

Integrability of central extensions of the Poisson Lie algebra via prequantization

BAS JANSSENS[†] AND CORNELIA VIZMAN[‡]

We present a geometric construction of central S^1 -extensions of the quantomorphism group of a prequantizable, compact, symplectic manifold, and explicitly describe the corresponding lattice of integrable cocycles on the Poisson Lie algebra. We use this to find nontrivial central S^1 -extensions of the universal cover of the group of Hamiltonian diffeomorphisms. In the process, we obtain central S^1 -extensions of Lie groups that act by exact strict contact transformations.

1. Introduction

Central Lie group extensions can be obtained by pullback of the prequantization central extension. The ingredients are a connected Lie group G with Lie algebra \mathfrak{g} , a connected prequantizable symplectic manifold (\mathcal{M}, Ω) , and a Hamiltonian action of G on \mathcal{M} .

As \mathcal{M} is prequantizable, it has a prequantum S^1 -bundle $\mathcal{P} \rightarrow \mathcal{M}$ with connection 1-form Θ , giving rise to the *quantomorphism group* $\text{Diff}(\mathcal{P}, \Theta)$ of connection-preserving automorphisms of this bundle. Since its identity component $\text{Diff}(\mathcal{P}, \Theta)_0$ is a central S^1 -extension of the Hamiltonian diffeomorphism group $\text{Ham}(\mathcal{M}, \Omega)$, its pullback by the Hamiltonian action $G \rightarrow \text{Ham}(\mathcal{M}, \Omega)$ yields a central S^1 -extension \widehat{G} of G ,

$$(1.1) \quad \begin{array}{ccc} \widehat{G} & \longrightarrow & \text{Diff}(\mathcal{P}, \Theta)_0 \\ \downarrow & & \downarrow \\ G & \longrightarrow & \text{Ham}(\mathcal{M}, \Omega). \end{array}$$

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If the manifold \mathcal{M} and the Lie group G are infinite dimensional, then this construction remains valid; the pullback \widehat{G} is still a Lie group, even though this may not be the case for $\text{Ham}(\mathcal{M}, \Omega)$ and $\text{Diff}(\mathcal{P}, \Theta)$ (cf. [19]).

We apply this construction in the following setting. Suppose that $\pi : P \rightarrow M$ is a prequantum S^1 -bundle over a compact, symplectic manifold (M, ω) of dimension $2n$, and that θ is a connection 1-form with curvature ω . The identity component $G = \text{Diff}(P, \theta)_0$ of the quantomorphism group is then a Fréchet Lie group, with Lie algebra \mathfrak{g} isomorphic to the Poisson Lie algebra $C^\infty(M)$. The infinite dimensional symplectic manifold \mathcal{M} on which G acts will be a connected component of the *nonlinear Grassmannian* $\text{Gr}_{2n-1}(P)$ of codimension two closed, oriented, embedded submanifolds of P . This Fréchet manifold is prequantizable by [6, 7], and the natural action of G on \mathcal{M} is Hamiltonian. In this way, we obtain central S^1 -extensions $\widehat{G} \rightarrow G$ of the identity component $G = \text{Diff}(P, \theta)_0$ of the quantomorphism group.

For any (Fréchet) Lie group G , the central extensions of G by S^1 play a pivotal role in the theory of *projective unitary G -representations*. Every such representation gives rise to a central S^1 -extension \widehat{G} , together with a *linear* unitary \widehat{G} -representation [8, 22, 28]. Passing to the infinitesimal level, one obtains information on the projective G -representations from the (often more accessible) linear representation theory of the corresponding central Lie algebra extensions $\widehat{\mathfrak{g}}$.

In the passage to the infinitesimal level, however, one important piece of information is lost: not every Lie algebra extension $\widehat{\mathfrak{g}} \rightarrow \mathfrak{g}$ integrates to a group extension $\widehat{G} \rightarrow G$. The ones that do determine a lattice $\Lambda \subseteq H^2(\mathfrak{g}, \mathbb{R})$ in the continuous second Lie algebra cohomology of \mathfrak{g} , called the *lattice of integrable classes*.

In the context of quantomorphism groups, the continuous second Lie algebra cohomology of the Poisson Lie algebra $C^\infty(M)$ has been explicitly determined; in [9], we proved that

$$H^2(C^\infty(M), \mathbb{R}) \simeq H^1(M, \mathbb{R}).$$

To the best of our knowledge, it remains an open problem to determine the full lattice $\Lambda \subseteq H^1(M, \mathbb{R})$ of integrable classes; it appears that the period homomorphism governing integrability (cf. [18, Thm. 7.9]) is not easy to calculate in the setting of quantomorphism groups.

In the present paper, we contribute towards a solution by explicitly determining the sublattice $\Lambda_0 \subseteq \Lambda$ corresponding to the group extensions $\widehat{G} \rightarrow G$

described above. We find that

$$\Lambda_0 = \frac{n+1}{2\pi \text{vol}(M)} \pi_! (H^2(P, \mathbb{R})_{\mathbb{Z}}),$$

where $H^2(P, \mathbb{R})_{\mathbb{Z}}$ is the lattice of integral classes in de Rham cohomology, and $\pi_!$ denotes fiber integration. This formula is easily evaluated in concrete situations. If M is a compact surface, then $\Lambda_0 \subseteq H^1(M, \mathbb{R})$ is of full rank. On the other extreme, we find that $\Lambda_0 = \{0\}$ if M is a compact Kähler manifold of dimension $2n \geq 4$. Intermediate behavior is displayed by *nilmanifolds*. Thurston's nilmanifold M^4 , for example, affords a lattice Λ_0 that is of rank 2 in the 3-dimensional vector space $H^1(M^4, \mathbb{R})$.

We expect that in the representation theory of the Poisson Lie algebra $C^\infty(M)$, the lattice Λ of integrable classes will play the same role as the *integral level condition* in the representation theory of loop algebras and affine Kac-Moody algebras [10, §12]. From a differential geometric point of view, integrality of the level for a loop algebra \mathfrak{g} (possibly twisted, over a simple Lie algebra) is precisely the condition that the induced class in $H^2(\mathfrak{g}, \mathbb{R}) \simeq H^1(S^1, \mathbb{R})$ corresponds to a Lie group extension, cf. [22, §4].

2. Prequantization central extension

Let G be a connected Lie group with Lie algebra \mathfrak{g} , and let (\mathcal{M}, Ω) be a connected, prequantizable, symplectic manifold with a Hamiltonian G -action. Both G and \mathcal{M} are allowed to be infinite dimensional manifolds, modeled on locally convex spaces. Let $\pi: \mathcal{P} \rightarrow \mathcal{M}$ be a *prequantum bundle*, i.e. a principal S^1 -bundle with principal connection 1-form Θ and curvature Ω . In particular, the identity $\pi^*\Omega = d\Theta$ holds.

2.1. The Kostant-Souriau extension

The prequantum bundle $\mathcal{P} \rightarrow \mathcal{M}$ gives rise to the *prequantization central extension* [12, 25]

$$(2.1) \quad S^1 \rightarrow \text{Diff}(\mathcal{P}, \Theta)_0 \rightarrow \text{Ham}(\mathcal{M}, \Omega),$$

with $\text{Ham}(\mathcal{M}, \Omega)$ the group of Hamiltonian diffeomorphisms, and $\text{Diff}(\mathcal{P}, \Theta)_0$ the identity component of the *qantomorphism group*

$$\text{Diff}(\mathcal{P}, \Theta) = \{\varphi \in \text{Diff}(\mathcal{P}) : \varphi^*\Theta = \Theta\}.$$

Note that $\varphi^*\Theta = \Theta$ implies $\varphi_*E = E$, where $E \in \mathfrak{X}(P)$ is the infinitesimal generator of the S^1 -action. In particular, every quantomorphism is a bundle automorphism.

The infinitesimal counterpart of (2.1) is the central extension

$$(2.2) \quad \mathbb{R} \rightarrow \mathfrak{X}(\mathcal{P}, \Theta) \rightarrow \mathfrak{X}_{\text{ham}}(\mathcal{M}, \Omega)$$

of the Lie algebra of *Hamiltonian* vector fields

$$\mathfrak{X}_{\text{ham}}(\mathcal{M}, \Omega) := \{X_f \in \mathfrak{X}(\mathcal{M}) : i_{X_f}\Omega = -df\},$$

namely the *quantomorphism Lie algebra*

$$\mathfrak{X}(\mathcal{P}, \Theta) := \{X \in \mathfrak{X}(\mathcal{P}) : L_X\Theta = 0\}.$$

It is isomorphic to the Poisson Lie algebra $C^\infty(\mathcal{M})$ via the Lie algebra isomorphism

$$(2.3) \quad \zeta : C^\infty(\mathcal{M}) \rightarrow \mathfrak{X}(\mathcal{P}, \Theta), \quad \zeta_f := X_f^{\text{hor}} + (\pi^* f)E,$$

where Y^{hor} denotes the horizontal lift of the vector field $Y \in \mathfrak{X}(\mathcal{M})$. The central extension (2.2) can thus be identified with the *Kostant-Souriau* extension

$$(2.4) \quad \mathbb{R} \rightarrow C^\infty(\mathcal{M}) \rightarrow \mathfrak{X}_{\text{ham}}(\mathcal{M}, \Omega),$$

induced by the map $f \mapsto X_f$.

The central extensions of a locally convex Lie algebra \mathfrak{g} are classified by its *continuous second Lie algebra cohomology* $H^2(\mathfrak{g}, \mathbb{R})$, cf. e.g. [9, §2.3]. This is the cohomology of the cochain complex $C^n(\mathfrak{g}, \mathbb{R})$ of continuous, alternating, n -linear maps $\mathfrak{g}^n \rightarrow \mathbb{R}$, with differential $\delta : C^n(\mathfrak{g}, \mathbb{R}) \rightarrow C^{n+1}(\mathfrak{g}, \mathbb{R})$ defined by

$$(2.5) \quad \delta\psi(x_0, \dots, x_n) := \sum_{0 \leq i < j \leq n} (-1)^{i+j} \psi([x_i, x_j], x_0, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n).$$

A continuous linear splitting $s : \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ of the central extension $\mathbb{R} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g}$ gives rise to the 2-cocycle $\psi(X, Y) := [s(X), s(Y)] - s([X, Y])$. Given a point

$x_0 \in \mathcal{M}$, we split (2.4) by mapping $X_f \in \mathfrak{X}_{\text{ham}}(\mathcal{M}, \Omega)$ to the unique Hamiltonian function f^{x_0} vanishing on x_0 . The splitting

$$(2.6) \quad s_{x_0} : X_f \mapsto f^{x_0}$$

yields the *Kostant-Souriau cocycle* ψ_{KS} on $\mathfrak{X}_{\text{ham}}(\mathcal{M})$ that is given by

$$(2.7) \quad \psi_{KS}(X_f, X_g) = \{f, g\}(x_0) = \Omega(X_f, X_g)(x_0).$$

2.2. Group extensions from Hamiltonian actions

If \mathcal{M} is a compact (hence finite dimensional) manifold, then $\text{Ham}(\mathcal{M}, \Omega)$ and $\text{Diff}(\mathcal{P}, \Theta)$ are both infinite dimensional Lie groups, with Lie algebras $\mathfrak{X}_{\text{ham}}(\mathcal{M}, \Omega)$ and $\mathfrak{X}(\mathcal{P}, \Theta)$, respectively, see [23, §3], [21, §VIII.4]. Even though this need no longer be the case if \mathcal{M} is infinite dimensional, it is still true that the pullback \widehat{G} of (2.1) under the Hamiltonian action of a Lie group G on \mathcal{M} has a smooth Lie group structure.

Theorem 2.1. [19, Thm 3.4] *Let (\mathcal{M}, Ω) be a prequantizable, symplectic manifold with a Hamiltonian action of a connected Lie group G . Then the pullback of the prequantization central extension (2.1) by the action $G \rightarrow \text{Ham}(\mathcal{M}, \Omega)$ provides a central Lie group extension*

$$(2.8) \quad S^1 \rightarrow \widehat{G} \rightarrow G.$$

The derived Lie algebra extension $\mathbb{R} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g}$ is given by the pullback of the Kostant-Souriau extension (2.4) along the infinitesimal action $\mathfrak{g} \rightarrow \mathfrak{X}_{\text{ham}}(\mathcal{M}, \Omega)$. The linear splitting (2.6) therefore induces a linear splitting of $\widehat{\mathfrak{g}} \rightarrow \mathfrak{g}$, and the corresponding 2-cocycle σ on \mathfrak{g} is the pullback by $\mathfrak{g} \rightarrow \mathfrak{X}_{\text{ham}}(\mathcal{M}, \Omega)$ of the Kostant-Souriau cocycle. It is given explicitly by

$$(2.9) \quad \sigma(\xi, \eta) = \Omega(\xi_{\mathcal{M}}, \eta_{\mathcal{M}})(x_0),$$

where $\xi_{\mathcal{M}}$ denotes the fundamental vector field on \mathcal{M} for $\xi \in \mathfrak{g}$.

3. Exact volume preserving diffeomorphisms

In order to obtain central extensions of the Lie group $G = \text{Diff}_{\text{ex}}(M, \nu)$ of exact volume preserving diffeomorphisms of a compact manifold M endowed with volume form ν , we consider its Hamiltonian action on the non-linear Grassmannian of codimension 2 embedded submanifolds of M .

3.1. Non-linear Grassmannians

Let M be a closed, connected manifold of dimension m . The *non-linear Grassmannian* $\text{Gr}_k(M)$ consists of k -dimensional, closed, oriented, embedded submanifolds $N \subseteq M$. It is a Fréchet manifold in a natural way, cf. [5, 13]. The tangent space of $\text{Gr}_k(M)$ at N can naturally be identified with the space of smooth sections of the normal bundle $TN^\perp := (TM|_N)/TN$.

For every $r \geq 0$, the transgression map $\tau: \Omega^{k+r}(M) \rightarrow \Omega^r(\text{Gr}_k(M))$ is defined by

$$(\tau\alpha)_N([Y_1], \dots, [Y_r]) := \int_N i_{Y_r} \cdots i_{Y_1}(\alpha|_N).$$

Here all $[Y_j]$ are tangent vectors at $N \in \text{Gr}_k(M)$, i.e. sections of TN^\perp . The expression above is independent of the vector fields Y_j on M along N chosen to represent $[Y_j]$.

The natural group action of $\text{Diff}(M)$ on $\text{Gr}_k(M)$, defined by $(\varphi, N) \mapsto \varphi(N)$, is smooth, since it descends from the action of $\text{Diff}(M)$ on the manifold of embeddings into M defined by composition $(\varphi, f) \mapsto \varphi \circ f$. It differentiates to the Lie algebra action $\mathfrak{X}(M) \rightarrow \mathfrak{X}(\text{Gr}_k(M))$ given by $X \mapsto \tau_X$ with $\tau_X(N) = [X|_N]$. The transgression enjoys the following functorial properties:

$$(3.1) \quad \begin{aligned} d \circ \tau &= \tau \circ d, & \varphi^* \circ \tau &= \tau \circ \varphi^*, \\ i_{\tau_X} \circ \tau &= \tau \circ i_X, & L_{\tau_X} \circ \tau &= \tau \circ L_X. \end{aligned}$$

Theorem 3.1. [7, §25.3] [6, Thm. 1] *Let $\alpha \in \Omega^{k+2}(M)$ be a closed differential form with integral cohomology class. Then the non-linear Grassmannian $\text{Gr}_k(M)$ endowed with the closed 2-form $\Omega = \tau\alpha$ is prequantizable, i.e. there exist an S^1 -bundle $\mathcal{P} \rightarrow \text{Gr}_k(M)$ with connection form $\Theta \in \Omega^1(\mathcal{P})$ and curvature Ω .*

3.2. Lichnerowicz central extensions

Let $\nu \in \Omega^m(M)$ be a volume form on M , normalized so that $\text{vol}_\nu(M) = 1$. It induces a symplectic form $\Omega = \tau\nu$ on the codimension two non-linear Grassmannian $\text{Gr}_{m-2}(M)$ [7]. This is the higher dimensional version of the natural symplectic form on the space of knots in \mathbb{R}^3 [17]. The natural action of the group $\text{Diff}(M, \nu)$ of volume preserving diffeomorphisms on $\text{Gr}_{m-2}(M)$ is symplectic, as $\varphi^*\tau\nu = \tau\varphi^*\nu = \tau\nu$ for all $\varphi \in \text{Diff}(M, \nu)$ by (3.1). To get a Hamiltonian action, we have to restrict to the subgroup $\text{Diff}_{\text{ex}}(M, \nu)$ of *exact volume preserving diffeomorphisms*.

Its Lie algebra $\mathfrak{X}_{\text{ex}}(M, \nu)$ of *exact divergence free vector fields* is the kernel of the infinitesimal flux homomorphism, defined on the Lie algebra $\mathfrak{X}(M, \nu)$ of divergence free vector fields by

$$(3.2) \quad \mathfrak{X}(M, \nu) \rightarrow H^{m-1}(M, \mathbb{R}), \quad X \mapsto [i_X \nu].$$

We denote by X_α the exact divergence free vector field with potential $\alpha \in \Omega^{m-2}(M)$, i.e. $i_{X_\alpha} \nu = d\alpha$.

The Lie algebra homomorphism (3.2) is integrated by *Thurston's flux homomorphism*. On the universal cover of the identity component of the group of volume preserving diffeomorphisms, we define

$$(3.3) \quad \widetilde{\text{Flux}} : \widetilde{\text{Diff}}(M, \nu)_0 \rightarrow H^{m-1}(M, \mathbb{R}) \quad \text{by} \quad \widetilde{\text{Flux}}([\varphi_t]) = \int_0^1 [i_{X_t} \nu] dt,$$

where φ_t is a volume preserving isotopy from the identity to φ , and X_t is the time dependent vector field such that $\frac{d}{dt} \varphi_t = X_t \circ \varphi_t$. By [1, Thm. 3.1.1], this is a well defined group homomorphism.

For any codimension one submanifold $N \subset M$, the integral $\int_N \widetilde{\text{Flux}}([\varphi_t])$ is the volume swept out by N under φ_t . Therefore, the monodromy subgroup $\Gamma := \widetilde{\text{Flux}}(\pi_1(\text{Diff}(M, \nu)_0))$ of $H^{m-1}(M, \mathbb{R})$ is discrete. It follows that equation (3.3) factors through a Lie group homomorphism

$$(3.4) \quad \text{Flux} : \text{Diff}(M, \nu)_0 \rightarrow H^{m-1}(M, \mathbb{R})/\Gamma, \quad \text{Flux}(\varphi) = \int_0^1 [i_{X_t} \nu] dt \quad \text{mod } \Gamma.$$

The group of exact volume preserving diffeomorphisms $\text{Diff}_{\text{ex}}(M, \nu)$ is now defined as the kernel of the Flux homomorphism; it is a Lie group with Lie algebra $\mathfrak{X}_{\text{ex}}(M, \nu)$ [1, 13].

Since $\text{vol}_\nu(M)$ is normalized to 1, the cohomology class $[k\nu] \in H^m(M, \mathbb{R})$ is integral for every $k \in \mathbb{Z}$. By Theorem 3.1, this implies that the manifold $\text{Gr}_{m-2}(M)$ with symplectic form $\Omega = k\tau\nu$ is prequantizable. The natural action of $\text{Diff}_{\text{ex}}(M, \nu)$ on $\text{Gr}_{m-2}(M)$ is Hamiltonian, as $i_{\tau X_\alpha} \tau\nu = \tau i_{X_\alpha} \nu = \tau d\alpha = d\tau\alpha$ for all $X_\alpha \in \mathfrak{X}_{\text{ex}}(M, \nu)$ by (3.1). Now we can apply Theorem 2.1 to this Hamiltonian action on a connected component \mathcal{M} of $\text{Gr}_{m-2}(M)$. This yields the central Lie group extension

$$(3.5) \quad S^1 \rightarrow \widehat{\text{Diff}}_{\text{ex}}(M, \nu) \rightarrow \text{Diff}_{\text{ex}}(M, \nu)$$

of the group of exact volume preserving diffeomorphisms.

To obtain the corresponding Lie algebra 2-cocycle, we fix a point $Q \in \mathcal{M}$, that is, a codimension two submanifold $Q \subset M$ in the connected component

\mathcal{M} of the nonlinear Grassmannian. By (2.9), the Lie algebra extension of $\mathfrak{X}_{\text{ex}}(M, \nu)$ corresponding to (3.5) for $k = 1$, is described by the Lie algebra 2-cocycle

$$(3.6) \quad \lambda_Q^\nu(X, Y) = (\tau\nu)_Q(\tau_X, \tau_Y) = \int_Q i_Y i_X \nu$$

on $\mathfrak{X}_{\text{ex}}(M, \nu)$, which we call the *singular Lichnerowicz cocycle*. If the class $[k\nu]$ is used to construct the extension, then the corresponding 2-cocycle is $k\lambda_Q^\nu$.

Theorem 3.2. [7, §25.5][6, Thm. 2] *Let ν be a volume form on M with $\text{vol}_\nu(M) = 1$ and Q a codimension two embedded submanifold of M . Then the Lie algebra extensions defined by integral multiples of the cocycle λ_Q^ν of equation (3.6) integrate to central Lie group extensions of the group of exact volume preserving diffeomorphisms $\text{Diff}_{\text{ex}}(M, \nu)$.*

Recall that two classes $[Q] \in H_{m-k}(M, \mathbb{R})$ and $[\eta] \in H^k(M, \mathbb{R})$ are called *Poincaré dual* if $\int_Q \gamma = \int_M \eta \wedge \gamma$ for all closed $\gamma \in \Omega^{m-k}(M)$. If the class $[\eta] \in H^2(M, \mathbb{R})$ is Poincaré dual to $[Q] \in H_{m-2}(M, \mathbb{R})$, then by [29, Prop. 2] the cocycle λ_Q^ν is cohomologous to the *Lichnerowicz cocycle* [15]

$$(3.7) \quad \lambda_\eta^\nu(X, Y) = \int_M \eta(X, Y)\nu = \int_M \eta \wedge i_Y i_X \nu.$$

If $\dim M \geq 3$, the map $[\eta] \mapsto [\lambda_\eta^\nu]$ is believed to be an isomorphism between $H^2(M, \mathbb{R})$ and the second Lie algebra cohomology group $H^2(\mathfrak{X}_{\text{ex}}(M, \nu), \mathbb{R})$, see [24, §10] for the outline of a proof.

Remark 3.3. If $[\eta]$ is Poincaré dual to $[Q]$ with $Q \in \text{Gr}_{m-2}(M)$, then in particular, it is an integral cohomology class. Conversely, every integral cohomology class $[\eta] \in H^2(M, \mathbb{R})$ is the Poincaré dual of a closed submanifold of codimension two in M ; it can be obtained (cf. [4, Prop. 12.8]) as the zero set of a section transversal to the zero section in a rank two vector bundle with Euler class $[\eta]$.

We infer that the Lichnerowicz cocycle (3.7) gives rise to an *integrable* Lie algebra extension if $[\eta] \in H^2(M, \mathbb{R})$ is an *integral* class in de Rham cohomology (in the sense that on integral singular 2-cycles, it evaluates to an integer). We denote by $H^2(M, \mathbb{R})_{\mathbb{Z}}$ the space of integral de Rham classes. It follows that the image of $H^2(M, \mathbb{R})_{\mathbb{Z}}$ by the map $[\eta] \mapsto [\lambda_\eta^\nu]$, which coincides

with the image of $H_{m-2}(M, \mathbb{Z})$ by the map $[Q] \mapsto [\lambda_Q^\nu]$, lies in the lattice of integrable classes in $H^2(\mathfrak{X}_{\text{ex}}(M, \nu), \mathbb{R})$.

4. Strict contactomorphisms

Let P be a compact manifold of dimension $2n + 1$, equipped with a contact 1-form θ . The group $\text{Diff}(P, \theta)$ of strict contactomorphisms is a subgroup of the volume preserving diffeomorphism group $\text{Diff}(P, \mu)$, where the volume form μ is a constant multiple of $\theta \wedge (d\theta)^n$. The group of *exact strict contactomorphisms* is defined as

$$(4.1) \quad \text{Diff}_{\text{ex}}(P, \theta) := \text{Diff}(P, \theta)_0 \cap \text{Diff}_{\text{ex}}(P, \mu).$$

We use Theorem 3.2 to investigate central extensions of locally convex Lie groups G that act on P by exact strict contactomorphisms.

The motivating example is the case where $P \rightarrow M$ is a prequantum bundle over a compact symplectic manifold (M, ω) ; in this case $\text{Diff}(P, \theta)$ is the quantomorphism group, while $\text{Diff}_{\text{ex}}(P, \theta)$ coincides with the identity component $\text{Diff}(P, \theta)_0$ of the quantomorphism group. Since the latter is a locally convex Lie group (it is even an ILH-Lie group by [21, VIII.4]), we thus obtain central Lie group extensions of $G = \text{Diff}(P, \theta)_0$ by S^1 .

4.1. Strict contactomorphisms

The contact form θ gives rise to the volume form $\theta \wedge (d\theta)^n$. We will use the following two normalizations of this form:

$$(4.2) \quad \mu := \frac{1}{(n + 1)!} \theta \wedge (d\theta)^n, \quad \text{and} \quad \nu := \frac{1}{\text{vol}_\mu(P)} \mu.$$

The *Reeb vector field* $E \in \mathfrak{X}(P)$ is uniquely determined by $i_E \theta = 1$ and $i_E d\theta = 0$. We define the *strict contactomorphism group*

$$\text{Diff}(P, \theta) := \{\varphi \in \text{Diff}(P); \quad \varphi^* \theta = \theta\} \subset \text{Diff}(P, \mu)$$

and the *Lie algebra of strict contact vector fields*

$$\mathfrak{X}(P, \theta) = \{X \in \mathfrak{X}(P); \quad L_X \theta = 0\} \subset \mathfrak{X}(P, \mu).$$

For every isotopy $\varphi_t \in \text{Diff}(P)$, starting at the identity and corresponding to the time dependent vector field $X_t \in \mathfrak{X}(P)$, we have $\varphi_t \in \text{Diff}(P, \theta)$ for all t if and only if the vector field X_t lies in $\mathfrak{X}(P, \theta)$ for all t .

A Hamiltonian function $f \in C^\infty(P)^E = \{f \in C^\infty(P) : L_E f = 0\}$ defines a unique strict contact vector field $\zeta_f \in \mathfrak{X}(P, \theta)$ by

$$i_{\zeta_f} \theta = f, \quad i_{\zeta_f} d\theta = -df.$$

The corresponding map

$$(4.3) \quad \zeta : C^\infty(P)^E \rightarrow \mathfrak{X}(P, \theta)$$

is an isomorphism of Fréchet Lie algebras, if $C^\infty(P)^E$ is equipped with the Lie bracket

$$(4.4) \quad \{f, g\} = d\theta(\zeta_f, \zeta_g) = L_{\zeta_f} g.$$

Proposition 4.1. *Let Q be a codimension two submanifold of the contact manifold (P, θ) . Then the pullback to $C^\infty(P)^E$ of the singular Lichnerowicz cocycle λ_Q^μ on $\mathfrak{X}(P, \mu)$ by the map ζ in (4.3) is given by*

$$(\zeta^* \lambda_Q^\mu)(f, g) = \sigma_Q(f, g) + \frac{1}{n+1} \delta \rho_Q(f, g),$$

where the 2-cocycle σ_Q and the 1-cochain ρ_Q on $C^\infty(P)^E$ are given by

$$(4.5) \quad \sigma_Q(f, g) := \int_Q gdf \wedge (d\theta)^{n-1} / (n-1)!,$$

$$(4.6) \quad \rho_Q(h) := - \int_Q h\theta \wedge (d\theta)^{n-1} / (n-1)!.$$

Proof. We calculate $\zeta^* \lambda_Q^\mu(f, g) = \lambda_Q^\mu(\zeta_f, \zeta_g) = \int_Q i_{\zeta_g} i_{\zeta_f} (\theta \wedge (d\theta)^n) / (n+1)!$. First, note that

$$(4.7) \quad \begin{aligned} i_{\zeta_g} i_{\zeta_f} (\theta \wedge (d\theta)^n / n!) &= i_{\zeta_g} (f(d\theta)^n / n! + \theta \wedge df \wedge (d\theta)^{n-1} / (n-1)!) \\ &= -(fdg - gdf) \wedge (d\theta)^{n-1} / (n-1)! \\ &\quad + \{f, g\} \theta \wedge (d\theta)^{n-1} / (n-1)! \\ &\quad - \theta \wedge df \wedge dg \wedge (d\theta)^{n-2} / (n-2)!. \end{aligned}$$

Expanding $d(\theta \wedge (fdg - gdf) \wedge (d\theta)^{n-2})$, we obtain

$$\begin{aligned} \theta \wedge df \wedge dg \wedge (d\theta)^{n-2} / (n-2)! &= \frac{1}{2}(n-1)(fdg - gdf) \wedge (d\theta)^{n-1} / (n-1)! \\ &\quad - \frac{1}{2}d(\theta \wedge (fdg - gdf) \wedge (d\theta)^{n-2} / (n-2)!). \end{aligned}$$

Inserting this into (4.7) yields

$$\begin{aligned}
 i_{\zeta_g} i_{\zeta_f} (\theta \wedge (d\theta)^n / n!) &= -(n + 1) \frac{1}{2} (fdg - gdf) \wedge (d\theta)^{n-1} / (n - 1)! \\
 &\quad + \{f, g\} \theta \wedge (d\theta)^{n-1} / (n - 1)! \\
 &\quad + \frac{1}{2} d (\theta \wedge (fdg - gdf) \wedge (d\theta)^{n-2} / (n - 2)!).
 \end{aligned}$$

Since the value of $\lambda_Q^\mu(\zeta_f, \zeta_g)$ is obtained by integrating the above expression over Q and dividing by $n + 1$, the last term vanishes (Q is closed), the middle term yields a multiple of the coboundary $\delta\rho_Q(f, g) = \rho_Q(-\{f, g\})$, and the first term yields the cocycle $\sigma_Q(f, g)$. □

In particular the classes $\zeta^*[\lambda_Q^\mu]$ and $[\sigma_Q]$ in $H^2(C^\infty(P)^E, \mathbb{R})$ coincide.

Remark 4.2 (Regular contact manifolds). For us, the motivating example is the total space (P, θ) of a prequantum S^1 -bundle $\pi: P \rightarrow M$ over a compact, symplectic manifold (M, ω) . These are called *regular* or *Boothby-Wang* contact manifolds [2]. The top form

$$\mu = \frac{1}{(n + 1)!} \theta \wedge (d\theta)^n = \frac{1}{(n + 1)!} \theta \wedge \pi^* \omega^n$$

is a volume form, and the Reeb vector field $E \in \mathfrak{X}(P)$ coincides with the infinitesimal generator of the principal S^1 -action. The group of strict contactomorphisms coincides with the quantomorphism group, hence it is a Fréchet Lie group [21, 23].

The pullback by π is an isomorphism between the Poisson Lie algebra $C^\infty(M)$ and the Lie algebra $C^\infty(P)^E$ with Lie bracket (4.4). Under the identification $C^\infty(M) \simeq C^\infty(P)^E$, the isomorphisms (2.3) and (4.3), both denoted by ζ , coincide. Moreover, the cocycle σ_Q in (4.5) can be identified with the cocycle ψ_{π_*Q} on the Poisson Lie algebra, determined by the singular $2n - 1$ cycle $C = \pi_*Q$ in M by the formula [9]:

$$\psi_C(f, g) = \int_C gdf \wedge \omega^{n-1} / (n - 1)!, \quad f, g \in C^\infty(M).$$

More details will be given in Section 5.

4.2. Exact strict contactomorphisms

Suppose that G is a locally convex Lie group that acts smoothly and effectively on P by exact strict contact transformations. Its Lie algebra \mathfrak{g} is then a subalgebra of $\mathfrak{X}(P, \theta) \simeq C^\infty(P)^E$. We investigate the integrability of

the pullback $\iota^*[\sigma_Q] \in H^2(\mathfrak{g}, \mathbb{R})$ along the inclusion $\iota: \mathfrak{g} \hookrightarrow C^\infty(P)^E$, where σ_Q is the cocycle $\sigma_Q(f, g) = \int_Q gdf \wedge (d\theta)^{n-1}/(n-1)!$ of (4.5). For general contact manifolds, we have to impose the condition of exactness so that we can make use of Theorem 3.2. However, in the important special case of regular contact manifolds, we will show that the exactness condition is automatically satisfied.

Analogous to the group $\text{Diff}_{\text{ex}}(P, \theta) := \text{Diff}(P, \theta)_0 \cap \text{Diff}_{\text{ex}}(P, \mu)$ of exact strict contact transformations, we define the Lie algebra of *exact strict contact vector fields* by

$$\mathfrak{X}_{\text{ex}}(P, \theta) := \mathfrak{X}(P, \theta) \cap \mathfrak{X}_{\text{ex}}(P, \mu).$$

For every isotopy $\varphi_t \in \text{Diff}(P)$, starting at the identity and determined by the time dependent vector field $X_t \in \mathfrak{X}(P)$, we have $\varphi_t \in \text{Diff}_{\text{ex}}(P, \theta)$ for all t if and only if $X_t \in \mathfrak{X}_{\text{ex}}(P, \theta)$ for all t .

Lemma 4.3. *The function space*

$$C_0^\infty(P)^E := \{f \in C^\infty(P)^E; \quad f(d\theta)^n \in d\Omega^{2n-1}(P)\}$$

is a Lie subalgebra of $C^\infty(P)^E$ of finite codimension $\leq \dim H^{2n}(P, \mathbb{R})$, isomorphic under $f \mapsto \zeta_f$ to the Lie algebra $\mathfrak{X}_{\text{ex}}(P, \theta)$ of exact strict contact vector fields.

Proof. As $f(d\theta)^n$ is closed for all $f \in C^\infty(P)^E$, we can define the linear map

$$(4.8) \quad C^\infty(P)^E \rightarrow H^{2n}(P, \mathbb{R}), \quad f \mapsto \frac{1}{n!}[f(d\theta)^n]$$

with kernel $C_0^\infty(P)^E$. This is a Lie algebra homomorphism, as $\{f, g\}(d\theta)^n = ndf \wedge dg \wedge (d\theta)^{n-1}$ is exact. It coincides, under the identification (4.3), with the flux homomorphism (3.2) for the volume form μ restricted to $\mathfrak{X}(P, \theta) \simeq C^\infty(P)^E$, as

$$\begin{aligned} (n+1)![i_{\zeta_f}\mu] &= [i_{\zeta_f}(\theta \wedge (d\theta)^n)] \\ &= [f(d\theta)^n - n(df) \wedge \theta \wedge (d\theta)^{n-1}] \\ &= [f(d\theta)^n - nd(f\theta \wedge (d\theta)^{n-1}) + nf(d\theta)^n] \\ &= (n+1)[f(d\theta)^n]. \end{aligned}$$

It follows that the kernel $C_0^\infty(P)^E$ of (4.8) is identified under ζ with the Lie algebra $\mathfrak{X}(P, \theta) \cap \mathfrak{X}_{\text{ex}}(P, \mu) = \mathfrak{X}_{\text{ex}}(P, \theta)$. □

Proposition 4.4. *If the contact manifold (P, θ) is regular, i.e. the total space of a prequantum bundle $\pi : P \rightarrow M$, then the Lie algebras of strict contact and exact strict contact vector fields coincide: $\mathfrak{X}(P, \theta) = \mathfrak{X}_{\text{ex}}(P, \theta)$. Moreover, the group of exact strict contact diffeomorphisms is precisely the connected component of the quantomorphism group:*

$$\text{Diff}_{\text{ex}}(P, \theta) = \text{Diff}(P, \theta)_0.$$

Proof. Any $f \in C^\infty(P)^E$ is of the form $\pi^* \bar{f}$ for a smooth function \bar{f} on the compact symplectic manifold M . If we write $\bar{f} = \bar{f}_0 + c$ with $\int_M \bar{f}_0 \omega^n = 0$, then $\bar{f}_0 \omega^n = d\gamma$ is exact, so that the form $f(d\theta)^n = c(d\theta)^n + \pi^*(\bar{f}_0 \omega^n) = cd(\theta \wedge (d\theta)^{n-1}) + d\pi^* \gamma$ is exact as well. We thus find $C_0^\infty(P)^E = C^\infty(P)^E \simeq C^\infty(M)$ and the conclusion follows. □

The following example shows that for contact manifolds that are not regular, the Lie algebra $C_0^\infty(P)^E$ can be strictly smaller than $C^\infty(P)^E$.

Example 4.5. An example of a non-regular contact form on the 3-torus $P = \mathbb{T}^3$ is $\theta = \cos z dx + \sin z dy$. The orbits of the Reeb vector field $E = \cos z \partial_x + \sin z \partial_y$ determine constant slope foliations on each 2-torus of constant z . We show that the Lie algebra $C_0^\infty(P)^E \simeq \mathfrak{X}_{\text{ex}}(P, \theta)$ has codimension two in $C^\infty(P)^E \simeq \mathfrak{X}(P, \theta)$.

We use the inclusion $\mathfrak{X}(P, \theta) \subset \mathfrak{X}(P, \mu)$. Any divergence free vector field $X \in \mathfrak{X}(P, \mu)$ is the sum $X = X_0 + X_\alpha$ of an exact divergence free vector field X_α with potential 1-form $\alpha = A dx + B dy + C dz$ and a constant vector field $X_0 = a \partial_x + b \partial_y + c \partial_z$. With volume form $\mu = \frac{1}{2} \theta \wedge d\theta = -\frac{1}{2} dx \wedge dy \wedge dz$, we have $X_\alpha = 2(B_z - C_y) \partial_x + 2(C_x - A_z) \partial_y + 2(A_y - B_x) \partial_z$. The vector field X is strict contact if $L_X \theta = 0$, which amounts to

$$\begin{aligned} \sin z(C_{xx} - A_{xz} - A_y + B_x - c) + \cos z(B_{xz} - C_{xy}) &= 0 \\ \cos z(B_{yz} - C_{yy} + A_y - B_x + c) + \sin z(C_{xy} - A_{yz}) &= 0 \\ \cos z(B_{zz} - C_{yz}) + \sin z(C_{xz} - A_{zz}) &= 0. \end{aligned}$$

Thus $a, b \in \mathbb{R}$ are arbitrary, $c = 0$ (as can be seen by integrating the above equations over x and y), and X_α is a strict contact vector field. We find that $\mathfrak{X}(P, \theta)$ is isomorphic to the semidirect product $\text{Span}\{\partial_x, \partial_y\} \ltimes \mathfrak{X}_{\text{ex}}(P, \theta)$, and that the flux homomorphism (3.2) restricted to $\mathfrak{X}(P, \theta)$ has 2-dimensional image generated by $[dy \wedge dz], [dx \wedge dz] \in H^2(P, \mathbb{R})$.

We apply Theorem 3.2 to the contact manifold P with integral volume form $k\nu = \frac{k}{\text{vol}_\mu(P)} \mu$ for $k \in \mathbb{Z}$ and we obtain the following central result.

Theorem 4.6. *Let G be a Lie group acting smoothly and effectively on (P, θ) by exact strict contact transformations. Then the restriction to \mathfrak{g} of the class $\frac{k}{\text{vol}_\mu(P)}[\sigma_Q]$ is integrable to a central Lie group extension of G .*

Proof. Since $G \subseteq \text{Diff}_{\text{ex}}(P, \theta)$ and $\text{Diff}_{\text{ex}}(P, \theta) \subseteq \text{Diff}_{\text{ex}}(P, k\nu)$, the action of G on the connected component \mathcal{M} of $\text{Gr}_{2n-1}(P)$ is Hamiltonian. Theorem 2.1 then yields a Lie group extension

$$S^1 \rightarrow \widehat{G} \rightarrow G.$$

By Theorem 3.2, the corresponding class in $H^2(\mathfrak{g}, \mathbb{R})$ is the pullback along the inclusion $\iota: \mathfrak{g} \hookrightarrow \mathfrak{X}_{\text{ex}}(P, \nu)$ of $[\lambda_Q^k] = \frac{k}{\text{vol}_\mu(P)}[\lambda_Q^\mu]$. By Proposition 4.1, this is the restriction to \mathfrak{g} of the class

$$\frac{k}{\text{vol}_\mu(P)}[\sigma_Q] \in H^2(C_0^\infty(P)^E, \mathbb{R}),$$

where σ_Q is the cocycle $\sigma_Q(f, g) = \int_Q gdf \wedge (d\theta)^{n-1}/(n-1)!$ of (4.5). □

5. The quantomorphism group

Let $P \rightarrow M$ be a prequantum bundle over a compact symplectic manifold (M, ω) , and let $\theta \in \Omega^1(P)$ be a connection 1-form with curvature ω . We apply Theorem 4.6 to the identity component $G = \text{Diff}(P, \theta)_0$ of the quantomorphism group. As its Lie algebra is isomorphic to the Poisson Lie algebra $\mathfrak{g} = C^\infty(M)$, an explicit description of the second Lie algebra cohomology is available [9]. With the above construction, we obtain a lattice of integrable classes in $H^2(C^\infty(M), \mathbb{R})$.

5.1. Cohomology of the Poisson Lie algebra

We describe the second Lie algebra cohomology $H^2(C^\infty(M), \mathbb{R})$ in two different ways: using *Roger cocycles* related to $H^1(M, \mathbb{R})$, and using *singular cocycles* related to $H_{2n-1}(M, \mathbb{R})$. The two pictures are linked by Poincaré duality.

Definition 5.1. The *Roger cocycle* [24, §9] associated to a closed 1-form α on the $2n$ -dimensional symplectic manifold (M, ω) is defined by

$$(5.1) \quad \psi_\alpha(f, g) := \int_M f\alpha(X_g)\omega^n/n! = - \int_M \alpha \wedge fdg \wedge \omega^{n-1}/(n-1)!.$$

They were first defined for surfaces in [11]. The Roger cocycles link the first de Rham cohomology of M to the second Lie algebra cohomology of the Poisson Lie algebra $C^\infty(M)$.

Theorem 5.2. [24, §9][9, §4] *The Roger cocycles ψ_α and $\psi_{\alpha'}$ are cohomologous if and only if $\alpha - \alpha'$ is exact, and the corresponding map $[\alpha] \mapsto [\psi_\alpha]$ is an isomorphism*

$$H^1(M, \mathbb{R}) \xrightarrow{\sim} H^2(C^\infty(M), \mathbb{R}).$$

This shows (cf. [9, §2]) that every (locally convex) central extension $\mathbb{R} \rightarrow \widehat{\mathfrak{g}} \rightarrow C^\infty(M)$ corresponds to a Roger cocycle (5.1) with respect to some linear splitting $C^\infty(M) \rightarrow \widehat{\mathfrak{g}}$. However, the cocycles that come from the Hamiltonian action of $\text{Diff}(P, \theta)$ on $\text{Gr}_{2n-1}(P)$, using splittings of type (2.6), are more closely related to *singular* homology.

Definition 5.3. The *singular cocycle* ψ_C on $C^\infty(M)$, associated to a singular $(2n - 1)$ -cycle C on M , is defined by

$$(5.2) \quad \psi_C(f, g) := \int_C gdf \wedge \omega^{n-1} / (n - 1)!.$$

The Lie algebra 2-cocycles ψ_C and $\psi_{C'}$ are cohomologous if and only if $C - C'$ is a boundary.

Proposition 5.4. *The singular cocycle ψ_C is cohomologous to the Roger cocycle ψ_α if and only if the class $[C] \in H_{2n-1}(M, \mathbb{R})$ is Poincaré dual to $[\alpha] \in H^1(M, \mathbb{R})$. In particular, the map $[C] \mapsto [\psi_C]$ is an isomorphism*

$$(5.3) \quad H_{2n-1}(M, \mathbb{R}) \xrightarrow{\sim} H^2(C^\infty(M), \mathbb{R}).$$

Proof. In view of the fact that $C^\infty(M) \simeq \mathbb{R} \oplus \mathfrak{X}_{\text{ham}}(M, \omega)$ for compact M , this follows from the discussion at the end of [9, §5]. □

5.2. An integrable lattice in $H^2(C^\infty(M), \mathbb{R})$

By applying Theorem 4.6 to the regular contact manifold (P, θ) , we obtain a lattice of integrable classes in the Lie algebra cohomology $H^2(C^\infty(M), \mathbb{R})$. In the following, we denote the lattice of integral classes by $H_*(M, \mathbb{R})_{\mathbb{Z}}$ (homology) or $H^*(M, \mathbb{R})_{\mathbb{Z}}$ (cohomology).

Corollary 5.5 (Singular version). *Let $[C] \in H_{2n-1}(M, \mathbb{R})_{\mathbb{Z}}$ be in the image under π_* of $H_{2n-1}(P, \mathbb{R})_{\mathbb{Z}}$. Then the Lie algebra extension corresponding to the class*

$$\frac{n + 1}{2\pi \text{vol}(M)} [\psi_C]$$

integrates to a central extension of $\text{Diff}(P, \theta)_0$ by S^1 . In this expression, $\text{vol}(M) = \int_M \omega^n/n!$ is the Liouville volume of M , and $[\psi_C] \in H^2(C^\infty(M), \mathbb{R})$ is the singular class (5.2).

Proof. We apply Theorem 4.6 to the unit component $G = \text{Diff}(P, \theta)_0$ of the quantomorphism group. By [21, VIII.4], this is a Fréchet Lie group, with Lie algebra \mathfrak{g} isomorphic to the Poisson Lie algebra $C^\infty(M)$. By Proposition 4.4, the group $\text{Diff}(P, \theta)_0$ coincides with the group $\text{Diff}_{\text{ex}}(P, \theta)$ of exact strict contactomorphisms.

Recall that (P, θ) is a regular contact manifold, with $d\theta = \pi^*\omega$ and $C^\infty(P)^E = \{\pi^*f; f \in C^\infty(M)\}$. If $Q \subseteq P$ is an embedded, closed, oriented submanifold, then the Hamiltonian action of $\text{Diff}(P, \theta)_0$ on the connected component $\mathcal{M} \subseteq \text{Gr}_{2n-1}(P)$ of Q gives rise to a central Lie group extension with Lie algebra cocycle on $C^\infty(M)$ given by

$$\psi(f, g) = \frac{1}{\text{vol}_\mu(P)} \sigma_Q(\pi^*f, \pi^*g) = \frac{1}{\text{vol}_\mu(P)} \int_Q \pi^*(gdf \wedge \omega^{n-1}) / (n - 1)!$$

Expressing the volume of P as $\text{vol}_\mu(P) = \frac{2\pi}{n+1} \text{vol}(M)$, we find

$$\psi(f, g) = \frac{n + 1}{2\pi \text{vol}(M)} \psi_{\pi_*Q}(f, g),$$

where π_*Q is the pushforward along $\pi: P \rightarrow M$ of the singular $(2n - 1)$ -cycle represented by the embedded closed submanifold $Q \subseteq P$, and ψ_{π_*Q} is the singular cocycle of (5.2). By Remark 3.3, every class in $H^2(P, \mathbb{R})_{\mathbb{Z}}$ can be represented by an oriented, embedded submanifold Q , so with $[C] = \pi_*[Q]$ in $H_{2n-1}(M, \mathbb{R})_{\mathbb{Z}}$, the result follows. □

Remark 5.6 (Triviality of Lie algebra extensions). Note that from the above proof, it follows that the Lie algebra extension corresponding to the Hamiltonian action of $\text{Diff}(P, \theta)_0$ on the connected component \mathcal{M} of Q in $\text{Gr}_{2n-1}(P)$ is trivial if and only if $[\psi_{\pi_*Q}] \in H^2(C^\infty(M), \mathbb{R})$ is zero. By Proposition 5.4, this is the case if and only if $\pi_*[Q] \in H_{2n-1}(M, \mathbb{R})$ vanishes.

Using Poincaré duality, we translate this to Roger cocycles and de Rham cohomology. For a smooth map $f: M \rightarrow N$, we denote by

$$f_! : H^*(M, \mathbb{Z}) \rightarrow H^*(N, \mathbb{Z})$$

the map that corresponds to $f_*: H_*(M, \mathbb{Z}) \rightarrow H_*(N, \mathbb{Z})$ under Poincaré duality. For the prequantum bundle $\pi: P \rightarrow M$, the induced map $\pi_! : H^k(P, \mathbb{R})_{\mathbb{Z}} \rightarrow H^{k-1}(M, \mathbb{R})_{\mathbb{Z}}$ on integral classes in de Rham cohomology is fiber integration.

Corollary 5.7 (de Rham version). *For every class $[\alpha]$ in the sublattice $\pi_!(H^2(P, \mathbb{R})_{\mathbb{Z}})$ of $H^1(M, \mathbb{R})$, the Lie algebra extension corresponding to the class*

$$\frac{n+1}{2\pi \text{vol}(M)} [\psi_{\alpha}]$$

in $H^2(C^\infty(M), \mathbb{R})$ integrates to a central extension of $\text{Diff}(P, \theta)_0$ by S^1 .

Proof. The Roger cocycle ψ_{α} is cohomologous to the singular cocycle ψ_C if $[C] \in H_{2n-1}(M, \mathbb{R})_{\mathbb{Z}}$ is Poincaré dual to $[\alpha] \in H^1(M, \mathbb{R})_{\mathbb{Z}}$. If $[C] = \pi_*[Q]$ for $[Q] \in H_{2n-1}(Q, \mathbb{R})_{\mathbb{Z}}$, then $[\alpha] = \pi_![\eta_Q]$ for the Poincaré dual $[\eta_Q]$ of Q , as

$$\int_M \pi_! \eta_Q \wedge \gamma = \int_P \eta_Q \wedge \pi^* \gamma = \int_Q \pi^* \gamma = \int_C \gamma.$$

The result now follows from Corollary 5.5. □

The lattice $\pi_!(H^2(P, \mathbb{R})_{\mathbb{Z}})$, which yields the integrable classes in Lie algebra cohomology, is contained in the lattice $(\pi_! H^2(P, \mathbb{R}))_{\mathbb{Z}}$ of integral classes in $\pi_! H^2(P, \mathbb{R})$. Note however that it can be strictly smaller, cf. §5.3.3. The following proposition is helpful in determining this lattice.

Proposition 5.8. *The image of $\pi_! : H^2(P, \mathbb{Z}) \rightarrow H^1(M, \mathbb{Z})$ is the kernel of taking the cup product with the Euler class $[P] \in H^2(M, \mathbb{Z})$ of the bundle P ,*

$$(5.4) \quad \pi_! H^2(P, \mathbb{Z}) = \{[\alpha] \in H^1(M, \mathbb{Z}); \quad [P] \smile [\alpha] = 0\}.$$

Note that the image of $[P]$ in $H^2(M, \mathbb{R})$ is the class $[\omega]$ of the symplectic form.

Proof. This follows immediately from the Gysin long exact sequence in integral cohomology, associated to the principal S^1 -bundle $P \rightarrow M$,

$$\dots \rightarrow H^2(P, \mathbb{Z}) \xrightarrow{\pi_!} H^1(M, \mathbb{Z}) \xrightarrow{[P] \smile} H^3(M, \mathbb{Z}) \xrightarrow{\pi^*} H^3(P, \mathbb{Z}) \rightarrow \dots,$$

see e.g. [3, §VI.13]. □

Similarly, it follows from Poincaré duality (or the Gysin sequence in homology, [26, §9.3]), that

$$(5.5) \quad \pi_*(H_{2n-1}(P, \mathbb{Z})) = \{[C] \in H_{2n-1}(M, \mathbb{Z}); [C_P] \frown [C] = 0\},$$

where $[C_P] \in H_{2n-2}(M, \mathbb{Z})$, Poincaré dual to $[P]$, is the zero set of a transversal section of the prequantum line bundle $P \times_{S^1} \mathbb{C} \rightarrow M$.

5.3. Examples

We calculate the integrable classes in $H^2(C^\infty(M), \mathbb{R})$ that correspond to our group extensions for a number of explicit examples. For compact surfaces, they span the second Lie algebra cohomology, whereas for compact Kähler manifolds of $\dim_{\mathbb{R}} \geq 4$, they are all trivial. For non-Kähler symplectic manifolds, our method yields non-trivial integrable classes, but they do not necessarily span $H^2(C^\infty(M), \mathbb{R})$. We illustrate this at the hand of Thurston’s nilmanifold, which was historically the first example of a non-Kähler symplectic manifold.

5.3.1. Compact surfaces. Let M be a compact orientable 2-dimensional manifold of genus g , with generators $[a_1], \dots, [a_g]$ and $[b_1], \dots, [b_g]$ of the first homology group $H_1(M, \mathbb{Z})$. A symplectic form $\omega \in \Omega^2(M)$ is prequantizable if and only if $\text{vol}(M) \in \mathbb{Z}$. As $H_3(M, \mathbb{Z}) = \{0\}$, equation (5.5) shows that $\pi_*H_1(P, \mathbb{Z}) = H_1(M, \mathbb{Z})$. From Corollary 5.5, we thus obtain:

Corollary 5.9. *For $k_i, l_i \in \mathbb{Z}$, the Lie algebra cocycles on the Poisson Lie algebra $C^\infty(M)$*

$$(5.6) \quad \psi(f, h) = \frac{1}{\pi \text{vol}(M)} \left(\sum_{i=1}^g k_i \int_{a_i} h df + \sum_{i=1}^g l_i \int_{b_i} h df \right)$$

integrate to central S^1 -extensions of the group $\text{Diff}(P, \theta)_0$ of quantomorphisms.

By Theorem 5.2, the \mathbb{R} -span of this integrable lattice is the full second Lie algebra cohomology of the Poisson Lie algebra $C^\infty(M)$. The above result on integrable cocycles appears to be new.

5.3.2. Kähler manifolds. If M is a prequantizable compact Kähler manifold of dimension $2n$, $n \geq 2$, then the map

$$H^1(M, \mathbb{R}) \rightarrow H^{2n-1}(M, \mathbb{R}); \quad [\alpha] \mapsto [\omega]^{n-1} \wedge [\alpha]$$

is an isomorphism by the Hard Lefschetz Theorem. Since $n \geq 2$, the map $[\alpha] \mapsto [\omega] \wedge [\alpha]$ is injective, so Proposition 5.8 implies that $\pi_1 H^2(P, \mathbb{R})_{\mathbb{Z}} = \{0\}$. From Remark 5.6, we then obtain the following result.

Corollary 5.10. *If M is a compact Kähler manifold of real dimension ≥ 4 , then the central Lie group extension derived from the Hamiltonian action of $\text{Diff}(P, \theta)_0$ on the connected component \mathcal{M} of $\text{Gr}_{2n-1}(P)$ splits at the Lie algebra level.*

In particular, the Hamiltonian action of $\text{Diff}(P, \theta)_0$ on \mathcal{M} lifts to an action of the universal cover $\widetilde{\text{Diff}}(P, \theta)_0$ on the prequantum bundle $\mathcal{P} \rightarrow \mathcal{M}$. For compact Kähler manifolds of $\dim_{\mathbb{R}} \geq 4$, we thus obtain a linear representation of $\widetilde{\text{Diff}}(P, \theta)_0$ on the space of sections of the prequantum line bundle $\mathcal{L} \rightarrow \mathcal{M}$ associated to \mathcal{P} . This marks a qualitative difference with the case $\dim_{\mathbb{R}} = 2$, where the central Lie algebra deformation (5.6) occurs.

5.3.3. Thurston’s nilmanifold. A nilmanifold $M = \Gamma \backslash N$ is a compact homogeneous space for a connected nilpotent Lie group N . Without loss of generality, one may assume that N is 1-connected, and that $\Gamma \subseteq N$ is discrete and co-compact [16]. If \mathfrak{n} is the Lie algebra of N , then by [20], the inclusion $\bigwedge \mathfrak{n}^* \hookrightarrow \Omega(M)$ as left invariant forms yields an isomorphism between the Lie algebra cohomology $H^*(\mathfrak{n}, \mathbb{R})$ of \mathfrak{n} and the de Rham cohomology $H^*(M, \mathbb{R})$ of M . This remains true over rings of integers localized at small primes [14].

To illustrate that nontrivial lattices of integrable cocycles for $C^\infty(M)$ exist in dimension ≥ 4 , we consider the quotient $M_r = \Gamma \backslash N$ with $N = \text{Heis}(\mathbb{R}, r) \times \mathbb{R}$ and $\Gamma = \text{Heis}(\mathbb{Z}, r) \times \mathbb{Z}$, where $\text{Heis}(R, r)$ is the Heisenberg group over the ring R at level $r \in \mathbb{N}$,

$$\text{Heis}(R, r) := \left\{ \begin{pmatrix} 1 & u & h/r \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}; u, v, h \in R \right\}.$$

For $r = 1$, this is Thurston’s symplectic manifold M^4 [27].

The Lie algebra \mathfrak{n} is generated by x, p, h and z , with h, z central and $[x, p] = rh$. The left invariant forms corresponding to the dual basis are $x^* = du, p^* = dv, z^* = dz$ and $h^* = dh + rudv$. The differential $\delta: \bigwedge \mathfrak{n}^* \rightarrow \bigwedge \mathfrak{n}^*$ in (2.5) is determined by its action on the generators: $\delta h^* = rx^* \wedge p^*$ and $\delta x^* = \delta p^* = \delta z^* = 0$.

The cohomology of the Eilenberg-MacLane space $M_r \simeq K(\Gamma, 1)$ is readily calculated from $H^*(B\Gamma, \mathbb{Z})$. From [14, §6.1] and the Künneth formula, one finds

$$H^*(M_r, \mathbb{Z}) \simeq H^*(\mathfrak{n}, \mathbb{Z}) = \bigwedge [x^*, p^*, z^*, x^* \wedge h^*, p^* \wedge h^*] / (rx^* \wedge p^*).$$

For $a, b \in \mathbb{Z} - \{0\}$, we define the (integral) symplectic form $\omega_{ab} \in \Omega^2(M_r)$ by

$$\omega_{ab} = ah^* \wedge x^* + bz^* \wedge p^* = a(dh \wedge du - rudu \wedge dv) + bdz \wedge dv.$$

It determines the Euler class $[P_{abc}] \in H^2(M_r, \mathbb{Z})$ of the prequantum line bundle only up to torsion;

$$[P_{abc}] = ah^* \wedge x^* + bz^* \wedge p^* + cx^* \wedge p^*,$$

where $c \in \{0, \dots, r - 1\}$ labels the different prequantum line bundles with the same curvature class $[\omega_{ab}] \in H^2(M_r, \mathbb{R})$. To determine the kernel

$$K := \text{Ker} \left([P_{abc}] \smile \cdot : H^1(M_r, \mathbb{Z}) \rightarrow H^3(M_r, \mathbb{Z}) \right) = \pi_! H^2(P_{abc}, \mathbb{Z})$$

of the cup product with the Euler class, note that for $\gamma = tx^* + sp^* + \lambda z^*$,

$$[P_{abc}] \smile \gamma = (\lambda c - tb)z^* \wedge x^* \wedge p^* + (as)h^* \wedge x^* \wedge p^* + (\lambda a)h^* \wedge x^* \wedge z^*.$$

Since $a \neq 0$ and $\delta h^* = rx^* \wedge p^*$ represents the zero class in cohomology, we find that $[P_{abc}] \smile \gamma = 0$ if and only if $\lambda = 0$ and $[as] = [tb] = 0$ in $\mathbb{Z}/r\mathbb{Z}$, or, equivalently, if there exist $k_x, k_p \in \mathbb{Z}$ with $t = k_x r / \text{gcd}(r, b)$ and $s = k_p r / \text{gcd}(r, a)$. It follows that

$$K = \{k_x(r/\text{gcd}(r, b))x^* + k_p(r/\text{gcd}(r, a))p^*; \quad k_x, k_y \in \mathbb{Z}\}.$$

From Corollary 5.7 with $n = 2$ and $\text{vol}(M_r) = ab$, we then obtain the following lattice of integrable classes in second Lie algebra cohomology:

Corollary 5.11. *For (M_r, ω_{ab}) and $k_x, k_p \in \mathbb{Z}$, the 2-cocycles*

$$\begin{aligned} \psi(f, g) &= \frac{3rk_p}{2\pi b \operatorname{gcd}(r, a)} \int_M f dg \wedge du \wedge dv \wedge dh \\ &+ \frac{3rk_x}{2\pi a \operatorname{gcd}(r, b)} \int_M f dg \wedge du \wedge dv \wedge dz \end{aligned}$$

for the Poisson Lie algebra $C^\infty(M_r)$ are integrable to the identity component of the quantomorphism group $\operatorname{Diff}(P_{abc}, \theta)_0$.

Since $H^2(C^\infty(M_r), \mathbb{R}) \simeq H^1(M_r, \mathbb{R})$ by Theorem 5.2, we find a 2-dimensional plane spanned by integrable classes in this 3-dimensional cohomology space.

6. The Hamiltonian group

In this final section, we briefly describe how the central extensions of the quantomorphism group $\operatorname{Diff}(P, \theta)_0$, obtained in Corollary 5.7, can be pulled back by a homomorphism $\operatorname{Ham}(M, \omega) \rightarrow \operatorname{Diff}(P, \theta)_0$. This yields central S^1 -extensions of the universal covering group $\widetilde{\operatorname{Ham}}(M, \omega)$.

The homomorphism $\widetilde{\operatorname{Ham}}(M, \omega) \rightarrow \operatorname{Diff}(P, \theta)_0$ is obtained as follows. Since M is compact, the Kostant-Souriau extension (2.4) is split ([9, Corollary 3.5]) by the Lie algebra homomorphisms

$$\mathbb{R} \xleftrightarrow{\rho} C^\infty(M) \xleftrightarrow{\kappa} \mathfrak{X}_{\operatorname{ham}}(M),$$

defined by

$$\rho(f) := \frac{1}{\operatorname{vol}(M)} \int_M f \omega^n / n!, \quad \kappa(X_f) := f - \rho(f).$$

By Lie's Second Theorem for regular Lie groups [13, Thm. 40.3], these Lie algebra homomorphisms integrate to group homomorphisms

$$\mathbb{R} \xrightarrow{R} \widetilde{\operatorname{Diff}}(P, \theta)_0 \xrightarrow{K} \widetilde{\operatorname{Ham}}(M, \omega)$$

on the universal covering groups. This yields the following commutative diagram:

$$\begin{array}{ccccc}
 & & & & \widehat{\text{Diff}}(P, \theta)_0 \\
 & & & & \downarrow \\
 \mathbb{R} & \xleftarrow{R} & \widetilde{\text{Diff}}(P, \theta)_0 & \xrightarrow{\text{Pr}} & \text{Diff}(P, \theta)_0 \\
 & & \downarrow \uparrow K & \nearrow & \downarrow \\
 & & \widehat{\text{Ham}}(M, \omega) & \longrightarrow & \text{Ham}(M, \omega).
 \end{array}$$

The pullback along the homomorphism $\text{Pr} \circ K: \widehat{\text{Ham}}(M, \omega) \rightarrow \widetilde{\text{Diff}}(P, \theta)_0$ of a central S^1 -extension $\widehat{\text{Diff}}(P, \theta)_0 \rightarrow \text{Diff}(P, \theta)_0$ then yields a central Lie group extension $\widehat{H} \rightarrow \widehat{\text{Ham}}(M, \omega)$ by S^1 ,

$$\begin{array}{ccc}
 \widehat{H} & \dashrightarrow & \widehat{\text{Diff}}(P, \theta)_0 \\
 \downarrow & & \downarrow \\
 \widehat{\text{Ham}}(M, \omega) & \xrightarrow{\text{Pro}K} & \text{Diff}(P, \theta)_0.
 \end{array}$$

If ψ is the cocycle of $C^\infty(M)$ corresponding to the central extension $\widehat{\text{Diff}}(P, \theta)_0$ of $\text{Diff}(P, \theta)_0$, then $\kappa^*\psi$ is the corresponding Lie algebra cocycle on $\mathfrak{X}_{\text{ham}}(M)$. The pullback by κ of the Roger cocycle (5.1) is

$$(\kappa^*\psi_\alpha)(X_f, X_g) = \int_M f\alpha(X_g)\omega^n/n!,$$

and the pullback of the singular cocycle (5.2) is

$$(\kappa^*\psi_C)(X_f, X_g) = \int_C gdf \wedge \omega^{n-1}/(n-1)!.$$

Both expressions are independent of the choice of Hamiltonian functions.

Proposition 6.1. *If the 2-cocycle ψ on $C^\infty(M)$ can be integrated to a central extension of the group of quantomorphisms $\text{Diff}(P, \theta)_0$, then $\kappa^*\psi$ can be integrated to a central extension of the universal covering group $\widehat{\text{Ham}}(M, \omega)$ of the group of Hamiltonian diffeomorphisms.*

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DELFT INSTITUTE OF APPLIED MATHEMATICS
DELFT UNIVERSITY OF TECHNOLOGY
2628 XE DELFT, THE NETHERLANDS
E-mail address: B.Janssens@tudelft.nl

DEPARTMENT OF MATHEMATICS, WEST UNIVERSITY OF TIMIȘOARA
300223–TIMIȘOARA ROMANIA
E-mail address: cornelia.vizman@e-uvt.ro

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