

# An infinite-rank summand of knots with trivial Alexander polynomial

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We show that there exists a  $\mathbb{Z}^\infty$ -summand in the subgroup of the knot concordance group generated by knots with trivial Alexander polynomial. We use the  $\Upsilon$ -invariant introduced by Ozsváth, Stipsicz and Szabó. For our computation, we give a sufficient condition for two satellite knots to have the identical  $\Upsilon$ -invariant.

## 1. Introduction

The celebrated theorems on 4-dimensional topology due to Donaldson [Don83] and Freedman [Fre82b] have an interesting consequence on the study of knot concordance. Freedman proved that knots with trivial Alexander polynomial are topologically slice [Fre82a, FQ90, GT04]. Using Donaldson's diagonalization theorem, Casson and Akbulut had noticed that there exists a knot with trivial Alexander polynomial, which is not smoothly slice. (Their results are unpublished. See the paper of Cochran and Gompf [CG88].) Topologically slice knots which are not smoothly slice measure the subtle difference between topological and smooth category in the study of 4-dimensional topology. For example, it is well-known that a topologically slice knot which is not smoothly slice gives an exotic  $\mathbb{R}^4$  [GS99, Exercise 9.4.23].

Following these results, topologically slice knots (modulo smooth concordance) have become of great interest and been studied extensively. We would like to review some results related to this. We first define certain subgroups of the knot concordance group.

Let  $\mathcal{C}$  and  $\mathcal{C}^{top}$  be the smooth and topological knot concordance group, respectively. Let  $\mathcal{C}_T$  be the subgroup of  $\mathcal{C}$  consisting of topologically slice knots. That is,  $\mathcal{C}_T$  is the kernel of the natural map  $\mathcal{C} \rightarrow \mathcal{C}^{top}$ . Let  $\mathcal{C}_\Delta$  be the subgroup of  $\mathcal{C}$  generated by knots with trivial Alexander polynomial. Then  $\mathcal{C}_\Delta$  is a subgroup of  $\mathcal{C}_T$  by the aforementioned result of Freedman.

Using Furuta's results [Fur90] on Fintushel-Stern invariants (defined in [FS85]), Endo [End95] proved that there is a  $\mathbb{Z}^\infty$ -subgroup in  $\mathcal{C}_\Delta$  consisting of certain Pretzel knots. In [HK11, HK12], Hedden and Kirk introduced

similar gauge theoretic invariants and proved that there is a  $\mathbb{Z}^\infty$ -subgroup in  $\mathcal{C}_\Delta$  consisting of the Whitehead doubles of certain torus knots.

Using Ozsváth-Szabó's Heegaard Floer correction terms [OS03a], it is known that  $\mathcal{C}_T/\mathcal{C}_\Delta$  is highly non-trivial. Hedden, Livingston and Ruberman [HLR12] proved that  $\mathcal{C}_T/\mathcal{C}_\Delta$  contains a  $\mathbb{Z}^\infty$ -subgroup, and Hedden, Kim and Livingston [HKL16] showed that it also contains a  $\mathbb{Z}_2^\infty$ -subgroup.

Despite these developments, the splitting problem of  $\mathcal{C}_T$  (for example, the existence of a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_T$ ) was poorly understood and had been regarded as a difficult but interesting problem. The first result is due to Livingston. In [Liv04], he found a  $\mathbb{Z}$ -summand in  $\mathcal{C}_\Delta$  using Ozsváth-Szabó's  $\tau$ -invariant [OS04b]. By the development of concordance homomorphisms to the integers, including the  $\tau$ -invariant, Rasmussen's  $s$ -invariant [Ras10] and Manolescu-Owens'  $\delta$ -invariant [MO07], it is known that there exists a  $\mathbb{Z}^3$ -summand in  $\mathcal{C}_\Delta$  [Liv08].

By generalizing the  $\delta$ -invariant, Jabuka [Jab12] obtained infinitely many knot concordance homomorphisms on  $\mathcal{C}_T$ . However, in general, Jabuka's homomorphisms are difficult to calculate simultaneously.

Recently, using her  $\epsilon$ -invariant, Hom [Hom15a] proved the existence of a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_T$ . After Hom's work, Ozsváth, Stipsicz and Szabó [OSS17] defined the  $\Upsilon$ -invariant and found a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_T$  using  $\Upsilon$ . We will discuss later that Hom's knots and Ozsváth-Stipsicz-Szabó's knots have non-trivial Alexander polynomials. It is natural to ask whether  $\mathcal{C}_\Delta$  contains a  $\mathbb{Z}^\infty$ -summand or not. Our main result addresses this question.

**Theorem A.** *There exists a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_\Delta$ .*

Let  $D$  be the positively-clasped untwisted Whitehead double of the right-handed trefoil knot, and  $K_{p,q}$  be the  $(p, q)$ -cable of a knot  $K$ . To prove Theorem A, we consider  $\{D_{n,1}\}_{n=2}^\infty$  and their Upsilon invariants  $\Upsilon_{D_{n,1}}$ . Before we discuss the computation of  $\Upsilon_{D_{n,1}}$ , we compare our knots with the knots which are recently known to generate a  $\mathbb{Z}^\infty$ -summand of  $\mathcal{C}_T$ . The following are the sets of topologically slice knots considered by Hom [Hom15a] and Ozsváth, Stipsicz and Szabó [OSS17] respectively:

$$HOM := \{D_{n,n+1}\# -T_{n,n+1}\}_{n=2}^\infty \text{ and } OSS := \{D_{n,2n-1}\# -T_{n,2n-1}\}_{n=2}^\infty,$$

where  $T_{p,q}$  is the  $(p, q)$ -torus knot and  $-K$  denotes the mirror image of a knot  $K$  with the opposite orientation. Observe that the Alexander polynomials of knots in  $HOM$  and  $OSS$  are non-trivial. Nonetheless, their knots might generate a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_\Delta$ , but it seems hard to determine if they are not concordant to any knots with trivial Alexander polynomial. See [HLR12]

for a method to do such. We also point out that their knots are composite but ours are prime. Moreover, it will be shown in Section 5 that the knots in  $OSS$  are linearly independent to a subset of our knots, which also generate a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_\Delta$ .

We are back to discuss our computation of  $\Upsilon_{D_{n,1}}$ . Instead of computing  $\Upsilon_{D_{n,1}}$  directly, we first obtain a condition for two satellite knots to have identical  $\Upsilon$ , using another concordance invariant from knot Floer homology,  $\nu^+$  due to Hom and Wu [HW14]. We define two knots  $K_1$  and  $K_2$  to be  $\nu^+$ -equivalent if  $\nu^+(K_1 \# -K_2) = \nu^+(-K_1 \# K_2) = 0$ . It is known that  $\nu^+$ -equivalent knots have the same  $\Upsilon$ -invariants [OSS17, Proposition 4.7]. Here we develop further as follows:

**Theorem B.** *Let  $P$  be a pattern with nonzero winding number. If two knots  $K_1$  and  $K_2$  are  $\nu^+$ -equivalent, then  $P(K_1)$  and  $P(K_2)$  are  $\nu^+$ -equivalent, and consequently  $\Upsilon_{P(K_1)} \equiv \Upsilon_{P(K_2)}$ .*

The proof of Theorem B hinges on the results of Cochran-Franklin-Hedden-Horn [CFHH13], Levine-Ruberman [LR14] and Ozsváth-Szabó [OS03a].

In fact  $D$  and  $T_{2,3}$  are  $\nu^+$ -equivalent. See Example 3.2. Therefore it is enough to compute  $\Upsilon_{T_{2,3;n,1}}$  for  $\Upsilon_{D_{n,1}}$ . In Section 4 we compute a part of  $\Upsilon_{T_{2,3;n,1}}$  by understanding the infinity-version of the knot Floer chain complex of  $T_{2,3;n,1}$  using the hat-version of knot Floer chain complex of  $T_{2,3;n,1}$  given by Hom [Hom15b].

**Theorem C.** *Let  $T_{2,3;n,1}$  be the  $(n, 1)$ -cable of the right-handed trefoil knot. Then*

$$\Upsilon_{T_{2,3;n,1}}(t) = \begin{cases} -nt & \text{if } t \leq \frac{2}{1+n} \\ t - 2 & \text{if } \frac{2}{1+n} \leq t \leq \frac{2}{1+n} + \epsilon \end{cases}$$

for some small  $\epsilon > 0$ . See Figure 1.

For thin knots (see Definition 2.1), we have the following corollary of Theorem B (see Example 3.3).

**Corollary D.** *Suppose  $K_1$  and  $K_2$  are thin knots. If  $\tau(K_1) = \tau(K_2)$ , then  $\Upsilon_{P(K_1)} \equiv \Upsilon_{P(K_2)}$  for any pattern  $P$  with non-zero winding number.*

We conclude this section presenting questions naturally arisen during this work.

**Question 1.** Are  $\{D_{n,1}\}$  and Hom's knots linearly independent in  $\mathcal{C}$ ?

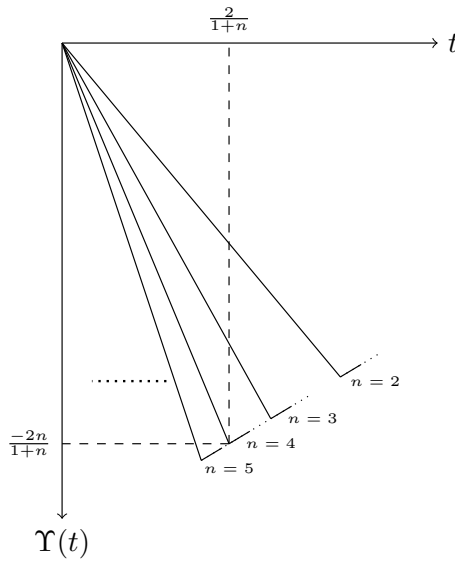


Figure 1: The graphs of  $\Upsilon_{T_{2,3;n,1}}$  ( $\equiv \Upsilon_{D_{n,1}}$ ) for  $n = 2, \dots, 5$ .

As mentioned above, it follows from our computation of  $\Upsilon$  that our knots and Ozsváth-Stipsicz-Szabó’s knots are linearly independent in  $\mathcal{C}$ . See the end of Section 5. However, by using Theorem B and a recent result of Wang [Wan18], it is easy to see that the first singularities of the  $\Upsilon$ -invariants of Hom’s knots occur at the same  $t$ ’s as ours. Therefore, the part of  $\Upsilon$  we computed cannot determine the linear independence of ours and Hom’s.

**Question 2.** Are Hom’s knots or Ozsváth-Stipsicz-Szabó’s knots not concordant to any knots with trivial Alexander polynomial? Do they generate a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_T/\mathcal{C}_\Delta$ ?

As far as the authors know, the only known method to answer the question is to use the obstruction of Hedden-Livingston-Ruberman [HLR12] based on Heegaard Floer correction terms. Matt Hedden pointed out that, using the obstruction, one can show that  $D_{2,3}\# - T_{2,3}$  (Hom and Ozsváth-Stipsicz-Szabó’s knots for  $n = 2$ ) is not concordant to any knot with trivial Alexander polynomial. However it might need much more work to show those knots represent nontrivial elements in  $\mathcal{C}_T/\mathcal{C}_\Delta$ .

**Question 3.** What is the behavior of the  $\Upsilon$ -invariant under cabling? Is  $\Upsilon_{K_{p,q}}$  determined by some invariants of  $K$ ?

The behavior of the  $\tau$ -invariant (the negative slope of  $\Upsilon$  at  $t = 0$ ) under the cabling operation has been studied intensively. For example, see [Hed09, VC10, Pet13, Hom14]. In particular  $\tau(K_{p,q})$  is determined by  $\tau(K)$ ,  $\epsilon(K)$ ,  $p$  and  $q$  [Hom14]. Therefore one can expect analogous results for the  $\Upsilon$ -invariant. However the results for  $\tau$  have relied on various computation techniques for the hat-version (simple version) of knot Floer homology. Even if we luckily compute a part of  $\Upsilon$  for some specific knots using the hat-version of knot Floer homology, one might need better computation techniques of the full knot Floer chain complex, to answer this question.

**Remark 1.1.** Preparing for this article, Chen informed us of a partial answer for Question 3 [Che16]. In particular he obtained an inequality for  $\Upsilon_{K_{p,q}}$  at  $t \in [0, \frac{2}{p}]$  in terms of  $\Upsilon_K$ ,  $p$  and  $q$ , which is analogous to Van Cott's formula for  $\tau$  [VC10].

**Question 4.** Is there a  $\mathbb{Z}_2^\infty$ -subgroup or a  $\mathbb{Z}_2^\infty$ -summand in  $\mathcal{C}_\Delta$ ?

As we mentioned above, it is known that there exists a  $\mathbb{Z}_2^\infty$ -subgroup in  $\mathcal{C}_T/\mathcal{C}_\Delta$  [HKL16] using Heegaard Floer correction terms. Therefore, a natural question to ask is if there exists a  $\mathbb{Z}_2^\infty$ -subgroup in  $\mathcal{C}_\Delta$ , or a  $\mathbb{Z}_2^\infty$ -summand in  $\mathcal{C}_T$ ,  $\mathcal{C}_\Delta$  or  $\mathcal{C}_T/\mathcal{C}_\Delta$ . However many concordance invariants such as  $\tau$  and  $\Upsilon$  do not work effectively to detect torsion elements in  $\mathcal{C}$ .

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## 2. Background on Heegaard Floer homology

In this section we briefly recall the Heegaard Floer homology for closed oriented 3-manifolds and the knot Floer homology for knots in  $S^3$ , and various smooth knot concordance invariants coming from them, and introduce some properties of those invariants. The purpose of this section is to set up notations and collect the background materials, which will be used later in the following sections. We refer the readers to a recent survey paper of Hom [Hom17] for more detailed expositions on this subject.

## 2.1. Heegaard Floer homology and Knot Floer homology

Let  $Y$  be a closed oriented 3-manifold and  $\mathfrak{t}$  be a  $\text{spin}^c$  structure over  $Y$ . Heegaard Floer homology, introduced by Ozsváth and Szabó [OS04b], associates to  $(Y, \mathfrak{t})$  a relatively graded chain complex  $CF^\infty(Y, \mathfrak{t})$  of finitely generated free modules over  $\mathbb{F}[U, U^{-1}]$ , where  $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ . We call the homological grading of the chain complex the *Maslov grading*, and the multiplication of  $U$  lowers the grading by 2.

From the definition of the boundary map, the negative power of  $U$  naturally induces a filtration on the chain complex, called the *algebraic filtration*. The filtered chain homotopy type of  $CF^\infty(Y, \mathfrak{t})$  is an invariant of  $(Y, \mathfrak{t})$ . Hence, the homology of  $CF^\infty(Y, \mathfrak{t})$ , denoted by  $HF^\infty(Y, \mathfrak{t})$ , is an invariant of  $(Y, \mathfrak{t})$ , called *Heegaard Floer homology* of  $(Y, \mathfrak{t})$ . The algebraic filtration of  $CF^\infty$  allows us to define various versions of Heegaard Floer homology groups as follows:

$$\begin{aligned} HF^- &:= H_*(CF^\infty\{i < 0\}), \\ HF^+ &:= H_*(CF^\infty\{i \geq 0\}), \\ \widehat{HF} &:= H_*(CF^\infty\{i = 0\}), \end{aligned}$$

where  $CS$  denotes the sub or quotient complex of  $C$  generated by the elements in the filtration levels in  $S$ .

Ozsváth and Szabó [OS04a] and independently Rasmussen [Ras03] observed that any knot  $K$  in  $S^3$  induces an additional filtration on a Heegaard Floer chain complex of  $S^3$ . We call the induced filtration the *Alexander filtration*. Together with the algebraic filtration, the  $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain homotopy type of the chain complex is an invariant of  $K$ , denoted by  $CFK^\infty(K)$  and called the *full knot Floer chain complex*. The multiplication of  $U$  lowers Alexander and algebraic filtration by 1 in  $CFK^\infty$ . We usually use  $i$  for the algebraic filtration level and  $j$  for the Alexander filtration level of  $CFK^\infty$ , as describing a subset of the  $\mathbb{Z} \oplus \mathbb{Z}$ -filtration level. The homology of the associated graded chain complex of  $CFK^\infty\{i = 0\}$  is denoted by  $\widehat{HFK}(K)$ , i.e.

$$\widehat{HFK}(K) := \bigoplus_{k \in \mathbb{Z}} H_*(C\{i = 0, j = k\}).$$

We define the  $\delta$ -grading to be the Maslov grading subtracted from the Alexander grading.

**Definition 2.1.** A knot  $K \subset S^3$  is called *thin* if  $\widehat{HFK}(K)$  is supported in a single  $\delta$ -grading.

It is convenient to depict a knot Floer chain complex as dots and arrows in an  $(i, j)$ -plane, in which a dot at  $(i, j)$ -coordinate indicates an  $\mathbb{F}$ -generator of the chain complex at the filtration level  $(i, j)$ , and an arrow stands for a non-trivial component of the ending dot in the differential of the starting dot. Note that an arrow starting from a dot in  $(k, l)$ -coordinate always maps to a dot in  $CFK^\infty\{i \leq k, j \leq l\}$ .

**2.2. Ozsváth-Szabó’s correction terms**

For a closed oriented 3-manifold  $Y$  with a torsion  $\text{spin}^c$ -structure  $\mathfrak{t}$  over  $Y$ , the relative Maslov grading of  $CF^\infty(Y, \mathfrak{t})$  can be lifted to an absolute  $\mathbb{Q}$ -grading [OS03a].

If  $Y$  is a rational homology 3-sphere, it is known that  $HF^+(Y, \mathfrak{t})$  can be decomposed as two parts: the image of the multiplication of sufficiently large power of  $U$ , which is isomorphic to  $\mathcal{T}^+ = \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$ , and the quotient of the image,  $HF_{red}^+(Y, \mathfrak{t})$ . Hence we have

$$HF^+(Y, \mathfrak{t}) \cong \mathcal{T}^+ \oplus HF_{red}^+(Y, \mathfrak{t}).$$

The *correction term*, or *d-invariant*, of  $(Y, \mathfrak{t})$  is defined by the lowest absolute grading of an element in the  $\mathcal{T}^+$ -tower in  $HF^+(Y, \mathfrak{t})$  [OS03a].

If  $H_1(Y; \mathbb{Z}) = \mathbb{Z}$  and  $\mathfrak{t}_0$  is the torsion  $\text{spin}^c$ -structure over  $Y$ , then it is known that  $HF^+(Y, \mathfrak{t}_0)$  has two  $\mathcal{T}^+$ -towers whose absolute grading is supported in  $2\mathbb{Z} + \frac{1}{2}$  and  $2\mathbb{Z} - \frac{1}{2}$  respectively. We define  $d_{\pm 1/2}(Y)$  to be the lowest absolute grading in each tower corresponding to the grading  $2\mathbb{Z} \pm \frac{1}{2}$  respectively.

**Example 2.2.** By [OS03b], it is known that

$$HF^+(S^3) = \mathcal{T}_{(0)}^+ \quad \text{and} \quad HF^+(S^2 \times S^1, \mathfrak{t}_0) = \mathcal{T}_{(1/2)}^+ \oplus \mathcal{T}_{(-1/2)}^+,$$

where the numbers in the parentheses indicate the lowest absolute Maslov grading of each tower  $\mathcal{T}^+$ . Hence  $d(S^3) = 0$ , and  $d_{\pm 1/2}(S^1 \times S^2) = \pm \frac{1}{2}$ .

We recall a special case of [OS03a, Proposition 4.12] for the later purpose. For a knot  $K \subset S^3$  and  $n \in \mathbb{Z}$ , let  $S_n^3(K)$  be the 3-manifold obtained by the  $n$ -surgery of  $S^3$  along  $K$ .

**Proposition 2.3** ([OS03a, Proposition 4.12]). *For a knot  $K \subset S^3$ ,  $d_{1/2}(S_0^3(K)) = \frac{1}{2} + d(S_1^3(K))$ .*

The correction term provides a  $\text{spin}^c$  rational homology cobordism invariant for  $\text{spin}^c$  3-manifolds with *standard*  $HF^\infty$  (which is true if  $b_1 \leq 2$ ). See [OS03a, Theorem 1.2] for the case of rational homology 3-spheres. For 3-manifolds with  $H_1 \cong \mathbb{Z}$ , this can be obtained from [OS03a, Theorem 9.15]. In [LR14], Levine and Ruberman generalize the correction terms to  $d(Y, \mathfrak{t}, V)$  for any 3-manifold  $Y$  with standard  $HF^\infty$ , a torsion  $\text{spin}^c$ -structure  $\mathfrak{t}$  over  $Y$  and a subspace  $V \subset H_1(Y; \mathbb{Z})$ . They showed the rational cobordism invariance of the generalized correction terms in [LR14, Proposition 4.5].

**Remark 2.4.** If  $H_1(Y; \mathbb{Z}) = \mathbb{Z}$  and  $\mathfrak{t}_0$  is the torsion  $\text{spin}^c$ -structure over  $Y$ , then it is immediate from the definitions that  $d_{1/2}(Y, \mathfrak{t}_0) = d(Y, \mathfrak{t}_0, 0)$ .

**Proposition 2.5** ([OS03a, Theorem 9.15], [LR14, Proposition 4.5]). *Suppose that  $Y_i$  is a closed, oriented 3-manifold such that  $H_1(Y_i; \mathbb{Z}) \cong \mathbb{Z}$  for  $i = 1, 2$ . Suppose that there exists a  $\text{spin}^c$ -rational homology cobordism  $(W, \mathfrak{s})$  from  $(Y_1, \mathfrak{t}_1)$  to  $(Y_2, \mathfrak{t}_2)$  where  $\mathfrak{t}_i$  is the torsion  $\text{spin}^c$ -structure on  $Y_i$ . Then  $d_{1/2}(Y_1) = d_{1/2}(Y_2)$ .*

*Proof.* By Remark 2.4, this is a special case of [LR14, Proposition 4.5] when  $V = V' = 0$ .  $\square$

### 2.3. Concordance invariants from the knot Floer homology

Let  $K$  be a knot in  $S^3$  and  $C$  be the full knot Floer chain complex of  $K$ ,  $CFK^\infty(K)$ . The concordance invariants  $\tau$ ,  $\nu^+$  and  $\Upsilon$  are induced from the natural maps between sub or quotient complexes of  $C$ . We use the coordinate  $(i, j)$  again for the (algebraic, Alexander)-filtration level of  $C$ .

Consider the sequence of the inclusion maps between the quotient complexes of  $C$ :

$$\iota_k : C\{i = 0, j \leq k\} \rightarrow C\{i = 0\}.$$

Then  $\tau(K)$  is defined by the minimum of  $k$  such that  $\iota_k$  induces a nontrivial map on the homology [OS03b].

The invariant  $\nu^+$  is defined in a similar manner. Now we consider the following sequence of projection maps,

$$v_k^+ : C\{\max\{i, j - k\} \geq 0\} \rightarrow C\{i \leq 0\}.$$

Then we define  $\nu^+(K)$  to be the minimum of  $k$  such that  $(v_k^+)_*(1) = 1$ , where 1 denotes the lowest graded generator of the  $\mathcal{T}^+$ -tower in the homology of each complex [HW14].



Instead of the original definition of the  $\Upsilon$ -invariant given in [OSS17], we use the alternative definition due to Livingston [Liv17]. Fix a real number  $t$  in  $[0, 2]$ . Consider the following 1-parameter family of inclusion maps

$$d_s : C\{j \leq \frac{2}{t}s + (1 - \frac{2}{t})i\} \rightarrow C.$$

The Upsilon invariant of  $K$  at  $t$ ,  $\Upsilon_K(t)$ , is defined by the maximum of  $-2s$  such that the image of  $(d_s)_*$  contains the nontrivial element of grading 0. The  $\Upsilon$  maps a knot  $K$  to a continuous piecewise-linear function on  $[0, 2]$  such that  $\Upsilon_K(0) = \Upsilon_K(2) = 0$  and  $\Upsilon_K(t) = \Upsilon_K(2 - t)$ .

We recall some properties of  $\tau$ ,  $\nu^+$  and  $\Upsilon$  that will be used in this paper.

**Proposition 2.6** ([OS04b, OSS17, HW14]). *Let  $K$  be a knot. The invariants  $\tau$ ,  $\nu^+$  and  $\Upsilon$  satisfy the following properties:*

- 1) *The invariants  $\tau$ ,  $\nu^+$  and  $\Upsilon$  are smooth concordance invariants. In particular,  $\tau$  and  $\Upsilon$  are homomorphisms from  $\mathcal{C}$  to  $\mathbb{Z}$  and  $Cont([0, 2])$  respectively.*
- 2)  *$|\Upsilon_K(t)| \leq t \max(\nu^+(K), \nu^+(-K))$  for any  $t \in [0, 2]$ .*
- 3)  *$\Upsilon'_K(0) = -\tau(K)$ .*
- 4)  *$\nu^+(K) = 0$  if and only if  $d(S_1^3(K)) = 0$ .*

Note that Proposition 2.6(4) is originally stated as  $\nu^+(K) = 0$  if and only if  $V_0(K) = 0$  in [HW14]. The concordance invariant  $V_0(K)$  is equal to  $-\frac{1}{2}d(S_1^3(K))$ .

### 3. $\nu^+$ -equivalence and proof of Theorem B

We define two knots  $K_1$  and  $K_2$  to be  $\nu^+$ -equivalent if

$$\nu^+(K_1 \# -K_2) = \nu^+(-K_1 \# K_2) = 0.$$

In this section, we give an equivalent condition of  $\nu^+$ -equivalence and examples of  $\nu^+$ -equivalent knots, and prove Theorem B. The following lemma is implicit in [Hom17].

**Lemma 3.1** ([Hom17, Proposition 3.11]). *Suppose  $K_1$  and  $K_2$  are two knots in  $S^3$ . Then,  $K_1$  and  $K_2$  are  $\nu^+$ -equivalent if and only if there is a filtered chain homotopy equivalence  $CFK^\infty(K_1) \oplus A_1 \simeq CFK^\infty(K_2) \oplus A_2$  for some acyclic complexes  $A_1$  and  $A_2$ .*

*Proof.* In [Hom17, Theorem 1], Hom proved that if  $K_1$  and  $K_2$  are concordant, then  $CFK^\infty(K_1) \oplus A_1 \simeq CFK^\infty(K_2) \oplus A_2$  for some acyclic complexes  $A_1$  and  $A_2$ . The same proof works under the weaker assumption that  $K_1$  and  $K_2$  are  $\nu^+$ -equivalent. In Hom's proof, the assumption (that  $K_1$  and  $K_2$  are concordant) is used just once to argue that

$$\nu^+(-K_1 \# K_2) = \nu^+(K_1 \# -K_2) = 0.$$

Compare [Hom17, Proposition 3.11].

For the opposite direction, assume that

$$CFK^\infty(K_1) \oplus A_1 \simeq CFK^\infty(K_2) \oplus A_2$$

for some acyclic  $A_1$  and  $A_2$ . By tensoring both sides by  $CFK^\infty(-K_2)$ , we obtain that

$$CFK^\infty(K_1 \# -K_2) \oplus A'_1 \simeq CFK^\infty(K_2 \# -K_2) \oplus A'_2 \simeq CFK^\infty(U) \oplus A''_2,$$

where the last filtered chain homotopy equivalence follows from [Hom17, Theorem 1].

As pointed out in [Pet10, Section 5.5],  $d(S_1^3(K))$  can be obtained by the direct summand containing the generator of  $H_*(CFK^\infty(K))$ . Hence,

$$d(S_1^3(K_1 \# -K_2)) = d(S_1^3(U)) = 0.$$

By Proposition 2.6 (4),  $\nu^+(K_1 \# -K_2) = 0$ . The same argument shows that  $\nu^+(-K_1 \# K_2) = 0$  and hence  $K_1$  and  $K_2$  are  $\nu^+$ -equivalent. This completes the proof.  $\square$

**Example 3.2.** In [HKL16, Proposition 6.1], it is shown that  $CFK^\infty(D) \simeq CFK^\infty(T_{2,3}) \oplus A$  for an acyclic summand  $A$ . Therefore,  $D$  and  $T_{2,3}$  are  $\nu^+$ -equivalent. Moreover,  $\#_{i=1}^k D$  is  $\nu^+$ -equivalent to  $T_{2,2k+1}$  by [HKL16, Theorem B.1].

**Example 3.3.** Suppose that  $K$  is a thin knot such that  $\tau(K) = \pm k$  for  $k \geq 0$ . By Petkova [Pet13, Section 3.1] and Lemma 3.1,  $K$  is  $\nu^+$ -equivalent to  $T_{2,\pm(2k+1)}$ .

We remark that if  $K_1$  and  $K_2$  are  $\nu^+$ -equivalent knots, then

$$|\Upsilon_{K_1}(t) - \Upsilon_{K_2}(t)| \leq t \max(\nu^+(K_1\# - K_2), \nu^+(-K_1\#K_2)) = 0$$

for any  $t \in [0, 2]$  by Proposition 2.6 (1) and (2), and hence  $\Upsilon_{K_1} \equiv \Upsilon_{K_2}$ . (This also follows from Lemma 3.1 and the definition of  $\Upsilon$ .) We further show that the  $\Upsilon$ -invariants of most satellites of  $\nu^+$ -equivalent knots coincide. More precisely, we show the following:

**Theorem B.** *Let  $P$  be a pattern with nonzero winding number. If two knots  $K_1$  and  $K_2$  are  $\nu^+$ -equivalent, then  $P(K_1)$  and  $P(K_2)$  are  $\nu^+$ -equivalent, and consequently  $\Upsilon_{P(K_1)} \equiv \Upsilon_{P(K_2)}$ .*

We will first prove Theorem B assuming the following lemma which hinges on the results of Cochran-Franklin-Hedden-Horn [CFHH13] and Levine-Ruberman [LR14].

**Lemma 3.4.** *Suppose that  $Q$  is a pattern with non-zero winding number such that  $Q(U)$  is slice where  $U$  is the unknot. Then  $d(S_1^3(Q(K))) = d(S_1^3(K))$  for any knot  $K$ .*

*Proof of Theorem B.* Suppose that  $P$  is a pattern with non-zero winding number and  $K_1$  and  $K_2$  are  $\nu^+$ -equivalent. By Proposition 2.6(4),  $K_1$  and  $K_2$  are  $\nu^+$ -equivalent if and only if  $d(S_1^3(K_1\# - K_2)) = d(S_1^3(-K_1\#K_2)) = 0$ .

As in [CDR14, Figure 5.2], let  $Q$  be the pattern described in Figure 2. (Here,  $\overline{K_2}$  is the mirror image of  $K_2$ .) Note that  $Q(U) = P(K_2)\# - P(K_2)$  is slice and

$$Q(K_1\# - K_2) = P(K_1\#K_2\# - K_2)\# - P(K_2).$$

Since  $K_2\# - K_2$  is slice,  $Q(K_1\# - K_2)$  is concordant to  $P(K_1)\# - P(K_2)$ . (Recall that if  $K$  and  $K'$  are concordant, then  $P(K)$  is concordant to  $P(K')$  for any pattern  $P$ .) The winding number of  $Q$  is equal to that of  $P$  and hence non-zero. By applying Lemma 3.4 to  $K = K_1\# - K_2$ , we can conclude that

$$d(S_1^3(P(K_1)\# - P(K_2))) = d(S_1^3(K_1\# - K_2)) = 0.$$

The last equality follows from the assumption that  $K_1$  and  $K_2$  are  $\nu^+$ -equivalent and by Proposition 2.6(4). The same argument shows that  $d(S_1^3(-P(K_1)\#P(K_2))) = 0$ . Therefore,  $P(K_1)$  and  $P(K_2)$  are  $\nu^+$ -equivalent. It follows from Proposition 2.6(2) that  $\Upsilon_{P(K_1)} \equiv \Upsilon_{P(K_2)}$ .  $\square$

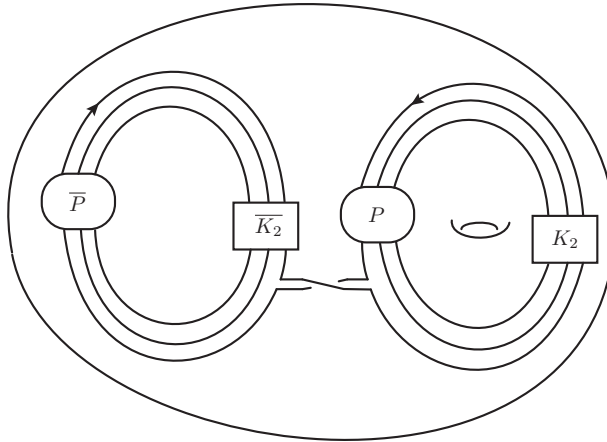


Figure 2: The pattern  $Q$  in the proof of Theorem B.

We give a proof of Lemma 3.4.

*Proof of Lemma 3.4.* Suppose  $Q$  is a pattern with non-zero winding number such that  $Q(U)$  is slice. It follows from [CFHH13, Theorem 2.1] that  $S_0^3(K)$  and  $S_0^3(Q(K))$  are rationally homology cobordant that is, there exists a compact 4-manifold  $W$  such that

- 1)  $\partial W = S_0^3(Q(K)) \sqcup -S_0^3(K)$ .
- 2)  $H_*(W, S_0^3(J); \mathbb{Q}) = 0$  for  $J = K$  and  $Q(K)$ .

Since  $W$  is a rational homology cobordism, the intersection form of  $W$  is trivial and every second cohomology class in  $W$  is characteristic. In particular, there is a  $\text{spin}^c$ -structure  $\mathfrak{s}$  on  $W$  such that  $c_1(\mathfrak{s}) = 0$ . The restriction of  $\mathfrak{s}$  to the boundary components are their unique torsion  $\text{spin}^c$ -structures. We can apply Propositions 2.3 and 2.5 to conclude that  $d(S_1^3(K)) = d(S_1^3(Q(K)))$ . This completes the proof.  $\square$

#### 4. Computation of $\Upsilon_{T_{2,3;n,1}}$

The goal of this section is to compute  $\Upsilon_{T_{2,3;n,1}}(t)$  for some small  $t$ . Our strategy is to determine a part of  $\Upsilon$  from some information about  $CFK^\infty$ , which can be obtained from  $\widehat{HFK}$ . The starting point is  $\widehat{HFK}(T_{2,3;n,1})$  computed by Hom in [Hom15b, Proposition 3.2]. We remark that Hom's result is based on Petkova's computation in [Pet13] using the bordered Floer homology.

**Remark 4.1.** After the first draft of this paper, Wenzhao Chen informed us that our arguments can be simplified after using his estimates on  $\Upsilon$  of cable knots [Che16].

As an  $\mathbb{F}[U, U^{-1}]$ -basis of  $CFK^\infty(T_{2,3;n,1})$ , we will take Hom’s generators of  $\widehat{HFK}(T_{2,3;n,1})$  multiplied by appropriate  $U$ -powers given in Table 2. For the reader’s convenience, we first recall Hom’s result on  $\widehat{HFK}(T_{2,3;n,1})$ .

**Proposition 4.2 ([Hom15b, Proposition 3.2]).** *The rank of the group  $\widehat{HFK}(T_{2,3;n,1})$  is  $6n - 5$ . The generators are listed in Table 1, and the non-zero higher differentials are*

$$\begin{aligned} \partial b_1 v_1 &= b_1 \mu_1 [n] \\ \partial b_j v_1 &= b_{2n-j-1} v_1 [n-j] & 2 \leq j \leq n-1 \\ \partial b_j v_2 &= b_{j+1} \mu_1 [1] & 1 \leq j \leq n-2 \\ \partial b_{n-1} v_2 &= b_n v_2 [1] \\ \partial b_j \mu_2 &= b_{2n-j-1} \mu_2 [n-j] & 1 \leq j \leq n-1, \end{aligned}$$

where the brackets denote the drop in Alexander filtration. For example, the Alexander filtration of  $b_1 \mu_1$  is  $n$  less than that of  $b_1 v_1$ .

Generator	$(M, A)$	
$au_1$	$(0, n)$	
$b_1 v_1$	$(-1, n-1)$	
$b_1 \mu_1$	$(-2, -1)$	
$b_j v_2$	$(-2j-1, -j)$	$1 \leq j \leq n-2$
$b_{j+1} \mu_1$	$(-2j-2, -j-1)$	$1 \leq j \leq n-2$
$b_{n-1} v_2$	$(-2n+1, -n+1)$	
$b_n v_2$	$(-2n, -n)$	
$b_j v_1$	$(-1, -j+n)$	$2 \leq j \leq n-1$
$b_{2n-1-j} v_1$	$(-2, 0)$	$2 \leq j \leq n-1$
$b_j \mu_2$	$(0, -j+n)$	$1 \leq j \leq n-1$
$b_{2n-1-j} \mu_2$	$(-1, 0)$	$1 \leq j \leq n-1$

Table 1: The generators of  $\widehat{HFK}(T_{2,3;n,1})$ .

Recall that  $CFK^\infty(K)$  is well-defined up to filtered chain homotopy equivalence. Using [Ras03, Lemma 4.5], we change  $CFK^\infty(T_{2,3;n,1})$  by filtered chain homotopy equivalence so that  $CFK^\infty(T_{2,3;n,1})\{i = 0\}$  is equal to  $\widehat{HFK}(T_{2,3;n,1})$ . From now on, the isomorphism type of the underlying module of  $CFK^\infty(T_{2,3;n,1})$  is fixed. Therefore,  $CFK^\infty(T_{2,3;n,1})$  is a free  $\mathbb{F}[U, U^{-1}]$ -module generated by

$$S = \{x_k, x'_k, y_l, y'_l, z_m, w_p \mid 2 \leq k \leq n, 1 \leq l \leq n, 2 \leq m \leq n - 1, 1 \leq p \leq n - 1\}.$$

Here, the elements of  $S$  are given in Table 2. They are appropriate  $U$ -translates of Hom's generators. It is easy to check the following facts about the elements in  $S$  (see Figure 3):

- 1) The Maslov gradings of  $x_k$  and  $x'_k$  are 1.
- 2) The Maslov gradings of  $y_k, y'_k,$  and  $z_k$  are 0.
- 3) The Maslov gradings of  $w_k$  are  $-1$ .
- 4) The (algebraic, Alexander) filtrations of  $x_k$  and  $y_k$  are  $(1, k)$  and  $(0, k)$ , respectively.
- 5) The (algebraic, Alexander) filtrations of  $x'_k$  and  $y'_k$  are  $(k, 1)$  and  $(k, 0)$ , respectively.
- 6) The (algebraic, Alexander) filtrations of  $z_k$  and  $w_k$  are  $(1, 1)$  and  $(0, 0)$ , respectively.

**Lemma 4.3.** *Let  $C_{\text{model}}$  denote the  $\mathbb{F}$ -module generated by*

$$S = \{x_k, x'_k, y_l, y'_l, z_m, w_p \mid 2 \leq k \leq n, 1 \leq l \leq n, 2 \leq m \leq n - 1, 1 \leq p \leq n - 1\}.$$

*Then  $C_{\text{model}}$  is a subcomplex that is,  $\partial C_{\text{model}} \subset C_{\text{model}}$ .*

Before proving this lemma, we first discuss its consequence.

**Remark 4.4.** As an  $\mathbb{F}[U, U^{-1}]$ -module,

$$CFK^\infty(T_{2,3;n,1}) \cong C_{\text{model}} \otimes \mathbb{F}[U, U^{-1}].$$

Since the differential is  $U$ -equivariant, Lemma 4.3 gives an isomorphism between chain complexes as  $CFK^\infty(T_{2,3;n,1}) \cong C_{\text{model}} \otimes \mathbb{F}[U, U^{-1}]$  where  $\mathbb{F}[U, U^{-1}]$  is endowed with trivial differential and the tensor product is over  $\mathbb{F}$ .

Generator	$(Alg, Alex, M)$	
$y_n = au_1$	$(0, n, 0)$	
$x_n = U^{-1}b_1v_1$	$(1, n, 1)$	
$y'_1 = U^{-1}b_1\mu_1$	$(1, 0, 0)$	
$x'_k = U^{-k}b_{k-1}v_2$	$(k, 1, 1)$	$2 \leq k \leq n - 1$
$y'_k = U^{-k}b_k\mu_1$	$(k, 0, 0)$	$2 \leq k \leq n - 1$
$x'_n = U^{-n}b_{n-1}v_2$	$(n, 1, 1)$	
$y'_n = U^{-n}b_nv_2$	$(n, 0, 0)$	
$x_k = U^{-1}b_{n-k+1}v_1$	$(1, k, 1)$	$2 \leq k \leq n - 1$
$z_k = U^{-1}b_{n+k-2}v_1$	$(1, 1, 0)$	$2 \leq k \leq n - 1$
$y_k = b_{n-k}\mu_2$	$(0, k, 0)$	$1 \leq k \leq n - 1$
$w_k = b_{n+1-k}\mu_2$	$(0, 0, -1)$	$1 \leq k \leq n - 1$

Table 2: The generators of  $CFK^\infty(T_{2,3;n,1})$ .

As usual, we may write the differential as  $\partial = \partial_V + \partial_D + \partial_H$ . Here,  $\partial_V$ ,  $\partial_D$  and  $\partial_H$  are the vertical differential, the diagonal differential and the horizontal differential, respectively. As the terminologies suggest,  $\partial_V$ ,  $\partial_D$  and  $\partial_H$  are defined via the following conditions:

- 1)  $\partial_V$  fixes algebraic filtration but strictly lowers Alexander filtration.
- 2)  $\partial_H$  fixes Alexander filtration but strictly lowers algebraic filtrations.
- 3)  $\partial_D$  strictly lowers both filtrations.

Note that there is no differential component between the same  $(i, j)$ -filtration level since  $CFK^\infty(T_{2,3;n,1})\{i = 0\} = \widehat{HFK}(T_{2,3;n,1})$ .

*Proof of Lemma 4.3.* The lemma follows from the consideration of the Maslov grading  $M$  and  $\mathbb{Z} \oplus \mathbb{Z}$ -filtration. Let  $C^\infty := CFK^\infty(T_{2,3;n,1})$ . Recall that  $C^\infty\{i = 0\} = \widehat{HFK}(T_{2,3;n,1})$ . Since  $U$  commutes with  $\partial$ , Proposition 4.2 determines  $\partial_V$  (see also Figure 3):

$$\begin{aligned}
 \partial_V x_n &= y'_1, \\
 \partial_V x_k &= z_k \quad 2 \leq k \leq n - 1, \\
 \partial_V x'_k &= y'_k \quad 2 \leq k \leq n, \\
 \partial_V y_k &= w_k \quad 1 \leq k \leq n - 1.
 \end{aligned}$$

We first observe that  $\partial y_k = \partial_V y_k = w_k$  for all  $k$ . This is equivalent to  $\partial_D y_k + \partial_H y_k = 0$ . Since  $M(y_k) = 0$ ,  $M(\partial_D y_k + \partial_H y_k) = -1$ . Since  $\partial_D$  and  $\partial_H$

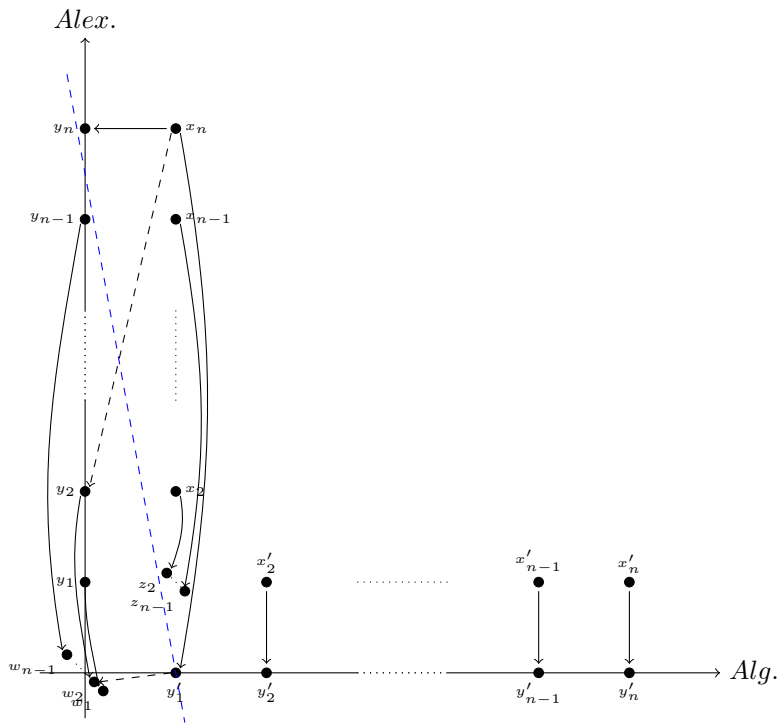


Figure 3: A summand  $C_{model}$  of the complex  $CFK^\infty(T_{2,3;n,1})$ .

strictly lower the algebraic filtration,  $\partial_H y_k + \partial_D y_k \in C^\infty\{i \leq -1\}$ . Since multiplying by  $U$  lowers the Maslov grading by 2, every element of  $C^\infty\{i \leq -1\}$  has the Maslov grading less than or equal to  $-2$ . For every  $k$ , it follows that  $\partial_D y_k + \partial_H y_k = 0$ , and  $\partial y_k = w_k \in C_{model}$ .

Since  $\partial^2 = 0$ ,  $\partial w_k = \partial^2 y_k = 0$  for all  $k$ . Recall that we have determined  $\partial_V$ . It follows that  $\partial_V z_k = 0$  for all  $k$ . Then,  $\partial z_k = \partial_D z_k + \partial_H z_k \in C^\infty\{i \leq 0, j \leq 1\}$  and  $M(\partial z_k) = -1$ . By the grading consideration, we can write

$$\partial_H z_k = \epsilon_k U x_2, \quad \partial_D z_k = \sum_{l=1}^{n-1} a_{k,l} w_l$$

where  $\epsilon_k, a_{k,l} \in \{0, 1\}$ . On the other hand, note that  $\partial_V U x_2 = U z_2$  and hence  $\partial U x_2 \neq 0$ . Since  $\partial w_l = 0$  for all  $l$ ,

$$\partial^2 z_k = \sum_{l=1}^{n-1} a_{k,l} \partial w_l + \epsilon_k \partial U x_2 = \epsilon_k \partial U x_2.$$



Since  $\partial^2 = 0$  and  $\partial Ux_2 \neq 0$ , it follows that  $\epsilon_k = 0$  and  $\partial z_k = \sum_{l=1}^{n-1} a_{k,l}w_l \in C_{\text{model}}$  for all  $k$ .

Now we prove that  $\partial x_k, \partial x'_k \in C_{\text{model}}$ . Since  $M(x_k) = M(x'_k) = 1$ ,  $M(\partial x_k) = M(\partial x'_k) = 0$ . Since the multiplication by  $U$  or  $U^{-1}$  changes the Maslov grading by 2, the Maslov grading 0 elements are linear combinations of  $y_k, y'_k$  and  $z_k$ . This shows that  $\partial x_k, \partial x'_k \in C_{\text{model}}$ .

Since  $M(y'_k) = 0$ ,  $M(\partial y'_k) = -1$ . Note that  $\partial y'_k \in C\{j \leq 0\}$ . By grading and filtration consideration, we can write

$$\partial y'_k = \sum_{l=2}^{k+1} b_{k,l}Ux'_l + \sum_{l=1}^{n-1} c_{k,l}w_l,$$

where  $b_{k,l}, c_{k,l} \in \{0, 1\}$ . It should be understood that  $b_{k,n+1} = 0$ . Since  $\partial w_l = 0$  for all  $l$ ,

$$(*) \quad 0 = \partial^2 y'_k = \sum_{l=2}^{k+1} b_{k,l}U\partial x'_l$$

for all  $k$ . If  $b_{k,l} = 0$  for all  $k, l$ , then  $\partial y'_k \in C_{\text{model}}$  and we are done. Suppose that there is  $k$  such that  $b_{k,l} \neq 0$  for some  $l$ . Fix such  $k$  and let  $m$  be the maximum among  $l$ 's such that  $b_{k,l} = 1$ . Note that  $U\partial x'_m = Uy'_m + U\partial_D x'_m + U\partial_H x'_m$ . Then, (\*) implies that

$$0 = U\partial x'_m + \sum_{l=2}^{m-1} b_{k,l}U\partial x'_l = Uy'_m + U\partial_D x'_m + U\partial_H x'_m + \sum_{l=2}^{m-1} b_{k,l}U\partial x'_l.$$

Recall that the algebraic filtration of  $Uy'_m$  is  $m - 1$ . On the other hand, the algebraic filtration of the remaining terms in the right hand side are less than or equal to  $m - 2$ . This is a contradiction and completes the proof.  $\square$

From the lemma above, we may write  $\partial y'_1 = \sum_{i=1}^{n-1} a_i w_i$  for some  $a_i \in \{0, 1\}$ .

**Lemma 4.5.** *In  $H_*(C^\infty)$ ,*

$$[y_n] = [y'_1 + \sum_{i=1}^{n-1} a_i y_i] \neq 0.$$

*Proof.* Note that  $M(y_n) = 0$ . Since  $y_n$  is null-homologous in  $C^\infty\{j = n\}$  and  $x_n$  is the unique element  $C^\infty\{j = n\}$  with grading 1,  $\partial_H x_n = y_n$ . See

[Hom15b, Lemma 3.3]. Therefore, we may write

$$\partial x_n = (\partial_V + \partial_D + \partial_H)x_n = y'_1 + \sum_{i=1}^{n-1} b_i y_i + y_n$$

for some  $b_i \in \{0, 1\}$ . Note that, by Lemma 4.3, it is enough to consider the elements in  $C_{model}$  to prove this Lemma.

Write  $\partial y'_1 = \sum_{i=1}^{n-1} a_i w_i$ . By considering  $\partial^2 x_n = 0$ , we obtain that  $a_i = b_i$  for all  $i$ :

$$0 = \partial^2 x_n = \partial \left( y'_1 + \sum_{i=1}^{n-1} b_i y_i + y_n \right) = \sum_{i=1}^{n-1} (a_i + b_i) w_i.$$

By considering  $\partial x_n$ , it follows that  $[y_n] = [y'_1 + \sum_{i=1}^{n-1} a_i y_i]$ .

We will prove that  $[y'_1 + \sum_{i=1}^{n-1} a_i y_i] \neq 0$  in  $H_*(C^\infty)$  by diagram chasing on the  $(i, j)$ -plane of  $C_{model}$  (see Figure 3). Suppose that  $y'_1 + \sum_{i=1}^{n-1} a_i y_i$  is a boundary. By grading considerations, we may write

$$(**) \quad \partial \left( \sum_{i=2}^n (c_i x_i + c'_i x'_i) \right) = y'_1 + \sum_{i=1}^{n-1} a_i y_i,$$

for some  $c_i, c'_i \in \{0, 1\}$ . We look at  $y'_1$  on the RHS of (\*\*). Observe that the arrows coming to  $y'_1$  are a vertical arrow from  $x_n$  and possible diagonal arrows from  $x'_k$ 's for some  $k = 2, \dots, n$ . It follows from the above equation that either  $c_n \neq 0$  or  $c'_k \neq 0$  for some  $k$ . However,  $c_n = 0$ , since  $\partial x_n$  contains  $y_n$ , to which no other arrow maps, and the RHS of (\*\*) does not have  $y_n$ .

Pick  $l_1$  such that  $c'_{l_1} \neq 0$ . Since  $\partial_V x'_{l_1} = y'_{l_1}$  and the RHS of (\*\*) does not have  $y'_{l_1}$ , there must be a diagonal arrow coming from  $x'_{l_2}$  to  $y'_{l_1}$  and  $c_{l_2} \neq 0$ . By continuing this argument, we have that  $c'_n \neq 0$ . However this is not possible since  $\partial x'_n$  contains  $y'_n$ , to which no arrow comes and the RHS of (\*\*) does not have  $y'_n$ . Therefore,  $y'_1 + \sum_{i=1}^{n-1} a_i y_i$  is not a boundary in  $C_{model}$ , and hence in  $C^\infty$ . □

We are ready to prove Theorem C. For the reader's convenience, we recall the statement of Theorem C.

**Theorem C.** *Let  $T_{2,3;n,1}$  be the  $(n, 1)$ -cable of the right-handed trefoil knot. Then*

$$\Upsilon_{T_{2,3;n,1}}(t) = \begin{cases} -nt & \text{if } t \leq \frac{2}{1+n} \\ t - 2 & \text{if } \frac{2}{1+n} \leq t \leq \frac{2}{1+n} + \epsilon \end{cases}$$

for some small  $\epsilon > 0$ .

*Proof.* Recall that the  $\Upsilon$ -invariant is determined by when an inclusion map from a subcomplex of  $C^\infty$  to  $C^\infty$  is nontrivial in homology at the Maslov grading 0. Hence it is enough to consider the direct-summand  $C_{model}$  in which all Maslov grading 0 elements of  $C^\infty$  are located.

Let  $L$  be a line on the  $(i, j)$ -plane of  $C_{model}$  with negative slope, and  $C_{model}(L)$  be the subcomplex of  $C_{model}$  generated by elements in the lower half-plane to  $L$  (including  $L$ ).

We first consider lines  $L$  with slope less than  $-n$ . For those lines, if the  $y$ -intercept of  $L$  is less than  $n$ ,  $H_*(C_{model}(L))_{(0)} = 0$ . If the  $y$ -intercept of  $L$  is  $\geq n$ , then  $[y_n]$  become nontrivial in  $H_*(C_{model}(L))_{(0)}$ . By Lemma 4.5,  $[y_n]$  is non-trivial in  $H_*(C_{model})$ . Since the equation for  $L$  is given as  $Alex = \frac{2}{t}s + (1 - \frac{2}{t})Alg$ , we get  $\Upsilon_{T_{2,3;n,1}}(t) = -nt$  for  $t \leq \frac{2}{1+n}$ .

Similarly, suppose that the slope of a line  $L$  is between  $-n$  and  $-n + \delta$  for some  $0 < \delta < 1$ . In this situation, if the  $x$ -intercept is less than 1, then  $H_*(C_{model}(L))_{(0)} = 0$ . If the  $x$ -intercept is equal to 1, then  $H_*(C_{model}(L))_{(0)}$  is generated by  $[y'_1 + \sum_{i=1}^{n-1} a_i y_i]$  which is non-trivial in  $H_*(C_{model})$  by Lemma 4.5. Hence  $\Upsilon_{T_{2,3;n,1}}(t) = t - 2$  for  $\frac{2}{1+n} \leq t \leq \frac{2}{1+n-\delta}$ . □

### 5. Proof of Theorem A

**Theorem A.** *There exists a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_\Delta$ .*

*Proof.* Let  $D_{n,1}$  be the  $(n, 1)$ -cable of positively-clasped untwisted Whitehead double of the trefoil knot,  $n \geq 2$ . By cabling formula,  $\Delta_{D_{n,1}}(t) = \Delta_D(t^n) = 1$ . Consider a homomorphism  $\phi : \mathcal{C} \rightarrow \mathbb{Z}^\infty$  defined by

$$\phi([K]) = \left( \frac{1}{1+k} \Delta \Upsilon'_K \left( \frac{2}{1+k} \right) \right)_{k=2}^\infty$$

where  $\Delta \Upsilon'_k(t_0) := \lim_{t \searrow t_0} \Upsilon'_K(t) - \lim_{t \nearrow t_0} \Upsilon'_K(t)$ . Note that  $\phi$  takes  $[K]$  to finitely many nonzero integers by [OSS17, Propositions 1.4 and 1.7]. Since  $D$  and  $T_{2,3}$  are  $\nu^+$ -equivalent, it follows from Theorems B and C that  $\phi([D_{n,1}]) = (*, \dots, *, 1, 0, 0, \dots)$ , where 1 is  $n - 1$ 's coordinate. Hence,  $\phi$  is surjective and Theorem A follows. □

In the introduction we claim that our knots are linearly independent to the knots introduced by Ozsváth, Stipsicz and Szabó, which generate a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_T$ . We prove the claim here.

**Proposition 5.1.** *Consider the following sets of knots:*

$$KP := \{D_{2n-1,1}\}_{n=2}^\infty \text{ and } OSS := \{D_{n,2n-1}\# - T_{n,2n-1}\}_{n=2}^\infty.$$

*The knots in  $KP$  and the knots in  $OSS$  are linearly independent in  $\mathcal{C}$ .*

*Proof.* By restricting the range of  $\phi$  (in the proof of Theorem A above) only to odd coordinates, one can easily see that the knots in  $KP$  generates a  $\mathbb{Z}^\infty$ -summand in  $\mathcal{C}_T$ . Recall that the  $\Upsilon$ -invariants of the knots in  $OSS$  have the first singularities at  $t = \frac{2}{2n-1}$  (see the proof of [OSS17, Theorem 1.20]), but the knots in  $KP$  have them at  $t = \frac{1}{n}$ .  $\square$

## References

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