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Orderability, contact non-squeezing, and Rabinowitz Floer homology

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We study Liouville fillable contact manifolds (Σ, ξ) with non-zero and spectrally finite Rabinowitz Floer homology and assign spectral numbers to paths of contactomorphisms. As a consequence we prove that $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ is orderable in the sense of Eliashberg and Polterovich. This provides a new class of orderable contact manifolds. If the contact manifold is in addition periodic or a prequantization space $M \times S^1$ for M a Liouville manifold, then we construct a contact capacity in the sense of Sandon [44]. This can be used to prove a general non-squeezing result, which amongst other examples in particular recovers the beautiful non-squeezing results from [24].

1. Introduction and results

Suppose (Σ, ξ) is a closed coorientable contact manifold. Denote by $\mathrm{Cont}_0(\Sigma, \xi)$ the identity component of the group of contactomorphisms, and denote by $\mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)$ the set of smooth paths of contactomorphisms starting at the identity. The universal cover $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ is then $\mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)/\sim$, where $\sim$ denotes the equivalence relation of being homotopic with fixed endpoints. Suppose $\alpha \in \Omega^1(\Sigma)$ is a contact form defining ξ, and θ^t its Reeb flow. To a path $\varphi = \{\varphi_t\}_{0 \le t \le 1} \in \mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)$ we can associate its contact Hamiltonian h_t

$$h_t \circ \varphi_t := \alpha\left(\frac{d}{dt}\varphi_t\right) : \Sigma \to \mathbb{R}, \tag{1.1}$$

which uniquely determines the path φ. In this article we are interested in four classes of contact manifolds, labelled **(A)**, **(A)**$^+$, **(B)**, and **(C)**. See Section 2 for precise definitions of the terms involved. In particular, see Definition 3.1 for the definition of a *spectrally finite* class.

Assumption (A): (Σ,ξ) admits a Liouville filling W such that there exists a spectrally finite class $Z \in \mathrm{RFH}_*(\Sigma, W)$.

Theorem 1.1. *Suppose (Σ,ξ) satisfies Assumption* **(A)**. *Then for any spectrally finite class $Z \in \mathrm{RFH}_*(\Sigma, W)$ there is a map*

$$c(\cdot, Z) : \mathcal{P}\mathrm{Cont}_0(\Sigma,\xi) \to \mathbb{R}$$

with the following properties.

1) *If $\varphi \sim \psi$ then $c(\varphi, Z) = c(\psi, Z)$. Thus $c(\cdot, Z)$ descends to define a map (denoted by the same symbol) $c(\cdot, Z) : \widetilde{\mathrm{Cont}}_0(\Sigma,\xi) \to \mathbb{R}$.*
2) *For any $T \in \mathbb{R}$, $c(t \mapsto \theta^{tT}, Z) = -T + c(\mathrm{id}_\Sigma, Z)$.*
3) *The map c is continuous with respect to the C^2-norm on $\mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)$.*
4) *If φ resp. ψ is generated by the contact Hamiltonian h_t resp. k_t with $h_t(x) \ge k_t(x)$ for all $x \in \Sigma$ and $t \in [0,1]$ then $c(\varphi, Z) \le c(\psi, Z)$.*

Corollary 1.2. *If there exists a constant $\delta > 0$ such that $h_t(x) \ge \delta$ for all $x \in \Sigma$ and $t \in [0,1]$ then $c(\varphi, Z) < c(\mathrm{id}_\Sigma, Z)$, where φ is generated by h_t.*

Proof. Note that the constant function δ generates the path $\{t \mapsto \theta^{t\delta}\}$ thus Theorem 1.1 (2) & (4) implies

$$c(\varphi, Z) \le c(t \mapsto \theta^{t\delta}, Z) = -\delta + c(\mathrm{id}_\Sigma, Z) < c(\mathrm{id}_\Sigma, Z). \tag{1.2}$$

$\square$

Corollary 1.3. *If (Σ,ξ) satisfies Assumption* **(A)** *then $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ is orderable in the sense of Eliashberg–Polterovich [25].*

Proof. Recall from [25, Criterion 1.2.C.] that $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ is orderable if and only if no contractible loop φ of contactomorphisms exists whose contact Hamiltonian h_t satisfies $h_t(x) > 0$ for all $x \in \Sigma$ and $t \in [0, 1]$. Let us assume, by contradiction, that φ is such a loop. Then (1) in Theorem 1.1 implies that $c(\varphi, Z) = c(\mathrm{id}_\Sigma, Z)$ since φ is contractible. On the other hand Corollary 1.2 implies that $c(\varphi, Z) < c(\mathrm{id}_\Sigma, Z)$. This contradiction proves the Corollary. □

Remark 1.4. Together with its companion article [8] this article is the first to establish Rabinowitz Floer homology (RFH), a tool for studying orderability and non-squeezing questions in contact geometry. The aim of the article [8] is very different from the present one since it is solely concerned with a link between the famous Weinstein conjecture and orderability. In this article we derive obstructions from Rabinowitz Floer homology to non-orderability and to squeezing phenomena. Since Rabinowitz Floer homology, defined originally by Cieliebak and Frauenfelder [15] is nowadays rather computable, see eg. [18], this delivers checkable criteria for orderability and non-squeezing. In particular, we reproduce many of the previously known examples of orderable contact manifolds and similarly for the non-squeezing results. At the same time our approach gives entirely new classes of orderable contact manifolds and an abstract non-squeezing results.

A precursor to this development is the article [7] by Frauenfelder and the first author in which a rather different version of Rabinowitz Floer homology is used to mimic Givental's construction of the non-linear Maslov index [32–34]. On unit cotangent bundles this also leads to an obstruction to a (strong form) of non-orderability.

The notion of contact capacity (see below) was introduced by Sandon in [44]. She was the first to discover a connection between translated points and orderability and other contact rigidity phenomena.

Corollary 1.3 has the following rephrasing.

Corollary 1.5. *Let (Σ, ξ) be a closed contact manifold for which $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ is not orderable. Then for any Liouville filling W of Σ with $\mathrm{RFH}(\Sigma, W)$ containing a spectrally finite class one has*

$$\mathrm{RFH}_*(\Sigma, W) = 0. \tag{1.3}$$

We illustrate the above with some examples.

Example 1.6.

- The sphere S^{2n-1} with its *standard* contact structure is not orderable by [24, Theorem 1.10]. The standard contact structure on S^{2n-1} is index positive, cf. Definition 3.2. Thus, for any Liouville filling W of Σ, $\mathrm{RFH}(\Sigma, W)$ contains only spectrally finite classes. Thus, $\mathrm{RFH}(S^{2n-1}, W) = 0$ for any Liouville filling W. The equivalent statement of vanishing symplectic homology of any Liouville filling of the standard contact sphere was proved before by Smith, see [48, Corollary 6.5].
- A new class of orderable contact manifolds is given by links of weighted homogeneous singularities with positive Milnor number. This includes certain *Brieskorn manifolds*, and in particular *non-standard* structures on spheres (the Ustilovsky spheres), as well as contact structures on *exotic* spheres. This was communicated to us by Otto van Koert, see Example 1.11 below, which also includes more examples.
- Let $\Sigma = S_g^*B$ be the unit cotangent bundle of the closed Riemannian manifold (B, g) equipped with its standard contact structure ξ. The Liouville filling by the unit codisk bundle D_g^*B always has $\mathrm{RFH}_*(S_g^*B, D_g^*B) \neq 0$ due to Cieliebak-Frauenfelder-Oancea [18], see also Abbondandolo-Schwarz [4]. Moreover there certainly exists spectrally finite classes; in fact the stronger Assumption $\mathbf{(A)}^+$ below is satisfied, see Example 1.7. Thus, $\widetilde{\mathrm{Cont}}_0(S_g^*B, \xi)$ is orderable, which was proved by Eliashberg-Kim-Polterovich [24] and Chernov-Nemirovski [14].

Recall that given a Reeb orbit γ, we denote by $\mu_{\mathrm{CZ}}^{\mathrm{tr}}(\gamma)$ its *transverse Conley Zehnder index*, and we denote by $\nu^{\mathrm{tr}}(\gamma)$ its *transverse nullity*. Let us define

$$\mu(\gamma) := \mu_{\mathrm{CZ}}^{\mathrm{tr}}(\gamma) - \frac{1}{2}\nu^{\mathrm{tr}}(\gamma). \tag{1.4}$$

Assumption $\mathbf{(A)}^+$: (Σ, ξ) admits a Liouville filling $(W^{2n}, d\lambda)$ such that $\alpha := \lambda|_\Sigma$ is Morse-Bott with non-zero Rabinowitz Floer homology. Moreover the Reeb flow $\theta^t : \Sigma \to \Sigma$ of α has no contractible Reeb orbits γ with

$$\mu(\gamma) \in [-n - \nu^{\mathrm{tr}}(\gamma), 1 - n]. \tag{1.5}$$

As proved in Lemma 3.16 below, Assumption $\mathbf{(A)}^+$ guarantees the existense of a non-zero class μ_Σ which is not only spectrally finite but in addition has spectral value 0: $c(\mathrm{id}_\Sigma, \mu_\Sigma) = 0$.

Example 1.7. If (Σ, ξ) admits a Liouville filling $(W, d\lambda)$ such that the Reeb vector field of $\alpha := \lambda|_\Sigma$ has no contractible Reeb orbits (e.g. $\mathbb{T}^3$ with its standard contact structure, which is filled by $D^*\mathbb{T}^2$. In this case the $\mathrm{RFH}_*(\Sigma, D^*\mathbb{T}^2)$ is equal to the singular homology of Σ, and thus Assumption $\mathbf{(A)}^+$ is trivially satisfied. More generally, recall that (Σ, α) is called *dynamically convex* if the transverse Conley-Zehnder index $\mu^{\mathrm{tr}}(\gamma)$ is at least $n+1$ for all such γ (in the case where Σ has dimension 3, this coincides with the original definition of dynamical convexity [35]). Since the transverse nullity is at most $2n-2$, this is implied by requiring that the quantity $\mu(\gamma)$ defined in (3.15) is at least 2. Therefore dynamical convexity is a stronger assumption than (1.5), and hence Assumption $\mathbf{(A)}^+$ is also satisfied for any Liouville fillable contact manifold which admits a filing with non-zero Rabinowitz Floer homology and which is dynamically convex. Similarly Assumption $\mathbf{(A)}^+$ is satisfied for any fibrewise convex hypersurface in a cotangent bundle of dimension at least four, since in this case the Conley-Zehnder index of a Reeb orbit can be identified with the Morse index of the corresponding critical point of the Lagrangian action functional associated to the Legendre dual Lagrangian. This result is essentially due to Duistermaat [23]; see for instance [3, Theorem 4.1] for a detailed modern proof.

The fact that this class μ_Σ has the property that $c(\mathrm{id}_\Sigma, \mu_\Sigma) = 0$ allows us to strengthen Statement (4) of Theorem 1.1 to the following statement, which is proved as Corollary 3.18 below.

Theorem 1.8. *Assume* (Σ, ξ) *satisfies Assumption* $\mathbf{(A)}^+$. *Suppose* $\varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ *has contact Hamiltonian* h_t. *Assume* $h_t \leq 0$ *and there exists* $x \in \Sigma$ *such that* $h_t(x) < 0$ *for all* $t \in [0, 1]$ *then* $c(\varphi, \mu_\Sigma) > 0$.

Remark 1.9. In Section 1.1 below we provide an example to show that the same implication with opposite inequalities in the above theorem does not hold. See Remarks 3.19 and 5.14, and Appendix A.

For us the main relevance of Theorem 1.8 is that it implies the contact capacity $\bar{c}(\cdot, \mu_\Sigma)$ we define below is non-trivial, see Remark 1.19 below.

Definition 1.10. We call a contact form α periodic if its Reeb flow θ^t is a 1-periodic loop: $\theta^1 = \mathrm{id}_\Sigma$.

Let us now assume that (Σ, ξ) satisfies the following condition:

Assumption (B): (Σ, ξ) admits a Liouville filling $(W, d\lambda)$ such that the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, W)$ contains a spectrally finite class and such that $\alpha := \lambda|_\Sigma$ is periodic.

Example 1.11 (Communicated to us by Otto van Koert). An interesting class of examples where our results apply is the following. Assume that (Q, ω) is a simply connected symplectic manifold. Assume in addition that $[\omega]$ is integral and (Q, ω) is monotone, with negative monotonicity constant. Let $K \subset Q$ denote a closed connected symplectic submanifold of codimension 2 such that K is Poincaré dual to $k[\omega]$ for some $k \in \mathbb{N}$. Such a hypersurface is known as a *Donaldson hypersurface*, since Donaldson showed that every symplectic manifold with an integral symplectic form admits a symplectic submanifold Poincaré dual to $k[\omega]$ for $k \in \mathbb{N}$ sufficently large [22]. Assume in addition that $\mathrm{H}_1(K; \mathbb{Z}) = 0$. Let $\nu(K)$ denote a collar neighborhood of K in Q. Then the complement $Q \setminus \nu(K)$ is the interior of a Liouville domain (W_1, λ_1) with the property that the Reeb flow on $\Sigma := \partial W_1$ is periodic (see for instance [21]). Denote by W the completion of W_1. If we assume that the inclusion $\Sigma \hookrightarrow W$ induces an injection on π_1 (e.g. if $\dim Q \geq 6$), then the symplectic homology $\mathrm{SH}_*(W)$ is non-zero (see below), and hence so is the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, W)$. Assumption **(B)** is satisfied by Lemma 3.5 and Lemma 3.7.

There are several ways to see that the symplectic homology of $\mathrm{SH}_*(W)$ is non-zero. The simplest one is an index argument, and goes as follows. Since (Q, ω) is monotone it follows easily that $c_1(TW)$ is torsion, and hence the Conley-Zehnder index is a well defined integer for contractible orbits. Next the monotonicity assumption and a suitable choice of Hamiltonian functions imply that the index of all contractible Reeb orbit is at most $n = \frac{1}{2} \dim W$. However there is a well defined map $\mathrm{H}_{*+n}(W_1, \Sigma) \to \mathrm{SH}_*(W)$, and the image of the fundamental class has degree n. This class therefore remains non-zero in $\mathrm{SH}_*(W)$ due to index reasons. Alternatively, one can argue using S^1-equivariant symplectic homology: the proof of Lemma 7.6 in [40] implies that $\mathrm{SH}_*^{S^1,+}(W)$ has no generators with large positive degree, since the index growth of non constant one periodic orbits is proportional to the (non-positive) monotonicity constant. However if $\mathrm{SH}_*(W) = 0$ then work of Bourgeois-Oancea [13] implies that $\mathrm{SH}_*^{S^1}(W) = 0$. The Viterbo long exact sequence (see [12, Lemma 4.8]) then implies $\mathrm{SH}^{S^1,+}(W) \cong \mathrm{H}_{*+n}(W, \Sigma) \otimes \mathrm{H}_*(\mathbb{CP}^\infty; \mathbb{Z})$, which has generators with arbitrary positive degree, which is a contradiction.

If we assume that $\pi_2(Q) = 0$ then it follows from the homotopy exact sequence of the fibration that all the Reeb orbits on Σ are non-contractible.

If in addition $\mathrm{H}_1(Q;\mathbb{Z})=0$ and $\mathrm{H}_1(K;\mathbb{Z})=0$ then the construction described above yields examples satisfying Assumption **(A)**$^+$ and **(B)** (cf. Example 1.7). Moreover, very explicit examples are the complement of a degree k-curve in $\mathbb{C}\mathrm{P}^2$ with $k\geq 3$ which admit Liouville fillings (even though $\mathrm{H}_1(K;\mathbb{Z})\neq 0$).

Finally, another more general class of examples where our results apply are links of weighted homogeneous singularities with positive Milnor number. The latter guarantees the existence of Lagrangian spheres which in turn implies non-vanishing of RFH, see [38]. In particular, this includes certain *Brieskorn manifolds*, see [38, Theorem 1.2] for a precise statement. We mention here only that these include *non-standard* contact structures on spheres (the Ustilovsky spheres [50]), as well as contact structures on *exotic* spheres.

The advantage of Assumption **(B)** is the following. As before Z denotes a spectrally finite class in $\mathrm{RFH}_*(\Sigma,W)$. The definitions and results that follow are based on Sandon's article [44].

Definition 1.12. We define for $\varphi\in\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ an integer $\overline{c}(\varphi,Z)$ by

$$\overline{c}(\varphi,Z):=\lceil c(\varphi,Z)\rceil\,. \tag{1.6}$$

Proposition 1.13. *The function $\overline{c}(\cdot,Z):\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)\to\mathbb{Z}$ is conjugation invariant: if $\psi\in\mathrm{Cont}_0(\Sigma,\xi)$ and $\varphi\in\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ then*

$$\overline{c}(\psi\varphi\psi^{-1},Z)=\overline{c}(\varphi,Z). \tag{1.7}$$

Remark 1.14. In contrast to spectral invariants in Hamiltonian Floer homology, Proposition 1.13 is a non-trivial result. See Remark 1.20 below.

Given a path φ of contactomorphisms, we define the *support* of φ,

$$\mathfrak{S}(\varphi):=\bigcup_{0\leq t\leq 1}\mathrm{supp}(\varphi_t), \tag{1.8}$$

where $\mathrm{supp}(\varphi_t):=\overline{\{x\in\Sigma\mid\varphi_t(x)\neq x\}}$.

Definition 1.15. For an open set $U\subset\Sigma$ we define the *contact capacity*

$$\overline{c}(U,Z):=\sup\Big\{\overline{c}(\varphi,Z)\mid\varphi\in\widetilde{\mathrm{Cont}}_0(\Sigma,\xi),\ \mathfrak{S}(\varphi)\subset U\Big\}\in\mathbb{Z}\cup\{\pm\infty\}, \tag{1.9}$$

where by convention we declare that $\sup\emptyset=-\infty$.

Proposition 1.13 has the following immediate corollary.

Corollary 1.16. *For all $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$, one has*

$$\bar{c}(\psi(U), Z) = \bar{c}(U, Z). \tag{1.10}$$

Using the contact capacity we obtain the following abstract non-squeezing results.

Theorem 1.17. *Let $U \subset V \subset \Sigma$ be open sets and assume that there exists $\varphi \in \mathrm{Cont}_0(\Sigma, \xi)$ with $\varphi(V) \subset U$. Then*

$$\bar{c}(U, Z) = \bar{c}(V, Z). \tag{1.11}$$

In particular, if $\bar{c}(U, Z) < \bar{c}(V, Z)$ then there exists no contact isotopy mapping V into U.

Proof. Suppose φ is as in the statement of the Theorem. Then trivially we have $\bar{c}(U, Z) \le \bar{c}(V, Z)$ and $\bar{c}(\varphi(V), Z) \le \bar{c}(U, Z)$. By Corollary 1.16 we also have $\bar{c}(\varphi(V), Z) = \bar{c}(V, Z)$, and hence $\bar{c}(U, Z) = \bar{c}(V, Z)$ as claimed. □

Remark 1.18. Of course, the contact capacities defined in [20, 26, 44, 52] satisfy analogues of Theorem 1.17.

Remark 1.19. If we assume both Assumptions **(A)**$^+$ and **(B)** then it follows from Theorem 1.8 that $\bar{c}(U, \mu_\Sigma) \ge 1$ for every non-empty open subset $U \subset \Sigma$, and hence the capacity $\bar{c}(\cdot, \mu_\Sigma)$ is non-trivial. In the next section we provide a class of examples (namely, contact manifold satisfying Assumption **(C)**) where $\bar{c}(\cdot, \mu_\Sigma)$ is computable, and thus derive applications of Theorem 1.17.

1.1. Prequantization spaces

Fix a Liouville manifold $(M, d\gamma)$ (i.e. the completion of Liouville domain, cf. Definition 2.1). The *prequantization space* of M is the contact manifold $\Sigma := M \times S^1$, equipped with the contact structure $\xi := \ker \alpha$, where

$$\alpha := \gamma + d\tau, \tag{1.12}$$

where τ is the coordinate on $S^1 = \mathbb{R}/\mathbb{Z}$. These contact manifolds are the last type we study in this paper. Note that unlike the other classes of contact manifolds we study, these contact manifolds are *not* compact.

Assumption (C): $(\Sigma, \xi = \ker \alpha)$ is a prequantization space $\Sigma = M \times S^1$, where $(M, d\gamma)$ is a Liouville manifold, and $\alpha = \gamma + d\tau$.

Let P_1 denote a 2-torus with a small disc removed, so that $\partial P_1 = S^1$. Equip P_1 with an exact symplectic form $d\beta_1$ such that $\beta_1|_{\partial P_1} = d\tau$. Let $(P, d\beta)$ denote the completion of the Liouville domain $(P_1, d\beta_1)$, and consider

$$W := M \times P, \tag{1.13}$$

equipped with the symplectic form $d\lambda$ where $\lambda := \gamma + \beta$. Even though Σ is periodic, W is *not* a Liouville filling of Σ, and in fact Σ does not satisfy either Assumptions **(A)** or **(B)** - for instance, as already mentioned Σ is non-compact. Nevertheless, it is still possible to define the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, W)$, and we prove that

$$\mathrm{RFH}_*(\Sigma, W) \cong \mathrm{HF}_*(M) \otimes \mathrm{H}_*(S^1; \mathbb{Z}_2). \tag{1.14}$$

Here

$$\mathrm{HF}_*(M) \cong \mathrm{H}^{n-*}(M; \mathbb{Z}_2) \tag{1.15}$$

denotes the Hamiltonian Floer homology of M, defined using *compactly supported* Hamiltonians (see Frauenfelder-Schlenk [30]). Moreover the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, W)$ constructed in this way satisfies the analogue of Assumption $\mathbf{(A)}^+$, that is, there is a suitable non-zero spectrally finite class $\mu_\Sigma \in \mathrm{RFH}_n(\Sigma, W)$. Indeed, in this case one simply takes μ_Σ to be the image of the class $\{\mathrm{pt}\} \times [S^1] \in \mathrm{H}^0(M; \mathbb{Z}_2) \otimes \mathrm{H}_1(S^1; \mathbb{Z}_2)$ under the isomorphisms (1.14) and (1.15). In fact, in this setting all non-zero classes are automatically spectrally finite, since all critical points are constant, and hence have critical value zero.

Since the Hamiltonian Floer homology is non-zero, one can associate a spectral number $c_M(f)$ to any $f \in \widetilde{\mathrm{Ham}}_c(M, d\gamma)$, the universal cover of the group of compactly supported Hamiltonian diffeomorphisms (see eg. Schwarz [47] or Frauenfelder-Schlenk [30]) . As in the contact case described above, c_M can then be used to define a *symplectic capacity* $c_M(\mathcal{O})$ for $\mathcal{O} \subset M$ open, by setting

$$c_M(\mathcal{O}) := \sup \{ c_M(f) \mid \mathfrak{S}(f) \subset \mathcal{O} \}. \tag{1.16}$$

Remark 1.20. In contrast to the contact case (see Proposition 1.13 and Remark 1.14), the proof that $c_M(f)$ is invariant under conjugation, that is, $c_M(hfh^{-1}) = c_M(f)$ for $f \in \widetilde{\mathrm{Ham}}_c(M, d\gamma)$ and $h \in \mathrm{Symp}_c(M, d\gamma)$ is immediate, since in this case the action spectrum of hfh^{-1} is the same as the action spectrum of f (see for instance [36, Chapter 5, Proposition 7]). This in turn immediately implies that c_M is a symplectic capacity, that is, $c_M(f(\mathcal{O})) = c_M(\mathcal{O})$ for any symplectomorphism f and any open set $\mathcal{O} \subset M$.

Going back to $\Sigma = M \times S^1$, let us denote by $\mathrm{Cont}_{0,c}(\Sigma, \xi)$ those contactomorphisms φ with compact support. There is a natural way to lift an element $f \in \widetilde{\mathrm{Ham}}_c(M, d\gamma)$ to obtain an element $\varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$, as we now recall. The equation

$$f_t^*\gamma - \gamma = da_t, \quad a_0 \equiv 0, \tag{1.17}$$

determines a smooth compactly supported function $a_t : M \to \mathbb{R}$. Define $\varphi_t : \Sigma \to \Sigma$ by

$$\varphi_t(y, \tau) := \big(f_t(y), \underbrace{\tau - a_t(y)}_{\mathrm{mod}\ 1}\big). \tag{1.18}$$

As explained in Section 5.1 below, one can define for any non-zero class Z the spectral numbers $c(\varphi, Z)$ for $\varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ in much the same way as before. Similarly one can define the capacity $\bar{c}(U)$ for $U \subset \Sigma$ open in the same way as before (if U is not precompact then one must again only use elements of $\widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ when defining $\bar{c}(U)$). Moreover most of the results stated thus far in the paper continue to hold (this statement is made more precise in Section 5.1). In particular, Parts (1), (3), and (4) of Theorem 1.1 remain true, and so do Proposition 1.13 and Theorem 1.17.

It is natural to ask the question: if $\varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ is the lift of $f \in \widetilde{\mathrm{Ham}}_c(M, d\gamma)$, how is $c(\varphi) := c(\varphi, \mu_\Sigma)$ related to $c_M(f)$? Note that $\mathfrak{S}(\varphi) = \mathfrak{S}(f) \times S^1$, and hence another question is how the c_M capacity of $\mathcal{O} \subset M$ is related to the c capacity of $\mathcal{O} \times S^1$. As with the case of $\mathbb{R}^{2n} \times S^1$, treated in [44], the following result answers these questions in the nicest possible way.

Theorem 1.21. *Suppose $f \in \widetilde{\mathrm{Ham}}_c(M, d\gamma)$, and let $\varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ denote the lift of f. Then*

$$c_M(f) = c(\varphi). \tag{1.19}$$

Moreover, if $\mathcal{O} \subset M$ is open and has compact closure then

$$c_M(\mathcal{O}) = c(\mathcal{O} \times S^1). \tag{1.20}$$

Theorem 1.21 allows us to prove non-squeezing results on Σ by making use of the known results on M. Let $l^t : M \to M$ denote the flow of the Liouville vector field on M and set $\zeta^t := l^{\log t}$. We will prove the following general result.

Theorem 1.22. *Suppose $\mathcal{O} \subset M$ is a non-empty open set with compact closure and unit capacity: $c_M(\mathcal{O}) = 1$. Suppose there exists a contact isotopy $\varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ such that*

$$\varphi_1\left(\zeta^{r_2}(\mathcal{O}) \times S^1\right) \subset \zeta^{r_1}(\mathcal{O}) \times S^1 \tag{1.21}$$

for $r_1, r_2 \in \mathbb{R}$. Then $\lceil r_2 \rceil \leq \lceil r_1 \rceil$. In addition if $\mathcal{O} \subset \mathcal{Q} \subset M$ are open sets with the property that there exists $\varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ with $\varphi_1(\mathcal{Q} \times S^1) \subset \mathcal{O} \times S^1$ then $\lceil c_M(\mathcal{Q}) \rceil = \lceil c_M(\mathcal{O}) \rceil$.

Proof. Note that for any $r \in \mathbb{R}$,

$$c_M(\zeta^r(\mathcal{O})) = r c_M(\mathcal{O}) \neq 0. \tag{1.22}$$

Thus

$$\bar{c}(\zeta^r(\mathcal{O}) \times S^1) = \lceil c_M(\zeta^r(\mathcal{O})) \rceil = \lceil r \rceil . \tag{1.23}$$

The result is now an immediate consequence of Theorem 1.17 (which, as remarked above, does indeed remain true in this setting). The last statement follows similarly. □

Remark 1.23. Theorem 1.22 recovers the beautiful non-squeezing result of [24, Theorem 1.2]. In this case one takes $M = \mathbb{R}^{2n}$ and U the unit ball. They prove that if $\lceil r_1 \rceil < \lceil r_2 \rceil$ then it is not possible to squeeze $B(r_2) \times S^1$ into the *cylinder*[1] $C(r_1) \times S^1$. This result was also recovered by Sandon [44] using generating functions.

A further applications of Theorem 1.21 is the following. Here we denote by c_{HZ} the *Hofer-Zehnder capacity* (see Definition 5.19 below or [36]).

[1]Note that whilst $C(r_1) := B^2(r_1) \times \mathbb{R}^{2n-2}$ does not have compact closure in $\mathbb{R}^{2n}$, and thus $\bar{c}(C(r_1) \times S^1)$ is not defined, since we only work with compactly supported contactomorphisms we can deduce this from the second statement of Theorem 1.22 by taking $\mathcal{O} = B(r_2)$ and $\mathcal{Q}$ a sufficiently large ellipse contained in $C(r_1)$.

Theorem 1.24. *Let $(M, d\gamma)$ denote a Liouville manifold. Equip $\mathbb{R}^{2m}$ with the standard symplectic form $d\lambda_{\mathrm{std}}$, and consider the contact manifold $(\widetilde{\Sigma}, \alpha + \lambda_{\mathrm{std}})$, where $\widetilde{\Sigma} := M \times \mathbb{R}^{2m} \times S^1$. Suppose $\mathcal{O} \subseteq M$ is open and $c_{\mathrm{HZ}}(\mathcal{O}, M) < \infty$. Choose $r_0 > 0$ such that*

$$\lceil \pi r_0^2 \rceil < \lceil c_{\mathrm{HZ}}(\mathcal{O}, M) \rceil \tag{1.24}$$

and set

$$r_1 := \sqrt{\tfrac{1}{\pi} c_{\mathrm{HZ}}(\mathcal{O}, M)} + 1 \tag{1.25}$$

Then there does not exist $\varphi \in \mathrm{Cont}_{0,c}(\widetilde{\Sigma}, \alpha + \lambda_{\mathrm{std}})$ such that

$$\varphi(\mathcal{O} \times B(r_1) \times S^1) \subset \mathcal{O} \times B(r_0) \times S^1. \tag{1.26}$$

The proof of Theorem 1.24 is given in Section 5.4. See also Corollary 5.22 for an application of Theorem 1.24.

Finally, following [24, Section 1.7] we investigate a rigidity phenomenon of positive contractible loops of contactomorphisms. Suppose now that $(M, d\gamma)$ is the completion of a Liouville domain $(M_1, d\gamma_1)$. Set $S := \partial M_1$ and $\kappa := \gamma|_S$, so that (S, κ) is a contact manifold. Abbreviate

$$M_r := \begin{cases} M_1 \backslash (S \times (r, 1)), & 0 < r < 1, \\ M_1 \cup_S (S \times [1, r]), & r \geq 1. \end{cases} \tag{1.27}$$

We can prove the following result, which roughly speaking says that if $\widetilde{\mathrm{Cont}}_0(S, \kappa)$ is *non-orderable*, so there exists a positive contractible loop $\chi = \{\chi_t\}_{t \in S^1} \subset \mathrm{Cont}_0(S, \kappa)$ of contactomorphisms, then it is *not* possible to homotope ζ through *positive* loops to id_S. In [24, Theorem 1.11] this was proved for $S = S^{2n-1}$.

Theorem 1.25. *Set $c := c_M(M_1)$ and assume that $c < \infty$. Suppose that $\chi = \{\chi_t\}_{t \in S^1} \subset \mathrm{Cont}_0(S, \kappa)$ is a positive contractible loop of contactomorphisms. Let $g_t : S \to (0, \infty)$ denote the contact Hamiltonian of χ, and set*

$$\varepsilon := \min_{(t,y) \in S^1 \times S} g_t(y) > 0. \tag{1.28}$$

Then if $\{\chi_{s,t}\}_{0 \leq s \leq 1}$ is any homotopy of loops of contactomorphisms such that $\chi_{1,t} = \chi_t$ and $\chi_{0,t} = \mathrm{id}_S$ with corresponding contact Hamiltonian $g_{s,t} : S \to \mathbb{R}$ then there exists $(s, t, y) \in [0, 1] \times S^1 \times S$ such that $g_{s,t}(y) \leq -(1 - \varepsilon)c$.

Proof. This follows directly from Theorem 1.22 and the material from [24, Section 2.1]. Indeed, suppose there exists $\delta > 0$ such that $g_{s,t}(y) > -(1-\varepsilon)(c-\delta)$ for all $(s,t,y) \in [0,1] \times S^1 \times S$. Set $a := \min\{\varepsilon, \varepsilon c\}$. Then as proved in [24, Section 2.1] for any $r < \frac{1}{c-\delta}$ it is possible to squeeze $M_r \times S^1$ into $M_{\frac{r}{1+ar}} \times S^1$. Fix $0 < \lambda < \min\{a, \delta\}$ and take $r = \frac{1}{c-\lambda}$. Then

$$\begin{aligned} \overline{c}(M_r \times S^1) &= \lceil c_M(M_r) \rceil \\ &= \lceil r c_M(M_1) \rceil \\ &= \left\lceil \tfrac{c}{c-\lambda} \right\rceil = 2. \end{aligned} \tag{1.29}$$

But

$$\begin{aligned} \overline{c}(M_{\frac{r}{1+ar}}) \times S^1) &= \left\lceil c_M(M_{\frac{r}{1+ar}}) \right\rceil \\ &= \left\lceil \frac{cr}{1+ar} \right\rceil \\ &= \left\lceil \frac{c}{c-\lambda+a} \right\rceil = 1. \end{aligned} \tag{1.30}$$

This contradicts Theorem 1.22. $\square$

Note added in March 2019: The research presented in this paper was completed several years ago. In the meantime, various exciting further developments have been made. In particular, we would like to draw the reader's attention to [19], in which a product is constructed on Rabinowitz-Floer homology.

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2. Preliminaries

2.1. Introductory definitions

Suppose (Σ, ξ) is a connected closed coorientable contact manifold. We denote by $\mathcal{P}\mathrm{Cont}_0(\Sigma, \xi)$ the set of all smoothly parametrized paths $\{\varphi_t\}_{0 \le t \le 1}$

with $\varphi_0 = \mathrm{id}_\Sigma$. We introduce an equivalence relation $\sim$ on $\mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)$ by saying that two paths φ and ψ are equivalent if $\varphi_1 = \psi_1$ and we can connect φ and ψ via a smooth family $\varphi^s = \{\varphi_t^s\}_{0\le s,t\le 1}$ of paths such that $\varphi^0 = \varphi$, $\varphi^1 = \psi$ and such that φ_1^s is independent of s. The universal cover $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ of $\mathrm{Cont}_0(\Sigma,\xi)$ is then $\mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)/\sim$. We now give the precise definition of a Liouville manifold, and what it means for Σ to be Liouville fillable.

Definition 2.1. A *Liouville domain* (W_1,λ_1) is a compact exact symplectic manifold with[2] $c_1|_{\pi_2(W_1)} = 0$ such that $\lambda_1|_{\partial W_1}$ is a positive contact form on ∂W_1. Equivalently the vector field Z_1 on W_1 defined by $\iota_{Z_1}\lambda_1 = d\lambda_1$ should be transverse to ∂W_1 and point outwards. Z_1 is called the *Liouville vector field*, and we denote by l^t the flow of Z_1, which is defined for all $t \le 0$, and thus induces an embedding $\partial W_1 \times (0,1] \to W_1$ defined by $(x,r) \mapsto l^{\log r}(x)$. Thus we can form the *completion* $(W, d\lambda)$ of (W_1,λ_1) by attaching $\partial W_1 \times [1,\infty)$ onto ∂W_1:

$$W := W_1 \cup_{\partial W_1} (\partial W_1 \times [1,\infty)). \tag{2.1}$$

We extend λ_1 to a 1-form λ on W by setting $\lambda = r\lambda_1|_{\partial W_1}$ on $\partial W_1 \times [1,\infty)$. Thus $d\lambda$ is a symplectic form on W. Similarly we extend Z_1 to a vector field Z on W by setting $Z = r\partial_r$ on $\partial W_1 \times [1,\infty)$. One calls $(W,d\lambda)$ a *Liouville manifold* - thus Liouville manifolds are exact non-compact symplectic manifolds obtained by completing a Liouville domain.

We say that a closed connected coorientable contact manifold (Σ,ξ) is *Liouville fillable* if there exists a Liouville domain $(W_1, d\lambda_1)$ such that $\Sigma = \partial W_1$ and such that if $\alpha := \lambda|_\Sigma$ then α is a positive contact form on Σ with $\ker\alpha = \xi$. By a slight abuse of notation we will generally refer to the Liouville manifold $(W,d\lambda)$ obtained from completing $(W_1,d\lambda_1)$ as "the" filling of Σ.

The *symplectization* $S\Sigma$ of a contact manifold $(\Sigma, \xi = \ker\alpha)$ is the symplectic manifold $\Sigma \times (0,\infty)$ equipped with the symplectic form $d(r\alpha)$. If Σ is Liouville fillable with filling $(W,d\lambda)$ then one can embed $S\Sigma \hookrightarrow W$ by using the flow l^t of the Liouville vector field Z of V. Next we recall how to lift a path $\varphi = \{\varphi_t\}_{0\le t\le 1}$ of contactomorphisms to a symplectic isotopy $\Phi = \{\Phi_t\}_{0\le t\le 1}$ on the symplectization $S\Sigma$. Write $\varphi_t^*\alpha = \rho_t\varphi_t$. Then define

[2]Some of the results proved in this paper do not require $c_1|_{\pi_2(W_1)} = 0$, but for the sake of a uniform presentation we will assume this throughout.

$\Phi_t : S\Sigma \to S\Sigma$ by

$$\Phi_t(x, r) := \left(\varphi_t(x), \frac{r}{\rho_t(x)}\right). \tag{2.2}$$

The path Φ_t is Hamiltonian (in fact it preserves λ) with Hamiltonian function

$$H_t(x, r) := rh_t(x) : S\Sigma \to \mathbb{R}. \tag{2.3}$$

We next define precisely what it means for a contact form α generating ξ to be of Morse-Bott type.

Definition 2.2. A contact 1-form $\alpha \in \Omega^1(\Sigma)$ generating ξ is said to be of *Morse-Bott type* if for each $T > 0$, the set $P_T \subset \Sigma$ of points $x \in \Sigma$ satisfying $\theta^T(x) = x$ (recall $\theta^t : \Sigma \to \Sigma$ is the Reeb flow of α) is a closed submanifold of Σ, with the property that $\operatorname{rank} d\alpha|_{P_T}$ is locally constant and

$$T_x P_T = \ker\left(D\theta^T(x) - \mathrm{id}_{T_x\Sigma}\right) \qquad \text{for all } x \in P_T. \tag{2.4}$$

A Liouville fillable contact manifold (Σ, ξ) is said to admit a *Morse-Bott Liouville filling* if there exists a filling $(W, d\lambda)$ such that $\alpha := \lambda|_\Sigma$ is of Morse-Bott type.

Let us now recall the definition of a translated point of a contactomorphism. This notion was introduced by Sandon in [44, 45].

Definition 2.3. Let (Σ, ξ) denote a closed connected coorientable contact manifold, and fix a contact form $\alpha \in \Omega^1(\Sigma)$ generating ξ. Fix $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$. We can write $\psi^*\alpha = \rho\alpha$ for a smooth positive function ρ on Σ. A *translated point* of ψ with respect to α is a point $x \in \Sigma$ with the property that there exists $\eta \in \mathbb{R}$ such that

$$\psi(x) = \theta^\eta(x), \quad \text{and} \quad \rho(x) = 1. \tag{2.5}$$

We call η the *time-shift* of x. Note that if the leaf $\{\theta^t(x)\}_{t\in\mathbb{R}}$ is closed (which is always the case when α is periodic) then the time-shift is not unique. Indeed, if the leaf $\{\theta^t(x)\}_{t\in\mathbb{R}}$ has period $T > 0$ then $\psi(x) = \theta^{\eta+\nu T}(x)$ for all $\nu \in \mathbb{Z}$.

Now let us define what it means for a translated point x of an element $[\varphi] \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ to be *contractible* with respect to a Liouville filling $(W, d\lambda)$.

Definition 2.4. Let $(W, d\lambda)$ denote a Liouville filling of (Σ, ξ), with $\alpha = \lambda|_{\Sigma}$. Suppose $[\varphi] \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ and x is a translated point of φ_1 of time-shift η. We say that the pair (x, η) is a *contractible* translated point if the continuous loop $l : \mathbb{R}/2\mathbb{Z} \to \Sigma$ obtained from concatenating the path $\{\varphi_t(x)\}_{0 \le t \le 1}$ with the path $\{\theta^{-\eta t}(x)\}_{0 \le t \le 1}$ is contractible in W. It is easy to see that this does not depend on path $\varphi = \{\varphi_t\}_{0 \le t \le 1} \in \mathcal{P}\mathrm{Cont}_0(\Sigma, \xi)$ representing $[\varphi]$.

For us, the usefulness of translated points stems from the fact that the translated points of φ are the generators of the Rabinowitz Floer homology associated to φ, when the Rabinowitz Floer homology is well defined; see Lemma 2.7 or [9] for more information.

2.2. The Rabinowitz action functional $\mathcal{A}_\varphi$ on $\Lambda(S\Sigma) \times \mathbb{R}$

Write $\Lambda(S\Sigma) := C^{\infty}_{\mathrm{contr}}(S^1, S\Sigma)$ for the component of the free loop space containing the contractible loops.

Definition 2.5. Fix a path $\varphi \in \mathcal{P}\mathrm{Cont}_0(\Sigma, \xi)$ as above, and let H_t denote the Hamiltonian (2.3). We define the *perturbed Rabinowitz action functional* (cf. [5, 15])

$$\mathcal{A}_\varphi : \Lambda(S\Sigma) \times \mathbb{R} \to \mathbb{R} \tag{2.6}$$

by

$$\mathcal{A}_\varphi(u, \eta) := \int_0^1 u^*\lambda - \eta \int_0^1 \beta(t)(r(t) - 1)dt - \int_0^1 \dot{\chi}(t) H_{\chi(t)}(u(t))dt, \tag{2.7}$$

where $\beta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is a smooth function with

$$\beta(t) = 0 \ \forall t \in [\tfrac{1}{2}, 1], \quad \text{and} \quad \int_0^1 \beta(t)dt = 1, \tag{2.8}$$

and $\chi : [0, 1] \to [0, 1]$ is a smooth monotone map with $\chi(\frac{1}{2}) = 0$, $\chi(1) = 1$, and $r(t)$ is the $\mathbb{R}$-component of the map $u : S^1 \to S\Sigma = \Sigma \times \mathbb{R}$. Denote by $\mathrm{Crit}(\mathcal{A}_\varphi)$ the set of critical points of $\mathcal{A}_\varphi$, and denote by $\mathrm{Spec}(\varphi) := \mathcal{A}_\varphi(\mathrm{Crit}(\mathcal{A}_\varphi))$.

Remark 2.6. In this paper we define the Rabinowitz action functional only on the component of contractible loops of the free loop space, as all the applications we have in mind here pertain only to this component. Nevertheless, it is possible to carry out all of our constructions on the full loop space without any changes. This is because the symplectic form (on $S\Sigma$ or on the Liouville filling W) is exact, and so there are no ambiguities in the definition of $\mathcal{A}_\varphi$ on non-contractible loops.

The following lemma explains why the perturbed Rabinowitz action functional is useful in detecting translated points. It is a minor variant of an argument originally due to the first author and Frauenfelder [5, Proposition 2.4]. For the convenience of the reader we recall the proof again here.

Lemma 2.7. *[9] A pair (u, η) is a critical point of $\mathcal{A}_\varphi$ only if, writing $u(t) = (x(t), r(t)) \in \Sigma \times (0, \infty)$, $p := x(\frac{1}{2})$ is a translated point of φ, with time-shift $-\eta$. Conversely every such pair (p, η) gives rise to a unique critical point of $\mathcal{A}_\varphi$. Moreover if (u, η) is a critical point of $\mathcal{A}_\varphi$ then*

$$\mathcal{A}_\varphi(u, \eta) = \eta. \tag{2.9}$$

If φ is an exact path of contactomorphisms then $r(t) \equiv 1$ for every critical point $(u = (x, r), \eta)$.

Proof. Denote by $\Phi_t : S\Sigma \to S\Sigma$ the symplectic isotopy (2.2). A pair (u, η) with $u = (x, r) : S^1 \to \Sigma \times (0, \infty)$ belongs to $\mathrm{Crit}(\mathcal{A}_\varphi)$ if and only if

$$\begin{cases} \dot{u}(t) = \eta\beta(t)R(x(t)) + \dot{\chi}(t)X_{H_{\chi(t)}}(u(t)), \\ \int_0^1 \beta(t)(r(t) - 1)dt = 0. \end{cases} \tag{2.10}$$

Thus for $t \in [0, \frac{1}{2}]$, we have $r(t) = 1$ and $\dot{x}(t) = -\eta R(x(t))$, and $x(1) = \Phi_1(x(\frac{1}{2}))$. Suppose $(u, \eta) \in \mathrm{Crit}(\mathcal{A}_\varphi)$. Thus $u(\frac{1}{2}) = (\theta^{-\eta}(x(0)), 1)$. Next, for $t \in [\frac{1}{2}, 1]$ we have $\dot{u}(t) = \dot{\chi}(t)X_{H_{\chi(t)}}(u(t))$. In particular, $\varphi(x(\frac{1}{2})) = \theta^{-\eta}(x(\frac{1}{2}))$, and thus $x(\frac{1}{2})$ is a translated point of φ. Moreover the time shift of x is given by $-\eta$ mod 1.

Next, we note that

$$\lambda(X_H(x, r)) = dH(x, r)\left(r\frac{\partial}{\partial r}\right) = H(x, r), \tag{2.11}$$

and hence

$$\begin{aligned} \mathcal{A}_\varphi(u, \eta) &= \int_0^{\frac{1}{2}} (r\alpha)(\eta\beta(u)R(x))dt + \int_{\frac{1}{2}}^1 \left[\lambda(\dot{\chi}X_{H_\chi}(u)) - \dot{\chi}H_\chi(u)\right] dt \\ &= \eta + 0. \end{aligned} \tag{2.12}$$

Finally if φ_t is exact for each t then the path Φ_t of symplectomorphisms defined in (2.2) is simply given by $\Phi_t(x,r) = (\varphi_t(x), r)$, from which the last statement immediately follows. $\square$

Remark 2.8. We emphasize again that if $(u = (x,r), \eta)$ is a critical point of $\mathcal{A}_\varphi$ then the time-shift of the translated point $x(\frac{1}{2})$ is the *negative* of the action value. This explains the Reeb flow will turn out to have a *negative* spectral number (cf. part (1) of Theorem 1.1).

We point out that there is a distinguished Morse-Bott component of $\mathrm{Crit}(\mathcal{A}_{\mathrm{id}_\Sigma})$ diffeomorphic to Σ corresponding to critical points $((x,1),0)$ for $x \in \Sigma$.

We now define what it means for φ to be non-degenerate. In the periodic case we also introduce the notion of being non-resonant.

Definition 2.9. A path φ is *non-degenerate* if $\mathcal{A}_\varphi : \Lambda(S\Sigma) \times \mathbb{R} \to \mathbb{R}$ is a Morse-Bott function. In the periodic case we say that φ is *non-resonant* if $\mathrm{Spec}(\varphi) \cap \mathbb{Z} = \emptyset$.

Remark 2.10. The identity id_Σ is non-degenerate if and only if α is of Morse-Bott type (see [15, Appendix B]). In [9] we explained why a generic path φ is non-degenerate (actually a stronger result is proved there: for generic φ the functional $\mathcal{A}_\varphi$ is even Morse). It is also easy to see that a generic φ is non-resonant. Finally we note that $\mathrm{Spec}(\varphi)$ depends only on the terminal map φ_1.

The following lemma is a minor variation on [47, Lemma 3.8]. The proof is included for completeness.

Lemma 2.11. *The set* $\mathrm{Spec}(\varphi)$ *is always a nowhere dense subset of* $\mathbb{R}$ *(even in the degenerate case).*

Proof. More generally, we will prove that any functional of the form

$$\mathcal{A} : \Lambda(W) \times \mathbb{R} \to \mathbb{R}, \quad \mathcal{A}(u,\eta) = \int_0^1 u^*\lambda - \eta \int_0^1 F_t(u(t))\,dt - \int_0^1 H_t(u(t))\,dt,$$

where $F, H \in C^\infty(S^1 \times W, \mathbb{R})$, has the property that $\mathrm{Spec}(\mathcal{A}) := \mathcal{A}(\mathrm{Crit}(\mathcal{A}))$ is a nowhere dense subset of $\mathbb{R}$. Let $\Lambda_2(W) := C^\infty(\mathbb{R}/2\mathbb{Z}, W)$, and consider

$\mathcal{A}_2 : \Lambda_2(W) \times \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{A}_2 : \Lambda(W) \times \mathbb{R} \to \mathbb{R}, \quad \mathcal{A}_2(w, \eta) = \int_0^2 w^*\lambda - \eta \int_0^1 F_t(w(t))\, dt - \int_0^1 H_t(w(t))\, dt$$

(note only the first term integrates over the whole domain $\mathbb{R}/2\mathbb{Z}$). Denoting by

$$K_\eta(t, x) := \eta F_t(x) + H_t(x),$$

we see that critical points (w, η) of $\mathcal{A}_2$ satisfy

$$\dot{w}(t) = \begin{cases} X_{K_\eta}(t, w(t)), & t \in [0, 1], \\ 0, & t \in [1, 2], \end{cases}$$

and

$$\int_0^1 F_t(w(t))\, dt = 0.$$

Thus there is a well defined map $i : \mathrm{Crit}(\mathcal{A}) \hookrightarrow \mathrm{Crit}(\mathcal{A}_2)$ given by $i(u, \eta) = (w, \eta)$, where $w(t) = u(t)$ for $t \in [0, 1]$ and $w(t) = u(1)$ for $t \in [1, 2]$. Now consider a neighborhood U of the zero section of TW small enough that the map $e(x, v) := (x, \exp_x(v))$ is a diffeomorphism of U onto a neighbourhood V of the diagonal Δ in $W \times W$.

Let Q denote the open set

$$Q := \Big\{ (x, \eta) \mid (x, \phi^1_{K_\eta}(x)) \in V \Big\} \subset W \times \mathbb{R},$$

and consider the map

$$c : Q \to \Lambda_2(W), \quad c(x, \eta)(t) := \begin{cases} \phi^t_{K_\eta}(x), & t \in [0, 1], \\ \exp_x\Big((2 - t) e^{-1}(x, \phi^1_{K_\eta}(x)) \Big), & t \in [1, 2]. \end{cases}$$

Then $c(u(0), \eta) = i(u, \eta)$ for all $(u, \eta) \in \mathrm{Crit}(\mathcal{A})$. Thus if we define $a : Q \to \mathbb{R}$ by $a(x, \eta) := \mathcal{A}_2(c(x, \eta))$, we see that $\mathrm{Crit}(\mathcal{A})$ can be identified with a subset of $\mathrm{Crit}(a)$. Since Q is finite-dimensional, the conclusion follows from Sard's theorem. $\square$

The next lemma explains why we pay particular attention to periodic contact manifolds. It is the analogue in the context of Rabinowitz Floer homology of the key idea in [44].It will prove crucial in the construction of the contact capacity (cf. Section 4, in particular Proposition 4.3). Fix $\varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ and fix a contactomorphism $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$.

Lemma 2.12. *Assume α is periodic. If $(u = (x, r), \eta) \in \mathrm{Crit}(\mathcal{A}_\varphi)$ with $\eta \in \mathbb{Z}$ then there exists a critical point $(u_1 = (x_1, r_1), \eta)$ of $\mathcal{A}_{\psi\varphi\psi^{-1}}$ with $x_1(\frac{1}{2}) = \psi(x(\frac{1}{2}))$. In particular,*

$$\mathrm{Spec}(\varphi) \cap \mathbb{Z} = \emptyset \quad \Leftrightarrow \quad \mathrm{Spec}(\psi\varphi\psi^{-1}) \cap \mathbb{Z} = \emptyset. \tag{2.13}$$

Moreover (u, η) is non-degenerate if and only if (u_1, η) is non-degenerate.

Proof. If $(u, \eta) \in \mathrm{Crit}(\mathcal{A}_\varphi)$ with $\eta \in \mathbb{Z}$ then since θ^t is 1-periodic, this means that if we write $u(t) = (x(t), r(t))$ then $x(\frac{1}{2})$ is a fixed point of φ. Thus $\psi(x(\frac{1}{2}))$ is a fixed point of $\psi\varphi\psi^{-1}$. Thus by Lemma 2.7 for each $\nu \in \mathbb{Z}$ there exists a critical point $(u_\nu = (x_\nu, r_\nu), \nu)$ of $\mathcal{A}_{\psi\varphi\psi^{-1}}$ with $x_\nu(\frac{1}{2}) = \psi(x_\nu(\frac{1}{2}))$. In particular, this is true for $\nu = \eta$. The final statement follows from the fact that the linearised equation is also conjugation invariant. □

2.3. Rabinowitz Floer homology

Let us now assume that (Σ, ξ) is Liouville fillable with a Morse-Bott Liouville filling $(W, d\lambda)$. We would like to extend $\mathcal{A}_\varphi$ to a functional defined on all of $\Lambda(W) \times \mathbb{R}$, where $\Lambda(W) := C^\infty_{\mathrm{contr}}(S^1, W)$ as before. In order to do this we must extend the function $(x, r) \mapsto r - 1$ and the Hamiltonian H_t to functions defined on all of W. At the same time, it is convenient to truncate them. As in [5, 18], we proceed as follows. Define $m : W \to \mathbb{R}$ so that

$$m(z) := \begin{cases} r - 1, & z = (x, r) \in \Sigma \times (\frac{1}{2}, \frac{3}{2}), \\ \frac{3}{4}, & z = (x, r) \in \Sigma \times (2, \infty), \\ -\frac{3}{4}, & z \in W \setminus S\Sigma. \end{cases} \tag{2.14}$$

Next, for $\kappa > 0$ let $\varepsilon_\kappa \in C^\infty([0, \infty), [0, 1])$ denote a smooth function such that

$$\varepsilon_\kappa(r) = \begin{cases} 1, & r \in [e^{-\kappa}, e^\kappa], \\ 0, & r \in [0, e^{-2\kappa}] \cup [e^\kappa + 1, \infty), \end{cases} \tag{2.15}$$

and such that

$$0 \le \varepsilon'_\kappa(r) \le 2e^{2\kappa} \quad \text{for } r \in [e^{-2\kappa}, e^{-\kappa}], \tag{2.16}$$

$$-2 \le \varepsilon'_\kappa(r) \le 0 \quad \text{for } r \in [e^\kappa, e^\kappa + 1]. \tag{2.17}$$

Then define $H^\kappa_t : W \to \mathbb{R}$ by setting $H^\kappa_t|_{W \setminus S\Sigma} = 0$ and

$$H^\kappa_t(x, r) := \varepsilon_\kappa(r) H_t(x, r) \quad \text{for } (x, r) \in S\Sigma. \tag{2.18}$$

We denote by $\mathcal{A}^\kappa_\varphi : \Lambda(W) \times \mathbb{R} \to \mathbb{R}$ the Rabinowitz action functional defined using the Hamiltonians m and H^κ_t:

$$\mathcal{A}^\kappa_\varphi(u,\eta) := \int_0^1 u^*\lambda - \eta \int_0^1 \beta(t)m(u(t))dt - \int_0^1 \dot\chi(t)H^\kappa_{\chi(t)}(u(t))dt. \tag{2.19}$$

Definition 2.13. Now let us recall the definition of the oscillation semi-norm on $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$. Firstly, suppose $\{\varphi_t\}_{0\le t\le 1} \in \mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)$. Let $h_t : \Sigma \to \mathbb{R}$ denote the contact Hamiltonian. The *oscillation semi-norm* $\|h\|_{\mathrm{osc}}$ is defined by

$$\|h\|_{\mathrm{osc}} := \|h\|_+ + \|h\|_- , \tag{2.20}$$

$$\|h\|_+ := \int_0^1 \max_{x\in\Sigma} h_t(x)dt, \quad \|h\|_- := -\int_0^1 \min_{x\in\Sigma} h_t(x)dt. \tag{2.21}$$

We then define the oscillation semi-norm $\|\varphi\|$ and its positive and negative parts $\|\varphi\|_\pm$ for $\varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ by taking the infimum of the oscillation semi-norms $\|h\|_{\mathrm{osc}}$ [resp. $\|h\|_\pm$] over all possible paths $\{\varphi_t\}_{0\le t\le1}$ representing φ (with corresponding contact Hamiltonians h_t). Note that the positive and negative parts $\|h\|_\pm$ can in fact have any sign, but their sum is always non-negative.

Definition 2.14. Suppose $\varphi \in \mathcal{P}\mathrm{Cont}_0(\Sigma,\xi)$. Let $\rho_t : \Sigma \to (0,\infty)$ is defined by $\varphi_t^*\alpha = \rho_t\alpha$. Define a constant $\kappa(\varphi) \ge 0$ by

$$\kappa(\varphi) := \max_{t\in[0,1]} \left| \int_0^t \max_{x\in\Sigma} \frac{\dot\rho_\tau(x)}{\rho_\tau(x)^2} d\tau \right| \tag{2.22}$$

Note that if φ is exact (i.e. $\varphi_t^*\alpha = \alpha$, so that $\rho_t = 1$) then $\kappa(\varphi) = 0$.

In [9, Proposition 2.5] we proved:

Lemma 2.15. *If $\kappa > \kappa(\varphi)$ then if $(u,\eta) \in \mathrm{Crit}(\mathcal{A}^\kappa_\varphi)$ then $u(S^1) \subseteq S\Sigma$, and moreover if we write $u(t) = (x(t), r(t))$ then $r(S^1) \subseteq (e^{-\kappa/2}, e^{\kappa/2})$.*

If φ is non-degenerate in the sense of Definition 2.9, then as explained in [9], Lemma 2.15 allows to define for $a, b \in [-\infty,\infty]\backslash\mathrm{Spec}(\varphi)$ the *Rabinowitz Floer homology*

$$\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W). \tag{2.23}$$

This is a semi-infinite dimensional Morse theory associated to the functional $\mathcal{A}^\kappa_\varphi$ (for some $\kappa > \kappa_0(\varphi)$), and we sketch the definition here and refer to e.g.

[5, 9] for more information. As indicated by the notation, the filtered Rabinowitz Floer homology is independent of the choice of κ, as explained in [7]. Indeed, since the critical points and values are independent of κ by Lemma 2.15, a continuation argument implies that $\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W)$ is independent of κ up to a chain complex isomorphism.

Definition 2.16. We denote by $\mathcal{J}(W)$ the set of smooth families

$$J = \{J_t(\cdot,\tau)\}_{(t,\tau)\in S^1\times\mathbb{R}} \tag{2.24}$$

of almost complex structures on W, which are compatible with $d\lambda$, meaning that for each $(t,z,\tau) \in S^1 \times W \times \mathbb{R}$, the bilinear form $d\lambda_z(J_t(z,\tau)\cdot,\cdot)$ defines a Riemannian metric on T_zW. In addition we require that

$$\sup_{(t,\tau)\in S^1\times\mathbb{R}} \|J_t(\cdot,\tau)\|_{C^k} < +\infty, \qquad \forall\, k \in \mathbb{N}, \tag{2.25}$$

where $\|\cdot\|_{C^k}$ is the norm taken with respect to some background metric on W. We denote by $\mathcal{J}_{\mathrm{conv}}(W) \subset \mathcal{J}(W)$ the subset consisting of those families J which are *contact type at infinity.* This means that there exists $S_0 > 0$ such that on $\Sigma \times [S_0, +\infty) \subset W$ one has

$$dr \circ J_t(\cdot,\cdot,\tau) = r\alpha, \quad \text{on} \quad \Sigma \times [S_0, \infty). \tag{2.26}$$

In particular, this means that J is independent of both $t \in S^1$ and $\tau \in \mathbb{R}$ on $\Sigma \times [S_0, +\infty)$.

Given $J \in \mathcal{J}_{\mathrm{conv}}(W)$ we can define an L^2-inner product $\langle\langle\cdot,\cdot\rangle\rangle_J$ on $\Lambda(W) \times \mathbb{R}$: for $(u,\eta) \in \Lambda(W) \times \mathbb{R}$, $\zeta, \zeta' \in \Gamma(u^*TW)$ and $b, b' \in \mathbb{R}$, set

$$\langle\langle(\zeta,b),(\zeta',b')\rangle\rangle_J := \int_0^1 d\lambda_{u(t)}\left(J_t(u(t),\eta)\zeta(t),\zeta'(t)\right)\,dt + bb'. \tag{2.27}$$

We denote by $\nabla_J\mathcal{A}_\varphi^\kappa$ the gradient of $\mathcal{A}_\varphi^\kappa$ with respect to $\langle\langle\cdot,\cdot\rangle\rangle_J$.

Assume that φ is non-degenerate and fix $\kappa > \kappa_0(\varphi)$ and $J \in \mathcal{J}_{\mathrm{conv}}(W)$. By assumption $\mathcal{A}_\varphi^\kappa$ is a Morse-Bott function. Pick a Morse function $g : \mathrm{Crit}(\mathcal{A}_\varphi^\kappa) \to \mathbb{R}$, and choose a Riemannian metric ϱ on $\mathrm{Crit}(\mathcal{A}_\varphi^\kappa)$ such that the negative gradient flow of $\nabla_\varrho g$ is *Morse-Smale*. Given two critical points $w^-, w^+ \in \mathrm{Crit}(g)$, with $w^\pm = (u^\pm, \eta^\pm)$, we denote by $\mathcal{M}_{w^-,w^+}(\mathcal{A}_\varphi^\kappa, g, J, \varrho)$ the moduli space of *gradient flow lines with cascades* of $-\nabla_J\mathcal{A}_\varphi^\kappa$ and $-\nabla_\varrho g$ running from w^- to w^+. See [27, Appendix A] or [15, Appendix A] for the precise definition.

Remark 2.17 (Transversality and Morse-Bott issues). In this paper we use the fact that the moduli spaces $\mathcal{M}_{w^-,w^+}(\mathcal{A}^\kappa_\varphi, g, J, \varrho)$ can generically be chosen to be smooth finite dimensional manifolds, whose zero-dimensional components have good compactness properties. Let us explain why this is true.

- Firstly, there exists a residual subset in the product of $\mathcal{J}_{\mathrm{conv}}(W)$ with the space of Riemannian metrics on $\mathrm{Crit}(\mathcal{A}^\kappa_\varphi)$ with the property that if (J, ϱ) belongs to this subset then the spaces $\mathcal{M}_{w^-,w^+}(\mathcal{A}^\kappa_\varphi, g, J, \varrho)$ are finite dimensional smooth manifolds. The proof of this does *not* quite follow from standard arguments in Floer theory. Nevertheless, a detailed proof is given in [2, Section 4.3]. This is the reason why the almost complex structures J we use in this article are required to additionally depend on $\tau \in \mathbb{R}$, rather than just being S^1-dependent.

- The compactness question is slightly more delicate, since at the time of writing there is no detailed treatment of the gluing analysis required to construct Morse-Bott Floer theory in general. This situation will be rectified shortly, but for now let us explain to what extent the constructions in this paper actually require this theory. In short, everything in Sections 2-4 can be redone *without* needing Morse-Bott theory. Section 5 is slightly different, and we will discuss this in Remark 5.9 below.

  In fact, in most cases of interest in this paper, we may assume that $\mathcal{A}^\kappa_\varphi$ is actually Morse (cf. Remark 2.10). In this case (without making any changes at all) we do not need any Morse-Bott theory: one can take the auxiliary Morse function g to be identically zero. The only case where we cannot assume this is when $\varphi = \mathrm{id}_\Sigma$. There are (at least) two different ways to deal with this which do not require Morse-Bott theory in full generality.

  For instance, as explained in [15, Appendix B] (see also [16]), we may assume that the contact form α generating ξ is not only Morse-Bott but also *transversely non-degenerate*, in the sense that for each $x \in P_T$ (cf. Definition 2.2), the linearisation $D\theta^T(x) - \mathrm{id}_{T_x\Sigma}|_{\xi_x} : \xi_x \to \xi_x$ does not have 1 in its spectrum. This implies that the critical manifolds of $\mathcal{A}_{\mathrm{id}}$ corresponding to closed Reeb orbits are all one-dimensional. In this case a detailed proof of the gluing analysis needed *is* available: this is due to Bourgeois-Oancea [11].

  Moreover, if the reader wishes to work without any Morse-Bott theory, this is also possible, at the expense of slightly changing the wording of the Assumptions **(A)**-**(B)** from the Introduction. Namely, one could simply *define* $\mathrm{RFH}_*(\Sigma, W)$ to be the homology of the perturbed Rabinowitz action functional $\mathcal{A}_\varphi$, where φ is a perturbation sufficiently close to the identity such that $\mathcal{A}_\varphi$ is a Morse function. This is a well-defined definition (i.e. the resulting homology is independent of the choice of perturbation), and agrees with the original definition if one uses Morse-Bott theory.

The fact that both of these approaches suffice to deal with the situation posited by Assumption **(B)** from the Introduction, i.e. when there exists a contact form α whose Reeb flow is periodic, follows immediately from the fact that the spectral value c defined in Definition 3.8 below depends Lipschitz continuously on the 1-form λ used to define the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, W)$. The proof of this statement can be extracted from the proof of Corollary 3.7 in [15], see also [6, Section 5].

Introduce a grading on $\mathrm{Crit}(g)$ by setting

$$(2.28)\quad \mu(u,\eta) := \begin{cases} \mu_{\mathrm{CZ}}(u) - \frac{1}{2}\dim_{(u,\eta)}\mathrm{Crit}(\mathcal{A}_\varphi^\kappa) + \mathrm{ind}_g(u,\eta), & \eta > 0, \\ \mu_{\mathrm{CZ}}(u) - \frac{1}{2}\dim_{(u,\eta)}\mathrm{Crit}(\mathcal{A}_\varphi^\kappa) + \mathrm{ind}_g(u,\eta) + 1, & \eta < 0, \\ 1 - n + \mathrm{ind}_g(u,\eta), & \eta = 0. \end{cases}$$

Here $\mu_{\mathrm{CZ}}(u)$ denotes the Conley-Zehnder index of the loop $t \mapsto u(t/\eta)$ and $\dim_{(u,\eta)}\mathrm{Crit}(\mathcal{A}_\varphi^\kappa)$ denotes the local dimension of $\mathrm{Crit}(\mathcal{A}_\varphi^\kappa)$ at (u,η), and $\mathrm{ind}_g(u,\eta)$ denotes the Morse index of (u,η) as a critical point of g. As mentioned above, in most cases of interest in this paper, we may assume that $\mathcal{A}_\varphi$ is actually Morse. In this case the Morse function g is taken to be identically zero, and (2.28) continues to hold, with $\mathrm{ind}_{g=0} := 0$. Equivalently, denoting by $\mu_{\mathrm{CZ}}^{\mathrm{tr}}(u)$ the *transverse Conley Zehnder index*, and denoting by $\nu^{\mathrm{tr}}(u,\eta)$ the *tranverse nullity* of the critical point (so that $\nu^{\mathrm{tr}}(u,\eta) = \dim_{(u,\eta)}\mathrm{Crit}(\mathcal{A}_\varphi^\kappa) - 1$), the formula (2.28) can be unified as

$$(2.29)\qquad \mu(u,\eta) := \mu_{\mathrm{CZ}}^{\mathrm{tr}}(u) - \frac{1}{2}\nu^{\mathrm{tr}}(u,\eta) + \mathrm{ind}_g(u,\eta)$$

in the case $\eta \neq 0$.

Remark 2.18. Our normalization convention for the Conley-Zehnder index is that if H is a C^2-small Morse function on W and x is a critical point of W then

$$\mu_{\mathrm{CZ}}(x) = n - \mathrm{ind}_H(x), \tag{2.30}$$

where $\mathrm{ind}_H(x)$ denotes the Morse index of x.

Given $-\infty < a < b < \infty$ denote by $\mathrm{RFC}_*^{(a,b)}(\mathcal{A}_\varphi^\kappa, g) := \mathrm{Crit}_*^{(a,b)}(g) \otimes \mathbb{Z}_2$, where $\mathrm{Crit}_*^{(a,b)}(g)$ denotes the set of critical points w of g with $a < \mathcal{A}_\varphi(w) < b$. We only do this when $a, b \notin \mathrm{Spec}(\varphi)$, even if this is not explicitly stated. Generically the moduli spaces $\mathcal{M}_{w^-,w^+}(\mathcal{A}_\varphi, g, J, \varrho)$ carry the structure of finite dimensional smooth manifolds, whose components of dimension zero are compact. One defines a boundary operator ∂ on $\mathrm{RFC}_*^{(a,b)}(\mathcal{A}_\varphi^\kappa, g)$ by counting the elements of the zero-dimensional parts of the moduli spaces $\mathcal{M}_{w^-,w^+}(\mathcal{A}_\varphi, g, J, \varrho)$.

The homology $\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W) := \mathrm{H}_*(\mathrm{RFC}_*^{(a,b)}(\mathcal{A}_\varphi^\kappa, g), \partial)$ does not depend on any of the auxiliary choices we made. We emphasize though that $\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W)$ depends on the choice of filling $(W, d\lambda)$. Finally we define

$$\mathrm{RFH}_*^{b}(\mathcal{A}_\varphi, W) := \varprojlim_{a\downarrow-\infty} \mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W), \tag{2.31}$$

and

$$\mathrm{RFH}_*(\mathcal{A}_\varphi, W) := \varinjlim_{b\uparrow+\infty} \mathrm{RFH}_*^{b}(\mathcal{A}_\varphi, W) = \varinjlim_{b\uparrow+\infty} \varprojlim_{a\downarrow-\infty} \mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W) \tag{2.32}$$

(the order of the limits in (2.32, cf. [17]) matters). As pointed out by Ritter [41], it follows from work of Cieliebak-Frauenfelder-Oancea [18] that the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, W)$ vanishes if and only if the symplectic homology $\mathrm{SH}_*(W)$ vanishes.

We briefly summarize now the key properties that we will need about the Rabinowitz Floer homology $\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W)$, which are all proved in [5, 15]:

1) The Rabinowitz Floer homology is independent of φ in the following strong sense. There is a universal object $\mathrm{RFH}_*(\Sigma, W)$ (which may be thought as corresponding to the case $\varphi = \mathrm{id}_\Sigma$) together with canonical isomorphisms

$$\zeta_\varphi : \mathrm{RFH}_*(\Sigma, W) \to \mathrm{RFH}_*(\mathcal{A}_\varphi, W). \tag{2.33}$$

Given two paths φ and ψ, there is a map $\zeta_{\varphi,\psi} : \mathrm{RFH}_*(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*(\mathcal{A}_\psi, W)$ with the property that

$$\zeta_\psi = \zeta_{\varphi,\psi} \circ \zeta_\varphi. \tag{2.34}$$

In particular, if $Z \in \mathrm{RFH}_*(\Sigma, W)$ is a non-zero class then $\mathrm{RFH}_*(\mathcal{A}_\varphi, W)$ contains a non-zero class Z_φ defined by

$$\zeta_{\varphi,\psi}\big(Z_\varphi\big) = Z_\psi \quad \text{and} \quad Z_{\mathrm{id}_\Sigma} = Z \in \mathrm{RFH}_*(\Sigma, W). \tag{2.35}$$

2) If $a \le b \le \infty$ there is a natural map

$$j_\varphi^{a,b} : \mathrm{RFH}_*^a(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*^b(\mathcal{A}_\varphi, W) \tag{2.36}$$

induced by the inclusion on the chain level, and similarly there is a natural map

$$p_\varphi^{a,b} : \mathrm{RFH}_*^b(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W) \tag{2.37}$$

induced by the restriction on the chain level. If $b = \infty$ we abbreviate $j_\varphi^{a,\infty} = j_\varphi^a$, and we write j^a for the map $\mathrm{RFH}_*^a(\Sigma, W) \to \mathrm{RFH}_*(\Sigma, W)$, with similar conventions for the maps $p_\varphi^{a,b}$. If $\mathrm{Spec}(\varphi) \cap [a,b] = \emptyset$ then the map

$$j_\varphi^{a,b} : \mathrm{RFH}_*^a(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*^b(\mathcal{A}_\varphi, W)$$

is an isomorphism and $p_\varphi^{a,b} : \mathrm{RFH}_*^b(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W)$ is the zero map (as $\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W) = 0$).

3) Moreover there is a filtered version of (2.34), which gives the existence of maps

$$\zeta_{\varphi,\psi}^a : \mathrm{RFH}_*^a(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*^{a+K(\varphi,\psi)}(\mathcal{A}_\psi, W) \tag{2.38}$$

for some constant $K(\varphi,\psi) \ge 0$. The maps (2.38) are a special case of [5, Lemma 2.7]. It will be important however to note that if the paths φ, ψ have contact Hamiltonians h_t and k_t then the constant $K(\varphi,\psi)$ satisfies

$$K(\varphi,\psi) \le e^{\max\{\kappa(\varphi),\kappa(\psi)\}} \max\big\{ \|h-k\|_+ , 0\big\}, \tag{2.39}$$

where we are using the notation from (2.20)-(2.21). Finally one has for all $Z \in \mathrm{RFH}_*^a(\mathcal{A}_\varphi, W)$ that

$$\zeta_{\varphi,\psi}\left(j_\varphi^a(Z)\right) = j_\psi^{a+K(\varphi,\psi)}\left(\zeta_{\varphi,\psi}^a(Z)\right). \tag{2.40}$$

4) We recall from Remark 2.8 that $\mathrm{Crit}(\mathcal{A}_{\mathrm{id}_\Sigma})$ contains Σ as a Morse-Bott component via the constants. For $\varepsilon > 0$ smaller than the smallest period of a contractible Reeb orbit, this gives rise to a canonical isomorphism

$$\mathrm{RFH}_*^{(-\varepsilon,\varepsilon)}(\Sigma, W) \cong \mathrm{H}_{*+n-1}(\Sigma;\mathbb{Z}_2). \tag{2.41}$$

Even though it is more or less standard, the estimate (2.39) is extremely important in all that follows, and hence we prove it here. To define the continuation homomorphism $\zeta_{\varphi,\psi}$ we denote by $H_t = rh_t$ and $K_t = rk_t$ the Hamiltonian functions of φ and ψ, respectively, and choose a linear homotopy

$$L_t^s := \nu(s)H_t + (1-\nu(s))K_t \tag{2.42}$$

for a smooth function $\nu : \mathbb{R} \to [0,1]$ with $\nu(s) = 1$ for $s \le -1$, $\nu(s) = 0$ for $s \ge 1$ and $\nu'(s) \le 0$. We define the (s-dependent) action functional $\mathcal{A}_s$ as in (2.7):
(2.43)

$$\mathcal{A}_s(u,\eta) = \int_0^1 u^*\lambda - \eta\int_0^1 \beta(t)m(u(t))dt - \int_0^1 \dot\chi(t)\varepsilon_\kappa(r)L^s_{\chi(t)}(u(t))dt.$$

where φ_s has corresponding Hamiltonian function L_t^s. Then counting solutions of

$$\partial_s(u,\eta) + \nabla_{J_s}\mathcal{A}_s(u,\eta) = 0 \tag{2.44}$$

with

$$(u_-,\eta_-) := \big(u(-\infty),\eta(-\infty)\big) \in \mathrm{Crit}(\mathcal{A}_\varphi)$$

and

$$(u_+,\eta_+) := \big(u(+\infty),\eta(+\infty)\big) \in \mathrm{Crit}(\mathcal{A}_\psi)$$

defines the continuation homomorphism. We recall that $\kappa > \max\{\kappa(\varphi), \kappa(\psi)\}$ and estimate

$$
\begin{aligned}
0 &\le \mathbb{E}_J(u,\eta) \\
&= \int_{-\infty}^{\infty}\int_0^1 |\partial_s(u,\eta)|^2_{J_s}\,dtds \\
&= -\int_{-\infty}^{\infty}\int_0^1 \langle\langle \nabla_{J_s}\mathcal{A}_s(u,\eta), \partial_s(u,\eta)\rangle\rangle_{J_s}\,dtds \\
&= -\int_{-\infty}^{\infty}\int_0^1 \frac{d}{ds}\mathcal{A}_s(u,\eta)dtds + \int_{-\infty}^{\infty}\int_0^1 \frac{\partial \mathcal{A}_s}{\partial s}(u,\eta)dtds \\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) - \int_{-\infty}^{\infty}\int_0^1 \dot\chi(t)\varepsilon_\kappa(r(t))\frac{\partial L^s_{\chi(t)}}{\partial s}(u(t))dtds \\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) \\
&\quad - \int_{-\infty}^{\infty}\int_0^1 \nu'(s)\varepsilon_\kappa(r(t))\dot\chi(t)\Big(H_{\chi(t)}(u(t)) - K_{\chi(t)}(u(t))\Big)dtds \\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) \\
&\quad + \int_{-\infty}^{\infty}\int_0^1 \underbrace{(-\nu'(s))}_{\ge 0}\underbrace{\varepsilon_\kappa(r(t))r(t)}_{0\le\cdot\le e^\kappa}\underbrace{\dot\chi(t)}_{\ge 0}\Big(h_{\chi(t)}(u(t)) - k_{\chi(t)}(u(t))\Big)dtds \\
&\le \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) \\
&\quad + \int_{-\infty}^{\infty}\int_0^1 \underbrace{(-\nu'(s))}_{\ge 0}\underbrace{\varepsilon_\kappa(r(t))r(t)}_{0\le\cdot\le e^\kappa}\underbrace{\dot\chi(t)}_{\ge 0}\max_{x\in\Sigma}\Big(h_{\chi(t)}(x) - k_{\chi(t)}(x)\Big)dtds \\
&\le \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) \\
&\quad + \int_{-\infty}^{\infty}\int_0^1 \underbrace{(-\nu'(s))}_{\ge 0}\underbrace{\varepsilon_\kappa(r(t))r(t)}_{0\le\cdot\le e^\kappa}\underbrace{\dot\chi(t)}_{\ge 0}\max\Big\{\max_{x\in\Sigma}\Big(h_{\chi(t)}(x) - k_{\chi(t)}(x)\Big),0\Big\}dtds \\
&\le \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) \\
&\quad + e^\kappa\int_{-\infty}^{\infty}\int_0^1 (-\nu'(s))\dot\chi(t)\max\Big\{\max_{x\in\Sigma}\Big(h_{\chi(t)}(x) - k_{\chi(t)}(x)\Big),0\Big\}dtds \\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) \\
&\quad + e^\kappa\underbrace{\int_{-\infty}^{\infty}(-\nu'(s))ds}_{=1}\int_0^1 \dot\chi(t)\max\Big\{\max_{x\in\Sigma}\Big(h_{\chi(t)}(x) - k_{\chi(t)}(x)\Big),0\Big\}dt \\
&= \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) + e^\kappa\int_0^1 \max\Big\{\max_{x\in\Sigma}\Big(h_t(x) - k_t(x)\Big),0\Big\}dt \\
&\le \mathcal{A}_\varphi(u_-,\eta_-) - \mathcal{A}_\psi(u_+,\eta_+) + e^\kappa\max\big\{\|h-k\|_+,0\big\}.
\end{aligned}
$$

This proves estimate (2.39).

3. Spectral invariants and orderability

We begin this section with the definition of the crucial property of being spectrally finite.

Definition 3.1. A non-zero class $Z \in \mathrm{RFH}_*(\Sigma, W)$ is said to be *spectrally finite* if

$$\inf \{a \in \mathbb{R} \mid Z \in j^a(\mathrm{RFH}^a_*(\Sigma, W))\} > -\infty. \tag{3.1}$$

We emphasise that we only define the notion of a spectrally finite class for the unperturbed Rabinowitz action functional.

Let $(\Sigma, \xi = \ker \alpha)$ denote a contact manifold such that α is Morse-Bott, see Definition 2.2. We recall that for $T \in \mathrm{Spec}(\alpha)$ the set

$$P_T := \left\{x \in \Sigma \mid \theta^T(x) = x\right\}$$

is a closed submanifold of Σ such that $d\alpha|_{P_T}$ has locally constant rank, and such that $T_x P_T = \ker(D\theta^T(x) - \mathrm{id})$ for all $x \in P_T$. Here as usual $\theta^t : \Sigma \to \Sigma$ denotes the Reeb flow of α. The Reeb flow induces an S^1-action on P_T, and we denote by $S_T := P_T/S^1$ the quotient of P_T by this action, which will in general be an orbifold. An orbit space S_T is called *simple* if there exists a point $x \in S_T$ with minimal period T.

Assume that (Σ, α) is Liouville fillable. This implies that the *mean index* of an orbit space, denoted by $\Delta(S_T)$, is a well defined real number (if P_T is not connected, one gets such a number for each connected component, but by a slight abuse of notation we will suppress this in the following). More precisely, fix $x \in P_T$ and let $\gamma(t) := \theta^t(x)$ for $t \in [0, T]$, so that $\gamma : [0, T] \to \Sigma$ is a closed Reeb orbit. The linearized Reeb flow preserves the symplectic form $d\alpha$ and thus gives rise to a family of symplectic maps $D\theta^t(x) : \xi_{\gamma(0)} \to \xi_{\gamma(t)}$ along γ. Trivialising ξ along γ we obtain a path $[0, T] \mapsto \mathrm{Sp}(2n - 2)$. We denote by $\Delta(S_T)$ mean index of this path (in the sense of [43, Section 5]), which, as the notation suggests, does not depend on $x \in P_T$.

If γ is not homologically trivial it may depend on the choice of trivialisation, and an additional assumption on the first Chern class of ξ is needed in order for this to be well defined. Nevertheless, in this paper we are only concerned with contractible Reeb orbits, and so in the remainder of this section we will implicitly only speak of the mean index $\Delta(S_T)$ when the component P_T contains contractible Reeb orbits.

The following two properties of the mean index Δ are important. Firstly, the mean index is linear with respect to iterations, that is,

$$\Delta(S_{kT}) = k\Delta(S_T), \qquad \forall\, k \in \mathbb{Z}. \tag{3.2}$$

Secondly if $(u, \eta) \in \mathrm{Crit}(\mathcal{A}_{\mathrm{id}})$ with $\eta \neq 0$ and γ is the $|\eta|$-periodic Reeb orbit given by $\gamma(t) := u(t/\eta)$ then there exists a constant c such that

$$|\Delta(S_{|\eta|}) - \mu_{\mathrm{CZ}}(\gamma)| \leq c, \tag{3.3}$$

cf. [43, Lemma 3.4] and [42]. The next definition was originally introduced by Ustilovsky [49] and Bourgeois [10].

Definition 3.2. We say that (Σ, α) is *index positive* if there exist constants $A > 0$ and $B \in \mathbb{R}$ such that for all $T \in \mathrm{Spec}(\alpha)$, one has

$$\Delta(S_T) \geq AT + B. \tag{3.4}$$

For us, the notion of index positivity is useful, since it implies that all non-zero classes in the Rabinowitz Floer homology are spectrally finite.

Lemma 3.3. *Suppose (Σ, α) is an index positive Liouville fillable contact manifold, with Liouville filling $(W, d\lambda)$. Then every non-zero class $Z \in \mathrm{RFH}_k(\Sigma, W)$ is spectrally finite.*

Proof. We claim that for any fixed degree $k \in \mathbb{Z}$, the assumption that (Σ, α) is index positive implies that there exists a constant $a(k) \in \mathbb{R}$ such that the chain group $\mathrm{RFC}_k^{(-\infty, a(k))} = \{0\}$, and thus the same is true in homology. This implies that for every non-zero $Z \in \mathrm{RFH}_k(\Sigma, W)$ the left-hand side of (3.1) is at least $a(k)$.

We now prove the claim. Fix $k \in \mathbb{Z}$, and suppose to the contrary that there exists a sequence $(a_l) \subset (-\infty, 0)$ with $a_l \to -\infty$ such that $\mathrm{RFC}_k^{(-\infty, a_l)} \neq \{0\}$. Thus there exist critical points $(u_l, \eta_l) \in \mathrm{Crit}(\mathcal{A}_{\mathrm{id}})$ such that $\mu(u_l, \eta_l) = k$ and $\eta_l = \mathcal{A}_{\mathrm{id}}(u_l, \eta_l) \leq a_l$.

Recall that by definition of $\mathcal{A}_{\mathrm{id}}$, since $\eta_l < 0$, the loop $\gamma_l(t) := u_l(t/\eta_l)$ is a closed contractible Reeb orbit of period $-\eta_l$. Hence $-\eta_l \in \mathrm{Spec}(\alpha)$ and

$\gamma_l \in S_{-\eta_l}$. By (2.28) and (3.3) there exists a constant c_0 such that

$$\begin{aligned}|\Delta(S_{-\eta_l}) + k| &= |\Delta(S_{-\eta_l}) + \mu(u_l, \eta_l)| \\ &\leq |\Delta(S_{-\eta_l}) - \mu_{\mathrm{CZ}}(\gamma_l)| + |\mu_{\mathrm{CZ}}(\gamma_l) + \mu(u_l, \eta_l)| \\ &\leq c_0\end{aligned} \tag{3.5}$$

However applying (3.4) with $T = -\eta_l \in \mathrm{Spec}(\alpha)$, we also have

$$\Delta(S_{-\eta_l}) \geq A(-\eta_l) + B \geq -Aa_l + B \to \infty, \tag{3.6}$$

which contradicts the previous inequality. □

Definition 3.4. We say that α is a *periodic* contact form if there exists $T \in \mathbb{R}$ such that $\theta^T = \mathrm{id}$. We say that α is a *Boothby-Wang* contact form if all simple Reeb orbits are closed and have the *same* period.

If α is periodic then there only finitely many simple orbit spaces $\{S_{T_j}\}_{j=1,\dots m}$. We order these so that $T_1 < T_2 < \cdots < T_m$, so that T_j divides T_m for each $1 \leq j \leq m$. Thus α is Boothby-Wang if $m = 1$. We denote by $\Delta(\alpha) := \Delta(S_{\tau T_m})$, where $\tau \in \mathbb{N}$ is the smallest positive integer such that the τth iteration of a principal orbit in S_{T_m} is contractible. Note that (3.2) implies that if $\Delta(S_{T_j}) = 0$ for some $1 \leq j \leq m$, then necessarily $\Delta(\alpha) = 0$.

In the periodic case, there is another way to prove that spectrally finite classes exist.

Lemma 3.5. *Suppose (Σ, α) is Liouville fillable contact manifold, with Liouville filling $(W, d\lambda)$, and assume that α is periodic. Assume $\Delta(\alpha) \neq 0$. Then every non-zero class $Z \in \mathrm{RFH}_k(\Sigma, W)$ is spectrally finite.*

Proof. We adopt the notation from Definition 3.4. As noted above, the assumption $\Delta(\alpha) \neq 0$ implies that $\Delta(S_{T_j}) \neq 0$ for all $1 \leq j \leq m$. We claim that this implies that for any integer $k \in \mathbb{Z}$, the chain group RFC_k has only finitely many generators, which in turn implies that every non-zero class is spectrally finite.

To prove the claim, suppose for contradiction there exists $k \in \mathbb{Z}$ such that RFC_k has infinitely many generators. By the pigeonhole principle, there exists a fixed $1 \leq j \leq m$ and a sequence $(u_l, n_l T_j) \in \mathrm{Crit}(\mathcal{A}_{\mathrm{id}})$ with $\mu(u_l, n_l T_j) = k$, such that $(n_l) \subset \mathbb{Z}$ is unbounded and $u_l\left(\frac{t}{|n_l| T_j}\right) \in S_{n_l T_j}$. As in the proof of (3.5), this implies that there exists a constant c_0 such that

$$|n_l \Delta(S_{T_j}) - k| \leq c_0, \qquad \forall\, l \in \mathbb{Z}.$$

Since (n_l) is unbounded, this forces $\Delta(S_{T_j}) = 0$, which contradicts the previous paragraph. □

Remark 3.6. The above arguments are presented in the Morse-Bott setting. An explicit perturbation scheme to the Morse situation where the above argument carries over is explained in [10, Section 2.2].

Up to rescaling, all Boothby-Wang contact forms arise in the following way, see [31, Section 7.2]. Take (Q, ω) to be a closed primitive integral symplectic manifold (i.e. the de Rham cohomology class $[\omega]$ has a primitive integral lift in $\mathrm{H}^2(Q; \mathbb{Z})$. Fix $k \in \mathbb{Z} \setminus \{0\}$. Then there is a circle bundle $q_k : \Sigma \to Q$ with Euler class $k[\omega]$ and connection 1-form α such that $q_k^*(k\omega) = d\alpha$. The 1-form α is then a Boothby-Wang contact form on Σ. One also calls (Σ, α) a *prequantisation space* over $(Q, k\omega)$.

The following computation is from [10, Chapter 9]. A very detailed proof can be found in [51].

Lemma 3.7. *Suppose (Σ, α) is a prequantisation space over $(Q, k\omega)$. Assume in addition that Q is simply connected and* monotone*, that is, there exists $c \in \mathbb{Z}$ such that $c_1(Q) = c[\omega]$. Then $\Delta(\alpha) = 2c$.*

Throughout the remainder of this section we require Assumption **(A)** from the Introduction to hold. More precisely, recall we say that a closed connected coorientable contact manifold (Σ, ξ) satisfies Assumption **(A)** if:

Assumption (A): (Σ, ξ) admits a Liouville filling $(W, d\lambda)$ such that $\alpha := \lambda|_\Sigma$ is Morse-Bott and such that there exists a spectrally finite class $Z \in \mathrm{RFH}_*(\Sigma, W)$.

Definition 3.8. Fix a spectrally finite class $Z \in \mathrm{RFH}_*(\Sigma, W)$ and let φ denote a non-degenerate path. We define its *spectral number* by

$$c(\varphi, Z) := \inf \left\{ a \in \mathbb{R} \mid Z_\varphi \in j^a_\varphi(\mathrm{RFH}^a_*(\mathcal{A}_\varphi, W)) \right\}, \tag{3.7}$$

where we use the notation Z_φ from (2.35). Note that $c(\varphi, Z) > -\infty$ for any φ; this follows directly from (2.39) and the fact that Z is a spectrally finite class.

Throughout the rest of the paper, the letter Z denotes a spectrally finite class in $\mathrm{RFH}_*(\Sigma, W)$.

Proposition 3.9. *Let φ and ψ be two non-degenerate paths. Then we have the estimate*

$$c(\psi, Z) \leq c(\varphi, Z) + K(\varphi, \psi) \tag{3.8}$$
$$\leq c(\varphi, Z) + e^{\max\{\kappa(\varphi), \kappa(\psi)\}} \max\left\{\|h - k\|_+, 0\right\}, \tag{3.9}$$

where h and k are the contact Hamiltonians of φ and ψ, respectively. In particular, we have

$$h_t(x) \leq k_t(x)\ \forall x \in \Sigma, t \in [0,1] \quad \Longrightarrow \quad c(\varphi, Z) \geq c(\psi, Z), \tag{3.10}$$

and so $c(\varphi, Z) > -\infty$ because Z is a spectrally finite.

Proof. This follows immediately from the definition of the spectral number together with (2.38) and the estimate (2.39). □

Lemma 3.10. *For any non-degenerate path $\varphi \in \mathcal{P}\mathrm{Cont}_0(\Sigma, \xi)$ the spectral numbers are all critical values of $\mathcal{A}_\varphi$, i.e. $c(\varphi, Z) \in \mathrm{Spec}(\varphi)$.*

Moreover $c(\cdot, Z)$ admits a unique extension to all of $\mathcal{P}\mathrm{Cont}_0(\Sigma, \xi)$: given a degenerate path φ, set

$$c(\varphi, Z) := \lim_k c(\varphi_k, Z), \tag{3.11}$$

where $\varphi_k \to \varphi$ is any sequence of non-degenerate paths converging to φ in C^2. The extension still satisfies $c(\varphi, Z) \in \mathrm{Spec}(\varphi)$ and the estimates (3.8) *and* (3.10)*. In particular, $c(\cdot, Z) : \mathcal{P}\mathrm{Cont}_0(\Sigma, \xi) \to \mathbb{R}$ is a continuous function when we equip $\mathcal{P}\mathrm{Cont}_0(\Sigma, \xi)$ with the C^2-topology.*

Proof. The assertion $c(\varphi, Z) \in \mathrm{Spec}(\varphi)$ follows immediately from the fact that $\mathrm{RFH}_*^c(\mathcal{A}_\varphi, W)$ only changes if c crosses a critical value of $\mathcal{A}_\varphi$, compare the discussion below (2.37).

To prove the existence of the extension we are required to prove that the limit exists and is independent of the choice of the approximating sequence φ_k. We denote by h_k the corresponding contact Hamiltonians. Since we assume that φ_k converges to φ in C^2 it follows that $\kappa(\varphi_k) \to \kappa(\varphi)$ and $h_k \to h$, the contact Hamiltonian of φ. From Proposition 3.9 we conclude that $(c(\varphi_k, Z))$ converges and in the same way independence of the approximating sequence (φ_k) is proved. That $c(\varphi, Z) \in \mathrm{Spec}(\varphi)$ and the estimates (3.8) and (3.10) hold follows from the definition of $c(\cdot, Z)$ as a limit. □

Lemma 3.11. *The map* $c(\cdot, Z) : \mathcal{P}\mathrm{Cont}_0(\Sigma, \xi) \to \mathbb{R}$ *descends to give a well defined map* $c(\cdot, Z) : \widetilde{\mathrm{Cont}}_0(\Sigma, \xi) \to \mathbb{R}$.

Proof. We recall from Remark 2.10 and Lemma 2.11 that $\mathrm{Spec}(\varphi) \subset \mathbb{R}$ is nowhere dense and actually only depends on the endpoint φ_1 of the path φ. Moreover, Lemma 3.10 implies that $c(\cdot, Z)$ is a continuous map. If we vary the path φ while keeping the endpoints fixed the continuous map $c(\cdot, Z)$ takes values in the fixed, nowhere dense set $\mathrm{Spec}(\varphi_1)$, thus is constant. This proves the Lemma. □

Lemma 3.12. *For any* $T \in \mathbb{R}$ *one has*

$$c(\theta^T, Z) = -T + c(\mathrm{id}_\Sigma, Z). \tag{3.12}$$

Proof. One has $\mathrm{Spec}(\theta^T) = -T + \mathrm{Spec}(\mathrm{id}_\Sigma)$. Since $\mathrm{Spec}(\mathrm{id}_\Sigma)$ is nowhere dense and $c(\cdot, Z)$ is continuous the result follows. □

Remark 3.13. Proposition 3.9 and Lemmata 3.10, 3.11, 3.12 constitute Theorem 1.1 from the introduction.

Given a path φ of contactomorphisms, we define the *support* of φ,

$$\mathfrak{S}(\varphi) := \bigcup_{0 \le t \le 1} \mathrm{supp}(\varphi_t), \tag{3.13}$$

where $\mathrm{supp}(\varphi_t) := \overline{\{x \in \Sigma \mid \varphi_t(x) \neq x\}}$.

Definition 3.14. For an open set $U \subset \Sigma$ we set

$$c(U, Z) := \sup \left\{ c(\varphi, Z) \mid \varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi),\ \mathfrak{S}(\varphi) \subset U \right\} \in (-\infty, \infty]. \tag{3.14}$$

Example 3.15. By Lemma 3.12 one has immediately that $c(\Sigma, Z) = \infty$ for any non-zero class Z.

Recall that given a Reeb orbit γ, we denote by $\mu^{\mathrm{tr}}_{\mathrm{CZ}}(\gamma)$ its *transverse Conley Zehnder index*, and we denote by $\nu^{\mathrm{tr}}(\gamma)$ its *transverse nullity*. Let us define

$$\mu(\gamma) := \mu^{\mathrm{tr}}_{\mathrm{CZ}}(\gamma) - \frac{1}{2}\nu^{\mathrm{tr}}(\gamma). \tag{3.15}$$

We now recall Assumption $\mathbf{(A)}^+$ from the Introduction.

Assumption (A)$^+$: (Σ, ξ) admits a Liouville filling $(W^{2n}, d\lambda)$ such that $\alpha := \lambda|_\Sigma$ is Morse-Bott with non-zero Rabinowitz Floer homology. Moreover the Reeb flow $\theta^t : \Sigma \to \Sigma$ of α has no contractible Reeb orbits γ with

$$\mu(\gamma) \in [-n - \nu^{\mathrm{tr}}(\gamma), 1 - n]. \tag{3.16}$$

The reason (3.16) is useful for us is given the following lemma.

Lemma 3.16. *Assume that the Reeb flow $\theta^t : \Sigma \to \Sigma$ has no contractible Reeb orbits γ with*

$$\mu(\gamma) \in [-n - \nu^{\mathrm{tr}}(\gamma), 1 - n]. \tag{3.17}$$

Then the fundamental class $[\Sigma] \in H_{2n-1}(\Sigma; \mathbb{Z}_2)$ defines a non-zero class

$$\mu_\Sigma \in \mathrm{RFH}_n(\Sigma, W),$$

which moreover satisfies

$$c(\mathrm{id}_\Sigma, \mu_\Sigma) = 0$$

Proof. Choose a Morse function g on $\mathrm{Crit}(\mathcal{A})$ with the property that the restriction of g to the component $\Sigma \subset \mathrm{Crit}(\mathcal{A})$ of constant loops has a unique maximum, say at a point $x_0 \in \Sigma$. Choose $\delta > 0$ smaller than the smallest period of a contractible Reeb orbit. Then the assumption that there are no contractible Reeb orbits satisfying (3.17) implies that there that the filtered complex in degree n consists

$$\mathrm{RFC}_n^{(-\infty,\delta)}(\mathcal{A}, g) = \{x_0\}. \tag{3.18}$$

Let $V \subset W$ denote the compact domain bounded by Σ, and denote by $\mathrm{SH}_*(V)$ and $\mathrm{SH}^*(V)$ the symplectic homology and cohomology of V, respectively. We now use [18, Proposition 1.4] which tells us there exists a commutative diagram

(3.19)

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{H}^0(V,\Sigma;\mathbb{Z}_2) \arrow[r] & \mathrm{H}^0(V;\mathbb{Z}_2) \arrow[r] & \mathrm{H}^0(\Sigma;\mathbb{Z}_2) \arrow[r] & \mathrm{H}^1(V,\Sigma;\mathbb{Z}_2) \arrow[r] & \dots \\
\dots \arrow[r] & \mathrm{SH}^{-n}(V) \arrow[r, "\varphi"] \arrow[u] & \mathrm{H}_{2n}(V,\Sigma;\mathbb{Z}_2) \arrow[r, "\psi"] \arrow[u, equal, "\mathrm{PD}"'] & \mathrm{RFH}_n^{(\infty,\delta)}(\Sigma, W) \arrow[r] \arrow[u] & \mathrm{SH}^{1-n}(V) \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

In fact, in [18, Proposition 1.4], instead of $\mathrm{RFH}_n^{(-\infty,\delta)}(\Sigma, W)$ in the third term on the bottom row, instead it is written $\check{\mathrm{SH}}_n^{(-\infty,\delta)}(V)$ (the so-called "V-shaped symplectic homology"). However by the main result of [18], Theorem 1.5, one has $\check{\mathrm{SH}}_*^{(a,b)}(V) \cong \mathrm{RFH}_*^{(a,b)}(\Sigma, W)$.

Since we are in degree $-n$, the map φ is identically zero, because by [18, Proposition 1.3] the map φ factors as a composition

$$\mathrm{SH}^{-n}(V) \to \mathrm{H}^0(V, \Sigma; \mathbb{Z}_2) \to \mathrm{H}_{2n}(V, \Sigma; \mathbb{Z}_2),$$

and $H^0(V, \Sigma; \mathbb{Z}_2) = 0$. Thus the map ψ is injective. We denote by $\mu_\Sigma := \psi([V])$, which is thus always a non-zero class in $\mathrm{RFH}_n^{(-\infty,\delta)}(\Sigma, W)$.

In general we cannot say much about that class μ_Σ. In particular, there is no reason why μ_Σ should define a non-zero class in the full Rabinowitz Floer homology $\mathrm{RFH}_n(\Sigma, W)$. However we show now that under our index assumption that there are no contractible Reeb orbits satisfying (3.17), not only can we can precisely identify the class μ_Σ, but in this case μ_Σ defines a class in the full Rabinowitz Floer homology $\mathrm{RFH}_n(\Sigma, W)$, and moreover if $\mathrm{RFH}_*(\Sigma, W) \neq 0$ then also $\mu_\Sigma \neq 0$.

Indeed, (3.18) implies that $\mu_\Sigma = [x_0]$ simply since there are no other generators. Thus μ_Σ is also a well defined class in $\mathrm{RFH}_n^{(-\delta,\infty)}(\Sigma, W)$ (it is obviously a class there). Let us now show that if $\mathrm{RFH}_*(\Sigma, W) \neq 0$ then μ_Σ is never a boundary in $\mathrm{RFH}_n^{(-\delta,\infty)}(\Sigma, W)$. For this we use the other commutative diagram of [18, Proposition 1.4], which reads:

(3.20)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}_{2n}(V,\Sigma;\mathbb{Z}_2) \arrow[r] \arrow[d, "c"] & \mathrm{H}_{2n-1}(\Sigma;\mathbb{Z}_2) \arrow[r] \arrow[d] & \mathrm{H}_{2n-1}(V;\mathbb{Z}_2) \arrow[r] \arrow[d, equal, "\mathrm{PD}"] & \dots \\
0 \arrow[r] & \mathrm{SH}_n(V) \arrow[r, "\zeta"] & \mathrm{RFH}_n^{(-\delta,\infty)}(\Sigma, W) \arrow[r] & \mathrm{H}^1(V,\Sigma;\mathbb{Z}_2) \arrow[r] & \dots,
\end{tikzcd}$$

where again we have used the fact that $\mathrm{H}_{2n}(V;\mathbb{Z}_2) = \mathrm{H}^0(V,\Sigma;\mathbb{Z}_2) = 0$ to put zeroes in the left-hand column, The class $u := c([V]) \in \mathrm{SH}_n(V)$ is the unit in the $\mathrm{SH}_*(V)$, and thus if $\mathrm{SH}_*(V)$ is non-zero then $u \neq 0$. Commutativity of the middle two squares of (3.19) and (3.20) tells us that $\zeta(u) = \mu_\Sigma$. Thus if $\mathrm{SH}_*(V) \neq 0$ then $\mu_\Sigma \neq 0$ in $\mathrm{RFH}_n^{(-\delta,\infty)}(\Sigma, W)$. Finally, since $\mathrm{SH}_*(V) = 0$ if and only if $\mathrm{RFH}_*(\Sigma, W) = 0$ – this is a theorem of Ritter [41, Theorem 96] and follows essentially from the argument we have just sketched – we conclude that under our index assumption that there are no contractible

Reeb orbits satisfying (3.17), if $\mathrm{RFH}_*(\Sigma, W) \neq 0$ then the critical point x_0 defines a non-zero class $\mu_\Sigma \in \mathrm{RFH}_n(\Sigma, W)$. This completes the proof. □

Let us show that $c(U, \mu_\Sigma) > 0$ for any non-empty set $U \subset \Sigma$.

Proposition 3.17. *Given any non-empty open set $U \subset \Sigma$, there exists $\psi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ such that $\mathfrak{S}(\psi) \subset U$ and $c(\psi, \mu_\Sigma) > 0$.*

Proof. We prove the proposition in three steps.

Step 1. We use an idea from Sandon [46]. Fix a C^2-small function $b : \Sigma \to \mathbb{R}$. We use b to build a contactomorphism $\Psi : T^*\Sigma \times \mathbb{R} \to T^*\Sigma \times \mathbb{R}$, where the 1-jet bundle $T^*\Sigma \times \mathbb{R}$ is equipped with the standard contact form $\lambda_0 + d\tau$ and $\lambda_0 = pdx$ in local coordinates. Namely, we set

$$\Psi(x, p, \tau) = (x, p - db(x), \tau + b(x)). \tag{3.21}$$

Note that critical points of b are in 1-1 correspondence with Reeb chords between the two Legendrians $\Sigma \times \{0\}$ and $\Psi(\Sigma \times \{0\})$ (where $\Sigma \subset T^*\Sigma$ is the zero section). Moreover, the contactomorphism Ψ is exact. Since b is assumed to be C^2-small, Ψ determines a contactomorphism of ψ of (Σ, α), defined as follows. Firstly, Weinstein's neighborhood theorem for Legendrian submanifolds (see [1, Theorem 2.2.4]) implies that there is an exact contactomorphism

$$\Xi : N \times (-\delta, \delta) \to Q \times (-\varepsilon, \varepsilon) \tag{3.22}$$

between a neighborhood $N \times (-\delta, \delta)$ of $\Sigma \times \{0\}$ inside $T^*\Sigma \times \mathbb{R}$ and a neighborhood $Q \times (-\varepsilon, \varepsilon)$ of $\Delta \times \{0\}$ inside $\Sigma \times \Sigma \times \mathbb{R}$, where Δ is the diagonal in $\Sigma \times \Sigma$. Here $\Sigma \times \Sigma \times \mathbb{R}$ is equipped with the contact form $e^r \mathrm{pr}_1^*\alpha - \mathrm{pr}_2^*\alpha$, where pr_j is the projection onto the jth factor. The contactomorphism ψ is then defined by looking at the restriction of $\Xi \circ \Psi \circ \Xi^{-1}$ to $\Delta \times \{0\}$ inside $Q \times (-\varepsilon, \varepsilon)$; we can write

$$\Xi \circ \Psi \circ \Xi^{-1}(x, x, 0) =: (x, \psi(x), 0), \tag{3.23}$$

for $\psi : \Sigma \to \Sigma$. Since Ψ and Ξ are exact contactomorphisms it follows that ψ is an exact contactomorphism, too.

Similarly, if we start with an isotopy $\{b_t\}_{0 \le t \le 1}$ with $b_0 = 0$ then we obtain a path $\psi = \{\psi_t\}_{0 \le t \le 1}$ of exact contactomorphisms with $\psi_0 = \mathrm{id}_\Sigma$. In this case one can check that the contact Hamiltonian of ψ is $-\frac{\partial b_t}{\partial t}$:

$$\alpha\left(\frac{\partial}{\partial t}\psi_t\right) = -\frac{\partial b_t}{\partial t} \circ \psi_t. \tag{3.24}$$

Note that the minus sign is due to the fact that $\psi(x)$ is the *second* entry in (3.23) and the contact form on $\Sigma \times \Sigma \times \mathbb{R}$ contains $-\mathrm{pr}_2^*\alpha$.

The key point now is that the translated points $x \in \Sigma$ of ψ_1 with time-shift $\eta \in (-\varepsilon, \varepsilon)$ are in 1-1 correspondence with the critical points of b_1: if $x \in \mathrm{Crit}(b_1)$ then

$$\psi_1(x) = \theta^{-b_1(x)}(x), \tag{3.25}$$

This follows from the following computations where we use that Ξ commutes with the Reeb flows on $T^*\Sigma \times \mathbb{R}$ and $\Sigma \times \Sigma \times \mathbb{R}$. The former is given by $(x, p, \tau) \mapsto (x, p, \tau + t)$ and the latter by $(x, y, a) \mapsto (x, \theta^{-t}(y), a)$.

$$\begin{aligned} (x, \psi_1(x), 0) &= \Xi \circ \Psi_1 \circ \Xi^{-1}(x, x, 0) \\ &= \Xi \circ \Psi_1(x, 0, 0) \\ &= \Xi(x, 0, b_1(x)) \\ &= (x, \theta^{-b_1(x)}(x), 0) \end{aligned} \tag{3.26}$$

Here, the third equality is true if and only if $x \in \mathrm{Crit}(b_1)$. Thus for each $x \in \mathrm{Crit}(b_1)$ there is a critical point $(u_x, b_1(x)) \in \mathrm{Crit}(\mathcal{A}_\psi)$, and any critical point (u, η) of $\mathcal{A}_\psi$ not of this form necessarily satisfies $|\eta| > \varepsilon$.

Step 2. Suppose now that we start with a C^2-small Morse-Bott function b on Σ. Define $b_t := tb$ for $t \in [0, 1]$, and let $\psi = \{\psi_t\}_{0 \le t \le 1}$ denote the corresponding path of contactomorphisms. If $x \in \mathrm{Crit}(b)$ then the critical point $(u_x, b(x))$ belongs to a Morse-Bott component of $\mathcal{A}_\psi$, and moreover we claim that

$$\mu(u_x, b(x)) = 1 - n + \mathrm{ind}_b(x), \tag{3.27}$$

where $\mathrm{ind}_b(x)$ denotes the maximal dimension of a subspace on which the Hessian $\mathrm{Hess}_b(x)$ of b at x is *strictly* negative definite.

To see this, we consider the Hamiltonian diffeomorphism Φ of $T^*\Sigma \times \mathbb{R} \times \mathbb{R}$ obtained by lifting Ψ, which as Ψ is exact, is given simply by

$$\Phi(q, p, \tau, \sigma) = (\Psi(q, p, \tau), \sigma)\,. \tag{3.28}$$

A translated point x of ψ_1 gives rise to the following path of Lagrangian subspaces:

$$L_t := \{(\hat{x}, -t\mathrm{Hess}_b(x)(\hat{x}), 0, \hat{\sigma}) \mid \hat{x} \in T_x\Sigma,\ \hat{\sigma} \in \mathbb{R}\} \subset T_{(x,0,0,\sigma)}(T^*\Sigma \times \mathbb{R} \times \mathbb{R}), \tag{3.29}$$

The desired index is then given by

$$\mu(u_x, b(x)) = 1 - n + \mu_{\mathrm{RS}}(L_0, L_1), \tag{3.30}$$

which in this case is just $1 - n + \mathrm{ind}_b(x)$ as claimed; note that the $1-n$ summand comes from the normalization used in the definition of the Rabinowitz index (2.28) above, and we are using the grading convention from Remark 2.18.

Step 3. We now prove the theorem. Suppose $U \subset \Sigma$ is open and non-empty. Choose a function $b : \Sigma \to [0, \infty)$ such that $\mathrm{supp}(b) \subset U$ and such that b is Morse on the interior of its support. Moreover we insist that b has a unique maximum $x_0 \in \Sigma$, with $0 < b(x_{\max}) < \varepsilon$, where ε is as in (3.22). Let ψ be as in Step 2. Since the contact Hamiltonian of ψ is $-b$ we can estimate

$$K(\mathrm{id}_\Sigma, \psi) \leq e^{\kappa(\psi)} \tfrac{1}{2} b(x_{\max}). \tag{3.31}$$

From Proposition 3.9 and Lemma 3.16 we obtain

$$c(\psi, \mu_\Sigma) \leq \underbrace{c(\mathrm{id}_\Sigma, \mu_\Sigma)}_{=0} + K(\mathrm{id}_\Sigma, \psi) \tag{3.32}$$

$$\leq e^{\kappa(\psi)} \tfrac{1}{2} b(x_{\max}). \tag{3.33}$$

We now assume in addition that $e^{\kappa(\psi)} \frac{1}{2} b(x_{\max}) < \epsilon$, too. Since the contact Hamiltonian $-b$ of ψ is non-positive, we have from (3.10) that

$$0 \leq c(\psi, \mu_\Sigma) < \epsilon. \tag{3.34}$$

We recall from Step 1 that any critical point (u, η) of $\mathcal{A}_\psi$ which is not of the form $(u_x, b(x))$ satisfies $|\eta| > \varepsilon$. Thus $c(\psi, \mu_\Sigma)$ is necessarily a critical value of b. Since μ_Σ has index n, and $x_{\max}$ is the only critical point of b of index $2n-1$ (so that the corresponding critical point $(u_{x_{\max}}, b(x_{\max}))$ has index $1 - n + 2n - 1 = n$), we see that

$$c(\psi, \mu_\Sigma) = b(x_{\max}) > 0. \tag{3.35}$$

The proof is complete. ∎

The following Corollary is Theorem 1.8 from the Introduction.

Corollary 3.18. *Suppose $\varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ has contact Hamiltonian h_t. Assume $h_t \leq 0$ and there exists $x \in \Sigma$ such that $h_t(x) < 0$ for all $t \in [0,1]$. Then $c(\varphi, \mu_\Sigma) > 0$.*

Proof. There exists a function $b : \Sigma \to [0, \infty)$ satisfying all the properties from the proof of Proposition 3.17 and in addition that

$$-tb(x) \geq h_t(x) \quad \forall x \in \Sigma, t \in [0,1]. \tag{3.36}$$

Let $\psi = \{\psi_t\}_{0 \leq t \leq 1}$ denote the contact isotopy whose contact Hamiltonian is $-tb$. Then Proposition 3.9 and Proposition 3.17 imply that

$$0 < b(x_{\max}) = c(\psi, \mu_\Sigma) \leq c(\varphi, \mu_\Sigma). \tag{3.37}$$

□

Remark 3.19. One might wonder whether the analogue of Corollary 3.18 continues to hold if instead we assume that h_t is non-negative and not identically zero. In the non-compact setting discussed in Section 5 we will see that this is false. See Remark 5.14 and Appendix A for more information.

4. Contact capacities

Let us now assume that (Σ, ξ) satisfies Assumption **(B)** from the Introduction:

> **Assumption (B):** (Σ, ξ) admits a Liouville filling $(W, d\lambda)$ such that the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma, W)$ contains a spectrally finite class and such that $\alpha := \lambda|_\Sigma$ is periodic.

As before Z denotes a non-zero spectrally finite class in $\mathrm{RFH}_*(\Sigma, W)$.

Definition 4.1. Following [44], we define for $\varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ an integer $\overline{c}(\varphi, Z)$ by

$$\overline{c}(\varphi, Z) := \lceil c(\varphi, Z) \rceil . \tag{4.1}$$

The reason periodicity is helpful is this function $\overline{c}(\cdot, Z)$ is *conjugation invariant*. We will prove this shortly in Proposition 4.3 below, but to begin with we present the following lemma. Recall from Definition 2.9 that we say φ is *non-resonant* if $\mathrm{Spec}(\varphi) \cap \mathbb{Z} = \emptyset$.

Lemma 4.2. *Suppose φ is degenerate with $c(\varphi, Z) \in \mathbb{Z}$ (and thus φ is necessarily resonant). Then there exists $\varphi_k \to \varphi$ such that φ_k is resonant and non-degenerate such that for all k sufficiently large one has $c(\varphi_k, Z) = c(\varphi, Z)$.*

Proof. Start with any sequence (φ_k) of non-degenerate paths such that $\varphi_k \to \varphi$, and set $c_k := c(\varphi_k, Z)$, so that $c(\varphi, Z) = \lim_k c_k$ (by definition). Thus there exists a translated point x_k of φ_k with time-shift c_k. Now set

$$\eta_k := c_k - c(\varphi, Z), \tag{4.2}$$

so that $\eta_k \to 0$. Now the key point is the following: since all the Reeb orbits are closed, η_k is necessarily *also* a time-shift of the translated point x_k, that is

$$\eta_k = c_k, \quad \mathrm{mod}\ 1, \tag{4.3}$$

see Figure 1.

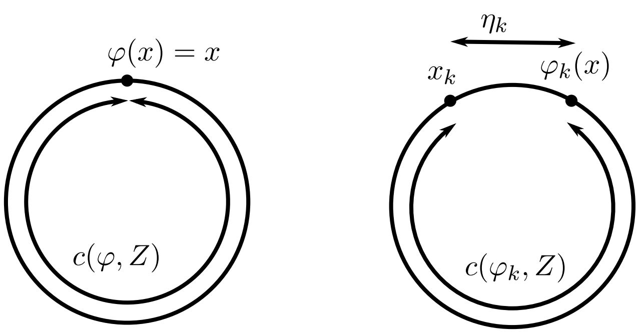

Figure 1.

The sequence $\theta^{-\eta_k} \circ \varphi_k$ still converges to φ, and it is easy to check that $\theta^{-\eta_k} \circ \varphi_k$ is still non-degenerate, and for all k sufficiently large one has that

$$c(\theta^{-\eta_k} \circ \varphi_k, Z) = c_k - \eta_k = c(\varphi, Z) \tag{4.4}$$

since

$$\mathrm{Spec}(\theta^T \circ \varphi) = T + \mathrm{Spec}(\varphi) \tag{4.5}$$

and $c(\cdot, Z)$ is continuous. $\square$

The following is Proposition 1.13 from the Introduction. A similar argument appears in [44, Section 3.4]. We carry out the proof of the degenerate case in detail (using Lemma 4.2 above), since this were not fully explained in [44].

Proposition 4.3. *The function* $\overline{c}(\cdot, Z) : \widetilde{\mathrm{Cont}}_0(\Sigma, \xi) \to \mathbb{Z}$ *is conjugation invariant: if* $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ *and* $\varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ *then*

$$\overline{c}(\psi\varphi\psi^{-1}, Z) = \overline{c}(\varphi, Z). \tag{4.6}$$

Proof. Assume firstly that φ is non-resonant, that is, $\mathrm{Spec}(\varphi) \cap \mathbb{Z} = \emptyset$ (see Definition 2.9). Fix $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ and let $\psi_s \in \mathrm{Cont}_0(\Sigma, \xi)$ be a path connecting id_Σ to ψ. Then we consider the map

$$s \mapsto c(\psi_s \varphi \psi_s^{-1}, Z). \tag{4.7}$$

Proposition 3.9 implies that this map is continuous. Lemma 2.12 implies that $\mathrm{Spec}(\psi_s\varphi\psi_s^{-1}) \cap \mathbb{Z} = \emptyset$ for all $s \in [0,1]$, and hence $\lceil c(\psi\varphi\psi^{-1}, Z)\rceil = \lceil c(\varphi, Z)\rceil$ as required.

There are now three cases to consider. Suppose that φ is resonant but that $c(\varphi, Z) \notin \mathbb{Z}$. Suppose $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$. Then for φ' non-resonant and sufficiently close to φ, one has that $\psi\varphi'\psi^{-1}$ is non-resonant and sufficiently close to $\psi\varphi\psi^{-1}$ and hence we have

$$\overline{c}(\psi\varphi\psi^{-1}, Z) = \overline{c}(\psi\varphi'\psi^{-1}, Z) = \overline{c}(\varphi', Z) = \overline{c}(\varphi, Z), \tag{4.8}$$

where the second equality used the step above. The next case is when φ is resonant and non-degenerate, with $c(\varphi, Z) \in \mathbb{Z}$. As before, given $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ we choose a path ψ_s connecting id_Σ to ψ. The key point now is that for any $s_0 \in [0,1]$, if (u_{s_0}, η_{s_0}) is a critical point of $\mathcal{A}_{\psi_{s_0}\varphi\psi_{s_0}^{-1}}$ with $\eta_{s_0} \in \mathbb{Z}$ then (u_{s_0}, η_{s_0}) is automatically non-degenerate by the last statement of Lemma 2.12. It follows that there exists $\varepsilon > 0$ such that

$$\mathrm{Spec}(\psi_s\varphi\varphi_s^{-1}) \cap [c(\varphi, Z) - \varepsilon, c(\varphi, Z) + \varepsilon] = \{c(\varphi, Z)\}, \tag{4.9}$$

and the result follows as above. The final case is when φ is both resonant and degenerate and $c(\varphi, Z) \in \mathbb{Z}$. In this case we employ Lemma 4.2 to find a sequence $\varphi_k \to \varphi$ such that φ_k is both resonant, non-degenerate, and such that for all large k one has $c(\varphi_k, Z) = c(\varphi, Z)$. The argument above then implies that for any $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$ and for all k sufficiently large, $c(\psi\varphi_k\psi^{-1}, Z) = c(\varphi_k, Z)$ is an integer. Since $c(\psi\varphi_k\psi^{-1}, Z) \to c(\psi\varphi\psi^{-1}, Z)$ the result follows. $\square$

Corollary 4.4. *One has* $\bar{c}(t \mapsto \theta^{tT}, Z) = \lceil -T + c(\mathrm{id}_\Sigma, Z) \rceil$ *for any* $T \in \mathbb{R}$.

Proof. Lemma 3.12. $\square$

We now define $\bar{c}(U, Z)$ in the same way as $c(U, Z)$ was defined in Definition 3.14.

Definition 4.5. For an open set $U \subset \Sigma$ we define the *contact capacity*

$$\bar{c}(U, Z) := \sup \Big\{ \bar{c}(\varphi, Z) \mid \varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi),\ \mathfrak{S}(\varphi) \subset U \Big\} \in \mathbb{Z} \cup \{\infty\}. \tag{4.10}$$

Remark 4.6. The notion of contact capacity was introduced by Sandon in [44]. She was the first to discover a connection between translated points and orderability and other contact rigidity phenomena.

The following is Corollary 1.16 from the Introduction.

Proposition 4.7. *For all* $\psi \in \mathrm{Cont}_0(\Sigma, \xi)$, *one has*

$$\bar{c}(\psi(U), Z) = \bar{c}(U, Z). \tag{4.11}$$

Proof. Since

$$\mathfrak{S}(\psi\varphi\psi^{-1}) = \psi(\mathfrak{S}(\varphi)), \tag{4.12}$$

we conclude from Proposition 4.3 that

$$\begin{aligned}
\bar{c}(U, Z) &= \sup \Big\{ \bar{c}(\varphi, Z) \mid \varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi),\ \mathfrak{S}(\varphi) \subset U \Big\} \\
&= \sup \Big\{ \bar{c}(\varphi, Z) \mid \varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi),\ \psi(\mathfrak{S}(\varphi)) \subset \psi(U) \Big\} \\
&= \sup \Big\{ \bar{c}(\varphi, Z) \mid \varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi),\ \mathfrak{S}(\psi\varphi\psi^{-1}) \subset \psi(U) \Big\} \\
&= \sup \Big\{ \bar{c}(\psi\varphi\psi^{-1}, Z) \mid \varphi \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi),\ \mathfrak{S}(\psi\varphi\psi^{-1}) \subset \psi(U) \Big\} \\
&= \sup \Big\{ \bar{c}(\mu, Z) \mid \mu \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi),\ \mathfrak{S}(\mu) \subset \psi(U) \Big\} \\
&= \bar{c}(\psi(U), Z). \qquad (4.13)
\end{aligned}$$

$\square$

For completeness we recall Theorem 1.17 which is proved in the Introduction.

Theorem 4.8. *Let $U \subset V \subset \Sigma$ be open sets and assume that there exists $\varphi \in \mathrm{Cont}_0(\Sigma, \xi)$ with $\varphi(V) \subset U$. Then*

$$\bar{c}(U, Z) = \bar{c}(V, Z). \tag{4.14}$$

In particular, if $\bar{c}(U, Z) < \bar{c}(V, Z)$ then there exists no contact isotopy mapping V into U.

Remark 4.9. If we assume that (Σ, ξ) satisfies both assumption **(A)**$^+$ and **(B)** then we know $\bar{c}(U, \mu_\Sigma) > 0$ whenever $U \subset \Sigma$ is a nonempty open subset. Unfortunately in general we do not know how to prove that $\bar{c}(U, Z)$ is ever finite. Nevertheless, in certain situations it *is* possible to prove finiteness of the capacities, for instance when the subset U is displaceable. In particular this is the case in the setting described in the next section, see Corollary 5.17.

5. Prequantization spaces

5.1. Hamiltonian Floer homology

Fix a Liouville domain $(M_1, d\gamma_1)$. Let $S := \partial M_1$ and $\kappa := \gamma_1|_S$, so that (S, κ) is a contact manifold. Let $(M, d\gamma)$ denote the completion of M_1, so that $M = M_1 \cup_S (S \times [1, \infty))$. It is convenient in this section to introduce the notation

$$M_\sigma := \begin{cases} M_1 \backslash \big(S \times (\sigma, 1)\big) & \text{if } 0 < \sigma < 1, \\ M_1 \cup_S \big(S \times [1, \sigma]\big) & \text{if } \sigma \geq 1. \end{cases} \tag{5.1}$$

Note here we are using σ to denote the $\mathbb{R}$-coordinate on the end of M - this is so as to avoid confusion in Section 5.3, when a second Liouville domain will come into play.

Denote by $\mathrm{Ham}_c(M, d\gamma)$ the group of Hamiltonian diffeomorphisms f on M with compact support. As before, a *path* $f = \{f_t\}_{0 \leq t \leq 1}$ of compactly supported Hamiltonian diffeomorphisms is assumed to be smoothly parametrized and begin at the identity: $f_0 = \mathrm{id}_M$. Given such a path $f = \{f_t\}_{0 \leq t \leq 1}$, let X_f denote the time-dependent vector field on M defined by

$$\frac{\partial}{\partial t} f_t = X_{f_t} \circ f_t. \tag{5.2}$$

The equation

$$f_t^* \gamma - \gamma = da_t, \quad a_0 \equiv 0 \tag{5.3}$$

determines a smooth compactly supported function $a_t : M \to \mathbb{R}$. If we define

$$F_t = i_{X_{f_t}}\gamma - \left(\frac{\partial}{\partial t} a_t\right) \circ f_t^{-1}, \tag{5.4}$$

then F_t generates f_t: $f_t = f_F^t$. We can recover a_t from F_t via

$$a_t = \int_0^t \left(i_{X_{f_s}}\gamma - F_s\right) \circ f_s ds \tag{5.5}$$

(see for instance [39, p294]).

We briefly explain the construction of the Hamiltonian Floer homology of f in this section. The setting we consider here is a special case of the one considered by Frauenfelder and Schlenk in [30], to which we refer to for more details. However it will be convenient for us to use the Morse-Bott framework developed by Frauenfelder [27], in order to make the link with the Rabinowitz Floer homology of $\Sigma := M \times S^1$ clearer in the next section.

Let us first note that for a given $F \in C_c^\infty(S^1 \times M, \mathbb{R})$, the flow f_F^t has many 1-periodic orbits, since f_F^t is compactly supported. Of course, constant 1-periodic orbits outside the support of f are uninteresting, and hence we introduce the following notation. Denote by

$$\sigma(F) := \inf \{\sigma > 0 \mid \mathfrak{S}(f_F) \subseteq M_\sigma\}. \tag{5.6}$$

Given a path $f = \{f_t\}_{0 \le t \le 1}$ in $\mathrm{Ham}_c(M, d\gamma)$, we set $\sigma(f) := \sigma(F)$, where F is given by (5.4). Next, we set

$$\mathcal{P}_F := \left\{y \in M_{\sigma(f)} \mid f_F^1(y) = y\right\}. \tag{5.7}$$

Definition 5.1. Define a subset $\mathcal{H}_c^{\mathrm{mb}} \subseteq C_c^\infty(S^1 \times M, \mathbb{R})$ (here the "mb" stands for Morse-Bott) to consist of those functions F with the property that $\mathcal{P}_F$ is either a closed submanifold of M or an open domain whose closure is a compact manifold, and for which

$$T_y\mathcal{P}_F = \ker(Df_F^1(y) - \mathbb{1}) \quad \text{for all } y \in \mathcal{P}_F. \tag{5.8}$$

It is well known that the subset $\mathcal{H}_c^{\mathrm{mb}}$ is generic in $C_c^\infty(S^1 \times M, \mathbb{R})$. We say that a path $f = \{f_t\}_{0 \le t \le 1}$ is *non-degenerate* if the function F defined in (5.4) belongs to $\mathcal{H}_c^{\mathrm{mb}}$.

We denote by R_κ the Reeb vector field of κ. Denote by $\widehat{\mathcal{H}}$ the set of time-dependent smooth functions $\widehat{F}$ on M with the property that there exists $C > 0$ such that $\widehat{F}_t|_{S\times[C,\infty)}$ is of the form $\widehat{F}_t(y,\sigma) = e(\sigma)$ for some smooth function $e : [C,\infty) \to \mathbb{R}$ satisfying

$$0 < e'(\sigma) < \wp(S,\kappa). \tag{5.9}$$

Here

$$\wp(S,\kappa) := \inf\{T > 0 \mid \exists \text{ a closed Reeb orbit of } R_\kappa \text{ of period } T > 0\}. \tag{5.10}$$

This ensures that if $\varphi^1_{\widehat{F}}$ denotes the flow of $\widehat{F}$ then $\varphi^1_{\widehat{F}}$ has no non-constant 1-periodic orbits on $S \times (C,\infty)$. Note that if $F \in C^\infty_c(S^1 \times M,\mathbb{R})$ then one can find $\widehat{F} \in \widehat{\mathcal{H}}$ such that $\widehat{F}|_{S^1\times M_{\sigma(F)}} = F$.

As a special case of this construction, consider a function O on M defined by requiring that $O = 0$ on the interior M_1° of M_1 and that

$$O(y,\sigma) = e(\sigma) \tag{5.11}$$

on $S \times [1,\infty)$, where $e(1) = 0$ and e satisfies (5.9). In this case one has

$$\mathcal{P}_O = M_1, \tag{5.12}$$

where points in M_1 are thought of as constant loops. In particular, $f^t_O|_{M_1^\circ} = \mathrm{id}_{M_1^\circ}$. Thus O is an extension of the zero function (generating the Hamiltonian diffeomorphism id_{M_1} to $\widehat{\mathcal{H}}$).

Definition 5.2. Fix a non-degenerate path $f = \{f_t\}_{0\le t\le 1}$, and let F denote the function defined in (5.4), and fix an extension $\widehat{F} \in \widehat{\mathcal{H}}$ such that $\widehat{F}|_{S^1\times M_{\sigma(F)}} = F$. Recall that $\Lambda(M) := C^\infty_{\mathrm{contr}}(S^1, M)$. Define the *Hamiltonian action functional* $\mathcal{A}_f : \Lambda(M) \to \mathbb{R}$ by

$$\mathcal{A}_f(v) := \int_0^1 v^*\gamma - \widehat{F}_t(v)dt. \tag{5.13}$$

Denote by $\mathrm{Crit}^\circ(\mathcal{A}_f)$ the set of critical points v of $\mathcal{A}_f$ with $v(S^1) \subseteq M_{\sigma(f)}$. Then $\mathrm{Crit}^\circ(\mathcal{A}_f)$ does not depend on the extension $\widehat{F}$ - in fact

$$\mathrm{Crit}^\circ(\mathcal{A}_f) \cong \mathcal{P}_F, \tag{5.14}$$

and hence the assumption (5.8) implies that each component of $\mathrm{Crit}^\circ(\mathcal{A}_f)$ is a Morse-Bott component for $\mathcal{A}_f$.

Fix a family J_t of $d\gamma$-compatible almost complex structures on M which are convex at infinity (cf. equation (2.26)). We define an L^2-inner product

$\langle\langle\cdot,\cdot\rangle\rangle_J$ on $\Lambda(M)$ as before (cf. equation (2.27), only this time there is no bb' term). We denote by $\nabla_J \mathcal{A}_F$ the gradient of $\mathcal{A}_F$ with respect to $\langle\langle\cdot,\cdot\rangle\rangle_J$. Pick a Morse function $g : \mathrm{Crit}^\circ(\mathcal{A}_f) \to \mathbb{R}$ and a Riemannian metric ϱ on $\mathrm{Crit}^\circ(\mathcal{A}_f)$ such that (g,ϱ) is a Morse-Smale pair. In the case where $\mathcal{P}_F$ is an open domain in M whose boundary is a compact manifold, g must be chosen so that $\langle dg, \mathbf{n}\rangle < 0$ on the boundary, where $\mathbf{n}$ is an outward pointing normal. As before we define moduli spaces $\mathcal{M}_{v^-,v^+}(\mathcal{A}_f, g, J, \varrho)$ of gradient flow lines with cascades for critical points $v^\pm \in \mathrm{Crit}(g)$. This time we grade $v \in \mathrm{Crit}(g)$ simply by $\mu(v) := \mu_{\mathrm{CZ}}(v) + \mathrm{ind}_g(v)$, where $\mu_{\mathrm{CZ}}(v)$ is the Conley-Zehnder index. A standard convexity argument gives the necessary compactness needed to define Floer homology - see Frauenfelder-Schlenk [30].

Given $-\infty < a < b < \infty$ denote by $\mathrm{CF}_*^{(a,b)}(\mathcal{A}_f, g) := \mathrm{Crit}_*^{(a,b)}(g) \otimes \mathbb{Z}_2$, where $\mathrm{Crit}_*^{(a,b)}(g)$ denotes the set of critical points v of g with $a < \mathcal{A}_f(v) < b$. As before one defines a boundary operator ∂ on $\mathrm{CF}_*^{(a,b)}(\mathcal{A}_f, g)$ by counting the elements of the zero-dimensional parts of the moduli spaces

$$\mathcal{M}_{v^-,v^+}(\mathcal{A}_f, g, J, \varrho)$$

for $v^- \neq v^+$. We denote by $\mathrm{HF}_*^{(a,b)}(\mathcal{A}_f)$ the associated homology, which as the notation suggests, is independent of the auxiliary data (g, J, ϱ) and of the extension $\widehat{F}$, see [30]. In fact, one can also show it is also independent of the choice of path f. We abbreviate $\mathrm{HF}_*^a(\mathcal{A}_f) := \mathrm{HF}_*^{(-\infty,a)}(\mathcal{A}_f)$ and $\mathrm{HF}_*(\mathcal{A}_f) := \mathrm{HF}_*^{(-\infty,\infty)}(\mathcal{A}_f)$. We denote the natural maps $\mathrm{HF}_*^a(\mathcal{A}_f) \to \mathrm{HF}_*(\mathcal{A}_f)$ by j_f^a in the same way as before. Under our grading convention explained in Remark 2.18, there is a canonical isomorphism

$$\mathrm{HF}_*(\mathcal{A}_f) \cong \mathrm{H}_{n+*}(M_1, \partial M_1; \mathbb{Z}_2) \cong \mathrm{H}^{n-*}(M_1; \mathbb{Z}_2). \tag{5.15}$$

See the proof of Lemma 5.3 below for one way to see this.

Next, the Floer homology $\mathrm{HF}_*(\mathcal{A}_f)$ carries the structure of a unital ring. The unit lives in degree n according to our sign conventions, and under the isomorphism (5.15), the unit corresponds to the fundamental class $[M_1] \in \mathrm{H}_{2n}(M_1, \partial M_1; \mathbb{Z}_2)$; see Lemma 5.3 below. We denote the unit by $\mathbf{1}_f \in \mathrm{HF}_n(\mathcal{A}_f)$. Since $\mathrm{HF}_*(\mathcal{A}_f)$ is necessarily non-zero, as usual one defines the *spectral number*

$$c_M(f) := \inf\left\{a \in \mathbb{R} \mid \mathbf{1}_f \in j_f^a(\mathrm{HF}_*^a(\mathcal{A}_f))\right\}. \tag{5.16}$$

As before, c_M is a well defined function

$$c_M : \widetilde{\mathrm{Ham}}_c(M, d\gamma) \to \mathbb{R}. \tag{5.17}$$

We can use c_M to define a capacity on open subsets $\mathcal{O} \subset M$,

$$c_M(\mathcal{O}) := \sup \{ c_M(f) \mid \mathfrak{S}(f) \subset \mathcal{O} \}, \tag{5.18}$$

in the same way as before. We use the subscript c_M to differentiate it from the function c associated to $\Sigma := M \times S^1$ that we will define shortly.

Lemma 5.3. *In the case of* $\mathrm{id} = \mathrm{id}_{M_1}$ *the unit* $\mathbf{1} = \mathbf{1}_{\mathrm{id}}$ *is simply given by the fundamental class* $[M_1]$*, and thus* $c_M(\mathrm{id}) = 0$*.*

Proof. We define $\mathcal{A}_{\mathrm{id}}$ using the function O defined in (5.11). Thus $\mathrm{Crit}^\circ(\mathcal{A}_{\mathrm{id}}) = M_1$, and every element of $\mathrm{Crit}^\circ(\mathcal{A}_{\mathrm{id}})$ has action value zero. Thus there are no gradient flow lines of $\mathcal{A}_{\mathrm{id}}$, and hence the Floer complex $\mathrm{CF}_*(\mathcal{A}_{\mathrm{id}}, g)$ reduces to the Morse complex of a Morse function g on M_1. Such a Morse function g can be chosen so that $g > 1$ on M_1° and such that g is the restriction of a Morse function $\widehat{g} : M \to \mathbb{R}$ such that $\widehat{g}(y, \sigma) = \frac{1}{\sigma}$ on $S \times [1, \infty)$. Thus this shows that

$$\mathrm{HF}_*(\mathcal{A}_{\mathrm{id}}) \cong \mathrm{HM}_{n+*}(g) \cong \mathrm{H}_{n+*}(M_1, \partial M_1; \mathbb{Z}_2), \tag{5.19}$$

which proves (5.15).

It is possible to prove directly using Morse-Bott techniques that the isomorphisms in (5.19) are ring maps, and thus the unit in $\mathrm{HF}_*(\mathcal{A}_{\mathrm{id}})$ is exactly the unit in Morse homology for g. The latter is of course the fundamental class $[M_1]$ under the isomorphism of the Morse homology of g with the relative homology of $(M_1, \partial M_1)$. In this situation however, we can simply make a degree argument: if the Morse function g has a unique maximum at a point $y_{\max}$ in M_1° then one necessarily has that $[y_{\max}]$ is a cycle in $\mathrm{HF}_n(\mathcal{A}_{\mathrm{id}})$, and that fact $\mathrm{HF}_n(\mathcal{A}_{\mathrm{id}}) = \mathbb{Z}_2[y_{\max}]$. Since the unit lives in degree n, it must therefore be precisely $[y_{\max}]$. □

5.2. The prequantization space $\boldsymbol{\Sigma = M \times S^1}$

The *prequantization space* of M is the contact manifold $\Sigma := M \times S^1$, equipped with the contact structure $\xi := \ker \alpha$, where

$$\alpha := \gamma + d\tau, \tag{5.20}$$

and τ is the coordinate on $S^1 \cong \mathbb{R}/\mathbb{Z}$. The last class of contact manifolds we study in this paper are these prequantization spaces, which for convenience we refer to as Assumption **(C)**:

Assumption (C): $(\Sigma, \xi = \ker\alpha)$ is a prequantization space $\Sigma = M \times S^1$, where $(M, d\gamma)$ is a Liouville manifold, and $\alpha = \gamma + d\tau$.

In this case Σ is obviously periodic, but it is *not* Liouville fillable in the previous sense. Aside from anything else, Σ is necessarily *non-compact.* However Σ does still retain enough of the properties needed above in order to define a Rabinowitz Floer homology, as will explain in the next section.

Let us denote by $\mathrm{Cont}_{0,c}(\Sigma, \xi)$ those contactomorphisms φ with compact support. There is a natural way to obtain a path $\varphi = \{\varphi_t\}_{0\le t\le 1}$ of compactly supported contactomorphisms on Σ from a path $f = \{f_t\}_{0\le t\le 1}$ of compactly supported Hamiltonian diffeomorphisms on M. Indeed, given such a path f, define $\varphi_t : \Sigma \to \Sigma$ by

$$\varphi_t(y,\tau) := \big(f_t(y), \underbrace{\tau - a_t(y)}_{\text{mod } 1}\big), \tag{5.21}$$

where a_t was defined in (5.3). One easily checks that φ_t is an *exact* contactomorphism. We say that the contact isotopy φ is the *lift* of the Hamiltonian isotopy f. In this case the contact Hamiltonian h_t associated to φ_t is simply F_t (thought of as a function on $M \times S^1$):

$$h_t \circ \varphi_t = \alpha\left(\frac{\partial}{\partial t}\varphi_t\right) = F_t \circ \varphi_t, \tag{5.22}$$

where F_t was defined in (5.4). Fix a function $\widehat{F} \in \widehat{\mathcal{H}}$ such that $\widehat{F} = F$ on $S^1 \times M_{\sigma(F)}$, and define $\widehat{H}_t : S\Sigma \to \mathbb{R}$ by $\widehat{H}_t := r\widehat{F}_t$.

Consider again the Rabinowitz action functional $\mathcal{A}_\varphi : \Lambda(S\Sigma) \times \mathbb{R} \to \mathbb{R}$ defined as in (2.7), using $\widehat{H}_t$. Suppose $(u, \eta) \in \mathrm{Crit}(\mathcal{A}_\varphi)$. Write

$$u(t) = (v(t), \tau(t), r(t)) \in M \times S^1 \times (0, \infty).$$

Then from (2.10) we have

$$\begin{aligned}
\big(f_1(v\left(\tfrac12\right), \underbrace{\tau\left(\tfrac12\right) - a_1\left(v\left(\tfrac12\right)\right)}_{\text{mod } 1}\big) &= \varphi_1\left(u\left(\tfrac12\right)\right),\\
&= \theta^{-\eta}\left(v\left(\tfrac12\right), \tau\left(\tfrac12\right)\right)\\
&= \big(v\left(\tfrac12\right), \underbrace{\tau\left(\tfrac12\right) - \eta}_{\text{mod } 1}\big)
\end{aligned} \tag{5.23}$$

and hence if $y := v(\frac12)$ then $f_1(y) = y$ and $a_1(y) = \eta$ mod 1. Thus from (5.5) one also has $\mathcal{A}_f(v) = \eta$ mod 1. Moreover since φ_t is exact one has $r(t) \equiv 1$ for all t (cf. the last statement of Lemma 2.7). Since we only consider *contractible* critical points of $\mathcal{A}_\varphi$, we deduce:

Lemma 5.4. *There exists a bijective map*

$$\pi : \mathrm{Crit}(\mathcal{A}_\varphi) \to \mathrm{Crit}(\mathcal{A}_f) \tag{5.24}$$

given by

$$\pi(u = (v, \tau, r), \eta) := \left(t \mapsto f_t\left(v\left(\tfrac{1}{2}\right)\right)\right). \tag{5.25}$$

Moreover

$$\mathcal{A}_\varphi(u, \eta) = \mathcal{A}_f(\pi(u, \eta)). \tag{5.26}$$

In particular, every critical point (u, η) *of* $\mathcal{A}_\varphi$ *has*

$$u(S^1) \subseteq M_{\sigma(f)} \times S^1 \times \{1\}. \tag{5.27}$$

Given a contactomorphism $\varphi \in \mathrm{Cont}_{0,c}(\Sigma, \xi)$, we denote by

$$\sigma(\varphi) = \inf\left\{\sigma > 0 \mid \mathfrak{S}(\varphi) \subseteq M_\sigma \times S^1\right\}. \tag{5.28}$$

Thus if φ is the lift of f then

$$\sigma(\varphi) = \sigma(f). \tag{5.29}$$

5.3. Rabinowitz Floer homology on Σ

Let P_1 denote a 2-torus with a disc removed, so that $\partial P_1 = S^1$. Equip P_1 with an exact symplectic form $d\beta_1$ such that $\beta_1|_{\partial P_1} = d\tau$ (the precise choice of β_1 is unimportant). Denote by $(P, d\beta)$ the completion of P_1, so that

$$\beta = r d\tau \quad \text{on} \quad \partial P_1 \times [1, \infty). \tag{5.30}$$

Consider

$$W := M \times P, \tag{5.31}$$

equipped with the symplectic form $d\lambda$ where $\lambda := \gamma + \beta$. Since $S^1 \times \mathbb{R}^+$ is naturally embedded in P, $S\Sigma = M \times S^1 \times \mathbb{R}^+$ can naturally be embedded inside of $W = M \times P$. Nevertheless, W is *not* a Liouville filling of Σ. Indeed, firstly W is not compact, and moreover when equipped with the symplectic form $d\lambda$, there is no embedding $(S\Sigma, d(r\alpha)) \hookrightarrow (W, d\lambda)$ that we can use in order to extend the Rabinowitz action functional $\mathcal{A}_\varphi$ to a functional defined on all on $\Lambda(W) \times \mathbb{R}$.

To circumvent this problem we will prove the following technical lemma.

Lemma 5.5. *There exists a family $\{\lambda_s\}_{s\geq 1} \subset \Omega^1(W)$ of 1-forms such that for all $s \geq 1$:*

1) *$\omega_s := d\lambda_s$ is a symplectic form on W.*

2) *Define*

$$W_s^+ := (M \backslash M_{2s-1} \times P) \cup (M \times P \backslash P_{2s-1}), \tag{5.32}$$

$$W_s^- := M_s \times P_{\frac{1}{2s-1}}. \tag{5.33}$$

Then

$$\lambda_s|_{W_s^+} = (2s-1)\gamma + \beta, \qquad \lambda_s|_{W_s^-} = \frac{1}{2s-1}\gamma + \beta. \tag{5.34}$$

Thus ω_s is split-convex at infinity in the sense of [30, Definition 3.1], and hence we can achieve compactness, see the proof of Theorem 5.7 below and also [30]. Moreover $\lambda_1 = \lambda$ everywhere.

3) *For $s > 1$, define*

$$V_s := M_s \times S^1 \times (\tfrac{1}{s}, s) \subset S\Sigma. \tag{5.35}$$

Then for each $s > 1$, the natural embedding

$$\iota_s : V_s \hookrightarrow W \tag{5.36}$$

satisfies $\iota_s^* \lambda_s = r\alpha$.

Proof. Define a family $\{f_s\}_{s\geq 1}$ of smooth functions, see Figure 2 :

$$f_s : [0, \infty) \times [0, \infty) \to (0, \infty) \tag{5.37}$$

such that

$$f_s(\sigma, r) = \begin{cases} r, & (\sigma, r) \in [0, s) \times (\frac{1}{s}, s), \\ \frac{1}{2s-1}, & (\sigma, r) \in [0, s) \times (0, \frac{1}{2s-1}), \\ 2s-1, & (\sigma, r) \in [0, s) \times (2s-1, \infty), \\ 2s-1, & (\sigma, r) \in [2s-1, \infty) \times (0, \infty). \end{cases} \tag{5.38}$$

and finally such that

$$\frac{\partial f_s}{\partial \sigma}(\sigma, r) \geq 0, \qquad \text{for all } (s, \sigma, r) \in [1, \infty) \times [0, \infty) \times [0, \infty). \tag{5.39}$$

The fact that such functions f_s exist is clear from Figure 2. On $M \setminus M_0 \times$

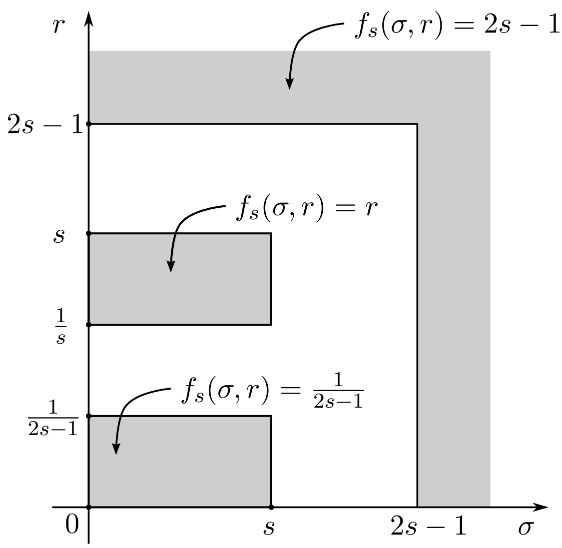

Figure 2: The function $f_s(\sigma, r)$.

$P \setminus P_0$, where both the σ and r-coordinates are defined, we set

$$\lambda_s := f_s \gamma + \beta. \tag{5.40}$$

The condition (5.39) guarantees that $\omega_s := d\lambda_s$ is symplectic where defined, and it is clear that statements (2) and (3) from the Lemma are satisfied. It remains to extend λ_s to all of W. This is done simply by "continuity":

$$\lambda_s = \begin{cases} f_s(0, r)\gamma + \beta, & \text{on } M_0 \times P \setminus P_0, \\ f_s(\sigma, 0)\gamma + \beta, & \text{on } M \setminus M_0 \times P_0, \\ \frac{1}{2s-1}\gamma + \beta, & \text{on } M_0 \times P_0. \end{cases} \tag{5.41}$$

□

Definition 5.6. Given a path $\varphi = \{\varphi_t\}_{0 \le t \le 1}$ of compactly supported contactomorphisms of Σ, we define the number $s_0(\varphi)$ by:

$$s_0(\varphi) := \max\{1, \kappa(\varphi), \sigma(\varphi)\}, \tag{5.42}$$

where $\kappa(\varphi)$ was defined in Definition 2.14 and $\sigma(\varphi)$ was defined in (5.28).

We now prove the following result.

Theorem 5.7. *For any non-degenerate path φ if $s > s_0(\varphi)$ then it is possible to define the Rabinowitz Floer homology* $\mathrm{RFH}_*(\mathcal{A}_\varphi, W, \omega_s)$ *(here the notation indicates that we are working with the symplectic structure ω_s on W). Moreover the Rabinowitz Floer homology is independent of the choice of $s > s_0(\varphi)$.*

Proof. Fix $s > s_0(\varphi)$, and consider the action functional $\mathcal{A}_\varphi : \Lambda(V_s) \times \mathbb{R} \to \mathbb{R}$ defined in the same way as before, only using the one-form λ_s. Let

$$\mathcal{J}^{\mathrm{split}}_{\mathrm{conv}}(W; \omega_s) \subset \mathcal{J}_{\mathrm{conv}}(W; \omega_s) \tag{5.43}$$

denote the set of families (where $\mathcal{J}_{\mathrm{conv}}(W; \omega_s)$ is defined as in Definition 2.16), such that in addition the restriction of J to the subset W_s^+ defined in (5.32) is *split* - that is, there exist almost complex structures $J' \in \mathcal{J}_{\mathrm{conv}}(M; (2s-1)d\gamma)$ and $J'' \in \mathcal{J}_{\mathrm{conv}}(P; d\beta)$ such that $J = J' \oplus J''$ on this set. Extend $\mathcal{A}_\varphi$ to a functional $\mathcal{A}^\kappa_\varphi$ defined on all of $\Lambda(W) \times \mathbb{R}$ in a similar way as before, by replacing $\widehat{H}_t$ with a truncated function $\widehat{H}^\kappa_t$ as in (2.18). As with the Hamiltonian Floer homology, we are now only interested in the set $\mathrm{Crit}^\circ(\mathcal{A}^\kappa_\varphi)$ of critical points $(u = (v, \tau, r), \eta)$ of $\mathcal{A}^\kappa_\varphi$ with $v(S^1) \subset M_{\sigma(\varphi)}$. Since $s > s_0(\varphi)$, all elements of $\mathrm{Crit}^\circ(\mathcal{A}^\kappa_\varphi)$ are contained in $\iota(V_s)$ (cf. Lemma 2.15) and $\widehat{H}^\kappa_t$ is constant outside $W^+_{2s-1+\varepsilon}$ for some small $\varepsilon > 0$. Thus if we work with an almost complex structure $J \in \mathcal{J}^{\mathrm{split}}_{\mathrm{conv}}(W; \omega_s)$, the maximum principle prohibits the cylinder part of flow lines of $-\nabla_J \mathcal{A}^\kappa_\varphi$ from ever entering $W^+_{2s-1+\varepsilon}$, see for instance [30, p18-19]. Thus the Rabinowitz Floer homology is well defined for this s. We point out that, since the cylinder part of flow lines stay in a compact subset of W, L^∞-bounds on the Lagrange multiplier are derived as in [15, Theorem 3.1].

In order to prove independence of s, first note that for

$$s > \max\{s_0(\varphi), s_0(\psi)\}$$

the continuation maps from points (1)-(4) on page 1505 show that

$$\mathrm{RFH}_*(\mathcal{A}_\varphi, W, \omega_s) \cong \mathrm{RFH}_*(\mathcal{A}_\psi, W, \omega_s). \tag{5.44}$$

Next we note that if $\mathrm{id} := \mathrm{id}_{M_1\times S^1}$ is the contactomorphism with contact Hamiltonian O as defined in (5.11) then $s_0(\mathrm{id}) = 1$. More generally, this is true for any *exact* path φ of contactomorphisms, since in this case for any $\varepsilon > 0$, every critical point of $\mathcal{A}_\varphi^\varepsilon$ is contained in $\Sigma \times \{0\}$ - see Lemma 2.15. Thus by (5.44) it suffices to show that $\mathrm{RFH}_*(\mathcal{A}_{\mathrm{id}}, W, \omega_s)$ is independent of $s > 1$. But this is clear, since every critical point of the Rabinowitz action functional $\mathcal{A}_{\mathrm{id}}$ has action value zero, as we are only looking at contractible critical points and we have filled S^1 with a punctured torus P_1 rather than a disc D^2. Thus $\mathrm{Crit}^\circ(\mathcal{A}_{\mathrm{id}}) = M_1 \times S^1 \times \{0\}$, and hence regardless of which symplectic structure we use, as in Lemma 5.3, the Rabinowitz complex reduces to the Morse complex of a Morse function $\tilde{g} : M_1 \times S^1 \to \mathbb{R}$. In particular, it does not depend on s. $\square$

We denote by $\mathrm{RFH}_*(\mathcal{A}_\varphi, W)$ the groups $\mathrm{RFH}_*(\mathcal{A}_\varphi, W, \omega_s)$ for any $s > s_0(\varphi)$.

Theorem 5.8. *If $\varphi = \{\varphi_t\}_{0\le t\le 1}$ is the lift of $f = \{f_t\}_{0\le t\le 1}$ then there exists a natural isomorphism*

$$\mathrm{RFH}_*(\mathcal{A}_\varphi, W) \cong \mathrm{HF}_*(\mathcal{A}_f) \otimes \mathrm{H}_*(S^1; \mathbb{Z}_2). \tag{5.45}$$

Proof. By naturality it suffices to prove the theorem in the case $f = \mathrm{id}_{M_1}$ and $\varphi = \mathrm{id} := \mathrm{id}_{M_1\times S^1}$. In this case as in the proof of the last part of Theorem 5.7, one has

$$\mathrm{RFH}_*(\mathcal{A}_{\mathrm{id}}, W) \cong \mathrm{HM}_{*+n}(\tilde{g}), \tag{5.46}$$

where $\tilde{g}$ is a Morse function on $M_1 \times S^1$. We choose $\tilde{g} = (g, g')$, where g is the Morse function considered in the proof of Lemma 5.3, and $g' : S^1 \to \mathbb{R}$ is a Morse function with two critical points $\tau_{\min}$ and $\tau_{\max}$. This gives

$$\begin{align} \mathrm{RFH}_*(\mathcal{A}_{\mathrm{id}}, W) &\cong \mathrm{HM}_{n+*}(\tilde{g}) \tag{5.47}\\ &\cong \mathrm{HM}_*(g) \otimes \mathrm{HM}_*(g') \tag{5.48}\\ &\cong \mathrm{HM}_{n+*}(M_1, \partial M_1) \otimes \mathrm{H}_*(S^1; \mathbb{Z}_2). \tag{5.49}\end{align}$$

This completes the proof. $\square$

Remark 5.9. As in Remark 2.17, once again the issue of using Morse-Bott methods crops up here. Unfortunately here it does not seem possible to entirely eliminate Morse-Bott theory. However the Morse-Bott theory needed here is comparatively "tame". Namely, the action functional has precisely one critical manifold on which it takes the critical value 0. Whilst as in Remark 2.17 one can easily define the two Floer theories without needing to use Morse-Bott theory, we are unaware of an easy way to prove Theorem 5.8 without using Morse-Bott methods. Nevertheless, we can at least reduce the situation to a special case of a result covered by Frauenfelder's Habilitationsschrift [28]. Namely, instead of choosing $f = \mathrm{id}_{M_1}$ in (5.45), one can instead choose f to be generated by a C^2-small Morse function on M_1. The isomorphism in Theorem 5.8 can then be obtained by making use of a correpondence theorem relating trajectories upstairs and downstairs, which is a (very) special case of [28, Theorem A].

Definition 5.10. As before, we denote by $\mu_\Sigma \in \mathrm{RFH}_n(\Sigma, W)$ the non-zero and spectrally finite class (in fact with spectral value $c(\mathrm{id}_\Sigma, \mu_\Sigma) = 0$) obtained under the isomorphisms from Theorem 5.8 and (5.15) from the class $[M_1] \otimes [S^1]$.

5.4. Relating the capacities

Definition 5.11. We define $c(\varphi) := c(\varphi, \mu_\Sigma)$ in the same way as before for $\varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$.

As long as we work with compactly supported contactomorphisms Proposition 3.9 remains true and its proof is literally the same.

Proposition 5.12. *Let φ, $\psi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ be two non-degenerate paths. Then we have the estimate*

$$
\begin{aligned}
&(5.50) \qquad & c(\psi) &\le c(\varphi) + K(\varphi, \psi) \\
&(5.51) \qquad & &\le c(\varphi) + e^{\max\{\kappa(\varphi), \kappa(\psi)\}} \|h - k\|_+,
\end{aligned}
$$

where h and k are the contact Hamiltonians of φ and ψ, respectively. In particular, we have In particular, we have

$$
(5.52) \qquad h_t(x) \le k_t(x)\ \forall x \in \Sigma, t \in [0,1] \quad \Longrightarrow \quad c(\varphi) \ge c(\psi)
$$

and the same implication with nonstrict inequalities.

The analogue of Corollary 3.18 remains true, too, again with the same proof.

Corollary 5.13. *Suppose $\varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ has contact Hamiltonian h_t. Assume $h_t \le 0$ and $h_t \ne 0$ for all $t \in [0, 1]$. Then $c(\varphi) > 0$.*

Remark 5.14. Recall in the closed case we proved that $c(t \mapsto \theta^{tT}, Z) = -T + c(\mathrm{id}_\Sigma, Z)$ for any $T \in \mathbb{R}$ (cf. Statement (2) of Theorem 1.1. In this setting the Reeb flow θ^t is of course *not* compactly supported, and thus its spectral value is not defined. Nevertheless it is still possible to define a "compactly supported Reeb flow" $\vartheta^t : \Sigma \to \Sigma$ which agrees with the normal Reeb flow on a neighborhood of a given closed Reeb orbit. For small T it is still possible to compute the spectral numbers $c(\vartheta^T)$, but it is no longer the case that $c(\vartheta^T) = -T$. Indeed, whilst for negative T one still has $c(\vartheta^T) = -T$, for positive T one has $c(\vartheta^T) = 0$. This shows that Corollary 5.13 fails if one instead assumes $h_t \ge 0$. Details are contained in Appendix A.

Definition 5.15. For an open non-empty set $U \subset \Sigma$ with compact closure we set

$$c(U) := \sup \left\{ c(\varphi) \mid \varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi),\ \mathfrak{S}(\varphi) \subset U \right\} \in (-\infty, \infty]. \tag{5.53}$$

and

$$\overline{c}(U) := \lceil c(U) \rceil. \tag{5.54}$$

Theorem 5.16. *Suppose $f \in \widetilde{\mathrm{Ham}}_c(M, d\gamma)$, and let $\varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ denote the lift of f. Then*

$$c_M(f) = c(\varphi). \tag{5.55}$$

Moreover, if $\mathcal{O} \subset M$ is open with compact closure then

$$c_M(\mathcal{O}) = c(\mathcal{O} \times S^1). \tag{5.56}$$

Proof. The first statement follows from Lemma 5.4 and Theorem 5.8. Thus clearly $c_M(\mathcal{O}) \le c(\mathcal{O} \times S^1)$. In order to complete the proof, we must show that given any $\psi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ with $\mathfrak{S}(\psi) \subset \mathcal{O} \times S^1$ there exists $f \in \widetilde{\mathrm{Ham}}_c(M, d\gamma)$ with $\mathfrak{S}(f) \subset \mathcal{O}$ and such that the lifted contactomorphism

φ satisfies

$$c(\psi) \leq c(\varphi). \tag{5.57}$$

This follows from Proposition 5.12: if h_t denotes the contact Hamiltonian of ψ we choose functions $F_t : M \to \mathbb{R}$ supported inside $\mathcal{O}$ satisfying

$$h_t \geq F_t. \tag{5.58}$$

The lift φ of the corresponding path f of Hamiltonian diffeomorphisms generated by F satisfies the required inequality. $\square$

In this setting, we can use the fact that c_M satisfies the triangle inequality to obtain more information on c. In particular, we obtain a criterion for $c(U)$ to be finite (cf. Remark 4.9).

Corollary 5.17. *Suppose $U \subset \Sigma$ is a non-empty open set with compact closure, and suppose that $\mathrm{pr}_M(U)$ is a Hamiltonian displaceable subset of M. Then $c(U) < \infty$.*

Proof. We have $c(U) \leq c(\mathrm{pr}_M(U) \times S^1) = c_M(\mathrm{pr}_M(U))$, and $c_M(\mathrm{pr}_M(U)) < \infty$ by Theorem 5.20 below. $\square$

We also have the following result:

Proposition 5.18. *Suppose that $\varphi \in \widetilde{\mathrm{Cont}}_{0,c}(\Sigma, \xi)$ has the property that $\mathrm{pr}_M(\mathfrak{S}(\varphi))$ is a Hamiltonian displaceable subset of M. Then $c(\varphi) \geq 0$.*

Proof. First assume that φ is the lift of an element $f \in \widetilde{\mathrm{Ham}}_c(M, d\gamma)$. The fact that $c_M(f) \geq 0$ whenever $\mathfrak{S}(f)$ is Hamiltonian displaceable is well known, but for the convenience of the reader we give the short argument here. Suppose that $g_1 \in \mathrm{Ham}_c(M, d\gamma)$ displaces $\mathfrak{S}(f)$. Let $\{g_t\}_{0 \leq t \leq 1}$ denote some path connecting g_1 to id_M. Choose a path of paths $f^s = \{f_t^s\}_{0 \leq s,t \leq 1}$ connecting $f = f^1$ with the constant path $f_t^0 \equiv \mathrm{id}_M$ such that g_1 displaces $\mathfrak{S}(f^s)$ for each $0 \leq s \leq 1$. Then we claim that

$$c_M(gf) = c_M(g). \tag{5.59}$$

Indeed, the point is that any fixed point of $g_1 f_1$ lies outside of $\mathfrak{S}(f)$, and hence is necessarily also a fixed point of g_1. The same is true if we replace f_1 with f_1^s for any $0 \leq s \leq 1$, and thus it follows that $\mathrm{Spec}(\mathcal{A}_{gf^s})$ is independent of s. Since the function $s \mapsto c_M(gf^s)$ is continuous and $\mathrm{Spec}(\mathcal{A}_{gf^s})$

is nowhere dense, it must be constant. This proves (5.59). We then argue as follows:

$$c_M(g) = c_M(gf^{-1}f) \tag{5.60}$$
$$\leq c_M(gf^{-1}) + c_M(f) \tag{5.61}$$
$$\leq c_M(g) + c_M(f) \tag{5.62}$$

where (5.61) used the triangle inequality for c_M and (5.62) used (5.59) applied to f^{-1}. This implies that $c_M(f) \geq 0$. Finally to prove the general case where φ is not necessarily the lift of a Hamiltonian path f, we use the same argument from the proof of Theorem 5.16. Namely, we can find a path f of Hamiltonians with support inside $\mathrm{pr}_M(\mathfrak{S}(\varphi))$ such that $c_M(f) \leq c(\varphi)$. Then the argument above shows that $c_M(f) \geq 0$, and hence the same is true of $c(\varphi)$. $\square$

Let us quickly recall the definition of the *Hofer-Zehnder capacity*. See for instance [36] for an in depth treatment.

Definition 5.19. Let $\mathcal{O}$ be an open subset of M. We define the *Hofer-Zehnder capacity* $c_{\mathrm{HZ}}(\mathcal{O}, M)$ of $\mathcal{O}$ to

$$c_{\mathrm{HZ}}(\mathcal{O}, M) := \sup \{\|H\| \mid H \text{ is admissible}\}, \tag{5.63}$$

where $H \in C_c^\infty(\mathcal{O}, \mathbb{R})$ is *admissible* if there exists an open set $O \subset \mathcal{O}$ such that $H|_O = \max H$, and if the flow φ_H^t has no non-constant periodic orbits of period ≤ 1.

We also define the *displacement energy* by

$$e(\mathcal{O}, M) := \inf \left\{\|H\| \mid \varphi_H^1(\mathcal{O}) \cap \mathcal{O} = \emptyset\right\}. \tag{5.64}$$

The following result is due to Frauenfelder and Schlenk [30, Corollary 8.3], see also [29, 47].

Theorem 5.20. *If* (M_1, γ_1) *is a Liouville domain then*

$$c_{\mathrm{HZ}}(\mathcal{O}, M) \leq c_M(\mathcal{O}) \leq e(\mathcal{O}, M). \tag{5.65}$$

Denote by $B(r)$ the open ball of radius r in $\mathbb{R}^{2m}$. Then $c_{\mathrm{HZ}}(B(r), \mathbb{R}^{2m}) = \pi r^2$. We can now prove the following result, which was stated as Theorem 1.24 in the Introduction.

Theorem 5.21. *Let $(M, d\gamma)$ denote a Liouville manifold. Equip $\mathbb{R}^{2m}$ with the standard symplectic form $d\lambda_{\mathrm{std}}$, and consider the contact manifold $(\widetilde{\Sigma}, \alpha + \lambda_{\mathrm{std}})$, where $\widetilde{\Sigma} := M \times \mathbb{R}^{2m} \times S^1$. Suppose $\mathcal{O} \subseteq M$ is open and $c_{\mathrm{HZ}}(\mathcal{O}, M) < \infty$. Choose $r_0 > 0$ such that*

$$\lceil \pi r_0^2 \rceil < \lceil c_{\mathrm{HZ}}(\mathcal{O}, M) \rceil \tag{5.66}$$

and set

$$r_1 := \sqrt{\tfrac{1}{\pi} c_{\mathrm{HZ}}(\mathcal{O}, M)} + 1 \tag{5.67}$$

Then there does not exist $\varphi \in \mathrm{Cont}_{0,c}(\widetilde{\Sigma}, \alpha + \lambda_{\mathrm{std}})$ such that

$$\varphi(\mathcal{O} \times B(r_1) \times S^1) \subset \mathcal{O} \times B(r_0) \times S^1. \tag{5.68}$$

Proof. We first prove that for $r > r_1$,

$$c_{\mathrm{HZ}}(\mathcal{O} \times B(r), M \times \mathbb{R}^{2m}) \geq c_{\mathrm{HZ}}(\mathcal{O}, M). \tag{5.69}$$

Fix $\varepsilon > 0$. We consider a cutoff function $\beta : [0, \infty) \to [0, 1]$ such that $\beta(s) = 1$ for $s \in [0, r - 1 - \varepsilon]$ and $\beta(s) = 0$ for $s > r$, and such that $-1 \leq \beta'(s) \leq 0$ for all $s \in [0, \infty)$. Now suppose H is any admissible function on $\mathcal{O}$. Define $H_\beta : M \times \mathbb{R}^{2m} \to \mathbb{R}$ by

$$H_\beta(x, y) := \beta(|y|) H(x). \tag{5.70}$$

The symplectic gradient of H_β with respect to $d\gamma \oplus d\lambda_{\mathrm{std}}$ is

$$X_{H_\beta}(x, y) = \left(\beta(|y|) X_H(x), H(x) X_\beta(y)\right). \tag{5.71}$$

Suppose $\gamma : \mathbb{R} \to M \times \mathbb{R}^{2m}$ is a non-constant periodic orbit of X_{H_β}, with $\gamma(t + T) = \gamma(t)$ for all $t \in \mathbb{R}$. We shall show that $T > 1$, so that H_β is admissible. Write $\gamma(t) = (\gamma_x(t), \gamma_y(t))$. Then

$$\dot{\gamma}_x = \beta(|\gamma_y|) X_H(\gamma_x), \quad \dot{\gamma}_y = H(\gamma_x) X_\beta(\gamma_y). \tag{5.72}$$

Since $|\beta'| \leq 1$ we see that if γ_x is non-constant then $T > 1$. But if γ_x is constant, say $\gamma_x(t) = x_0$, then we must have $H(x_0) \neq 0$. Since β' is non-zero only for $|\gamma_y| \in (r-1-\varepsilon, r)$ we necessarily have

$$T \geq \frac{1}{H(x_0)}\pi(r-1-\varepsilon)^2 \geq \frac{1}{c_{\mathrm{HZ}}(\mathcal{O},M)}\pi(r-1-\varepsilon)^2. \tag{5.73}$$

Thus as long as

$$\pi(r-1-\varepsilon)^2 > c_{\mathrm{HZ}}(\mathcal{O},M), \tag{5.74}$$

H_β is indeed admissible. Since clearly max $H_\beta =$ max H, we see that

$$c_{\mathrm{HZ}}(\mathcal{O}\times B(r), M\times\mathbb{R}^{2n}) \geq c_{\mathrm{HZ}}(U,M) \tag{5.75}$$

provided that (5.74) holds. Since ε was arbitrary we obtain (5.69). Moreover for any $r>0$ one always has

$$e(\mathcal{O}\times B(r), M\times\mathbb{R}^{2n}) \leq \pi r^2, \tag{5.76}$$

as can be checked directly. The remainder of the proof is an easy application of Theorem 5.16, Theorem 5.20 and Theorem 4.8. Indeed, we have

$$\begin{aligned}\overline{c}(\mathcal{O}\times B(r_0)\times S^1) &= \left\lceil e(\mathcal{O}\times B(r_0), M\times\mathbb{R}^{2m})\right\rceil \\ &\leq \left\lceil \pi r_0^2\right\rceil \\ &< \left\lceil c_{\mathrm{HZ}}(\mathcal{O},M)\right\rceil \\ &\leq \left\lceil c_{\mathrm{HZ}}(\mathcal{O}\times B(r_1), M\times\mathbb{R}^{2m})\right\rceil \\ &\leq \overline{c}_{M\times\mathbb{R}^{2m}}(\mathcal{O}\times B(r_1)) \\ &= \overline{c}(\mathcal{O}\times B(r_1)\times S^1). \end{aligned} \tag{5.77}$$

□

Here is an application of Theorem 5.21, which can be seen as a more quantitive (albeit weaker, and with more hypotheses) version of the infinitesimal result of [24, Theorem 1.18].

Corollary 5.22. *Suppose X is a closed connected oriented Riemannian manifold which admits a circle action $S^1\times X\to X$ such that the loop $t\mapsto t\cdot p$ is not contractible for some $p\in X$. Then if $\mathcal{O}\subset T^*X$ is any neighborhood of the zero section then the conclusion of Proposition 5.21 holds.*

Proof. A result of Kei Irie [37] proves that in this setting the Hofer-Zehnder capacity of the unit disc bundle $D^*X\subset T^*X$ is finite. Thus the same is true of any neighborhood $\mathcal{O}\subset T^*X$ of the zero section, and hence the hypotheses of Theorem 5.21 are satisfied. □

Appendix A. The "compactly supported Reeb flow"

In this Appendix we continue to work in the setting from the previous section. Thus $\Sigma = M \times S^1$ is a prequantisation space associated to the completion of a Liouville domain $(M_1, d\gamma_1)$. Our aim is to construct a "compactly supported Reeb flow" whose support is contained in a tubular neighborhood of a closed Reeb orbit, and explicitly compute the spectral value. This result has been alluded to in Remarks 3.19 and 5.14.

Theorem A.1. *Suppose $(\Sigma = M \times S^1, \xi)$ satisfies Assumption* **(C)**. *Let $x(t) = (y_0, t)$ denote a closed embedded Reeb orbit (for some fixed $y_0 \in M$.) Then there exists $\rho_0 > 0$ and a neighborhood B of y_0 in M with the following significance: For all $\rho \in \mathbb{R}$ with $|\rho| < \rho_0$, there exists an exact contactomorphism $\vartheta^\rho \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ with $\mathfrak{S}(\vartheta^\rho) \subset B \times S^1$ with the property that if $x \in \mathfrak{S}(\vartheta^\rho)$ is a translated point of ϑ^ρ then*

$$\vartheta^\rho(x) = \theta^\rho(x). \tag{A.1}$$

In other words, from the point of view of translated points, ϑ^ρ is "the Reeb flow supported on x". Moreover if $B' \subset B$ is any neigborhood of y_0 then for $|\rho|$ sufficiently small we have $\mathfrak{S}(\vartheta^\rho) \subset B' \times S^1$. The spectral value $c(\vartheta^\rho)$ is given by

$$c(\vartheta^\rho) = \begin{cases} 0, & 0 \leq \rho < \rho_0, \\ -\rho, & -\rho_0 < \rho \leq 0. \end{cases} \tag{A.2}$$

Convention: In this appendix we equip $\mathbb{R}^{2n} \backslash \{0\}$ with polar coordinates (s, ϕ) where $s \in (0, \infty)$ and $\phi = (\phi_1, \dots, \phi_{2n-1})$ with $\phi_j \in \mathbb{R}/2\pi\mathbb{Z}$. In these coordinates the standard contact form α_{std} is given by

$$\alpha_{\mathrm{std}} = \sum_j \tfrac{1}{2} s^2 d\phi_j + d\tau. \tag{A.3}$$

This has the slightly unfortunate consequence that τ is 1-periodic but the ϕ_j are 2π-periodic! These conventions are chosen so that $c_{\mathbb{R}^{2n}}(B(r)) = \pi r^2$ instead of $\frac{1}{2} r^2$.

Proof of Theorem A.1. The argument is local in M, and hence it is sufficient to prove the result in the special case $M = \mathbb{R}^{2n}$. Thus $\Sigma = \mathbb{R}^{2n} \times S^1$ and $\alpha = \alpha_{\mathrm{std}}$ is given by (A.3). The Reeb vector field R of α is just $\frac{\partial}{\partial \tau}$, and the

Reeb flow θ^t is given by

$$\theta^t(s,\phi,\tau) = (s,\phi,\underbrace{\tau + t}_{\text{mod } 1}). \tag{A.4}$$

Fix $\rho \in \mathbb{R}$ such that $0 < |\rho| < \pi r^2$. Let $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$ denote a smooth function with the following properties:

1) There exists $\varepsilon > 0$ such that $f(s) = \rho$ for $0 \le s \le \varepsilon$ and $f(s) = 0$ for $r - \varepsilon \le s \le r$.
2) If $\rho < 0$ then $f'(s) \ge 0$ for all s. If $\rho > 0$ then $f'(s) \le 0$ for all s.
3) If $\rho < 0$ then $2\pi s - f'(s) > 0$ for all $s > 0$. If $\rho < 0$ then $2\pi s + f'(s) < 0$ for all $s > 0$.

Note that such a function only exists because $|\rho| < \pi r^2$. Indeed, if $\rho < 0$ then since $2\pi s - f'(s) > 0$ one has

$$-\rho = \int_0^r f'(s)ds < \int_0^r 2\pi s ds = \pi r^2. \tag{A.5}$$

Conversely it is easy to see that when $|\rho| < \pi r^2$ such functions really do exist. Now consider the contactomorphism ϑ^ρ of $\mathbb{R}^{2n} \times S^1$ whose contact Hamiltonian $h_t : \mathbb{R}^{2n} \times S^1$ is given by

$$h_t(s,\zeta,\tau) = f(r). \tag{A.6}$$

The contact vector field X_t of h_t is defined by the equations

$$\alpha(X_t) = h_t, \quad i_{X_t} d\alpha = dh_t(R)\alpha - dh_t. \tag{A.7}$$

This gives

$$X_t(s,\phi,\tau) = \sum_j \frac{f'(s)}{s}\frac{\partial}{\partial \phi_j} + \left(f(s) - \frac{sf'(s)}{2}\right)\frac{\partial}{\partial \tau}. \tag{A.8}$$

We can integrate this to obtain
(A.9)
$$\vartheta_t^\rho(s,\phi,\tau) = \left(s, \phi_1 + \frac{f'(s)}{s}t, \dots, \phi_{2n-1} + \frac{f'(s)}{2}t, \tau + \left(f(s) - \frac{sf'(s)}{2}\right)t\right),$$

and hence translated points of ϑ_1^ρ are tuples (s, ϕ, τ) with

$$\frac{f'(s)}{s} \in 2\pi\mathbb{Z}, \tag{A.10}$$

and the time-shift is given by

$$\eta = f(s) - \frac{sf'(s)}{2}. \tag{A.11}$$

By assumption one never has $f'(s)/2\pi s \in \mathbb{Z}$ unless $f'(s) = 0$. In other words, translated points only occur when $0 \le s \le \varepsilon$ or when $r - \varepsilon \le s \le \infty$. In particular, the only translated points of ϑ^ρ that lie in the interior of the support of ϑ^ρ are the points in $B(\varepsilon) \times S^1$. Since $\vartheta^\rho = \theta^\rho$ on $B(\varepsilon) \times S^1$, this justifies our claim that 'from the point of view of translated points', ϑ^ρ is the Reeb flow.

To complete the proof let us compute the spectral value of ϑ^ρ. Note that the *contractible* action spectrum of $\mathcal{A}_{\vartheta^\rho}$ is just $\{0, -\rho\}$, and hence we certainly have $c(\vartheta^\rho) \in \{0, -\rho\}$. For $\rho < 0$, one has $h_t < 0$ on the interior of its support and hence by Corollary 5.13 one has $c(\vartheta^\rho) > 0$, which implies $c(\vartheta^\rho) = -\rho$. Thus for $\rho > 0$ we must have $c(\vartheta^\rho) = 0$. This completes the proof. $\square$

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