_\beta]$ in $H^2(B;\mathbb{R})$.

To prove the third statement, recall that for a given smooth manifold N, there is a one-to-one correspondence between the set of principal S^1-bundles over N and the set of homotopy classes of maps $[N, BS^1]$ where ES^1 is a contractible space on which S^1 acts freely, and $BS^1 = ES^1/S^1$ is the classifying space of S^1. By applying this argument to M, we have a map $f : B \to BS^1$ and an S^1-equivariant map $\widetilde{f} : M \to ES^1$ such that

$$\begin{CD} M @>{\widetilde{f}}>> ES^1 \\ @V{q}VV @VV{\widetilde{q}}V \\ B @>{f}>> BS^1 \end{CD}$$

commutes.

Now, let α_0 be a normalized connection form on ES^1. Since $\widetilde{f}$ is S^1-equivariant, the pull-back $\widetilde{f}^*\alpha_0$ is also a normalized connection form on M so that we have $f^*\Theta_{\alpha_0} = \Theta_{\widetilde{f}^*\alpha_0}$. Furthermore, the above diagram induces a bundle morphism

$$\begin{CD} M \times_{S^1} \mathbb{C} @>{\widetilde{f}_{\mathbb{C}}}>> ES^1 \times_{S^1} \mathbb{C} \\ @V{q_{\mathbb{C}}}VV @VV{\widetilde{q}_{\mathbb{C}}}V \\ B @>{f}>> BS^1 \end{CD}$$

for any fixed linear S^1-action on $\mathbb{C}$ where $q_{\mathbb{C}}$ ($\widetilde{q}_{\mathbb{C}}$, respectively) is an extension of q ($\widetilde{q}$, respectively). Therefore, by the naturality of characteristic classes, it is enough to show that $[\Theta_{\alpha_0}]$ is equal to the first Chern class of the complex line bundle

$$\mathcal{O}(1) := ES^1 \times_{S^1} \mathbb{C} \to BS^1$$

where S^1 acts on $ES^1 \times \mathbb{C}$ by

$$t \cdot (x, z) = (t \cdot x, tz)$$

for every $t \in S^1$ and $(x, z) \in ES^1 \times \mathbb{C}$. Then it follows from Corollary 3.9. □

Let us consider a fixed-point-free S^1-manifold M. Even though the action is not free, we can find a connection form as follows.

Proposition 3.3. *Let (M, ψ) be a closed fixed-point-free S^1-manifold. Then there exist an $\mathfrak{s}^1$-valued 1-form α, called a* connection form *on M, such that*

- $\alpha(\underline{X}) = X$ *for every* $X \in \mathfrak{s}^1$, *and*

- *α is S^1-invariant.*

Proof. Let ℓ be a least common multiple of the orders of the isotropy subgroups of the elements in M and let $\mathbb{Z}_\ell$ be the cyclic subgroup of S^1 of order ℓ. Then we have a quotient map $\pi_\ell : M \to M/\mathbb{Z}_\ell$ and the quotient space $M/\mathbb{Z}_\ell$ becomes an orbifold. Note that $S^1/\mathbb{Z}_\ell$ acts on the quotient space $M/\mathbb{Z}_\ell$ freely so that $M/\mathbb{Z}_\ell$ is a principal $S^1/\mathbb{Z}_\ell$-bundle over $B = M/S^1$. The slice theorem 2.1 implies that the quotient space B is an orbifold, in particular, B is paracompact, see [Sa] for the detail. Since any principal S^1-bundle over a paracompact space admits a connection form(c.f. [KN, Chap II]), there exists a connection form α' on $M/\mathbb{Z}_\ell$. Then it is not hard to check that

$$\alpha = \frac{1}{\ell}\pi_\ell^*\alpha'$$

is our desired 1-form. □

Lemma 3.4. *Let α be a normalized connection form on M. There exists a unique closed 2-form Θ_α on M/S^1 such that $q^*\Theta_\alpha = d\alpha$ where $q : M \to M/S^1$ is the quotient map. Moreover, $[\Theta_\alpha] \in H^2(M/S^1; \mathbb{R})$ does not depend on the choice of α.*

Proof. The proof is exactly same as in the proof of Proposition 3.2. □

Now, we define the first Chern class of a fixed-point-free S^1-manifold as follows.

Definition 3.5. Let (M, ψ) be a closed fixed-point-free S^1-manifold. Let α be a normalized connection form on M. Then we call $[\Theta_\alpha] \in H^2(M/S^1; \mathbb{R})$ *the first Chern class (or the Euler class) of* (M, ψ) and we denote by $c_1(M, \psi)$.

Remark 3.6. [CdS, page 194] The reader should keep in mind that a connection form α is an $\mathfrak{s}^1$-valued 1-form, and we need to identify $\mathfrak{s}^1$ with $\mathbb{R}$ via ε to regard α as a usual $\mathbb{R}$-valued differential form. For example, Audin [Au, Example V.4.4] used an identification map $\varepsilon(\frac{\partial}{\partial\theta}) = 2\pi$ and defined the Chern class by $[\frac{1}{2\pi}\Theta_\alpha]$. In [CdS], Cannas da Silva used the same identification map as in our paper.

Note that since α is normalized, we have $\int_{S^1} \alpha = 1$. Thus if M is of dimension $2n+1$, then we have

$$\langle c_1(M,\psi)^n, [B]\rangle = \int_B \Theta_\alpha \wedge \Theta_\alpha \wedge \cdots \wedge \Theta_\alpha = \int_M \alpha \wedge (d\alpha)^n$$

where $B = M/S^1$ and $[B] \in H_{2n}(B;\mathbb{Z})$ is the fundamental homology class of B.

Remark 3.7. The theory of characteristic classes of orbibundles is well established in the case of *good orbibundles*. In fact, they are defined as elements of *orbifold cohomology*. In our case, B is the quotient space of M by a pseudo-free S^1-action and the orbifold cohomology $H^*_{orb}(B)$ is the same as the equivariant cohomology $H^*_{S^1}(M)$ (see [ALR, Proposition 1.51]). Also, the fibration

$$q_{S^1} : M \times_{S^1} ES^1 \to M/S^1 = B$$

induces an isomorphism $q^*_{S^1} : H^*(B) \to H^*_{S^1}(M)$ with coefficients in a field (see [ALR, Proposition 2.12]). With this identification, one can see that the Chern class $c_1(M,\psi) \in H^2(B;\mathbb{R})$ defined above is actually the same as the Chern class $c_1(\widetilde{E}) \in H^2(M \times_{S^1} ES^1) = H^2_{orb}(B;\mathbb{R})$ (defined as in [ALR, page 45]) of the associated line bundle

$$\widetilde{E} = (M \times \mathbb{C}) \times_{S^1} ES^1 \to M \times_{S^1} ES^1$$

In other words, we have

$$q^*_{S^1}(c_1(M,\psi)) = c_1(\widetilde{E}).$$

The following proposition gives an explicit computation of the Chern numbers of odd-dimensional spheres equipped with linear S^1-actions, which we use crucially to prove Proposition 1.3 and Theorem 1.4.

Proposition 3.8. *Suppose that an S^1-action ψ on $S^{2n-1} \subset \mathbb{C}^n$ is given by*

$$t \cdot (z_1, z_2, \ldots, z_n) = (t^{p_1} z_1, t^{p_2} z_2, \ldots, t^{p_n} z_n)$$

for some $(p_1, p_2, \ldots, p_n) \in (\mathbb{Z} \setminus \{0\})^n$. Then

$$\langle c_1(S^{2n-1},\psi)^n, [S^{2n-1}/S^1]\rangle = \frac{1}{\prod_{i=1}^n p_i}.$$

Proof. We will use real coordinates $(x_j, y_j) = z_j = x_j + iy_j$ for $j = 1, 2, \dots, n$. Recall that $\mathfrak{s}^1$ is identified with $\mathbb{R}$ which is parametrized by θ and $t = e^{2\pi i\theta}$. For $X = \frac{\partial}{\partial\theta}$, we have

$$\underline{X} = 2\pi \sum_j p_j \left(-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j} \right).$$

Define a connection form α on S^{2n-1} such that

$$\alpha = \frac{1}{2\pi} \sum_j \frac{1}{p_j} (-y_j dx_j + x_j dy_j).$$

Then we can easily check that α is a normalized connection form on S^{2n-1}. By differentiating α, we have

$$\begin{aligned} d\alpha &= \frac{1}{2\pi} \sum_j \frac{1}{p_j} (-dy_j \wedge dx_j + dx_j \wedge dy_j) \\ &= \frac{1}{2\pi} \sum_j \frac{2}{p_j} dx_j \wedge dy_j \\ &= \frac{1}{\pi} \sum_j \frac{1}{p_j} dx_j \wedge dy_j. \end{aligned}$$

Therefore, we have

$$\int_{S^{2n-1}} \alpha \wedge (d\alpha)^{n-1} = \int_{D^{2n}} (d\alpha)^n = \pi^{-n} \frac{n!}{\prod_j p_j} \mathrm{Vol}(D^{2n}) = \frac{1}{\prod_j p_j}$$

where the first equality comes from the Stoke's theorem. □

Corollary 3.9. *Let $\pi : ES^1 \to BS^1$ be the universal S^1-bundle and let α_0 be a normalized connection form on ES^1. Then the curvature form Θ_{α_0} on BS^1 represents the first Chern class of the complex line bundle*

$$\mathcal{O}(1) = ES^1 \times_{S^1} \mathbb{C}$$

where S^1 acts on $ES^1 \times \mathbb{C}$ by $t \cdot (x, z) = (t \cdot x, tz)$ for every $t \in S^1$ and $(x, z) \in ES^1 \times \mathbb{C}$.

Proof. Recall that the universal bundle $\widetilde{q} : ES^1 \to BS^1$ can be constructed as an inductive limit of the sequence of Hopf fibrations

$$\begin{array}{ccccccccc} S^3 & \hookrightarrow & S^5 & \hookrightarrow & \cdots & S^{2n+1} & \cdots & \hookrightarrow & ES^1 \sim S^\infty \\ \downarrow & & \downarrow & & & \downarrow & & & \downarrow \\ \mathbb{C}P^1 & \hookrightarrow & \mathbb{C}P^2 & \hookrightarrow & \cdots & \mathbb{C}P^n & \cdots & \hookrightarrow & BS^1 \sim \mathbb{C}P^\infty, \end{array}$$

where S^1 acts on $S^{2n-1} \subset \mathbb{C}^n$ by

$$t \cdot (z_1, z_2, \ldots, z_n) = (tz_1, tz_2, \ldots, tz_n).$$

Since $\mathcal{O}(1)$ is the dual bundle of the tautological line bundle $\mathcal{O}(-1)$ over BS^1, we have

$$c_1(\mathcal{O}(1)) = u \in H^2(BS^1; \mathbb{Z})$$

where u is the positive generator of $H^2(BS^1; \mathbb{Z}) \cong H^2(\mathbb{C}P^\infty; \mathbb{Z})$. Thus it is enough to show that

$$\left\langle [\Theta_{\alpha_0}], [\mathbb{C}P^1] \right\rangle = \int_{S^3} \alpha \wedge d\alpha = 1.$$

This follows from Proposition 3.8. $\square$

Remark 3.10. In [Ka, page 245], Kawasaki described a cohomology ring structure (over $\mathbb{Z}$) of the quotient space S^{2n+1}/S^1 where S^1-action ψ on S^{2n+1} is given by

$$t \cdot (z_0, z_1, z_2, \ldots, z_n) = (t^{p_0} z_0, t^{p_1} z_1, \ldots, t^{p_n} z_n)$$

for any positive integers $p_0, p_1, \ldots, p_n$ such that $\gcd(p_0, p_1, \ldots, p_n) = 1$. The ring structure of $H^*(S^{2n+1}/S^1; \mathbb{Z})$ is as follows. Let γ_k be the positive generator of $H^{2k}(S^{2n+1}/S^1; \mathbb{Z}) \cong \mathbb{Z}$. Then

$$\gamma_1 \cdot \gamma_k = \frac{\ell_1 \ell_k}{\ell_{k+1}} \gamma_{k+1}$$

where

$$\ell_k = \operatorname{lcm} \left\{ \frac{p_{i_0} p_{i_1} \cdots p_{i_k}}{\gcd(p_{i_0}, p_{i_1}, \ldots, p_{i_k})} \;\middle|\; 0 \le i_0 < \cdots < i_k \le n \right\}.$$

In particular, we have $\ell_1 = \mathrm{lcm}(p_0, p_1, \ldots, p_n)$ and $\ell_n = p_0p_1 \cdots p_n$ since the action is effective. Then it is not hard to show that

$$\gamma_1^n = \frac{\ell_1^n}{\ell_n}\gamma_n.$$

On the other hand, Godinho [Go, Proposition 2.15] proved that the action has the first Chern class

$$c_1(S^{2n+1}, \psi) = \frac{\gamma_1}{\mathrm{lcm}(p_0, p_1, \ldots, p_n)} = \frac{\gamma_1}{\ell_1}.$$

Consequently, the Chern number is

$$\langle c_1(S^{2n+1}, \psi)^n, [S^{2n+1}/S^1]\rangle = \frac{1}{\ell_n}\langle \gamma_n, [S^{2n+1}/S^1]\rangle = \frac{1}{p_0p_1 \cdots p_n}$$

which coincides with Proposition 3.8.

Remark 3.11. In [Lia], Liang studied the Chern number of a $(2n+1)$-dimensional homotopy sphere Σ^{2n+1} equipped with a differentiable pseudo-free S^1-action

$$\phi : S^1 \times \Sigma^{2n+1} \to \Sigma^{2n+1}$$

under certain assumption. More precisely, he proved that if there are exactly k exceptional orbits $C_1, \ldots, C_k$ in Σ^{2n+1} with isotropy subgroups $\mathbb{Z}_{q_1}, \ldots, \mathbb{Z}_{q_k}$ for some positive integers $q_1, \ldots, q_k$ such that $\gcd(q_i, q_j) = 1$ for each i, j with $i \neq j$, then

$$\langle c_1(\Sigma^{2n+1}, \phi)^n, [\Sigma^{2n+1}/S^1]\rangle = \pm\frac{1}{q_1 \cdots q_k}.$$

His result does not involve the condition "modulo $\mathbb{Z}$" since the proof relies on the fact [MY] that there exists an S^1-equivariant map of degree ± 1 from Σ^{2n+1} to S^{2n+1} where an S^1-action ϕ' on S^{2n+1} is given by

$$t \cdot (z_1, \ldots, z_{n+1}) = (t^{q_1 \cdots q_k} z_1, tz_2, \ldots, tz_{n+1}).$$

Thus we can obtain

$$\langle c_1(\Sigma^{2n+1}, \phi)^n, [\Sigma^{2n+1}/S^1]\rangle = \pm\langle c_1(S^{2n+1}, \phi')^n, [S^{2n+1}/S^1]\rangle = \pm\frac{1}{q_1 \cdots q_k}$$

where the equality on the right hand side comes from Proposition 3.8. Consequently, we cannot extend Liang's result (without "module $\mathbb{Z}$") to a general case by the lack of such an S^1-equivariant map to S^{2n+1}.

4. Proofs of Proposition 1.3 and Theorem 1.4

In this section, we give the complete proofs of Proposition 1.3 and Theorem 1.4. Throughout this section, for a given oriented manifold M, we denote M with the opposite orientation by $-M$.

Definition 4.1. Let (M, ψ) be a compact oriented fixed-point-free S^1-manifold with free S^1-boundary ∂M, i.e., ψ is free on ∂M. A *resolution* $\mathbf{N}$ of (M, ψ) is a triple (N, ϕ, h) consisting of a compact oriented free S^1-manifold (N, ϕ) with boundary ∂N and an orientation-preserving S^1-equivariant diffeomorphism $h : \partial N \to \partial M$ with respect to ϕ and ψ.

Remark 4.2. Suppose that M and N are given as in Definition 4.1. Then M/S^1 has singularities, while N/S^1 does not. If W is a singular space and if there exists a subset of W which is diffeomorphic to M/S^1, then we can always remove M/S^1 and glue N/S^1 along $\partial M/S^1$. In this manner, we can think of

$$\widetilde{W} := \left(W \setminus (\mathring{M}/S^1)\right) \bigsqcup N/S^1$$

as a resolution of W. This is the reason why we use the terminology 'resolution' in Definition 4.1.

Let (M, ψ) be an oriented compact fixed-point-free S^1-manifold with free S^1-boundary ∂M, and let $\mathbf{N} = (N, \phi, h)$ be a resolution of (M, ψ). Then we can glue M and N along their boundaries ∂M and ∂N by using h as follows.

By the equivariant collar neighborhood theorem [K, Theorem 3.5], there exist closed S^1-equivariant neighborhoods of ∂M and ∂N which are S^1-equivariantly diffeomorphic to $\partial M \times [0, \epsilon]$ and $\partial N \times [0, \epsilon]$, respectively, where S^1 acts on the left factors. Then we may extend h to a map $\overline{h}$ on $\partial N \times [0, \epsilon]$ to $\partial M \times [0, \epsilon]$ as

$$\begin{array}{cccc} \overline{h} & : & \partial N \times [0, \epsilon] & \to & \partial M \times [0, \epsilon] \\ & & (x, t) & \mapsto & (h(x), \epsilon - t). \end{array}$$

Note that the extended map $\overline{h}$ is S^1-equivariant and orientation-reversing. Thus we can glue $\mathring{M}$ and $-\mathring{N}$ along $\partial M \times (0, \epsilon)$ and $\partial N \times (0, \epsilon)$ via $\overline{h}$. Thus we get a closed fixed-point-free S^1-manifold

$$M_{\mathbf{N}} = \mathring{M} \sqcup_{\overline{h}} -\mathring{N},$$

where the S^1-action $\psi_{\mathbf{N}}$ on $M_{\mathbf{N}}$ is given by

$$\psi_{\mathbf{N}} = \psi \sqcup_{\overline{h}} \phi.$$

Notice that if ψ is pseudo-free, then so is $\psi_{\mathbf{N}}$.

Lemma 4.3. *Let (M, ψ) be a compact fixed-point-free S^1-manifold with free S^1-boundary ∂M. Suppose that there exists a resolution $\mathbf{N}$ of (M, ψ). Then $e(M_{\mathbf{N}}, \psi_{\mathbf{N}})$ is independent of the choice of a resolution $\mathbf{N}$.*

Proof. Suppose that there are two resolutions $\mathbf{N}_1 = (N_1, \phi_1, h_1)$ and $\mathbf{N}_2 = (N_2, \phi_2, h_2)$ of (M, ψ) so that we have two closed fixed-point-free S^1-manifolds

$$\begin{aligned}(M_{\mathbf{N}_1}, \psi_{\mathbf{N}_1}) &= (\mathring{M} \sqcup_{\overline{h}_1} -\mathring{N}_1, \psi \sqcup_{\overline{h}_1} \phi_1), \text{ and} \\ (M_{\mathbf{N}_2}, \psi_{\mathbf{N}_2}) &= (\mathring{M} \sqcup_{\overline{h}_2} -\mathring{N}_2, \psi \sqcup_{\overline{h}_2} \phi_2).\end{aligned}$$

For the sake of simplicity, we denote by $\overline{M}_i = M_{\mathbf{N}_i}$ and $\overline{\psi}_i = \psi_{\mathbf{N}_i}$ for each $i = 1, 2$. Then our aim is to prove that

$$e(\overline{M}_1, \overline{\psi}_1) = e(\overline{M}_2, \overline{\psi}_2).$$

Now, let α_∂ be a connection form on ∂M. Then α_∂ can be extended to a connection form, which we still denote by α_∂, on $\partial M \times [0, \epsilon]$ via the projection $\partial M \times [0, \epsilon] \to \partial M$. Let α be a connection form on M such that the restriction of α to a closed collar neighborhood $\partial M \times [0, \epsilon]$ is α_∂. Such an α always exists by the existence of a partition of unity (see [KN, Theorem 2.1]). Similarly, for each $i = 1, 2$, we can construct a connection form α_i on N_i such that the restriction of α_i on $\partial N_i \times [0, \epsilon]$ is the pull-back $\overline{h}_i^* \alpha_\partial$.

Since α and α_i agree on $\partial M \times [0, \epsilon]$ and $\partial N_i \times [0, \epsilon]$ via $\overline{h}_i$, we can define a connection form $\overline{\alpha}_i$ on $\overline{M}_i$ by gluing α and α_i via $\overline{h}_i$. Then we have

$$\begin{aligned} e(\overline{M}_1, \overline{\psi}_1) - e(\overline{M}_2, \overline{\psi}_2) &\equiv \int_{\overline{M}_1} \overline{\alpha}_1 \wedge (d\overline{\alpha}_1)^n - \int_{\overline{M}_2} \overline{\alpha}_2 \wedge (d\overline{\alpha}_2)^n \\ &\equiv \int_{(\mathring{M} \setminus \partial M \times (0,\epsilon)) \sqcup -\mathring{N}_1} \overline{\alpha}_1 \wedge (d\overline{\alpha}_1)^n \\ &\quad - \int_{(\mathring{M} \setminus \partial M \times (0,\epsilon)) \sqcup -\mathring{N}_2} \overline{\alpha}_2 \wedge (d\overline{\alpha}_2)^n \\ &\equiv \int_{N_2} \alpha_2 \wedge (d\alpha_2)^n - \int_{N_1} \alpha_1 \wedge (d\alpha_1)^n \pmod{\mathbb{Z}}. \end{aligned}$$

On the other hand, $h := h_2^{-1} \circ h_1 : \partial N_1 \to \partial N_2$ is an orientation-preserving S^1-equivariant diffeomorphism so that (N_2, ϕ_2, h) is a resolution of (N_1, ϕ_1). Thus we can glue N_1 and N_2 along the collar neighborhoods of their boundaries via $\overline{h}$. If we let

$$(\overline{N}, \overline{\phi}) = (\mathring{N}_2 \sqcup_{\overline{h}} -\mathring{N}_1, \phi_2 \sqcup_{\overline{h}} \phi_1),$$

then $(\overline{N}, \overline{\phi})$ becomes a closed free S^1-manifold. In particular, α_i on $\partial N_i \times [0, \epsilon]$ agree with $\overline{h}_i^* \alpha_\partial$ so that there exists a connection form $\overline{\alpha}$ on $\overline{N}$ such that $\overline{\alpha}|_{\mathring{N}_i} = \alpha_i|_{\mathring{N}_i}$ for each $i = 1, 2$. Consequently, since $\overline{\phi}$ is free, we have

$$\begin{aligned} 0 &\equiv \int_{\overline{N}} \overline{\alpha} \wedge (d\overline{\alpha})^n \ (\mathrm{mod}\, \mathbb{Z}) = \int_{\mathring{N}_2 \setminus \partial N_2 \times (0,\epsilon) \sqcup -\mathring{N}_1} \overline{\alpha} \wedge (d\overline{\alpha})^n \\ &= \int_{N_2} \alpha_2 \wedge (d\alpha_2)^n - \int_{N_1} \alpha_1 \wedge (d\alpha_1)^n - \int_{\partial N_2 \times (0,\epsilon)} \alpha_2 \wedge (d\alpha_2)^n. \end{aligned}$$

Since α_2 on $\partial N_2 \times (0, \epsilon)$ is the same as $\overline{h}_2^* \alpha_\partial$ and $\alpha_\partial \wedge (d\alpha_\partial)^n = 0$, the last term vanishes. Therefore

$$\int_{N_2} \alpha_2 \wedge (d\alpha_2)^n - \int_{N_1} \alpha_1 \wedge (d\alpha_1)^n \equiv 0 \ (\mathrm{mod}\, \mathbb{Z})$$

which completes the proof. □

In general, for a compact fixed-point-free S^1-manifold with free S^1-boundary, we do not know whether a resolution always exists. However, if we consider a closed tubular neighborhood of an isolated exceptional orbit, then a resolution always exists (see Proposition 1.3). To show this, suppose that there exists a closed S^1-manifold (M, ψ) having only one exceptional orbit C. Then the local data of (M, ψ) is given by

$$\mathcal{L}(M, \psi) = \{(C, (p; \overrightarrow{q}))\},$$

for some $p = p(C) \in \mathbb{N}$ and $\overrightarrow{q} = \overrightarrow{q}(C) = (q_1, q_2, \ldots, q_n) \in \left(\mathbb{Z}_p^\times\right)^n$. Then by Proposition 1.1, there exists a tubular neighborhood $\mathcal{U} \cong S^1 \times \mathbb{C}^n$ of C such that the S^1-action is given by

$$t \cdot (w, z_1, z_2, \ldots, z_n) = (t^p w, t^{q_1} z_1, t^{q_2} z_2, \ldots, t^{q_n} z_n)$$

for every $t \in S^1$ and $(w, z_1, z_2, \ldots, z_n) \in S^1 \times \mathbb{C}^n$. Observe that the complement $M \setminus \mathcal{U}$ of $\mathcal{U}$ defines a resolution of $\overline{\mathcal{U}}$. Thus Lemma 4.3 implies that $e(M, \psi)$ depends only on $(\overline{\mathcal{U}}, \psi|_{\overline{\mathcal{U}}})$, or equivalently, the local invariant $(p; \overrightarrow{q})$ of C. We first show the existence of such an (M, ψ) in the case where $n = 1$.

Notation 4.4. From now on, we denote the closed unit disk in $\mathbb{C}$ by D and identify $\overline{\mathcal{U}}$ with $S^1 \times D^n$. Moreover, we denote $(S^1 \times D^n, (p; q_1, q_2, \ldots, q_n))$ the space $S^1 \times D^n$ equipped with an S^1-action given by

$$t \cdot (w, z_1, z_2, \ldots, z_n) = (t^p w, t^{q_1} z_1, t^{q_2} z_2, \ldots, t^{q_n} z_n)$$

for every $t \in S^1$ and $(w, z_1, z_2, \ldots, z_n) \in S^1 \times D^n$.

Lemma 4.5. *Let $p > 1$ be an integer and let $q \in \mathbb{Z}_p^\times$. Then there exists a 3-dimensional closed pseudo-free S^1-manifold (M, ψ) having exactly one exceptional orbit C of $(p; q)$-type. Furthermore, we have*

$$e(M, \psi) = \frac{q^{-1}}{p} \pmod{\mathbb{Z}}$$

where $q^{-1} q \equiv 1$ in $\mathbb{Z}_p^\times$.

Proof. By Proposition 1.1 and Definition 2.4, there exists an S^1-equivariant closed tubular neighborhood of C isomorphic to $(S^1 \times D, (p; q))$. Let $m = q^{-1}$ be the inverse of q modulo p and a be an integer satisfying

$$pa + mq = 1.$$

Now, let us consider a linear S^1-action $\overline{\psi}$ on $S^3 = \partial(D \times D) \subset \mathbb{C}^2$ given by

$$t \cdot (z_1, z_2) = (t^p z_1, t z_2).$$

We first claim that $S^3/\mathbb{Z}_m$ with the induced $S^1/\mathbb{Z}_m$-action, namely ψ, is our desired manifold (M, ψ) where $\mathbb{Z}_m \subset S^1$ is the cyclic subgroup of S^1 of order m.

Observe that $S^3 = D \times S^1 \cup S^1 \times D$ so that

$$S^3/\mathbb{Z}_m = D \times_{\mathbb{Z}_m} S^1 \cup S^1 \times_{\mathbb{Z}_m} D.$$

Since the S^1-action on $D \times S^1$ is free, the induced $S^1/\mathbb{Z}_m$-action ψ on $D \times_{\mathbb{Z}_m} S^1$ is also free. Thus it is enough to show that the action ψ on $S^1 \times_{\mathbb{Z}_m} D$ has only one exceptional orbit of type $(p; q)$.

We apply Lemma 2.2 with m, $x_0 = m_0 = p$, and $x_1 = m_1 = 1$. Then we can choose $a_1 = a$ and $s_1 = q$ and we have a S^1-equivariant diffeomorphism

$$\Phi : S^1 \times_{\mathbb{Z}_m} D \to S^1 \times D$$

where the target admits the residual S^1-action given by

$$t \cdot (w, z) = (t^{x_0} w, t^{s_1} z) = (t^p w, t^q z).$$

Therefore $(S^1 \times_{\mathbb{Z}_m} D, \psi)$ is S^1-equivariantly diffeomorphic to $(S^1 \times D, (p; q))$ and so $(S^3/\mathbb{Z}_m, \psi)$ has exactly one exceptional orbit of $(p; q)$-type as desired.

On the other hand, by Proposition 3.8, we have

$$e(S^3, \overline{\psi}) = \frac{1}{p}.$$

Let $\underline{X}$ ($\underline{X}^m$, respectively) be the fundamental vector field on S^3 ($S^3/\mathbb{Z}_m$, respectively) with respect to $\overline{\psi}$ (ψ, respectively). Then the quotient map $q : S^3 \to S^3/\mathbb{Z}_m$ maps the fundamental vector field $\underline{X}$ to $m\underline{X}^m$. Thus if we choose any connection form α on $S^3/\mathbb{Z}_m$, then $\frac{1}{m} q^* \alpha$ is a connection form on S^3. Therefore, we have

$$\begin{aligned} e(S^3/\mathbb{Z}_m, \psi) &= \int_{S^3/\mathbb{Z}_m} \alpha \wedge d\alpha = \frac{1}{m} \int_{S^3} q^*\alpha \wedge d(q^*\alpha) \\ &= m \int_{S^3} \frac{1}{m} q^*\alpha \wedge \frac{1}{m} d(q^*\alpha) \\ &= \frac{m}{p} \equiv \frac{q^{-1}}{p} \pmod{\mathbb{Z}}. \end{aligned}$$

□

Remark 4.6. In Lemma 4.5, (M, ψ) is not unique. For example, if (M, ψ) is given in Lemma 4.5 and if we perform an S^1-equivariant Dehn surgery along a free orbit in (M, ψ), then we get a new pseudo-free S^1-manifold $(\widetilde{M}, \widetilde{\psi})$ having exactly one exceptional orbit of $(p; q)$-type.

To prove Proposition 1.3, we need the following series of lemmas.

Lemma 4.7. *Suppose that (M, ψ) be a $(2n-1)$-dimensional closed pseudo-free S^1-manifold having only one exceptional orbit C of $(p; q_1, q_2, \ldots, q_{n-1})$-type where $p \in \mathbb{N}$ and $(q_1, \ldots, q_{n-1}) \in (\mathbb{Z}_p^\times)^{n-1}$. Then there exists a $(2n+1)$-dimensional closed pseudo-free S^1-manifold $(\widetilde{M}, \widetilde{\psi})$ having only one orbit $\widetilde{C}$ of $(p; q_1, q_2, \ldots, q_{n-1}, 1)$-type. Moreover, we have $e(M, \psi) = e(\widetilde{M}, \widetilde{\psi})$.*

Proof. Recall that D is the unit disk in $\mathbb{C}$. Let us consider a manifold $M \times D$ with an S^1-action $\overline{\psi}$ given by

$$t \cdot (x, z) := (t \cdot x, tz)$$

for every $t \in S^1$ and $(x, z) \in M \times D$. Then it is obvious that $\overline{\psi}$ has only one exceptional orbit $C \times \{0\}$ of $(p; q_1, q_2, \ldots, q_{n-1}, 1)$-type. Thus it is enough to construct a resolution of $(M \times D, \overline{\psi})$.

Let $E = M \times_{S^1} D$ with an S^1-action ϕ given by

$$t \cdot [x, z] = [t \cdot x, z] = [x, t^{-1}z]$$

for every $t \in S^1$ and $[x, z] \in M \times_{S^1} D$. Then we have $\partial E = M \times_{S^1} S^1 = M$ and ϕ on ∂E coincides with ψ. Thus the product space $N = E \times S^1$ with an S^1-action $\overline{\phi}$ given by

$$t \cdot ([x, z], w) = (t \cdot [x, z], tw)$$

has a boundary $\partial N = M \times S^1$ such that $\overline{\phi}|_{\partial N} = \overline{\psi}|_{M \times S^1}$ via the canonical identification map $h : \partial(M \times D) \to \partial N = \partial(E \times S^1)$. Obviously, $\overline{\phi}$ is free on N so that $(N, \overline{\phi}, h)$ is a resolution of $(M \times D, \overline{\psi})$ if N is smooth. However, the problem is that E is not smooth and neither is N in general. In fact, there is only one isolated singularity $C \times_{S^1} \{0\}$ on the zero section $M \times_{S^1} \{0\} \subset E$ where C is the unique exceptional orbit of (M, ψ). Locally, a neighborhood of $C \times_{S^1} \{0\}$ is S^1-equivariantly diffeomorphic to

$$(S^1 \times \mathbb{C}^{n-1}) \times_{S^1} \mathbb{C} \cong \mathbb{C}^{n-1} \times_{\mathbb{Z}_p} \mathbb{C},$$

where S^1-action ϕ on $\mathbb{C}^{n-1} \times_{\mathbb{Z}_p} \mathbb{C}$ is given by

$$t \cdot [z_1, z_2, \ldots, z_{n-1}, z] = [z_1, z_2, \ldots, z_{n-1}, t^{-1}z]$$

for every $t \in S^1$ and $[z_1, z_2, \ldots, z_{n-1}, z] \in \mathbb{C}^{n-1} \times_{\mathbb{Z}_p} \mathbb{C}$. In other words, $C \times_{S^1} \{0\}$ corresponds to the origin $\mathbf{0}$ in $\mathbb{C}^{n-1} \times_{\mathbb{Z}_p} \mathbb{C}$ which is a cyclic quotient singularity fixed by ϕ. Furthermore, it is a toroidal singularity, i.e., $\mathbb{C}^{n-1} \times_{\mathbb{Z}_p} \mathbb{C}$ is an affine toric variety with the isolated singularity $\mathbf{0}$ equipped with a $(\mathbb{C}^*)^n$-action given by

$$(t_1, t_2, \ldots, t_n) \cdot [z_1, z_2, \ldots, z_{n-1}, z] = [t_1 z_1, t_2 z_2, \ldots, t_{n-1} z_{n-1}, t_n z]$$

for every $(t_1, t_2, \ldots, t_n) \in (\mathbb{C}^*)^n$ and $[z_1, z_2, \ldots, z_{n-1}, z] \in \mathbb{C}^{n-1} \times_{\mathbb{Z}_p} \mathbb{C}$ such that ϕ acts as a subgroup of $(\mathbb{C}^*)^n$. Therefore, by [KKMS, Theorem 11], there exists a $(\mathbb{C}^*)^n$-equivariant resolution of $\mathbb{C}^{n-1} \times_{\mathbb{Z}_p} \mathbb{C}$. Consequently, there exists a ϕ-equivariant resolution E' of E with an extended S^1-action ϕ'. Thus $E' \times S^1$ admits a free S^1-action $\overline{\phi}'$ given by

$$t \cdot (x, w) = (t \cdot x, tw)$$

for every $t \in S^1$ and $(x, w) \in E' \times S^1$. Since $\partial E' = \partial E$ via the canonical identification map, say h', we have a triple $(E' \times S^1, \overline{\phi}', h')$ which is a resolution of $(M \times D, \overline{\psi})$. Therefore, we get a $(2n+1)$-dimensional closed pseudo-free S^1-manifold $(\widetilde{M}, \widetilde{\psi})$

$$\begin{aligned} \widetilde{M} &= M \times D \sqcup_{\overline{h}'} E' \times S^1, \\ \widetilde{\psi} &= \overline{\psi} \sqcup_{\overline{h}'} \overline{\phi}' \end{aligned}$$

where $\overline{h}' : \partial(E' \times S^1) \times [0, \epsilon] \to \partial(M \times D) \times [0, \epsilon]$ is an S^1-equivariant orientation-reversing diffeomorphism defined by h' as before. Obviously, $(\widetilde{M}, \widetilde{\psi})$ has exactly one exceptional orbit of type $(p; q_1, q_2, \ldots, q_{n-1}, 1)$.

Now, it remains to show that $e(\widetilde{M}, \widetilde{\psi}) = e(M, \psi)$. Let $\beta = d\theta$ be the normalized connection form on $D \setminus \{0\}$ with respect to an S^1-action on D given by $t \cdot z = tz$ where

$$D = \{re^{2\pi i\theta} | r, \theta \in [0, 1]\}.$$

We consider the pull-back of $\beta|_{\partial D = S^1}$ along the natural projection $E' \times S^1 \to S^1$ and denote by β again. Then β becomes a normalized connection form on $(E' \times S^1, \overline{\phi}')$.

We will construct a global normalized connection form on $(\widetilde{M}, \widetilde{\psi})$ as follows. Let α be a normalized connection form on (M, ψ) and let $f : [0, 1] \to [0, 1]$ be a smooth function such that $f(r) \equiv 0$ near $r = 0$ and $f(r) \equiv 1$ near $r = 1$. Let

$$\widehat{\alpha} = (1 - f(r))\alpha + f(r)\beta$$

be a one-form on $M \times D$ where $r = |z|$ for $z \in D$. Though β is not defined on the whole $M \times D$, the one-form $\widehat{\alpha}$ is well-defined on the whole $M \times D$ since $f \equiv 0$ near $r = 0$. Moreover, it is obvious that $\widehat{\alpha}$ is a normalized connection form on $M \times D$. In particular, $\widehat{\alpha}$ coincides with β on a neighborhood of $\partial(E' \times S^1) = M \times S^1 = \partial(M \times D)$. Thus we can glue $\widehat{\alpha}$ and β so that we get a global normalized connection form $\widetilde{\alpha}$, i.e. $\widetilde{\alpha}$ is a connection form on

$(\widetilde{M},\widetilde{\psi})$ such that

$$\widetilde{\alpha}|_{\partial(E'\times S^1)} = \beta \quad \text{and} \quad \widetilde{\alpha}|_{M\times D} = \widehat{\alpha}.$$

Since $d\beta = 0$ on $E'\times S^1$, we have

$$\begin{aligned} e(\widetilde{M},\widetilde{\psi}) &\equiv \int_{\widetilde{M}} \widetilde{\alpha}\wedge (d\widetilde{\alpha})^n \pmod{\mathbb{Z}} \\ &= \int_{M\times D} \widehat{\alpha}\wedge (d\widehat{\alpha})^n + \int_{E'\times S^1} \beta\wedge (d\beta)^n - \int_{\partial(E'\times S^1)\times(0,\epsilon)} \beta\wedge (d\beta)^n \\ &= \int_{M\times D} \widehat{\alpha}\wedge (d\widehat{\alpha})^n + 0 + 0 \\ &= \int_{M\times D} \left((1-f)^{n+1}\right)' \beta\wedge dr\wedge \alpha\wedge (d\alpha)^n \\ &= \int_0^1 -\left((1-f)^{n+1}\right)' dr \int_{\partial D}\beta \int_M \alpha\wedge (d\alpha)^n \\ &= e(M,\psi) \end{aligned}$$

which completes the proof. □

Lemma 4.8. *Let (M,ψ) be a $(2n+1)$-dimensional closed pseudo-free S^1-manifold with exactly one exceptional orbit C of $(p;\overrightarrow{q})$-type where $p\in\mathbb{N}$ and $\overrightarrow{q}\in(\mathbb{Z}_p^\times)^n$. Then for any $r\in\mathbb{N}$ with* $\gcd(p,r)=1$, *the quotient space $M/\mathbb{Z}_r$ with the induced $S^1/\mathbb{Z}_r$-action ψ_r is also a pseudo-free S^1-maifold with exactly one exceptional orbit of type $(p;r^{-1}\overrightarrow{q})$ where r^{-1} is the inverse of r in $\mathbb{Z}_p^\times$. Moreover, we have*

$$e(M/\mathbb{Z}_r,\psi_r) = r^n\cdot e(M,\psi) \pmod{\mathbb{Z}}.$$

Proof. Since $\gcd(p,r)=1$, it is straightforward that the $\mathbb{Z}_r$-action on M is free so that $M/\mathbb{Z}_r$ is a smooth manifold. Let $\mathcal{U}$ be an S^1-equivariant neighborhood of C. It is also obvious that ψ is free on $M\setminus\mathcal{U}$ and therefore the induced $S^1/\mathbb{Z}_r$-action ψ_r is also free on $(M\setminus\mathcal{U})/\mathbb{Z}_r$. Therefore, there is no exceptional orbit in $M\setminus\mathcal{U}$ so that we need only to care about a neighborhood $\mathcal{U}$ of C.

We apply Lemma 2.2 with parameters $m=r$, $x_0=m_0=p$ and $x_i=m_i=q_i$ since $\mathbb{Z}_r$ is a subgroup of S^1. Then $\mathcal{U}/\mathbb{Z}_r$ is S^1-equivalently diffeomorphic to $S^1\times\mathbb{C}^n$ such that the induced $S^1/\mathbb{Z}_r$-action on $S^1\times\mathbb{C}^n$ is given by

$$t \cdot (w, z_1, \dots, z_n) = (t^p w, t^{s_1} z_1, \dots, t^{s_n} z_n) = (t^p w, t^{r^{-1}q_1} z_1, \dots, t^{r^{-1}q_n} z_n).$$

where r^{-1} is the inverse of r in $\mathbb{Z}_p^\times$.

Now, it remains to show that

$$e(M/\mathbb{Z}_r, \psi_r) = r^n \cdot e(M, \psi) \pmod{\mathbb{Z}}.$$

Let $\underline{X}$ and $\underline{X}^r$ be the fundamental vector fields of (M, ψ) and $(M/\mathbb{Z}_r, \psi_r)$, respectively. Then the quotient map $q : M \to M/\mathbb{Z}_r$ maps $\underline{X}$ to $r\underline{X}^r$. Let α be a normalized connection form on $M/\mathbb{Z}_r$. Then we can easily check that $\frac{1}{r} q^*\alpha$ is a normalized connection form on M. Then,

$$\begin{aligned} e(M/\mathbb{Z}_r, \psi_r) &= \int_{M/\mathbb{Z}_r} \alpha \wedge (d\alpha)^n = \frac{1}{r} \int_M q^*\alpha \wedge (q^* d\alpha)^n \\ &= r^n \int_M \frac{1}{r} q^*\alpha \wedge (\frac{1}{r} q^* d\alpha)^n \\ &= r^n \cdot e(M, \psi) \end{aligned}$$

which completes the proof. □

Now we are ready to prove Proposition 1.3.

Proposition 4.9 (Proposition 1.3). *Let $p > 1$ be an integer and let $\overrightarrow{q} = (q_1, \dots, q_n) \in (\mathbb{Z}_p^\times)^n$. Then there exists a $(2n+1)$-dimensional oriented closed pseudo-free S^1-manifold (M, ψ) having exactly one exceptional orbit C of $(p; \overrightarrow{q})$-type. Moreover,*

$$e(M, \psi) = \frac{q_1^{-1} q_2^{-1} \cdots q_n^{-1}}{p} \pmod{\mathbb{Z}}$$

where q_j^{-1} is the inverse of q_j in $\mathbb{Z}_p^\times$.

Proof. Let $r_i = q_i q_{i+1}^{-1} \in \mathbb{Z}_p^\times$ for $i < n$ and let $r_n = q_n$. Then

$$q_i \equiv r_i r_{i+1} \cdots r_n \in \mathbb{Z}_p^\times$$

for every $i = 1, 2, \dots, n$. Thus C is of $(p; r_1 r_2 \cdots r_n, \dots, r_{n-1} r_n, r_n)$-type.

By Lemma 4.5, there exists a three-dimensional closed pseudo-free S^1-manifold $(\widetilde{M}_1, \widetilde{\psi}_1)$ having exactly one orbit of type $(p; r_1)$ and

$$e(\widetilde{M}_1, \widetilde{\psi}_1) = \frac{r_1^{-1}}{p} = \frac{q_1^{-1} q_2}{p} \ (\mathrm{mod}\, \mathbb{Z}).$$

By Lemma 4.7, there exists a five-dimensional closed pseudo-free S^1-manifold (M_2, ψ_2) having exactly one orbit of type $(p; r_1, 1)$ and

$$e(M_2, \psi_2) = e(\widetilde{M}_1, \widetilde{\psi}_1).$$

Then, by Lemma 4.8, $\widetilde{M}_2 = M_2/\mathbb{Z}_{r_2^{-1}}$ with the induced $S^1/\mathbb{Z}_{r_2^{-1}}$-action $\widetilde{\psi}_2$ is a five-dimensional closed pseudo-free S^1-manifold having exactly one orbit of type $(p; r_1 r_2, r_2)$ and

$$\begin{aligned} e(\widetilde{M}_2, \widetilde{\psi}_2) &= (r_2^{-1})^2 \cdot e(M_2, \psi_2) \\ &= (q_2^{-2} q_3^2) \cdot e(\widetilde{M}_1, \widetilde{\psi}_1) \\ &= \frac{q_1^{-1} q_2^{-1} q_3^2}{p} \ (\mathrm{mod}\, \mathbb{Z}) \end{aligned}$$

Inductively, we get a $(2n+1)$-dimensional closed pseudo-free S^1-manifold $(\widetilde{M}_n, \widetilde{\psi}_n)$ having exactly one orbit of type $(p; r_1 r_2 \cdots r_n, \dots, r_{n-1} r_n, r_n)$ and

$$\begin{aligned} e(\widetilde{M}_n, \widetilde{\psi}_n) &= (r_n^{-1})^n \cdot e(M_n, \psi_n) \\ &= (q_n^{-n}) \cdot e(\widetilde{M}_{n-1}, \widetilde{\psi}_{n-1}) \\ &= (q_n^{-n}) \cdot \frac{q_1^{-1} q_2^{-1} \cdots q_{n-1}^{-1} q_n^{n-1}}{p} \\ &= \frac{q_1^{-1} q_2^{-1} \cdots q_n^{-1}}{p} \end{aligned}$$

which completes the proof. □

Now, we state and prove our main theorem 1.4.

Theorem 4.10 (Theorem 1.4). *Suppose that (M, ψ) is a $(2n+1)$-dimensional oriented closed pseudo-free S^1-manifold with the set $\mathcal{E}$ of exceptional* *orbits. Then*

$$e(M,\psi) = \sum_{C\in\mathcal{E}} \frac{q_1(C)^{-1}q_2(C)^{-1}\cdots q_n(C)^{-1}}{p(C)} \pmod{\mathbb{Z}}$$

where $q_j(C)^{-1}$ is the inverse of $q_j(C)$ in $\mathbb{Z}_{p(C)}^\times$.

Proof. We use induction on the number of exceptional orbits. Suppose that $|\mathcal{E}| = 1$. Then it follows from Lemma 4.3 that Theorem 1.4 coincides with Proposition 1.3.

Now, let us assume that Theorem 1.4 holds for $|\mathcal{E}| = k-1$. Let (M,ψ) be a $(2n+1)$-dimensional closed pseudo-free S^1-manifold with $\mathcal{E} = \{C_1, C_2, \ldots, C_k\}$.

Assume that C_1 is of $(p;\overrightarrow{q})$-type for some integers $p \geq 2$ and $\overrightarrow{q} \in (\mathbb{Z}_p^\times)^n$. By Proposition 1.3, there exists a $(2n+1)$-dimensional closed pseudo-free S^1-manifold (N,ϕ) having exactly one exceptional orbit C' of $(p;\overrightarrow{q})$-type such that

$$e(N,\phi) = \frac{q_1^{-1}q_2^{-1}\cdots q_n^{-1}}{p} \pmod{\mathbb{Z}}.$$

Let $\mathcal{U}$ and $\mathcal{U}'$ be S^1-equivariant tubular neighborhoods of C_1 and C' respectively so that

$$\overline{\mathcal{U}} \cong (S^1 \times D^n, (p;\overrightarrow{q})) \cong \overline{\mathcal{U}'}$$

where D is the unit disk in $\mathbb{C}$. Let α be a normalized connection form on $S^1 \times D^n$ defined as a pull-back of a normalized connection form on (S^1, p) via the projection map $(S^1 \times D^n, (p;\overrightarrow{q})) \to (S^1, p)$. Let $\alpha_{\mathcal{U}}$ ($\alpha_{\mathcal{U}'}$, respectively) be the normalized connection form on $\mathcal{U}$ ($\mathcal{U}'$, respectively) induced by α via the identifications above. Let α_M (α_N, respectively) be an extension of $\alpha_{\mathcal{U}}$ ($\alpha_{\mathcal{U}'}$, respectively) to whole M (N, respectively) so that

- $d\alpha_M = 0$ on $\mathcal{U}$ and
- $d\alpha_N = 0$ on $\mathcal{U}'$.

Since there exists an obvious S^1-equivariant diffeomorphism $h : \partial\overline{\mathcal{U}'} \to \partial\overline{\mathcal{U}}$, we can easily check that the triple $(N \setminus \mathcal{U}', \phi|_{N\setminus\mathcal{U}'}, h|_{\partial\overline{\mathcal{U}}'})$ is a resolution of $(M \setminus \mathcal{U}, \psi|_{M\setminus\mathcal{U}})$ since ϕ is free on $N \setminus \mathcal{U}'$.

Now, let $(\overline{M}, \overline{\psi})$ be a closed S^1-manifold obtained by gluing $(M \setminus \mathcal{U}, \psi|_{M\setminus\mathcal{U}})$ and $(N \setminus \mathcal{U}', \phi|_{N\setminus\mathcal{U}'})$. Since $h^*(\alpha_M|_{\partial\mathcal{U}}) = \alpha_N|_{\partial\mathcal{U}'}$, we can glue $\alpha_M|_{M\setminus\mathcal{U}}$ and $\alpha_N|_{N\setminus\mathcal{U}'}$ so that we get a normalized connection form $\overline{\alpha}$ on $\overline{M}$ such that

- $\overline{\alpha}|_{M\setminus\mathcal{U}} = \alpha_M$ and

- $\overline{\alpha}|_{N\setminus\mathcal{U}'} = \alpha_N$.

Then,

$$\begin{aligned} e(M,\psi) - e(\overline{M},\overline{\psi}) &= \int_M \alpha_M \wedge (d\alpha_M)^n - \int_{\overline{M}} \overline{\alpha} \wedge (d\overline{\alpha})^n \\ &= \int_{\overline{\mathcal{U}}} \alpha_M \wedge (d\alpha_M)^n - \int_{-N\setminus\mathcal{U}'} \overline{\alpha_N} \wedge (d\overline{\alpha_N})^n \\ &= \int_N \alpha_N \wedge (d\alpha_N)^n \end{aligned}$$

where the last equality comes from the fact that

$$\int_{\overline{\mathcal{U}}} \alpha_M \wedge (d\alpha_M)^n = \int_{\overline{\mathcal{U}'}} \alpha_N \wedge (d\alpha_N)^n = 0.$$

Since $(\overline{M},\overline{\psi})$ has $(k-1)$ exceptional orbits $C_2,\dots,C_k$, by induction hypothesis, we have

$$\begin{aligned} e(M,\psi) &= e(\overline{M},\overline{\psi}) + \int_N \alpha_N \wedge (d\alpha_N)^n \\ &= e(\overline{M},\overline{\psi}) + e(N,\phi) = \sum_{C\in\mathcal{E}} \frac{q_1(C)^{-1}q_2(C)^{-1}\cdots q_n(C)^{-1}}{p(C)} \pmod{\mathbb{Z}} \end{aligned}$$

which completes the proof. □

5. Applications

In this section, we illustrate several applications of Theorem 1.4.

Let (M,ψ) be a $(2n+1)$-dimensional closed pseudo-free S^1-manifold such that $e(M,\psi) = 0$. Then 1.4 implies that

$$\sum_{C\in\mathcal{E}} \frac{q_1^{-1}(C)q_2^{-1}(C)\cdots q_n^{-1}(C)}{p(C)} \equiv 0 \pmod{\mathbb{Z}} \tag{1}$$

where $\mathcal{E}$ is the set of exceptional orbits of ψ. Thus the condition $e(M,\psi) = 0$ gives the constraint (1) on the local data $\mathcal{L}(M,\psi)$. We first give the proofs of Corollary 1.5 and Corollary 1.6 as we see below.

Corollary 5.1 (Corollary 1.5). *Suppose that (M,ψ) is a closed oriented pseudo-free S^1-manifold with $e(M,\psi) = 0$. If the action is not free, then M*

contains at least two exceptional orbits. If M contains exactly two exceptional orbits, then they must have the same isotropic subgroup.

Proof. Recall that the condition 'pseudo-free' implies that a numerator of each summand in (1) is never zero. Thus the first claim is straightforward by Theorem 1.4. If there are exactly two exceptional orbits C_1 and C_2, then

$$\frac{q(C_1)^{-1}}{p(C_1)} + \frac{q(C_2)^{-1}}{p(C_2)} \equiv 0 \pmod{\mathbb{Z}}$$

where

$$q(C_i)^{-1} = q_1(C_i)^{-1}q_2(C_i)^{-1} \cdots q_n(C_i)^{-1} \in \mathbb{Z}^{\times}_{p(C_i)}$$

for $i = 1, 2$. Then $\frac{p(C_2)q(C_1)^{-1}}{p(C_1)} \equiv 0 \pmod{\mathbb{Z}}$ and $\frac{p(C_1)q(C_2)^{-1}}{p(C_2)} \equiv 0 \pmod{\mathbb{Z}}$. Thus $p(C_1) \mid p(C_2)$ and $p(C_2) \mid p(C_1)$ so that $p(C_1) = p(C_2)$. □

Corollary 5.2 (Corollary 1.6). *Suppose that (M, ψ) is an oriented closed pseudo-free S^1-manifold with $e(M, \psi) = 0$. If C is an exceptional orbit with the isotropy subgroup $\mathbb{Z}_p$ for some $p > 1$, there exists an exceptional orbit $C' \neq C$ with the isotropy subgroup $\mathbb{Z}_{p'}$ for some integer p' such that* $\gcd(p, p') \neq 1$.

Proof. Let $C_1, C_2, \ldots, C_k$ be exceptional orbits. Suppose that

$$\gcd(p(C_1), p(C_i)) = 1 \text{ for every } i = 2, 3, \ldots, k.$$

By Theorem 1.4,

$$\frac{q(C_1)^{-1}}{p(C_1)} + \frac{K}{p(C_2)p(C_3)\cdots p(C_k)} \equiv 0 \pmod{\mathbb{Z}}$$

for some $K \in \mathbb{Z}$ where $q(C_1) = q_1(C_1)q_2(C_1)\cdots q_n(C_1)$. By multiplying both sides by $p(C_2)p(C_3)\cdots p(C_k)$, we get

$$\frac{q(C_1)^{-1} \cdot p(C_2)p(C_3)\cdots p(C_k)}{p(C_1)} \in \mathbb{Z}$$

which is a contradiction to the fact that

$$\gcd(p(C_1), q(C_1)) = \gcd(p(C_1), p(C_i)) = 1$$

for $i = 2, 3, \ldots, k$. Thus there exists some $C_j \neq C_1$ with $\gcd(p(C_1), p(C_j)) \neq 1$. □

Now, we illustrate two types of such manifolds. One is a product manifold as follows.

Proposition 5.3 (Proposition 1.7). *Let (M, ψ) be a $(2n+1)$-dimensional oriented closed fixed-point-free S^1-manifold. If $M = M_1 \times M_2$ for some closed S^1-manifolds M_1 and M_2 with positive dimensions, then $e(M, \psi) = 0$.*

Proof. Note that our assumption for ψ implies that the projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ are S^1-equivariant. Since ψ is fixed-point-free, either M_1 or M_2 cannot have a fixed point. Without loss of generality, we may assume that M_1 does not have a fixed point.

Let α_1 be a connection form on M_1. Then $\alpha := \pi_1^* \alpha_1$ becomes a connection form on $M_1 \times M_2$ and it satisfies

$$\int_{M_1 \times M_2} \alpha \wedge (d\alpha)^n = \int_{M_1} \alpha_1 \wedge (d\alpha_1)^n = 0$$

for a dimensional reason. In particular, we have

$$e(M_1 \times M_2, \psi) \equiv \int_{M_1 \times M_2} \alpha \wedge (d\alpha)^n = 0 \pmod{\mathbb{Z}}.$$

□

Now, we will show that Theorem 1.4 and Proposition 5.3 induce some well-known result on the fixed point theory of circle actions. Suppose that (M, J) is a $2n$-dimensional closed almost complex manifold equipped with an S^1-action with a discrete fixed point set M^{S^1}. Then for each fixed point $z \in M^{S^1}$, there exist non-zero integers $q_1(z), q_2(z), \ldots, q_n(z)$, called *weights* at z, such that the action is locally expressed by

$$t \cdot (z_1, z_2, \ldots, z_n) = (t^{q_1(z)} z_1, t^{q_2(z)} z_2, \ldots, t^{q_n(z)} z_n)$$

for any $t \in S^1$ where $(z_1, z_2, \ldots, z_n)$ is a local complex coordinates centered at z. Let us recall the *Atiyah-Bott-Berline-Vergne localization theorem* :

Theorem 5.4. *[AB][BV] For any equivariant cohomology class $\gamma \in H^*_{S^1}(M; \mathbb{R})$, we have*

$$\int_M \gamma = \sum_{z \in M^{S^1}} \frac{\gamma|_z}{\prod_{i=1}^n q_i(z) x}$$

where $\gamma|_z \in H^*_{S^1}(z; \mathbb{R}) \cong H^*(BS^1) = \mathbb{R}[x]$ is the restriction of γ onto z.

Note that if we apply Theorem 5.4 to $\gamma = 1 \in H^0_{S^1}(M)$, then Corollary 1.8 is straightforward. However, we give another proof of Corollary 1.8 by using Theorem 1.4 as we see below.

Corollary 5.5 (Corollary 1.8). *Let (M, J) be a closed almost complex S^1-manifold. Suppose that the action preserves J and that there are only isolated fixed points. Then,*

$$\sum_{z \in M^{S^1}} \frac{1}{\prod_{i=1}^n q_i(z)} = 0$$

where $q_1(z), \ldots, q_n(z)$ are the weights at z.

Proof. Let p be an arbitrarily large prime number such that $p > q_i(z)$ for every $z \in M^{S^1}$ and $i = 1, 2, \ldots, n$. Suppose that

$$\frac{a}{b} = \sum_{z \in M^{S^1}} \frac{1}{\prod_{i=1}^n q_i(z)} \neq 0 \tag{2}$$

for some integers a and b. Then

$$ab^{-1} = \sum_{z \in M^{S^1}} q_1(z)^{-1} q_2(z)^{-1} \cdots q_n(z)^{-1} \neq 0 \text{ in } \mathbb{Z}_p$$

by the assumption, where $q_i(z)^{-1}$ and b^{-1} are the inverses of $q_i(z)$ and b in $\mathbb{Z}_p^\times$, respectively, for every $i = 1, 2, \ldots, n$.

Let us consider the product space $M \times S^1$ with an S^1-action ψ given by

$$t \cdot (x, w) = (t \cdot x, t^p w)$$

for $t \in S^1$ and $(x, w) \in M \times S^1$. Then $(M \times S^1, \psi)$ is a pseudo-free S^1-manifold such that the set $\mathcal{E}$ of exceptional orbit is

$$\mathcal{E} = \left\{ \{z\} \times S^1 \subset M \times S^1 \mid z \in M^{S^1} \right\}.$$

For each exceptional orbit $\{z\} \times S^1$, the local invariant is given by

$$(p; q_1(z), q_2(z), \ldots, q_n(z)).$$

By Theorem 1.4 and Proposition 5.3, we have

$$\sum_{z \in M^{S^1}} \frac{q_1(z)^{-1} q_2(z)^{-1} \cdots q_n(z)^{-1}}{p} \equiv 0 \pmod{\mathbb{Z}}.$$

which is equivalent to

$$\sum_{z \in M^{S^1}} q_1(z)^{-1} q_2(z)^{-1} \cdots q_n(z)^{-1} = 0 \text{ in } \mathbb{Z}_p.$$

which leads a contradiction. □

The other type of manifolds having $e = 0$ comes from equivariant symplectic geometry. Here we give a brief introduction to the theory of circle actions on symplectic manifolds.

Let M be a $2n$-dimensional closed manifold. A differential 2-form ω on M is called a *symplectic form* if ω is closed and non-degenerate, i.e.,

- $d\omega = 0$, and
- ω^n is nowhere vanishing.

We call such a pair (M, ω) a *symplectic manifold.* A smooth S^1-action on (M, ω) is called *symplectic* if it preserves ω. Equivalently, an S^1-action is symplectic if $\mathcal{L}_{\underline{X}}\omega = di_{\underline{X}}\omega = 0$ where $\underline{X}$ is the fundamental vector field on M generated by the action. Thus if the action is symplectic, then $i_{\underline{X}}\omega$ is a closed 1-form so that it represents some cohomology class $[i_{\underline{X}}\omega] \in H^1(M; \mathbb{R})$. Now, let us assume that ω is integral so that $[\omega] \in H^2(M; \mathbb{Z})$. By a direct computation, we can easily check that $i_{\underline{X}}\omega$ is also integral. Thus we can define a smooth map $\mu : M \to \mathbb{R}/\mathbb{Z} \cong S^1$ such that

$$\mu(x) := \int_{\gamma_x} i_{\underline{X}}\omega \quad \mod \mathbb{Z}$$

where x_0 is a base point and γ_x is any path $\gamma_x : [0,1] \to M$ such that $\gamma_x(0) = x_0$ and $\gamma_x(1) = x$. We call μ a *generalized moment map.*

Lemma 5.6. *[McD][CKS, Proposition 2.2] Let μ be a generalized moment map. Then*

$$d\mu = i_{\underline{X}}\omega.$$

It is immediate consequences of Lemma 5.6 that μ is S^1-invariant and the set of critical points of μ is equal to M^{S^1}. Let $\theta \in \mathbb{R}/\mathbb{Z}$ be a regular value of μ. Then $F_\theta := \mu^{-1}(\theta)$ is a $(2n-1)$-dimensional closed fixed-point-free S^1-manifold. Note that the restriction $\omega|_{F_\theta}$ has one-dimensional kernel generated by $\underline{X}$ on F_θ. Thus $\omega|_{F_\theta}$ induces a symplectic structure ω_θ on the quotient $M_\theta := F_\theta/S^1$ and we call $(M_\theta, \omega_\theta)$ the *symplectic reduction at θ.*

If we choose $\epsilon > 0$ small enough so that $I_\theta := (\theta - \epsilon, \theta + \epsilon) \subset \mathbb{R}/\mathbb{Z}$ has no critical value, then

$$\mu^{-1}(I_\theta) \cong M_\theta \times I_\theta.$$

Thus we can compare $[\omega_\vartheta]$ with $[\omega_\theta]$ in $H^2(M;\mathbb{R})$ whenever $\vartheta \in I_\theta$. The following theorem due to Duistermaat and Heckman gives an explicit variation formula of reduced symplectic forms.

Theorem 5.7. *[DH] Let ψ_θ be the induced S^1-action on F_θ. Then*

$$[\omega_\vartheta] - [\omega_\theta] = (\vartheta - \theta) \cdot c_1(F_\theta, \psi_\theta)$$

for every $\vartheta \in I_\theta$.

Now, we can define a function, called the *Duistermaat-Heckman function*, on I_θ such that

$$\begin{aligned} \mathrm{DH} : I_\theta &\to \mathbb{R} \\ \vartheta &\mapsto \mathrm{Vol}(M_\vartheta, \omega_\vartheta) \end{aligned}$$

where $\mathrm{Vol}(M_\vartheta, \omega_\vartheta)$ is a symplectic volume given by

$$\mathrm{Vol}(M_\vartheta, \omega_\vartheta) = \int_{M_\vartheta} \omega_\vartheta^{n-1}.$$

By Theorem 5.7, the Duistermaat-Heckman function $\mathrm{DH}(\vartheta)$ is a locally polynomial of degree $n-1$ with the leading coefficient $\langle c_1(F_\theta, \psi_\theta)^{n-1}, [M_\theta] \rangle$. In other words,

$$\begin{aligned} \mathrm{DH}(\vartheta) &= \left(\int_{M_\theta} c_1(F_\theta, \psi_\theta)^{n-1} \right) (\vartheta - \theta)^{n-1} + \cdots + \int_{M_\theta} \omega_\theta^{n-1} \\ &= \left(\int_{M_\theta} c_1(F_\theta, \psi_\theta)^{n-1} \right) \vartheta^{n-1} + \cdots . \end{aligned}$$

Proposition 5.8 (Proposition 1.10). *Let (M, ω) be a closed symplectic manifold equipped with a fixed-point-free S^1-action ψ preserving ω. Let $\mu : M \to \mathbb{R}/\mathbb{Z}$ be a generalized moment map and let $F_\theta = \mu^{-1}(\theta)$ for $\theta \in \mathbb{R}/\mathbb{Z}$. Then $e(F_\theta, \psi|_{F_\theta}) = 0$.*

Proof. Since the S^1-action ψ on (M, ω) is fixed-point-free by assumption, the Duistermaat-Heckman function DH is a polynomial defined on the whole $\mathbb{R}/\mathbb{Z}$. Since any periodic polynomial is a constant function, all coefficients of

ϑ^{n-i} in $\mathrm{DH}(\vartheta)$ are zero for $1 \leq i < n$. Indeed, the coefficient of ϑ^{n-i} can be expressed as

$$\sum_{j=1}^{i} \binom{n-j}{i-j} (-\theta)^{i-j} \int_{M_\theta} c_1(F_\theta, \psi_\theta)^{n-j} \omega^{j-1}$$

for every $i = 1, \ldots, n$. In particular, we have $e(F_\theta, \psi_\theta) = 0$ when $i = 1$. □

Furthermore, we have the following corollary.

Corollary 5.9. *Let (M, ω) be a $(2n+2)$-dimensional closed symplectic manifold with a fixed-point-free symplectic S^1-action ψ. Assume that $[\omega] \in H^2(M; \mathbb{Z})$ and every submanifold fixed by some non-trivial finite subgroup of S^1 is of dimension two. Then we have*

$$\sum_{S \in \mathcal{J}} \omega(S) \cdot \frac{q_1^{-1}(S) q_2^{-1}(S) \cdots q_n^{-1}(S)}{p(S)} \equiv 0 \pmod{\mathbb{Z}}$$

where $\mathcal{J}$ is the set of connected submanifolds of M having non-trivial isotropy subgroups, $\omega(S)$ is the symplectic area of S, $p(S)$ is the order of the isotropy subgroup of S, $(q_1(S), \ldots, q_n(S))$ is the weight-vector of $\mathbb{Z}_{p(S)}$-representation on the normal bundle over S, and $q_i(S)^{-1}$ is the inverse of $q_i(S)$ in $\mathbb{Z}_{p(S)}^{\times}$ for every $i = 1, \ldots, n$.

Proof. Let $\mu : M \to \mathbb{R}/\mathbb{Z}$ be a generalized moment map for ψ. Without loss of generality, by scailing ω if necessary, we may assume that $\mu^* dt = i_{\underline{X}} \omega = d\mu$ where dt is a volume form on $\mathbb{R}/\mathbb{Z}$ such that $\int_{\mathbb{R}/\mathbb{Z}} dt = 1$ and $\underline{X}$ is the vector field generated by the S^1-action ψ, see [Au, p. 273] for more details. Since the action is fixed-point-free, there is no critical point of μ. Let $\theta \in \mathbb{R}/\mathbb{Z}$ and we denote ψ_θ the induced action on $F_\theta = \mu^{-1}(\theta)$.

Let $\mathcal{J} = \{S_1, \ldots, S_k\}$ be the set of connected symplectic submanifolds of (M, ω) having non-trivial isotropy subgroups. Since each S_i is two-dimensional and the induced action on $(S_i, \omega|_{S_i})$ is fixed-point-free and symplectic, we can easily see that S_i is diffeomorphic to T^2 and the restriction $\mu|_{S_i}$ becomes a generalized moment map for the induced symplectic S^1-action on $(S_i, \omega|_{S_i})$. Furthermore, each level set $(\mu|_{S_i})^{-1}(t)$ is the union of finite number of S^1-orbits for every $t \in \mathbb{R}/\mathbb{Z}$.

Thus $F_\theta \cap S_i = (\mu|_{S_i})^{-1}(\theta)$ is the union of finite number of S^1-orbits for each $i = 1, \ldots, k$. We denote the number of connected components of $F_\theta \cap S_i$ by n_i. Consequentely, there are exactly $n_1 + \cdots + n_k$ exceptional S^1-orbits in F_θ and hence (F_θ, ψ_θ) is a pseudo-free S^1-manifold. By Theorem 1.4, we have

$$\sum_{i=1}^{k} n_i \cdot \frac{q_1^{-1}(S_i) q_2^{-1}(S_i) \cdots q_n^{-1}(S_i)}{p(S_i)} \equiv 0 \pmod{\mathbb{Z}}.$$

Observe that $n_i = \omega(S_i)$ since if we choose a loop $\gamma_i : S^1 \to S_i \cong T^2$ generating a gradient-like vector field with respect to $\mu|_{S_i}$, then

$$\int_{S_i} \omega = \int_{\gamma_i} i_{\underline{X}} \omega = \langle dt, \mu_*[\gamma] \rangle = n_i$$

for every $1 \leq i \leq k$. This completes the proof. □

Remark 5.10. Any effective fixed-point-free symplectic circle action on a closed symplectic four manifold satisfies the condition in Corollary 5.9.

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