Complete integrability from Poisson-Nijenhuis structures on compact hermitian symmetric spaces

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Poisson-Nijenhuis (PN) structures have been proven to be relevant for the quantization of Poisson manifolds, through the notion of multiplicative integrable model on the symplectic groupoid. We study in this paper a class of PN structures defined by the compatible Bruhat-Poisson structure and KKS symplectic form on compact hermitian symmetric spaces. We determine the spectrum of the Nijenhuis tensor and prove complete integrability. In the case of Grassmannians, this leads to a bihamiltonian approach to Gelfand-Tsetlin variables. Our results provide a tool for the quantization of the Bruhat-Poisson structure on compact hermitian symmetric spaces.

1. Introduction

Flag manifolds can be considered as homogeneous spaces of compact matrix groups; when considered as coadjoint orbits they are endowed with the *Kirillov-Konstant-Souriau* symplectic form Ω_{kks}. By fixing the standard Poisson-Lie structure on the matrix group, the quotient map induces the *Bruhat-Poisson* structure π_0. It was shown in [11] that the two Poisson structures, the inverse of the KKS symplectic form and the Bruhat-Poisson, are compatible, *i.e.* their Schouten bracket vanishes, if and only if the flag manifold is a compact hermitian symmetric space. This fact implies that there exists a *Poisson-Nijenhuis* structure and most importantly there exists an integrable model admitting a *bihamiltonian* description. In this paper we compute the eigenvalues of the Nijenhuis operator $N = \pi_0 \circ \Omega_{kks}$ for the cases of classical groups; these eigenvalues give a specific choice of action variables.

In [11] and in [5] it was shown that for complex projective spaces these eigenvalues are given by the hamiltonians corresponding to fixing a certain basis of the torus; in particular it was noticed that they are actually the Gelfand-Tsetlin variables. Moreover it was announced but not proved that this is true for all Grassmannians. This paper aims to fill this gap and generalize to the other cases.

Our motivation for understanding the properties of this integrable model comes from the problem of quantizing the symplectic groupoid integrating the Bruhat-Poisson structure. This project was started in [1] for $\mathbb{C}P_1$ and developed in [2] for $\mathbb{C}P_n$. The main idea is that thanks to the groupoid structure we can use polarizations of the symplectic groupoid that are quite singular from the point of view of geometric quantization: indeed we can consider real polarizations that induce on the space of lagrangian leaves the structure of topological groupoid. This is enough for defining the convolution algebra from the groupoid of Bohr-Sommerfeld leaves (provided it admits a Haar system, which is true if, for instance, it is étale). This observation led us in [2] to introduce the notion *multiplicative integrability of the modular function*. The modular function is the groupoid cocycle that integrates the

modular vector field of the underlying Poisson manifold: it measures the non invariance of a given volume form with respect to hamiltonian transformations. The vector field (and so the integrated function) depends on the choice of a volume form but its cohomology class is independent. We require that the modular function be integrable in the usual dynamical sense but the hamiltonians in involution must be compatible with the groupoid structure in such a way that the contour level sets inherits the structure of topological groupoid. The bihamiltonian system on the projective space provides us with such a system: the modular vector field with respect to the symplectic volume form is the first hamiltonian vector field of the fundamental Lenard hierarchy. The hamiltonians can be lifted to the symplectic groupoid and give the multiplicative integrability of the modular function: the procedure is general but, in the form stated in [2], it requires that the eigenvalues be global smooth functions. This is true in the projective case but not in the general Grassmannians. This problem needs a more intrinsic understanding of the polarization and will be addressed in a separate publication.

Let us briefly describe the content of the paper. Let M_ϕ be a compact hermitian symmetric space that we see as a G-hamiltonian space (let $\mathfrak{g} = \mathrm{Lie}G$); ϕ denotes the non–compact root of the Dynkin diagram of $\mathfrak{g}$ associated to the symmetric space. Our strategy for diagonalizing the Nijenhuis tensor N_ϕ consists first in proving Proposition 6.1, where we show that the eigenvalues of every matrix valued function $\mathcal{M}$ solving the *master equation*

$$N_\phi^* d\mathcal{M} = d\mathcal{M}^- \mathcal{M} + \mathcal{M} d\mathcal{M}^+ + r d\mathcal{M},$$

define Nijenhuis eigenvalues. See the statement of Proposition 6.1 for the explanation of symbols. In Theorem 6.2 we introduce the basic solution of the master equation given by the moment map μ of the $\mathfrak{g}$-action in a representation R that is ϕ-decomposable (see Definition 6.1). This representation can be chosen as the fundamental representation in all cases but for $M_\phi = SO(n+2)/SO(n) \times SO(2)$, where we have to choose the spin representation. Since M_ϕ is a G-adjoint orbit, its eigenvalues are constant and we don't get Njienhuis eigenvalues directly from it. Nevertheless, we get the non–trivial solutions to the master equation by a reduction procedure. Indeed, in Subsection 6.3 we introduce case by case a chain of nested subalgebras

$$(1) \qquad\qquad \mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \cdots \supset \mathfrak{g}_n = 0$$

together with representation R_k of $\mathfrak{g}_k$, such that the moment map of $\mathfrak{g}_k$ in the representation R_k solves the master equation. Theorem 6.2 relies on

an explicit form of the contravariant connection that encodes the Poisson structure of vector bundles associated to the G-principal bundle on M_ϕ.

In order to show that the obtained eigenvalues are all and that the Nijenhuis operator is of maximal rank the essential ingredient is the concept of *collective complete integrability*. This is a method developed in [8–10] for constructing integrable models. One can consider the algebra of collective hamiltonians $F(\mathfrak{g}_1, \ldots, \mathfrak{g}_n)$ generated by the invariant functions on $\mathfrak{g}_i^*$ pulled back through the moment map. They are in involution and, if the above chain of nested subalgebras satisfies the hypothesis of Proposition 2.1, define an integrable model. The most famous integrable model of this form is the Gelfand-Tsetlin model on flag manifolds. The last step is then to prove integrability of the collective hamiltonians associated to the chain (1). Since the Nijenhuis eigenvalues are a specific choice of action variables for these integrable model, the Nijenhuis tensor is of maximal rank. We call the image of the Nijenhuis eigenvalues the *bihamiltonian polytope*. We determine these polytopes case by case. Moreover let us stress that, thanks to the bihamiltonian description, the collective hamiltonians are a commutative algebra also with respect to the Bruhat-Poisson structure. When M_ϕ is the Grassmannian $Gr(k, n)$ then we get the Gelfand-Tsetlin model whose integrability is well established since [10]. To the best of our knowledge, in the other cases $M_\phi = Sp(n)/U(n), SO(2n)/U(n), SO(n+2)/SO(n) \times SO(2)$ we get new integrable models and so considerable time is spent in proving integrability and describing the image of the moment map. This is the content of Theorems 7.2, 8.1, 9.1 and 10.1. These results should be compared with the equivalent problem in representation theory of finding Gelfand-Tsetlin bases in finite dimensional irreducible representations of classical Lie algebras. See the introduction of [16] for a general review of the subject, here we remark that the models considered in this paper correspond to those representations obtained by geometric quantization of compact hermitian symmetric spaces.

The plan of the paper is the following. In Section 2 we recall basic facts about Poisson geometry, Poisson-Lie groups; in particular we recall the notion of Poisson vector bundle that will be an important tool in our proof. We recall basic notions of Poisson-Nijenhuis structures as well and we briefly sketch the construction of collective integrable models. In Section 3 we recall basic facts of compact hermitian symmetric spaces and fix notations. In Section 4 we define the Bruhat-Poisson structure. In Section 5 we give an explicit expression of the contravariant connection defined on associated vector bundles. In Section 6 we introduce the Poisson Nijenhuis structure and develop the tools needed for the diagonalization. We introduce the master equation in Proposition 6.1 and prove that the moment map μ

solves it in Theorem 6.2. Finally, in Subsection 6.3 we introduce the chain of subalgebras giving the non–trivial solutions of the master equation. The proof that the collective hamiltonians associated to the chains of subalgebras define a completely integrable model is left to Section 7 for $Gr(k,n)$, Section 8 for $Sp(n)/U(n)$, Section 9 for $SO(2n)/U(n)$ and Section 10 for $SO(n+2)/SO(2) \times SO(2)$.

Notations. We will denote by $\mathfrak{g}$ the compact form of a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{t} = \mathfrak{t}_{\mathbb{C}} \cap \mathfrak{g}$, where $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is a choice of the Cartan subalgebra; let Φ denote the roots and $\mathfrak{g}_{\alpha}$ with $\alpha \in \Phi$ the root space. Let $\Phi^{\pm}$ be a choice of positive (negative) roots and $\Pi = \{\alpha_1, \ldots\}$ denote the simple roots. We denote by $\mathfrak{t}_+^*$ the fundamental Weyl chamber. When we consider the classical cases $\mathfrak{g} = \mathfrak{su}(n), \mathfrak{so}(n), \mathfrak{sp}(n)$, we identify $\mathfrak{g}$ with an algebra of matrices and we denote by $f_{\mathfrak{g}}$ the corresponding representation, and we refer to it as the fundamental representation. We denote by $0_{\mathfrak{g}}$ the one dimensional trivial representation. We denote by G the corresponding matrix group integrating it.

We denote by $a^{\dagger}$ the hermitian conjugate of a complex matrix a.

We recall that a simple root α_i is non–compact with respect to $(\mathfrak{g}, \mathfrak{t})$ if the positive roots are all of the form $\alpha = \sum_{j \neq i} c^j \alpha_j$ (of compact type) or $\alpha = \alpha_i + \sum_{j \neq i} c^j \alpha_j$ (of non–compact type). In the following, we list all possible non–compact roots.
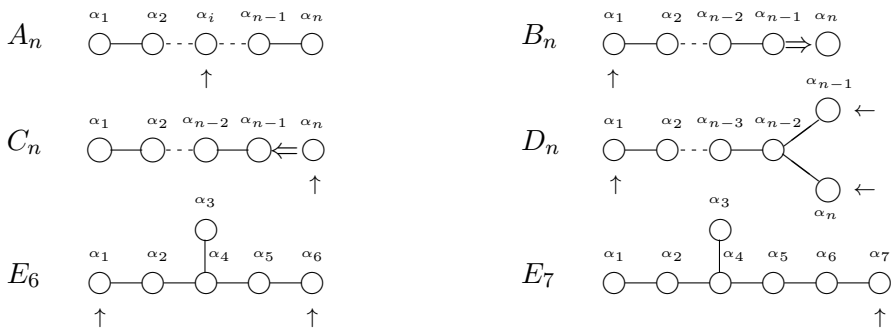

Figure 1: Dynkin diagrams with the non–compact roots marked.

2. Generalities

2.1. Poisson vector bundles

We fix in this Section the conventions and recall basic material about Poisson geometry. Let (M, π) be a Poisson manifold with $\pi \in C^\infty(\Lambda^2 TM)$ denoting the Poisson bivector and $\{f, g\} = \pi^{ij} \partial_i f \partial_j g$ denoting the Poisson bracket between $f, g \in C^\infty(M)$. The Jacobi identity for the Poisson bracket can be expressed as $[\pi, \pi] = 0$, where $[\ ,\ ]$ denotes the Schouten bracket between multivector fields. As a consequence the differential $d_{LP}(-) = [\pi, -]$ squares to zero and defines the Lichnerowicz-Poisson cohomology $H_{LP}(M, \pi)$. Given a volume form V on M, the *modular vector field* with respect to V is $\chi_V = \mathrm{div}_V(\pi)$: it satisfies $d_{LP}(\chi_V) = 0$ and its class in LP cohomology, that does not depend on the choice of the volume form, is called the *modular class*.

A Poisson structure defines an algebroid structure on T^*M that we denote by $T^*_\pi M$. The anchor is $\pi : T^*_m M \to T_m M$, $m \in M$, defined as $\langle \pi(\alpha_m), \beta_m \rangle = \langle \pi(m), \alpha_m \wedge \beta_m \rangle$, with $\alpha_m, \beta_m \in T^*_m M$; the bracket on $\Omega^1(M)$ is defined as

$$(2) \qquad \{\alpha, \beta\}_\pi = L_{\pi(\alpha)}\beta - L_{\pi(\beta)}\alpha - d\langle \pi, \alpha \wedge \beta \rangle, \qquad \alpha, \beta \in \Omega^1(M).$$

A Lie group (G, π) is called a *Poisson-Lie* group if it is a Poisson manifold such that the multiplication is a Poisson map (with the product Poisson structure on $G \times G$). As a consequence, $\delta_{\mathfrak{g}} : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ defined as $\delta_{\mathfrak{g}}(X) = \frac{d}{dt} r_{\exp(-tX)} \pi(\exp tX)|_{t=0}$, $X \in \mathfrak{g}$, defines a Lie algebra structure on $\mathfrak{g}^*$. We call $(\mathfrak{g}, \delta_{\mathfrak{g}})$ the *Lie bialgebra* of (G, π). Let us assume that G is connected and simply connected; let G^* be the connected and simply connected Lie group integrating $\mathfrak{g}^*$: it can be shown that there exists a canonical Poisson-Lie structure on it, such that $(\mathfrak{g}^*)^* = \mathfrak{g}$ as Lie algebras and G^* is said to be the Poisson-Lie dual of G. The action of (G, π_G) on (M, π_M) is a *Poisson action* if the action seen as a map from $(G \times M, \pi_G \oplus \pi_M)$ to (M, π_M) is a Poisson map. At the infinitesimal level this means that for each $X \in \mathfrak{g}$, denoting with $\ell : X \to \ell_X$ the map associating the corresponding fundamental vector field on M, we have that

$$L_{\ell_X}(\pi_M) = [\ell_X, \pi_M] = \ell(\delta_{\mathfrak{g}}(X)).$$

A subgroup $H \subset G$ is a Poisson-Lie subgroup if it is a Poisson submanifold. Let $\mathfrak{h}$ be the Lie algebra of H. It can be seen easily that, when connected, H is a Poisson-Lie subgroup if and only if $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is an ideal

of the dual Lie algebra $\mathfrak{g}^*$ or equivalently if and only if $\delta_\mathfrak{g}(\mathfrak{h}) \subset \wedge^2 \mathfrak{h}$. Finally, when H is a Poisson-Lie subgroup of (G, π), there is a unique Poisson structure on G/H such that the quotient map $G \to G/H$ is Poisson.

A vector bundle $\mathcal{E}$ over a Poisson manifold (M, π) is a *Poisson vector bundle* if there exists a bracket $\{,\}_\mathcal{E} : C^\infty(M) \otimes \Gamma^\infty(\mathcal{E}) \to \Gamma^\infty(\mathcal{E})$ which turns the smooth sections $\Gamma^\infty(\mathcal{E})$ into a Lie algebra module over $C^\infty(M)$, such that, for each $f, g \in C^\infty(M)$ and $\sigma \in \Gamma^\infty(\mathcal{E})$ we have

    i) $\{f, g\sigma\}_\mathcal{E} = \{f, g\}\sigma + g\{f, \sigma\}_\mathcal{E}$,

    ii) $\{fg, \sigma\}_\mathcal{E} = f\{g, \sigma\}_\mathcal{E} + g\{f, \sigma\}_\mathcal{E}.$

See [6] for a reference. These data can be equivalently encoded in the flat contravariant connection, $\nabla : \Omega^1(M) \otimes \Gamma^\infty(\mathcal{E}) \to \Gamma^\infty(\mathcal{E})$ defined as

$$\nabla_{df}(\sigma) = \{f, \sigma\}_\mathcal{E}, \qquad f \in C^\infty(M), \sigma \in \Gamma^\infty(\mathcal{E}).$$

Another equivalent way of stating the properties of Poisson vector bundle is by saying that ∇ defines a representation of the algebroid $T_\pi^* M$ canonically associated to (M, π) (see [4] for the definition of an algebroid representation). Let (K, π_K) be a Poisson-Lie group and

$$\begin{array}{ccc} (P, \pi_P) & \leftarrow & (K, \pi_K) \\ \downarrow & & \\ (M, \pi_M) & & \end{array}$$

be a *Poisson principal bundle*, that is a principal K-bundle P over M, such that the right action of (K, π_K) on (P, π_P) is a Poisson action and the projection $(P, \pi_P) \to (M, \pi_M)$ is a Poisson map. Let $R : K \to \text{End}V$ be a right representation of K on the vector space V denoted by $v \mapsto vR(k)$ and let $\mathcal{E}_R = P \times_R V$ be the associated vector bundle defined as the quotient of $P \times V$ with respect to the diagonal K–action. We can characterize sections of $\mathcal{E}_R$ as equivariant functions $C^\infty(P, V)^K$, *i.e.* $\sigma \in C^\infty(P, V)^K$ if $\sigma : P \to V$ is such that $\sigma(pk) = \sigma(p)R(k)$, $p \in P$, $k \in K$.

Lemma 2.1. *The bracket*

$$\{f, \sigma\}_{\mathcal{E}_R} \equiv \{f, \sigma\}_P$$

between $f \in C^\infty(M) = C^\infty(P)^K$ and $\sigma \in C^\infty(P, V)^K$ endows the associated vector bundle $\mathcal{E}_R$ of the structure of Poisson vector bundle over (M, π_M).

Proof. Since the right K action on P is Poisson, we have that for each $X \in \mathbf{k} = \mathrm{Lie}K$ (denoting with $r : X \to r_X$ the fundamental vector field of $X \in \mathbf{k}$)

$$r_X(\{f,\sigma\}_P) = \{r_X(f),\sigma\}_P + \{f,r_X(\sigma)\}_P + \langle r(\delta_{\mathbf{k}}(X)), df \wedge d\sigma\rangle$$
$$= \{f,r_X(\sigma)\}_P = \{f,\sigma\}_P R(X),$$

where $\delta_{\mathbf{k}}$ denotes the Lie bialgebra structure of $\mathbf{k}$; the first and the third term of the rhs of the first line vanish since f is invariant with respect to the $\mathbf{k}$ action and the last equality follows from the equivariance of σ with respect to the representation R. $\square$

2.2. Poisson-Nijenhuis structures

A $(1,1)$ tensor $N : TM \to TM$ is called a *Nijenhuis* tensor if it has vanishing Nijenhuis torsion, *i.e.* for any couple (v_1, v_2) of vector fields on M we have

$$T(N)(v_1,v_2) = [Nv_1, Nv_2] - N([Nv_1,v_2] + [v_1, Nv_2] - N[v_1,v_2]) = 0.$$

Given any bivector π, we recall that $\{,\}_\pi$ denotes the antisymmetric bracket on one forms defined in (2). A triple (M,π,N), where (M,π) is a Poisson manifold and N a Nijenhuis tensor is called a *Poisson-Nijenhuis* (PN) manifold if π and N are compatible, *i.e.*

$$N \circ \pi = \pi \circ N^*, \quad \{\alpha,\beta\}_{N\pi} = \{N^*\alpha,\beta\}_\pi + \{\alpha, N^*\beta\}_\pi - N^*\{\alpha,\beta\}_\pi,$$

for $\alpha,\beta \in \Omega^1(M)$, where N^* denotes the dual map.

We will consider the case, where $\pi = \Omega^{-1}$ is the inverse of a symplectic form and there exists a compatible Poisson structure π_0, *i.e.* such that $[\Omega^{-1},\pi_0] = 0$, or equivalently there is a pencil of Poisson structures $\pi_t = \pi_0 + t\Omega^{-1}$ for $t \in \mathbb{R}$. In this case $(M,\Omega^{-1}, N = \pi_0 \circ \Omega)$ is a Poisson-Nijenhuis structure (called also ωN-manifold). The PN structures are closely related to integrable systems, see [15] for a general reference to bihamiltonian systems, here we will recall few basic facts.

The *spectral problem* associated to the PN structure is the problem of determining the eigenvalues of N. The eigenspace of N corresponding to eigenvalue λ is the null space of $\pi_0 - \lambda\Omega^{-1}$; since the latter is anti-symmetric, the dimension of the null space is at least 2. We can then conclude that if $\dim M = 2n$ then N can have at most n distinct eigenvalues. We say that the rank is maximal if the distinct eigenvalues are exactly n on a dense open set

of M. We can define a map $J_N : M \to \mathbb{R}^n$ associating to every point $m \in M$ the eigenvalues $(\lambda_1(m), \ldots, \lambda_n(m)) \in \mathbb{R}^n$ and we call it the *bihamiltonian moment map*. There is not of course a unique way of defining it, according to how we enumerate the eigenvalues; one possibility is to order them, but in the examples considered in this paper other choices will be more natural. We call the image of J_N the *bihamiltonian polytope* and denote it with $\mathcal{C}(N)$. Note that for each $t \in \mathbb{R}$, the preimage along J_N of the union of hyperplanes $\mathcal{C}(t) = \bigcup_k \mathcal{C}^{(k)}(t)$, where $\mathcal{C}^{(k)}(t) = \{\lambda \in \mathcal{C}(N) | \lambda_k = -t\}$, is the set of points where the $\pi_t = \pi_0 + t\Omega^{-1}$ has not maximal rank.

A point m is *regular* if $\mathrm{rk}(dJ_N(m)) = n$. If $\{\lambda_i\}$ is a collection of functions that give the eigenvalues of N in a neighborhood of regular points then they satisfy the following equation

$$(3) \qquad\qquad N^* d\lambda_i = \lambda_i d\lambda_i.$$

It can be shown that the eigenvalues λ_i are in involution with respect to both Ω^{-1} and π_0. A collection of smooth functions $\{I_k\}$ satisfies the *Lenard recursion relations* if

$$(4) \qquad\qquad dI_{k+1} = N^* dI_k.$$

As a consequence, the I_k's are in involution with respect to both Ω^{-1} and π_0. A canonical collection of such functions is given by $I_k = \frac{1}{k}\mathrm{Tr}N^k$, $k = 1, \ldots, n$ (this is a consequence of (3)).

The modular vector field of $\pi_t = \pi_0 + t\Omega^{-1}$, $t \in \mathbb{R}$ with respect to the symplectic volume form is independent on t. It is a consequence of Theorem 3.5 of [3] that this modular vector field is the Ω^{-1} hamiltonian vector field of I_1, *i.e.*

$$\chi_\Omega = \mathrm{div}_\Omega \pi_t = \Omega^{-1}(d\mathrm{Tr}N).$$

In general, χ_Ω is only a Poisson vector field with respect to π_t. It is easy to show that $\log \det(N + t)$ is a local hamiltonian for χ_Ω with respect to π_t that is defined on all points such that $-t$ is not an eigenvalue of N.

2.3. Collective complete integrability

We recall here a general method for constructing integrable models, called Thimm method in [8], which we refer for details (see also [9]). Let M be an hamiltonian K-space with moment map $\Phi : M \to \mathbf{k}^*$, where $\mathbf{k} = \mathrm{Lie}K$. An hamiltonian of the form $\Phi^*(c)$ for $c \in C^\infty(\mathbf{k}^*)$ is called *collective*. Any

K-invariant function $f \in C^\infty(M)^K$ Poisson commutes with collective hamiltonians.

Definition 2.2. An hamiltonian K-space (M, Φ) is multiplicity free if one of the following equivalent conditions is satisfied:

   i) the algebra of K-invariant functions $C^\infty(M)^K$ is Poisson commutative;

  ii) for each $\alpha \in \mathbf{k}^*$, denoting with $K_\alpha \subset K$ its stabilizer subgroup with respect to the coadjoint action, the action of K_α on $\Phi^{-1}(\alpha)$ is transitive;

 iii) for each $\alpha \in \mathbf{k}^*$, denoting with $\mathcal{O}_\alpha$ the coadjoint orbit through α, the action of K on $\Phi^{-1}(\mathcal{O}_\alpha)$ is transitive.

The equivalence between properties $(i - iii)$ is shown in [8]. Let us consider the following chain of subalgebras

$$\mathbf{k} \equiv \mathbf{k}_1 \supset \mathbf{k}_2 \supset \cdots \mathbf{k}_k \supset \mathbf{k}_{k+1} = \{0\},$$

and let us denote by $K_i \supset K_{i+1}$ the corresponding chain of subgroups. Let $p_i : \mathbf{k}^* \to \mathbf{k}_i^*$ be the map dual to the inclusion $\mathbf{k}_i \subset \mathbf{k}$; it is easy to see that the invariant functions on $\mathbf{k}_i^*$ pulled back to M with $(p_i \circ \Phi)^*$ form an abelian subalgebra $F(\mathbf{k}_1, \ldots, \mathbf{k}_k) \subset C^\infty(M)$ of the Poisson algebra of functions on M.

Let us denote by $p_{ij} : \mathbf{k}_i^* \to \mathbf{k}_j^*$, $j > i$, the dual map of the inclusion of subalgebras. Every coadjoint orbit $\mathcal{O} \subset \mathbf{k}_i^*$ is a K_j-hamiltonian space with moment map p_{ij}. The following result is proven in [8].

Proposition 2.1. $F(\mathbf{k}_1, \ldots, \mathbf{k}_k)$ *defines a completely integrable model if and only if any coadjoint orbit $\mathcal{O} \subset \mathbf{k}_i^*$ in the image of $p_i \circ \Phi : M \to \mathbf{k}_i^*$ is multiplicity free with respect to the K_{i+1} coadjoint action for each $i = 1, \ldots, k$.*

If the subalgebras $\mathbf{k}_i$ are semisimple then action variables for such an integrable model can be defined as follows. Let $\beta_i : \mathbf{k}_i^* \to (\mathbf{t}_i^*)_+$ be the map that sends each point of $\mathbf{k}_i^*$ to the unique intersection of its K_i-coadjoint orbit with the positive Weyl chamber. This is a continuous map that is smooth in the preimage of the interior of the Weyl chamber. Let $\{\xi_i\}$ be a basis of integral lattice of $\mathbf{t}_i$: then the variables $\lambda_i = \langle \xi_i, \beta_i \circ \mu_{\mathbf{k}_i} \rangle$ are action variables.

The most important example of this construction is the so called *Gelfand-Tsetlin* integrable model on flag manifolds. We will discuss it in the case of Grassmannians in Section 7.

3. Compact hermitian symmetric spaces

Let us first fix the geometrical setting of compact hermitian symmetric spaces that we will need later, see [18].

Let $\phi \in \Pi$ be a non–compact root and Φ_c^+ and Φ_{nc}^+ the positive roots of compact and non–compact type. Let $\mathfrak{h}_\phi \subset \mathfrak{g}$ be the Lie subalgebra defined as

$$\mathfrak{h}_\phi = \mathfrak{t} \oplus_{\alpha \in \Phi_c^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}.$$

and let us denote $H_\phi \subset G$ the closed subgroup integrating it. We denote by $Z(\mathfrak{h}_\phi) \subset \mathfrak{h}_\phi$ the one dimensional center. Let $\rho_\phi \in Z(\mathfrak{h}_\phi)$ be normalized by $\phi(\rho_\phi) = i$. We denote by $\mathfrak{h}_\phi^\perp$ the orthogonal space to $\mathfrak{h}_\phi$ with respect to the Killing form. We have that

$$\mathfrak{h}_\phi^\perp = \oplus_{\alpha \in \Phi_{nc}^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}.$$

By identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ thanks to the Killing form, G/H_ϕ is identified as the adjoint orbit of ρ_ϕ. This fixes the KKS symplectic form Ω_{kks} in such a way that G acts hamiltonially with moment map given by

$$(5) \qquad \mu(g) = g\rho_\phi g^{-1} \quad g \in G.$$

The automorphism $A_\phi = \mathrm{Ad}_{K_\phi}$, where $K_\phi = e^{\pi \rho_\phi}$, satisfies $A_\phi^2 = \mathrm{id}$ so that $K_\phi^2 = e^{2\pi \rho_\phi} \in Z(G)$. We have accordingly that $\mathfrak{h}_\phi$ and $\mathfrak{h}_\phi^\perp$ are the eigenspaces of A_ϕ corresponding to eigenvalue 1 and -1 respectively.

We have also that $K_\phi^2 = e^{2ik_\phi} 1$ so that the fundamental representation decomposes as $V_+ \oplus V_-$ corresponding to the eigenvalues $\pm e^{ik_\phi}$ of K_ϕ. Let us denote by $n_\pm = \dim V_\pm$. Let $R_\pm$ denote both the left representation of H_ϕ on $V_\pm$ and the dual right representation on $V_\pm^*$. Let us consider the homogeneous principal bundle $G \to G/H_\phi$ and let $\mathcal{E}_\pm = G \times_{R_\pm} V_\pm^*$ denote the vector bundles associated to $R_\pm$. Concretely if $V_\pm = M_{n_\pm,1}(\mathbb{C})$ and H_ϕ acts by left matrix multiplication, then H_ϕ acts on $V_\pm^* = M_{1,n_\pm}(\mathbb{C})$ by right matrix multiplication.

For $g \in G \subset M_N(\mathbb{C})$ we get the corresponding decomposition

$$g = \begin{pmatrix} g^{++} & g^{+-} \\ g^{-+} & g^{--} \end{pmatrix},$$

and let

$$\sigma_+(g) = \begin{pmatrix} g^{++} \\ g^{-+} \end{pmatrix} \in M_{N,n_+}(\mathbb{C}), \qquad \sigma_-(g) = \begin{pmatrix} g^{+-} \\ g^{--} \end{pmatrix} \in M_{N,n_-}(\mathbb{C}).$$

We define $\epsilon_\pm(g) = \sigma_\pm(g)\sigma_\pm^\dagger(g) \in M_N(\mathbb{C})$.

Lemma 3.1. *One has $\epsilon_\pm^2 = \epsilon_\pm$. Moreover $\epsilon_+ + \epsilon_- = 1_N$ and*

$$(6) \qquad\qquad \mu = \sigma_+ R_+(\rho_\phi)\sigma_+^\dagger + \sigma_- R_-(\rho_\phi)\sigma_-^\dagger.$$

Proof. The first assertions follow from $\sigma_\pm^\dagger\sigma_\pm = 1_{n_\pm}$ that follows from $g^\dagger g = 1$, the last one is also clear. $\qquad\square$

The idempotents $\epsilon_\pm$ of Lemma 3.1 identify the vector bundles $\mathcal{E}_\pm$ as $\mathrm{Im}\,\epsilon_\pm \subset M_{1,N}(\mathbb{C}) = \mathbb{C}^N$, where $\epsilon_\pm$ acts by right matrix multiplication. Indeed the map $[g,v] \in \mathcal{E}_\pm \to v\,\sigma_\pm^\dagger(g)$, where $v \in M_{1,n_\pm}(\mathbb{C})$, gives the identification.

Let us discuss the various cases.

Example 3.2. **(AIII)** Let $G = SU(n)$ and let us choose as non–compact root the k-th root of the Dynkin diagram. If we choose as Cartan subalgebra the diagonal matrices we then get $H_\phi = S(U(k) \times U(n-k))$ embedded as block diagonal matrices. The symmetric space is the Grassmannian $Gr(k,n)$ of k-vector space inside $\mathbb{C}^n$. We have that

$$\rho_\phi = \frac{i}{n} \begin{pmatrix} (n-k)1_k & 0 \\ 0 & -k1_{n-k} \end{pmatrix}, \qquad K_\phi = e^{i\pi(1-\frac{k}{n})} \begin{pmatrix} 1_k & 0 \\ 0 & -1_{n-k} \end{pmatrix}.$$

Then clearly we get that $R_+ = f_{\mathfrak{u}(k)} \times 0_{n-k}$ and $R_- = 0_k \times f_{\mathfrak{u}(n-k)}$, where f denotes the fundamental representation and 0 the trivial one; $\mathcal{E}_+$ is the rank k tautological vector bundle over $Gr(k,n)$ and $\mathcal{E}_-$ is the rank $n-k$ tautological vector bundle over $Gr(n-k,n) \sim Gr(k,n)$. From (6) we get

$$\mu = i\frac{n-k}{n}\epsilon_+ - i\frac{k}{n}\epsilon_- = i\epsilon_+ - i\frac{k}{n}.$$

Remark 3.3. One can equally write $Gr(k,n) = U(n)/U(k) \times U(n-k)$. Then one can pick

$$\rho_\phi = \frac{i}{2} \begin{pmatrix} 1_k & 0 \\ 0 & -1_{n-k} \end{pmatrix},$$

which shortens some calculations.

Example 3.4. (BDI odd) Let $G = SO(2n + 1)$ and let us consider the first root in the Dynkin diagram, being the unique non–compact root. Let us choose the Cartan subalgebra as n copies of $\mathfrak{so}(2)$, each copy of them embedded in diagonal 2-dimensional block and having zero in the first diagonal entry. Then we have that H_ϕ is $SO(2n - 1) \times SO(2)$ embedded as block diagonal matrices with $SO(2)$ sitting in the lowest right block. The hermitian space is the Grassmannian of oriented real 2-dimensional subspaces of $\mathbb{R}^{2n+1}$. We then have

$$(7) \qquad \rho_\phi = \begin{pmatrix} 0_{2n-1} & 0 \\ 0 & \sigma \end{pmatrix}, \qquad K_\phi = \begin{pmatrix} 1_{2n-1} & 0 \\ 0 & -1_2 \end{pmatrix},$$

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then clearly $R_+ = f_{\mathfrak{so}(2n-1)} \times 0_{\mathfrak{so}(2)}$ and $R_- = 0_{\mathfrak{so}(2n-1)} \times f_{\mathfrak{so}(2)}$. The vector bundle $\mathcal{E}_-$ is the rank 2 tautological vector bundle. From (6) we get that

$$\mu = \sigma_- \sigma \sigma_-^\dagger,$$

so that $\mu^2 = \epsilon_-$.

Example 3.5. (CI) Let $G = Sp(n)$ be the compact symplectic group (denoted sometimes as $USp(n) = Sp(2n, \mathbb{C}) \cap U(2n)$). In this case the only non–compact root is the last one in the Dynkin diagram. The algebra is described as

$$\mathfrak{sp}(n) = \left\{ \begin{pmatrix} A & B \\ -B^\dagger & -A^t \end{pmatrix}, \ A = -A^\dagger, \ B = B^t, \ A, B \in M_n(\mathbb{C}) \right\}.$$

and the stabilizer subgroup is $H_\phi = U(n)$. By choosing the Cartan subalgebra $\mathfrak{t} = \mathbb{R}^n$ embedded as $a \in \mathbb{R}^n \to \mathrm{diag}(ia, -ia)$ we see that $U(n)$ is embedded as

$$H_\phi = U(n) = \left\{ \begin{pmatrix} X & 0 \\ 0 & \bar{X} \end{pmatrix}, \ X \in U(n) \right\}.$$

We then have

$$\rho_\phi = \begin{pmatrix} \frac{i}{2}1_n & 0 \\ 0 & -\frac{i}{2}1_n \end{pmatrix}, \qquad K_\phi = \begin{pmatrix} i\,1_n & 0 \\ 0 & -i\,1_n \end{pmatrix}.$$

The representations $R_+ = f_{\mathfrak{u}(n)}$ and $R_- = \bar{f}_{\mathfrak{u}(n)}$ and

$$\mu = \frac{i}{2}\epsilon_+ - \frac{i}{2}\epsilon_- = i\epsilon_+ - \frac{i}{2}.$$

Example 3.6. (DIII) Let $G = SO(2n)$ and let us consider as non–compact root the last root in the Dynkin diagram. The subgroup is then $U(n)$ and the symmetric space $SO(2n)/U(n)$ is the space of orthogonal complex structures on $\mathbb{R}^{2n}$. Let us choose as Cartan subalgebra

$$\mathfrak{t} = \{ \begin{pmatrix} 0_n & a \\ -a & 0_n \end{pmatrix}, \quad a = \mathrm{diag}(a_1, \dots, a_n), a_i \in \mathbb{R} \}.$$

The subgroup $H_\phi = U(n)$ is then embedded as

$$A + iB \in U(n) \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

We then have

$$\rho_\phi = \begin{pmatrix} 0_n & \frac{1}{2}1_n \\ -\frac{1}{2}1_n & 0 \end{pmatrix}, \quad K_\phi = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0 \end{pmatrix}.$$

The eigenspaces $V_\pm = \langle (a, \pm ia), a \in \mathbb{C}^n \rangle$. By direct computation we see that $R_+ = f_{\mathfrak{u}(n)}$ and $R_- = \bar{f}_{\mathfrak{u}(n)}$. By a direct computation we see that

$$\mu = \frac{i}{2}\epsilon_+ - \frac{i}{2}\epsilon_- = i\epsilon_+ - \frac{i}{2}.$$

Example 3.7. (BDI even) Let $G = SO(2n)$ and let us consider the first root in the Dynkin diagram as the non–compact root. The subgroup is then $H_\phi = SO(2(n-1)) \times SO(2)$. Let us choose now the Cartan as $\mathfrak{t} = \oplus_{k=1}^n \mathfrak{so}(2)$ embedded as a diagonal 2×2 block matrix. We then have

$$\rho_\phi = \begin{pmatrix} 0_{2n-2} & 0 \\ 0 & \sigma \end{pmatrix}, \quad K_\phi = \begin{pmatrix} 1_{2n-2} & 0 \\ 0 & -1_2 \end{pmatrix}.$$

Analogously to the (BDI odd) case, we have that that $R_+ = f_{\mathfrak{so}(2(n-1))} \times 0_{\mathfrak{so}(2)}$ and $R_- = 0_{\mathfrak{so}(2(n-1))} \times f_{\mathfrak{so}(2)}$ and $\mu = \sigma_- \sigma \sigma_-^\dagger$.

4. The Bruhat-Poisson structure

We recall here the definition of the Bruhat-Poisson structure on compact hermitian symmetric spaces G/H_ϕ. It is obtained from the so called standard Poisson structure on G, that we are going to define first.

Let G be the compact form of the complex classical group $G_\mathbb{C} \subset SL(N, \mathbb{C})$, $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{C}$ be their Lie algebras. Recall that $\mathfrak{g} = \{X \in \mathfrak{g}_\mathbb{C} \mid X^\dagger = -X\}$. Let us fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and the set of simple roots Π; we denote by $\Phi^\pm$ the positive (negative) roots. For each root α we denote by $\mathfrak{g}_\alpha \subset \mathfrak{g}_\mathbb{C}$ the root space. Let us define $J : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ as

$$(8) \qquad J(\mathfrak{t}) = 0, \quad J(E_\alpha) = \pm i E_\alpha \quad \alpha \in \Phi^\pm,$$

where $E_\alpha \in \mathfrak{g}_\alpha$. Let us remark that if $\mathfrak{h}_\phi$ denotes the subalgebra associated to the non–compact root ϕ, as described in the previous section, we have that

$$(9) \qquad J|_{\mathfrak{h}_\phi^\perp} = \mathrm{ad}_{\rho_\phi}.$$

Let us define

$$(10) \qquad C_\pm = i \pm J.$$

The Iwasawa decomposition is defined as $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus \mathfrak{b}_\pm$, where $\mathfrak{b}_\pm = C_\pm(\mathfrak{g})$; $\mathfrak{g}$ and $\mathfrak{b}_\pm$ are lagrangian subalgebras with respect to the non degenerate pairing $\langle A, B \rangle = \mathrm{ImTr}[AB]$. The triple $(\mathfrak{g}_\mathbb{C}, \mathfrak{g}, \mathfrak{b}_\pm)$ is a *Manin triple*. Let us denote by $(\mathrm{pr}_{\mathfrak{g}_+}, \mathrm{pr}_{\mathfrak{b}_+})$ and $(\mathrm{pr}_{\mathfrak{g}_-}, \mathrm{pr}_{\mathfrak{b}_-})$ the projections defined by the decomposition $\mathfrak{g} \oplus \mathfrak{b}_+$ and $\mathfrak{g} \oplus \mathfrak{b}_-$ respectively. We get in particular an identification of $\mathfrak{g}^*$ with $\mathfrak{b}_\pm$. If we use Tr to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ then one can check that $C_\pm : \mathfrak{g} \to \mathfrak{b}_\pm$ connects these two realizations of $\mathfrak{g}^*$.

Example 4.1. If $\mathfrak{g} = \mathfrak{su}(n)$ then $\mathfrak{b}_\pm = \mathfrak{a} \oplus \mathfrak{n}_\pm$, where $\mathfrak{a}$ denotes the algebra of real diagonal matrices and $\mathfrak{n}_+$ ($\mathfrak{n}_-$) the strictly upper (lower) diagonal complex matrices. The isomorphism $C_+ : \mathfrak{su}(n) \to \mathfrak{b}_+$ reads

$$(11) \qquad C_+(X)_{rs} = \begin{cases} 2iX_{rs} & r < s \\ iX_{rr} & r = s \\ 0 & r > s \end{cases} \qquad X \in \mathfrak{su}(n)$$

and analogously for $C_- = C_+^\dagger$. □

The standard Poisson-Lie structure π_G on G (see [13, 14] for a general reference) is defined as

$$
\begin{aligned}
\langle r_{g^{-1}}\pi_G(g), \xi \wedge \eta \rangle &= \langle \mathrm{pr}_{\mathfrak{g}_-}(\mathrm{Ad}_{g^{-1}} C_-(\xi)), \mathrm{pr}_{\mathfrak{b}_-}(\mathrm{Ad}_{g^{-1}} C_-(\eta)) \rangle \\
&= -\langle \mathrm{pr}_{\mathfrak{g}_+}(\mathrm{Ad}_{g^{-1}} C_+(\xi)), \mathrm{pr}_{\mathfrak{b}_+}(\mathrm{Ad}_{g^{-1}} C_+(\eta)) \rangle,
\end{aligned}
\tag{12}
$$

where $C_\pm$ are defined in (10) and $\xi, \eta \in \mathfrak{g} \equiv \mathfrak{g}^*$. It can be shown that π_G defines a Poisson-Lie structure. According to the different descriptions of $\mathfrak{g}^*$ described above, the dual Lie algebra can be described as the subalgebra $\mathfrak{b}_+ \equiv \mathfrak{b}_-^{op} \subset \mathfrak{g}_\mathbb{C}$ or as the following bracket on $\mathfrak{g}$

$$
[X, Y]_{\mathfrak{g}^*} = [J(X), Y] + [X, J(Y)], \quad X, Y \in \mathfrak{g},
\tag{13}
$$

where J is defined in (8). The dual Poisson-Lie group is the subgroup $B_+ \subset G_\mathbb{C}$ integrating the Lie algebra $\mathfrak{b}_+ = \mathfrak{b}_-^{op}$. The Iwasawa decomposition of $G_\mathbb{C}$ consists in the global decomposition $G_\mathbb{C} = GB_+ = GB_-$ and defines the left (right) dressing transformation of G^* on G, $(\gamma, g) \to {}^\gamma g$ and $(g, \gamma) \to g^\gamma$, where $g \in G, \gamma \in G^*$, as follows:

$$
\gamma g = {}^\gamma g \gamma^g, \qquad g\gamma = {}^g\gamma g^\gamma.
\tag{14}
$$

From the definition of the Poisson bivector π_G one can show the following expression for the dressing vector field associated to $\xi \in \mathfrak{b}_+$

$$
s_\xi(g) = \frac{d}{dt}(e^{t\xi} g)|_{t=0} = -\pi_G(r_{g^{-1}}^* \xi).
\tag{15}
$$

Let us remark that, since $G \subset M_N(\mathbb{C})$ is a matrix group and then $T_g G = g\mathfrak{g} \subset M_N(\mathbb{C})$, the vector field s_ξ evaluated in $g \in G$ is computed by acting s_ξ on the matrix valued function $g : G \to M_N(\mathbb{C})$.

Lemma 4.2. *The matrix adjoint* $\dagger : B_+ \to B_-^{op}$, *satisfies*

$$
\gamma^\dagger g = {}^{\gamma^{-1}} g,
\tag{16}
$$

for each $g \in G$ and $\gamma \in B_+$. The fundamental vector field of the left dressing action of $\xi \in \mathfrak{b}_+$ is

$$
s_\xi(g) = \xi g - g \, \mathrm{Ad}_{g^{-1}}^* \xi = g(\mathrm{Ad}_{g^{-1}}^* \xi)^\dagger - \xi^\dagger g = -s_{\xi^\dagger}(g).
\tag{17}
$$

Proof. Since $C_+^\dagger = C_-$, the matrix adjoint sends $\mathfrak{b}_+$ in $\mathfrak{b}_-$; the statement for the groups follows because they are exponential groups. From the above

definition of dressing transformation, we get

$$\gamma^\dagger g(\gamma^\dagger)^g = \gamma^\dagger g = (g^{-1}\gamma)^\dagger = (^{g^{-1}}\gamma(g^{-1})^\gamma)^\dagger$$

so that $^{\gamma^\dagger}g = ((g^{-1})^\gamma)^{-1} = g^{g^{-1}\gamma}$, where the last equality follows from $1 = (gg^{-1})^\gamma$. Analogously, we apply the same rules to exchange twice the order of $g^{g^{-1}}\gamma$ and find that $g^{g^{-1}\gamma} = {}^{\gamma^{-1}}g$, from which we get (16).

We see that

$$\xi g = \frac{d}{dt}(e^{t\xi}g)|_{t=0} = \frac{d}{dt}(^{e^{t\xi}}g)|_{t=0} + g\frac{d}{dt}(e^{t\xi})^g|_{t=0} = s_\xi(g) + g\operatorname{Ad}^*_{g^{-1}}\xi,$$

from which the first equality of (17) follows. In the last step we used the fact that the coadjoint action is the derivative of the dressing action at the identity. The second equality comes by using (16). $\qquad\square$

Let us consider now a non–compact root ϕ and let $\mathfrak{h}_\phi$ be the subalgebra associated to it and $H_\phi \subset G$ be the subgroup integrating it, as described in Section 3.

Lemma 4.3. *The subgroup $H_\phi \subset G$ is a Poisson-Lie subgroup.*

Proof. We have to show that $\mathfrak{h}_\phi^\perp \subset \mathfrak{g} \equiv \mathfrak{g}^*$ is an ideal of the Lie bracket (13). Since

$$[\mathfrak{h}_\phi, \mathfrak{h}_\phi^\perp] \subset \mathfrak{h}_\phi^\perp, \ [\mathfrak{h}_\phi^\perp, \mathfrak{h}_\phi^\perp] \subset \mathfrak{h}_\phi$$

and J preserves $\mathfrak{h}_\phi$ and $\mathfrak{h}_\phi^\perp$, it is enough to check that $\mathfrak{h}_\phi^\perp$ is an (abelian) subalgebra of $\mathfrak{g}^*$. Indeed, let $E_\alpha, E_\beta \in (\mathfrak{h}_\phi^\perp)_\mathbb{C}$ be root vectors, then

$$[E_\alpha, E_\beta]_{\mathfrak{g}^*} = i(\operatorname{sign}\alpha + \operatorname{sign}\beta)[E_\alpha, E_\beta] = 0 \,,$$

because if α and β are both non–compact positive (or negative) roots then $\alpha + \beta$ is not a root. $\qquad\square$

A Poisson structure is then induced on G/H_ϕ that we will denote by π_0. The quotient map $(G, \pi_G) \to (G/H_\phi, \pi_0)$ is a Poisson map and the homogeneous G action on G/H_ϕ is a Poisson action. By applying Lemma 2.1 we conclude that the associated bundles $\mathcal{E}_\pm$ are Poisson vector bundles, or alternatively that there exists a flat contravariant connection. We will discuss an explicit formula for this connection in the next section.

5. The contravariant connection

Let $M_\phi = G/H_\phi$ denote the compact hermitian symmetric space associated to the non–compact simple root ϕ (see Section 3 for notations). We have seen at the end of the previous section that, if we consider the Bruhat-Poisson structure, the vector bundles $\mathcal{E}_\pm$ are Poisson vector bundles. In this section we are going to describe an explicit formula for their contravariant connection.

Let $\nabla : C^\infty(\mathcal{E}_\pm) \to C^\infty(TM_\phi \otimes \mathcal{E}_\pm)$ be the flat contravariant connection that we define for later convenience as $\nabla_{df}(\sigma) = -\{f, \sigma\}$, with $f \in C^\infty(M_\phi)$ and $\sigma : G \to V_\pm^*$ equivariant, *i.e.* with the opposite sign with respect to Subsection 2.1. Let us define $\nabla_{N_\phi} = \Omega_{kks} \circ \nabla : C^\infty(\mathcal{E}_\pm) \to C^\infty(T^*M_\phi \otimes \mathcal{E}_\pm) \equiv \Omega^1(\mathcal{E}_\pm)$, where Ω_{kks} is the Kirillov-Konstant-Souriau symplectic form determined by the identification of M_ϕ with the adjoint orbit of ρ_ϕ. The label N_ϕ stands for the Nijenhuis tensor to be introduced later in Subsection 6.1.

We recall that μ is the moment map of the hamiltonian G-action defined in (5). We recall the notations given in Section 3. Let $g \in G \subset M_N(\mathbb{C})$ be written as $g = (\sigma_+, \sigma_-)$ with $\sigma_\pm \in M_{N,n_\pm}(\mathbb{C})$. If we denote by the same symbol the map $g : G \to M_N(\mathbb{C})$, we see that the i-th row $g_i : G \to \mathbb{C}^N$ is equivariant with respect to the right H_ϕ multiplication and defines a section of the trivial vector bundle $\mathcal{E}_+ \oplus \mathcal{E}_-$. Analogously, $(\sigma_\pm)_i$ denotes the i-th row and defines a section of $\mathcal{E}_\pm$.

We finally recall the definition of the Poisson structure on the vector bundles $\mathcal{E}_\pm$ associated to the Poisson principal bundle $G \to M_\phi = G/H_\phi$, as discussed at the end of Subsection 2.1. Indeed, for every equivariant function $\sigma : G \to \mathbb{C}^{n_\pm}$ the bracket is defined as $\{f, \sigma\}_{\mathcal{E}_\pm} = \{f, \sigma\}_G$, where $f \in C^\infty(M_\phi) \subset C^\infty(G)$ and $\{,\}_G$ is the standard Poisson-Lie structure on G defined in (12).

The main result is given by the following proposition.

Proposition 5.1. *The flat contravariant connection on the bundle $\mathcal{E}_\pm$ reads as*

$$\text{(18)} \qquad \nabla_{N_\phi}(\sigma_\pm) = (-J(d\mu) + [\mu, d\mu])\sigma_\pm.$$

Moreover, in the cases (AIII, CI, DIII), the above formula implies for $\mathcal{E}_\pm$

$$\nabla_{N_\phi}(\sigma_\pm) = \mp C_\pm(d\mu)\sigma_\pm,$$

where $C_\pm$ is defined in (10).

Proof. Let us compute $\nabla_{df}(g) = \iota_{df}(\nabla(g)) = -\{f, g\}_G$ for $f \in C^\infty(M_\phi) \subset C^\infty(G)$. We see that

$$\{f, g\}_G = \pi_G(df)(g) = -s_{\xi_f}(g)$$

where $\xi_f : G \to \mathfrak{g}^*$ is defined as $\xi_f(g) = r_g^* df$ and the expression of the dressing transformation is given in (15). It is easy to check that $\xi_f(gh) = \xi_f(g)$, for each $h \in H_\phi$ and $\xi_f'(g) \equiv l_g^* df = \mathrm{Ad}_{g^{-1}}^* \xi_f(g) \in \mathfrak{h}_\phi^\perp$. By using formula (17) in Lemma 4.2 and the identification of $\mathfrak{g}^*$ with $\mathfrak{b}_\pm$ given by $C_\pm$ defined in (10), we see that

$$s_{\xi_f}(g) = -C_-(\xi_f)g + gC_-(\xi_f') = C_+(\xi_f)g - gC_+(\xi_f')$$
$$= [-C_-(\xi_f) + gC_-(\xi_f')g^{-1}]g = [C_+(\xi_f) - gC_+(\xi_f')g^{-1}]g.$$

We have to characterize $\xi_f(g)$ and $\xi_f'(g)$. Since the ring of function of M_ϕ is generated by the matrix elements of the moment map μ, it is enough to consider $f = \mu(X)$, for any $X \in \mathfrak{g}$. We are going to show that

$$\xi_{\mu(X)} = r_g^* d\mu(X) = \{\mu(X), \mu\}_{kks} = -\langle d\mu, v_X \rangle \in \mathfrak{g}^* \equiv \mathfrak{g},$$

where v_X is the fundamental vector field of X. Indeed, let us evaluate both sides of the above equation with $Y \in \mathfrak{g}$: it is easy to check that the result is $\mu([X, Y])$ on both sides. Analogously, we evaluate

$$\xi_{\mu(X)}' = [\rho_\phi, g^{-1}Xg] \in \mathfrak{h}_\phi^\perp \subset \mathfrak{g}^* \equiv \mathfrak{g}.$$

Using (9), we then get

$$g\, C_\pm(\xi_{\mu(X)}')\, g^{-1} = g([i\rho_\phi, g^{-1}Xg] \pm [\rho_\phi, [\rho_\phi, g^{-1}Xg]])g^{-1}$$
$$= i[\mu, X] \pm [\mu, [\mu, X]] = \langle -id\mu \mp [\mu, d\mu], v_X \rangle.$$

By collecting all terms and recalling that $v_X = -\Omega_{kks}^{-1}(d\mu(X))$, we get

$$\nabla_{d\mu(X)}(g) = \langle \Lambda, v_X \rangle g = -\langle \Lambda, \Omega_{kks}^{-1}(d\mu(X)) \rangle g = \langle d\mu(X), \Omega_{kks}^{-1}(\Lambda) \rangle g,$$

where

$$\Lambda = \pm(-C_\pm(d\mu) + id\mu \pm [\mu, d\mu]) \in \Omega^1(M_\phi) \otimes \mathfrak{g}.$$

In the cases (AIII, CI, DIII), observe from the discussion around Lemma 3.1 that $d\mu = \pm id\epsilon_\pm$ and that $\sigma_\pm^\dagger \sigma_\mp = 0$ so that

$$
\begin{aligned}
(id\mu + [\mu, d\mu])\sigma_+ &= (d\epsilon_- - [\epsilon_-, d\epsilon_-])\sigma_+ \\
&= (d(\sigma_-\sigma_-^\dagger) - \sigma_-\sigma_-^\dagger d(\sigma_-\sigma_-^\dagger)))\sigma_+ \\
&= (\sigma_- d\sigma_-^\dagger - \sigma_-\sigma_-^\dagger \sigma_- d\sigma_-^\dagger)\sigma_+ = 0,
\end{aligned}
$$

and analogously for σ_-. $\qquad\square$

We are mostly interested in the following corollary of the above formula.

Corollary 5.2. *The moment map μ defined in (5) satisfies*

$$(19) \qquad\qquad N_\phi^* d\mu = [-J(d\mu), \mu] + d\mu.$$

Proof. By using the formula (18) for the contravariant connection and denoting with $p: G \to M_\phi$ the quotient map, we see that

$$
\begin{aligned}
N_\phi^* d\mu = \Omega_{kks} p_*(\pi_G d(g\rho_\phi g^\dagger)) &= \nabla_{N_\phi}(g)\rho_\phi g^\dagger + g\rho_\phi \nabla_{N_\phi}(g^\dagger) \\
&= [-J(d\mu), \mu] + [[\mu, d\mu], \mu],
\end{aligned}
$$

where we used the fact that $g = \sigma_+ \oplus \sigma_-$ as a section of $\mathcal{E}_+ \oplus \mathcal{E}_-$ and $\nabla_{N_\phi}(g) = \Omega_{kks}(p_*(\pi_G(dg)))$.

Let us show that $[[\mu, d\mu], \mu] = d\mu$. Indeed, for each $X \in \mathfrak{g}$ we see that

$$
\begin{aligned}
\langle [[\mu, d\mu], \mu], v_X \rangle = -[[\mu, [\mu, X]], \mu] &= g\, \mathrm{ad}_{\rho_\phi}^2 [\rho_\phi, g^{-1} X g] g^{-1} \\
&= -g[\rho_\phi, g^{-1} X g] g^{-1} = -[\mu, X] = \langle d\mu, v_X \rangle,
\end{aligned}
$$

since $[\rho_\phi, g^{-1} X g] \in \mathfrak{h}_\phi^\perp$ and $\mathrm{ad}_{\rho_\phi}^2 |_{\mathfrak{h}_\phi^\perp} = -1$ (easily seen from the discussion at the beginning of Section 3). $\qquad\square$

6. The bihamiltonian system

We recall in this Section the definition of the bihamiltonian system on compact hermitian symmetric spaces and discuss the diagonalization of the Nijenhuis tensor N_ϕ.

It is useful to review here our strategy for finding eigenvalues of the Nijenhuis tensor N_ϕ, as discussed in the Introduction. We will first prove in Proposition 6.1 that the non constant eigenvalues of the solutions of the *master equation* (21) are Nijenhuis eigenvalues. We will then find solutions

thanks to a method that is a sort of refinement of the Thimm method, discussed in Subsection 2.3 and based on the hamiltonian action of subalgebras of $\mathfrak{g}$. In Subsection 6.2 we will prove that a solution of (21) is given by the moment map μ_R of $\mathfrak{g}$ evaluated in a suitable representation R that we call *decomposable*. Since the eigenvalues of μ_R are constant, no Nijenhuis eigenvalues is produced out of it, but we will prove in Subsection 6.3 that the relevant solutions of (21) are given by minors of μ_R that represent the moment maps of certain subalgebras of $\mathfrak{g}$. In the remaining Sections we will finally show case by case that these solutions provide all the Nijenhuis eigenvalues by proving complete integrability.

6.1. Definition of the Poisson-Nijenhuis structure

It was proved in [11] that the Bruhat and the KKS Poisson structures on compact hermitian symmetric spaces are compatible. The following argument can be found in [5]. The G action on $M_\phi = G/H_\phi$ is Poisson with respect to the Bruhat-Poisson structure and leaves Ω_{kks}^{-1} invariant, so that, if we denote by $v : \mathfrak{g} \to \Gamma(TM_\phi)$ the map that associates to $X \in \mathfrak{g}$ the fundamental vector field v_X, we see that

$$L_{v_X}[\pi_0, \Omega_{kks}^{-1}] = [L_{v_X}\pi_0, \Omega_{kks}^{-1}] = [v(\delta_{\mathfrak{g}}(X)), \Omega_{kks}^{-1}] = 0.$$

For compact hermitian spaces, the only $\mathfrak{g}$-invariant 3-vector field is 0, so that we conclude that the Poisson structures π_0 and Ω_{kks}^{-1} are compatible, *i.e.* they satisfy

$$[\pi_0, \Omega_{kks}^{-1}] = 0.$$

The following are direct and fundamental consequences of this fact: $i)$ there is a pencil of homogeneous Poisson structures $\pi_t = \pi_0 + t\Omega_{kks}^{-1}$, $t \in \mathbb{R}$, on M_ϕ. $ii)$ The $(1,1)$ tensor

$$(20) \qquad\qquad N_\phi = \pi_0 \circ \Omega_{kks} : TM_\phi \to TM_\phi$$

is Nijenhuis, *i.e.* it has vanishing Nijenhuis torsion, so that $(M_\phi, \Omega_{kks}^{-1}, N_\phi)$ is a PN structure.

6.2. Diagonalization of the Nijenhuis tensor

The following Proposition gives the basic tool for producing eigenvalues of N_ϕ^*.

Proposition 6.1. *Let $\mathcal{M}$ be a matrix valued function on an open subset $U \subset M_\phi$. Assume that the eigenvalue m of $\mathcal{M}$ is a smooth non constant function on U with constant multiplicity. Consider an equation of type*

$$(21) \qquad N_\phi^* d\mathcal{M} = d\mathcal{M}^- \mathcal{M} + \mathcal{M} \, d\mathcal{M}^+ + r d\mathcal{M},$$

with $d\mathcal{M}^\pm$ being matrix valued one forms such that $d\mathcal{M}^+ + d\mathcal{M}^- = k d\mathcal{M}$ and $k, r \in \mathbb{C}$. Then

$$(22) \qquad N_\phi^* dm = \big(k\, m + r\big) dm,$$

i.e. $k\, m + r$ is an eigenvalue of N_ϕ^*.

Proof. Let $x \in C^\infty(U)$ and let $P(x, \mathcal{M}) = \det(Ix - \mathcal{M})$. We have that

$$N_\phi^* dP = (N_\phi^* dx)\partial_x P - P \operatorname{Tr}[(Ix - \mathcal{M})^{-1} N_\phi^* d\mathcal{M}],$$

where we think P as the restriction of a function defined on $\mathbb{R} \times M$ to the graph of x and we used the formula $d \det A = \det A \operatorname{Tr}[A^{-1} dA]$. We use (21) and we get

$$N_\phi^* dP = (N_\phi^* dx)\partial_x P - P \operatorname{Tr}\big[(Ix - \mathcal{M})^{-1}\big(d\mathcal{M}^- \mathcal{M} + \mathcal{M} \, d\mathcal{M}^+ + r d\mathcal{M}\big)\big].$$

We write the first term of the trace as

$$\begin{aligned}
\operatorname{Tr}\big[\mathcal{M}(Ix - \mathcal{M})^{-1} d\mathcal{M}^-\big] &= \operatorname{Tr}\big[(\mathcal{M} - Ix + Ix)(Ix - \mathcal{M})^{-1} d\mathcal{M}^-\big] \\
&= -\operatorname{Tr}\big[d\mathcal{M}^-\big] + x\operatorname{Tr}\big[(Ix - \mathcal{M})^{-1} d\mathcal{M}^-\big].
\end{aligned}$$

We do the same for the second term, and the two combine into

$$N_\phi^* dP = (N_\phi^* dx)\partial_x P + (xk + r)d_{\mathcal{M}} P + kP \operatorname{Tr}[d\mathcal{M}] \ ,$$

where we denote by $d_{\mathcal{M}} P$ the differential of P keeping x fixed. Let α be the multiplicity of m so that $P(x, \mathcal{M}) = (x - m)^\alpha P_0(x, \mathcal{M})$ with $P_0(m, \mathcal{M}) \neq 0$. Now we evaluate the above equation at $x = m + \varepsilon$ with ε constant as

$$\begin{aligned}
\varepsilon^\alpha N_\phi^* dP_0 = {}& N_\phi^* dm(\alpha \varepsilon^{\alpha - 1} P_0 + \varepsilon^\alpha \partial_x P_0) - ((m + \varepsilon)k + r)(\alpha \varepsilon^{\alpha - 1} dm P_0 \\
& - \varepsilon^\alpha d_{\mathcal{M}} P_0) + k\varepsilon^\alpha P_0 \operatorname{Tr}[d\mathcal{M}].
\end{aligned}$$

The dominant term of order $\varepsilon^{\alpha - 1}$ in $\varepsilon \to 0$ gives the formula (22). $\square$

We call (21) the *master equation*. In order to find solutions, we need to evaluate (19) in a representation of $\mathfrak{g}$ compatible with the non–compact root ϕ. In the (AIII,CI,DIII) cases it is enough to consider the defining representation, while in the BDI case it is necessary to switch to the spinor one. The main difference between the (AIII, CI, DIII) and the BDI cases is that in the latter case the moment map is not a linear combination of the idempotents defining the vector bundles $\mathcal{E}_\pm$. This is essentially due to the fact that in the decomposition of the fundamental representation of $\mathfrak{g} = \mathfrak{so}(n)$ in eigenspaces of $\exp \pi \rho_\phi$ in the BDI case we get a reducible representation of $\mathfrak{h}_\phi$ where ρ_ϕ is not multiple of the identity. Since this fact plays a central role in our diagonalization of the Nijenhuis tensor, we have to consider the moment map in a representation where the decomposition is in irreducible components.

Let us consider now a representation R of $\mathfrak{g}$ on V_R. Let $V_R = V_{R_\phi^+} \oplus V_{R_\phi^-}$ be the decomposition in eigenspaces of $R(e^{\pi \rho_\phi})$; let us call $R_\phi^\pm$ the corresponding representations of $\mathfrak{h}_\phi$.

Definition 6.1. The representation R is decomposable with respect to the non–compact root ϕ if

$$R_\phi^\pm(\rho_\phi) = r_\phi^\pm 1_{V_{R_\phi^\pm}}. \tag{23}$$

It is easy to check that, since $\mathrm{ad}_{\rho_\phi}|^2_{\mathfrak{h}_\phi^\perp} = -\mathrm{id}$, $(r_\phi^+ - r_\phi^-)^2 = -1$ so that we can choose $V_{R_\phi^\pm}$ such that $r_\phi^+ - r_\phi^- = i$.

Example 6.2. We analyze this property case by case using the discussion of the examples of Section 3.

(AIII) The fundamental representation of $\mathfrak{su}(n)$ is decomposable with respect to any non–compact root α_k; in fact it decomposes into the fundamental representation of $\mathfrak{u}(k)$ and $\mathfrak{u}(n-k)$ so that $r_{\alpha_k}^+ = i(n-k)/n$ and $r_{\alpha_k}^- = -ik/n$. See Example 3.2.

(CI) The fundamental representation of $\mathfrak{sp}(n)$ is decomposable with respect to the unique non–compact root, and the resulting $R_\phi^\pm$ are the fundamental and anti-fundamental representation of $\mathfrak{u}(n)$ and $r_\phi^\pm = \pm i/2$. See Example 3.5.

(DIII) The fundamental representation of $\mathfrak{so}(2n)$ is decomposable with respect to the last root of the Dynkin diagram, where $R_\phi^\pm$ are the fundamental and anti-fundamental representation of $\mathfrak{u}(n)$ and $r_\phi^\pm = \pm i/2$. See Example 3.6.

(BDI) The fundamental representation of $\mathfrak{so}(n)$ is not decomposable with respect to the first root, as can be seen in Examples 3.4 and 3.7. Their spin representations are decomposable with respect to the first root of their Dynkin diagram: indeed the weights are $(\pm 1/2, \ldots, \pm 1/2)$ so that $r_{\alpha_1}^\pm = \pm i/2$. We will give additional details at the end of this section.

Let R be ϕ-decomposable and let $\mathcal{E}_{R_\phi^\pm} = G \times_{R_\phi^\pm} V_{R_\phi^\pm}^*$ be the vector bundles on M_ϕ associated to $R_\phi^\pm$. By applying to (6) the representation R of the simply connected group integrating $\mathfrak{g}$ and denoting $\mu_R \equiv R(\mu)$ the moment map in this representation, we get

$$(24) \qquad \mu_R = r_\phi^+ \epsilon_{R_\phi^+} + r_\phi^- \epsilon_{R_\phi^-} = i\epsilon_{R_\phi^+} + r_\phi^-,$$

where $\epsilon_{R_\phi^\pm}$ are idempotents defining $\mathcal{E}_{R_\phi^\pm}$, see the definition given in Lemma 3.1.

Theorem 6.2. *Let R be a representation of $\mathfrak{g}$ decomposable with respect to ϕ. We have that*

$$(25) \qquad N_\phi^* d\mu_R = \pm d\mu_R^\mp \mu_R \pm \mu_R d\mu_R^\pm \mp 2ir_\phi^\pm d\mu_R,$$

where $d\mu_R^\pm \equiv R(C_\pm(d\mu))$.

Proof. By using the definition of $C_\pm = i \pm J$, we write $J(d\mu)$ as $C_+(d\mu) - i\,d\mu$, and place (19) in the representation R

$$N_\phi^* d\mu_R = -d\mu_R^+ \mu_R + \mu_R d\mu_R^+ + i\,d\mu_R^2 - 2i\mu_R\,d\mu_R + d\mu_R$$
$$= -d\mu_R^+ \mu_R - \mu_R d\mu_R^- + i\,d\mu_R^2 + d\mu_R.$$

By using (24) and the idempotency of $\epsilon_{R_\phi^+}$ we get $d\mu_R^2 = 2r_\phi^- d\mu_R + i\,d\mu_R$, and so the result. $\qquad\square$

By Theorem 6.2 μ_R satisfies the master equation with $d\mathcal{M}^\pm = d\mu_R^\pm$ and $k = 2i$, $r = -2ir_\phi^+$. Of course the eigenvalues of μ_R are constant and Proposition 6.1 does not apply. We will see in the following Subsection the general strategy to produce the Nijenhuis eigenvalues.

6.3. Reduction to a chain of subalgebras

In order to build the eigenvalues of the Nijenhuis tensor N_ϕ we will pick a chain of subalgebras

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_k \ ,$$

where each $\mathfrak{g}_i$ is equipped with a representation R_i such that the moment map in this representation $\mu_{\mathfrak{g}_i R_i}$ solves the master equation (21). With these data, we will get the eigenvalues by applying Proposition 6.1 at each step.

In this subsection we will show how to define these data case by case. The proof that we get all the eigenvalues from this construction is postponed to the next sections where we will use the results about integrability of the collective hamiltonians defined by the above chain of subalgebras.

AIII. Let $M_\phi = Gr(k, n)$; from the discussion in Example 6.2, we can conclude that Equation (25) is valid with R being the fundamental representation so that $\mu_R = \mu$ and

$$N_{\alpha_k}^* d\mu = C_-(d\mu)\mu + \mu\, C_+(d\mu) - 2ir_{\alpha_k}^+ d\mu,$$

where $r_{\alpha_k}^+ = i(n - k)/n$. Since $C_+(d\mu)$ and $C_-(d\mu)$ can be chosen as upper and lower triangular matrices respectively, it is easy to check that every $(n - s) \times (n - s)$ upper left minor $\mu^{(s)}$ solves the master equation (21) with $d\mathcal{M}^\pm = dC_\pm(\mu)^{(s)}$, $k = 2i$ and $r = -2ir_{\alpha_k}^+$.

In order to read these minors as moment maps of a chain of subalgebras, it is better to look at $Gr(k, n)$ as a $\mathfrak{u}(n)$ hamiltonian space rather than $\mathfrak{su}(n)$ and consider the chain of subalgebras

$$(26) \qquad\qquad \mathfrak{u}(n) \supset \mathfrak{u}(n - 1) \cdots \supset \mathfrak{u}(1)$$

with $\mathfrak{g}_s = \mathfrak{u}(n - s)$ embedded as the upper-left corner of $\mathfrak{g}_{s-1} = \mathfrak{u}(n - s + 1)$. It is clear that the minor $\mu^{(s)}$ is the moment map of $\mathfrak{u}(n - s)$ in the fundamental representation.

The eigenvalues of $\mu^{(s)}$ are the classical Gelfand-Tsetlin variables. In Section 7 we will review their properties and show that they exhaust all the possible Nijenhuis eigenvalues.

CI *and* **DIII.** From the discussion in Example 6.2 we know that in both cases the fundamental representation of $\mathfrak{g} = \mathfrak{so}(2n), \mathfrak{sp}(n)$ is decomposable with respect to the non–compact root ϕ. Equation (25) is then valid in the fundamental representation with $r_\phi^\pm = \pm i/2$ in both cases.

We pick the chain of subalgebras as

$$\mathfrak{sp}(n) \supset \mathfrak{u}(n) \supset \mathfrak{u}(n-1) \cdots \supset \mathfrak{u}(1), \tag{27}$$

$$\mathfrak{so}(2n) \supset \mathfrak{u}(n) \supset \mathfrak{u}(n-1) \cdots \supset \mathfrak{u}(1), \tag{28}$$

where $\mathfrak{g}_k = \mathfrak{u}(n+1-k)$ is embedded as upper left block of $\mathfrak{g}_{k-1}$ and is considered in the fundamental representation.

We will show first that the master equation is valid for the first $\mathfrak{u}(n)$ step. We need the following general discussion. Let R be a representation of $\mathfrak{g}$ decomposable with respect to the non–compact root ϕ and let $V_R = V_{R_\phi^+} \oplus V_{R_\phi^-}$. Since $R(\mathfrak{h}_\phi^\perp) : V_{R_\phi^\pm} \to V_{R_\phi^\mp}$, the moment map $\mu = \mu_{\mathfrak{h}_\phi} + \mu_{\mathfrak{h}_\phi^\perp}$ in the representation R accordingly decomposes as

$$\mu_R = \begin{pmatrix} \mu_{\mathfrak{h}_\phi R_\phi^+} & \mu_{\mathfrak{h}_\phi^\perp}^{+-} \\ \mu_{\mathfrak{h}_\phi^\perp}^{-+} & \mu_{\mathfrak{h}_\phi R_\phi^-} \end{pmatrix},$$

where $\mu_{\mathfrak{h}_\phi R_\phi^\pm}$ is the moment map of $\mathfrak{h}_\phi$ in the representation $R_\phi^\pm$. The following Lemma states that in this decomposition the non–compact positive roots are upper diagonal and the negative ones lower diagonal.

Lemma 6.3. *Let R be a representation decomposable with respect to the non–compact root ϕ. For each ϕ non–compact positive root α we have that*

$$R(E_\alpha)V_{R_\phi^+} = R(E_{-\alpha})V_{R_\phi^-} = 0.$$

Proof. Let α be a positive ϕ-non compact root. We see that for each $v_+ \in V_{R_\phi^+}$

$$r_\phi^- R(E_\alpha)v_+ = R(\rho_\phi)R(E_\alpha)v_+ = R(E_\alpha)R(\rho_\phi)v_+ + R([\rho_\phi, E_\alpha])v_+$$
$$= (r_\phi^+ + i)R(E_\alpha)v_+ = (r_\phi^- + 2i)R(E_\alpha)v_+,$$

so that $R(E_\alpha)v_+ = 0$. Analogously one can show that $R(E_{-\alpha})v_- = 0$. $\qquad\square$

As a consequence the matrices representing $C_+(\mathfrak{h}_\phi^\perp)$ and $C_-(\mathfrak{h}_\phi^\perp)$ are concentrated in the $(+-)$ and $(-+)$ block respectively. This has the consequence that the $(++)$ and $(--)$ block of the equation (25) satisfied by μ_R is the master equation for $\mu_{\mathfrak{h}_\phi R_\phi^\pm}$. If we consider the $(++)$ component in our case of $\mathfrak{g} = \mathfrak{so}(2n), \mathfrak{sp}(n)$, we get that the $\mathfrak{u}(n)$ moment map $\mu_{\mathfrak{u}(n)}$ in the

fundamental representation satisfies the master equation

$$N_\phi^* d\mu_{\mathfrak{u}(n)} = C_-(d\mu_{\mathfrak{u}(n)})\mu_{\mathfrak{u}(n)} + \mu_{\mathfrak{u}(n)}C_+(d\mu_{\mathfrak{u}(n)}) + d\mu_{\mathfrak{u}(n)}.$$

The subsequent reductions will proceed exactly the same as in the AIII case. In Sections 8 and 9, we will carry out the remaining details, including establishing the independence and the range of the eigenvalues.

BDI. This is the case where we have to use the moment map in a representation different from the fundamental. As it was observed in Example 6.2, the spin representation S is decomposable with respect to the non–compact root $\phi = \alpha_1$ with $r_\phi^\pm = \pm i/2$ so that equation (25) for μ_S means

$$N^* d\mu_S = d\mu_S^- \mu_S + \mu_S d\mu_S^+ + d\mu_S,$$

where $d\mu_S^\pm = S(C_\pm(d\mu))$.

Let $\mathfrak{g} = \mathfrak{so}(n+2)$ where $n+2 = 2N, 2N+1$. Let us recall a few basic facts of the spin representation S. We label coordinates of $\mathbb{R}^{2N}$ as $\{x_i, i = 1, \ldots 2N\}$, and that of $\mathbb{R}^{2N+1}$ as $\{x_0, x_i, i = 1, \ldots 2N\}$. We introduce complex coordinates $z_i = (x_{2i-1} + ix_{2i})/2$, $i = 1, \ldots, N$ and gamma matrices Γ_i. The action of the gamma matrices on $V_S = \wedge\langle d\bar{z}\rangle_{i=1}^N$ is defined as $\Gamma_i = d\bar{z}_i\wedge$, $\Gamma_{\bar{i}} = \iota_{\partial_{\bar{z}_i}}$ and $\Gamma_0 = (-1)^{\deg}$. Recalling that $S(X) = \frac{1}{8}X_{ij}[\Gamma_i, \Gamma_j]$ for $X \in \mathfrak{g}$ we easily see that

$$(29) \qquad\qquad S(\rho_\phi) = i\left(\Gamma_{\bar{N}}\Gamma_N - \frac{1}{2}\right),$$

so that $S(\rho_\phi)|_{V_S^\pm} = r_\phi^\pm 1_{V_S^\pm}$, where $V_S^+ = \wedge\langle d\bar{z}_i.i = 1, \ldots, N-1\rangle$ and $V_S^- = V_S^+ \otimes d\bar{z}_N$ and $S^\pm$ is the representation $(S, \pm i/2)$ of $\mathfrak{so}(n) \oplus \mathfrak{so}(2)$. Pay attention that $S^\pm$ is not to be confused with the chirality in the even case. With our choice of the Cartan subalgebra, the positive root vectors are represented as

$$\Delta_{2N+1}^+ = \{-dz^{\bar{j}}\, \iota_{d\bar{z}_i}, \ i > j; \quad \iota_{d\bar{z}_i}(-1)^{\deg}; \quad \iota_{d\bar{z}_i}\iota_{d\bar{z}_j}\},$$
$$\Delta_{2N}^+ = \{-dz^{\bar{j}}\, \iota_{d\bar{z}_i}, \ i > j; \quad \iota_{d\bar{z}_i}\iota_{d\bar{z}_j}\}.$$

A basis of V_S is given by the words $d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \in V_S$, $i_1 < \cdots < i_p$. We pick an ordering of the words such that $d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \prec d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, if $i_p < j_q$, or in case $i_p = j_q$ then $i_{p-1} < j_{q-1}$ and so on. In this basis, positive root vectors are upper diagonal so that $C_+(\mathfrak{g})$ are upper triangular matrices. We can again use the same logic as for the AIII case and conclude that every

upper left minor of μ_S satisfies the master equation with $k = 2i$ and $r = 1$. In particular the upper left 2^{N-1} minor $\mu_S^{(N-1)}$ is the moment map for the subalgebra $\mathfrak{so}(n) \oplus \mathfrak{so}(2)$ in the representation $(S, i/2)$. By iterating the procedure we can conclude that the upper left 2^{N-s} minor $\mu_S^{(N-s)}$ is the moment map of

$$\mathfrak{g}_s = \mathfrak{so}(n + 2 - 2s) \oplus \underbrace{\mathfrak{so}(2) \oplus \cdots \mathfrak{so}(2)}_{s}$$

in the representation $(S, i/2, \ldots, i/2)$.

To summarize, denoting with $\mathfrak{t} = \mathfrak{so}(2) \oplus \mathfrak{so}(2) \cdots$ the Cartan subalgebra of $\mathfrak{so}(n + 2)$, we proved that we produce Nijenhuis eigenvalues considering the eigenvalues of the moment map of the subalgebras appearing in the following chain

$$(30) \qquad \mathfrak{so}(n + 2) \supset \mathfrak{so}(n) \oplus \mathfrak{so}(2) \supset \mathfrak{so}(n - 2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2) \supset \cdots \supset \mathfrak{t}$$

considered in the representation $(S, i/2, \ldots, i/2)$. The proof of their independence and description of their range will be given in Section 10.

7. $M_\phi = Gr(k, n) = SU(n)/S(U(k) \times U(n - k)))$

Let us consider $M_\phi = Gr(k, n) = SU(n)/S(U(k) \times U(n - k))$. We showed in Subsection 6.3 that the moment map $\mu_{\mathfrak{u}(n-s)}$ of the subalgebra $\mathfrak{u}(n - s)$ appearing in the chain (26) in the fundamental representation solves the master equation (21). By applying Proposition 6.1 we get the Nijenhuis eigenvalues. The eigenvalues of these moment maps are the so called Gelfand-Tsetlin variables. Their integrability has been established in [8, 10]; let us briefly recall the construction.

The result is a consequence of the following proposition proved in [8].

Proposition 7.1. *Let $\mathcal{O}$ be a coadjoint orbit of $\mathfrak{u}(n)$ and let us consider $\mathfrak{u}(n - 1) \subset \mathfrak{u}(n)$ (embedded in the upper left corner, for instance). Then $\mathcal{O}$ is multiplicity free as hamiltonian $U(n - 1)$-space.*

By applying Proposition 2.1 we conclude that the chain

$$(31) \qquad \mathfrak{u}(n - 1) \supset \mathfrak{u}(n - 2) \cdots \supset \mathfrak{u}(1) \supset 0$$

defines an integrable model on any $U(n)$ coadjoint orbit.

Let us consider the $U(n)$ orbit $\mathcal{O}_{\tilde{\lambda}}$ of $i\tilde{\lambda}$, where $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n) \in \mathbb{R}^n$ satisfies $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_n$ and let again μ be the $\mathfrak{u}(n)$ moment map.

We recall that the moment map $\mu_{\mathfrak{u}(n-s)}$ is the upper left $(n-s) \times (n-s)$ minor $\mu^{(s)}$ of μ. It follows from the mini-max principle (see [10]) that the eigenvalues $i\tilde{\lambda}_j^{(s)}$ of $\mu_{\mathfrak{u}(n-s)}$ satisfy the Gelfand-Tsetlin inequalities

$$(32) \qquad \tilde{\lambda}_i^{(s)} \leq \tilde{\lambda}_i^{(s+1)} \leq \tilde{\lambda}_{i+1}^{(s)},$$

with $i = 1, \ldots, n - s$ and $\tilde{\lambda}_i^{(0)} = \tilde{\lambda}_i$. The *Gelfand-Tsetlin polytope* is defined as the subset $\mathcal{C}_{GT}(\lambda) \subset \mathbb{R}^{N(\tilde{\lambda})}$, with $N(\tilde{\lambda}) = \dim \mathcal{O}_{\tilde{\lambda}}/2$, of independent solutions of the inequalities (32). The $\tilde{\lambda}_i^{(s)}$ are a choice of action variables of the integrable system defined by the chain (31).

Here we are interested to the case of the Grassmannian $Gr(k, n)$ where

$$\tilde{\lambda} = (\underbrace{-k/n, \ldots,}_{n-k} \underbrace{1 - k/n, \ldots}_{k}),$$

i.e. the ordered eigenvalues of ρ_ϕ defined in Example 3.2.

Then $-i\mu^{(1)}$ has only one non-constant eigenvalue $\tilde{\lambda}_{n-k}^{(1)} \in [-k/n, 1 - k/n]$. This procedure can be iterated to the subsequent subalgebras, e.g. $-i\mu^{(2)}$ has two non-constant eigenvalues $\tilde{\lambda}_{n-k-1}^{(2)}, \tilde{\lambda}_{n-k}^{(2)}$ within the range $-k/n \leq \tilde{\lambda}_{n-k-1}^{(2)} \leq \tilde{\lambda}_{n-k}^{(1)} \leq \tilde{\lambda}_{n-k}^{(2)} \leq 1 - k/n$, and so on.

As an example, for $Gr(2, 4)$, we have the pattern

$$\begin{bmatrix} -\frac{1}{2} & & -\frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} \\ & -\frac{1}{2} & & \tilde{\lambda}_2^{(1)} & & \frac{1}{2} & \\ & & \tilde{\lambda}_1^{(2)} & & \tilde{\lambda}_2^{(2)} & & \\ & & & \tilde{\lambda}_1^{(3)} & & & \end{bmatrix}.$$

From [10] we know that the Gelfand-Tsetlin variables are independent and define a completely integrable system. As a consequence they exhaust all the possible eigenvalues of the Nijenhuis tensor N_ϕ. We have then shown the following result.

Theorem 7.2. *The Nijenhuis tensor (20) on $Gr(k, n)$ is of maximal rank and its eigenvalues are written in terms of the Gelfand-Tsetlin variables as*

$$(33) \qquad \lambda_j^{(s)} = -2 \left(\tilde{\lambda}_j^{(s)} - \frac{n-k}{n} \right), \qquad 1 \leq j \leq n - s \leq n - 1.$$

The bihamiltonian polytope coincides with the Gelfand-Tsetlin polytope

$$\mathcal{C}_{GT}(\underbrace{0,\ldots,0}_{k},\underbrace{2,\ldots,2}_{n-k}).$$

Remark 7.1. The case $k = 1$, the complex projective plane $\mathbb{C}P_n$, was solved in [5]. There is one non constant eigenvalue $\tilde{\lambda}^{(s)}_{n-s}$ for each $\mathfrak{u}(n-s)$. As it was observed in [2], these eigenvalues correspond to a specific basis of the Cartan subalgebra. In fact, since $\tilde{\lambda}^{(s)}_{n-s}$ is the unique non constant eigenvalue of $\mu^{(s)}$, we have that $id\tilde{\lambda}^{(n-s)} = d\mathrm{Tr}\mu^{(s)}$. One then checks that $d\lambda^{(s)}_{n-s} = d\mu(H_s)$ where

$$H_s = 2i \ \mathrm{diag}(\underbrace{1,\ldots,1}_{n-s},\underbrace{0,\ldots,0}_{s}) \, .$$

In particular these eigenvalues are global smooth functions. The result for the general case $Gr(k,n)$ was only conjectured in [5]. The eigenvalues are only continuous functions; by repeating the above logic one can show that for each s, $\sum_j \lambda^{(s)}_j$ is $\mu(H_s)$ up to a constant and is in particular smooth.

8. $M_\phi = Sp(n)/U(n)$

Consider now $M_\phi = Sp(n)/U(n)$. We showed in Section 6.3 that the moment map in the fundamental representation of $\mathfrak{g}_k = \mathfrak{u}(n+1-k) \subset \mathfrak{sp}(n)$ appearing in the chain (27) solves the master equation (21). By applying Proposition 6.1 we define the Njienhuis eigenvalues. In the following theorem we show that they are independent by proving the complete integrability of the collective hamiltonians defined by the chain (27).

Theorem 8.1. *The collective hamiltonians $F(\mathfrak{u}(n)\ldots\mathfrak{u}(1))$ define a completely integrable model. The Nijenhuis tensor (20) on $Sp(n)/U(n)$ has maximal rank. Its eigenvalues are all obtained as*

$$\lambda^{(k)}_i = -2\tilde{\lambda}^{(k)}_i + 1, \quad 1 \leq i \leq n+1-k \leq n,$$

where $i\tilde{\lambda}^{(k)}_i$ are the eigenvalues of the moment map of the hamiltonian $\mathfrak{g}_k = \mathfrak{u}(n+1-k)$ action.

The image of the bihamiltonian moment map is then described as the following polytope $\mathcal{C}(N_\phi) \subset \mathbb{R}^{\frac{n(n+1)}{2}}$, where $\left(\lambda^{(k)}_i\right) \in \mathcal{C}(N_\phi)$ if

$$0 \leq \lambda^{(1)}_1 \leq \cdots \leq \lambda^{(1)}_n \leq 2, \quad \lambda^{(k)}_i \leq \lambda^{(k+1)}_i \leq \lambda^{(k)}_{i+1} \, .$$

Proof. We want to apply Proposition 2.1. It is enough to show the multiplicity freeness of $U(n)$-orbits in M_ϕ; the other steps involve orbits of $U(k)$ contained in $\mu_{\mathfrak{u}(k)}(M_\phi)$ with respect to the $U(k-1)$ action, that are always multiplicity free as a consequence of Proposition 7.1. In order to use condition *ii)* of Definition 2.2 we shall show that for almost all $i\tilde{\lambda} \in \mathfrak{t}_n$, the diagonal $n \times n$ matrices of $\mathfrak{u}(n)$, the action of $U(n)_{i\tilde{\lambda}}$ on $\mu_{\mathfrak{u}(n)}^{-1}(i\tilde{\lambda})$, where $U(n)_{i\tilde{\lambda}} \subset U(n)$ is the stabilizer subgroup of $i\tilde{\lambda}$, is transitive.

If we parametrize $g \in Sp(n)$ as

$$(34) \qquad g = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \qquad \begin{array}{ll} A^\dagger A + B^t \bar{B} = 1, & A^\dagger B = B^t \bar{A}, \\ AA^\dagger + BB^\dagger = 1, & AB^t = BA^t, \end{array}$$

we compute

$$X = g\rho_\phi g^{-1} = \begin{pmatrix} i(AA^\dagger - 1/2) & -iAB^t \\ -i\bar{B}A^\dagger & -i(\bar{A}A^t - 1/2) \end{pmatrix},$$

so that the moment map for the $\mathfrak{u}(n)$ action is

$$\mu_{\mathfrak{u}(n)}(X) = i(AA^\dagger - 1/2),$$

and

$$(35) \qquad \mu_{\mathfrak{u}(n)}^{-1}(i\tilde{\lambda}) = \{\Omega = -iAB^t, AA^\dagger = 1/2 + \tilde{\lambda}, A, B \text{ satisfying } (34)\}.$$

This constraints $1/2 \pm \tilde{\lambda} \geq 0$. It is easy to see that the action of $k \in U(n)_{i\tilde{\lambda}}$ on Ω reads $k\Omega k^t$. Let us define

$$A_0 = \sqrt{1/2 + \tilde{\lambda}}, \quad B_0 = \sqrt{1/2 - \tilde{\lambda}},$$

so that $\Omega_0 = -iA_0 B_0 = -i\sqrt{1/4 - \tilde{\lambda}^2} \in \mu_{\mathfrak{u}(n)}^{-1}(i\tilde{\lambda})$. We are going to prove that in the dense open subset where $1/2 \pm \tilde{\lambda} > 0$ any $\Omega \in \mu_{\mathfrak{u}(n)}^{-1}(i\tilde{\lambda})$ is of the form $\Omega = k\Omega_0 k^t$, leading to the multiplicity freeness. From the restriction on $\tilde{\lambda}$, the matrix $AA^\dagger$ is invertible and we can write unambiguously the polar decomposition

$$A = A_0 U_A, \quad B = B_0 U_B, \quad \Omega = -iA_0 U_A U_B^t B_0$$

If we insert this decomposition in the relations of (34) we get that

$$U_A U_B^t \in U(n)_{i\tilde{\lambda}}, \quad U_A U_B^t = (U_A U_B^t)^t.$$

We want to show that we can find $k \in U(n)_{i\tilde{\lambda}}$ such that $U_A U_B^t = kk^t$ so that $\Omega = A_0 U_A U_B^t B_0 = A_0 kk^t B_0 = kA_0 B_0 k^t = k\Omega_0 k^t$. Indeed, $U_A U_B^t$ can be diagonalized as $U_A U_B^t = V u_0 V^\dagger$ with u_0 diagonal unitary matrix and V unitary. By ordering the eigenvalues of $\tilde{\lambda}$, $U_A U_B^t$ and so V are block diagonal; in particular V commutes with $\tilde{\lambda}$. Since $U_A U_B^t$ is symmetric $V^t V$ commutes with u_0. It is always possible to choose a unitary square root $\sqrt{u_0}$ commuting with $V^t V$. Then it is easy to check that $k = V\sqrt{u_0}V^\dagger$ is such that $U_A U_B^t = k^2$ with $k = k^t$ and $k \in U(n)_{i\tilde{\lambda}}$. $\qquad\square$

Example 8.1. Let us describe more explicitly the bihamiltonian polytope and the singularity locus in low dimension. We recall that the eigenvalues are globally continuous functions and their derivative becomes singular on the border of the Weyl chamber of each subalgebra $\mathfrak{g}_k$ appearing in (27).

If $M_\phi = Sp(1)/U(1)$ then $\mathcal{C}(N_\phi) = \{\lambda \in \mathbb{R}\,|\,0 \leq \lambda \leq 2\} \sim \Delta_1$, the one dimensional simplex; λ defines a smooth function. If $M_\phi = Sp(2)/U(2)$ then $\mathcal{C}(N_\phi) = \{(\lambda_i^{(k)})_{1 \leq i \leq 3-k \leq 2} \in \mathbb{R}^3\,|\,0 \leq \lambda_1^{(1)} \leq \lambda_1^{(2)} \leq \lambda_2^{(1)} \leq 2\} \sim \Delta_3$, the three dimensional simplex. The Nijenhuis eigenvalues $\lambda_i^{(1)}$ are singular when they reach the boundary of the positive Weyl chamber of $\mathfrak{g}_1 = \mathfrak{u}(2)$ that happens when $\lambda_1^{(1)} = \lambda_2^{(1)}$.

Let $M_\phi = Sp(3)/U(3)$; then

$$\mathcal{C}(N_\phi) = \{(\lambda_i^{(k)}) \in \mathbb{R}^6\,|\,0 \leq \lambda_1^{(1)} \leq \lambda_1^{(2)} \leq \lambda_2^{(1)} \leq \lambda_2^{(2)} \leq \lambda_3^{(1)} \leq 2,$$
$$\lambda_1^{(2)} \leq \lambda_1^{(3)} \leq \lambda_2^{(2)}\}.$$

The singularity locus is reached on the boundary of the Weyl chamber of $\mathfrak{g}_1 = \mathfrak{u}(3)$ and $\mathfrak{g}_2 = \mathfrak{u}(2)$, that is when $\lambda_1^{(1)} = \lambda_2^{(1)}$, $\lambda_2^{(1)} = \lambda_3^{(1)}$ or $\lambda_1^{(2)} = \lambda_2^{(2)}$.

9. $M_\phi = SO(2n)/U(n)$

Let us consider $M_\phi = SO(2n)/U(n)$. We showed in Section 6.3 that the moment map in the fundamental representation of $\mathfrak{g}_k = \mathfrak{u}(n+1-k) \subset \mathfrak{so}(2n)$ appearing in the chain (28) solves the master equation (21). By applying Proposition 6.1 we get the Njienhuis eigenvalues.

In the following theorem we prove that these are all the eigenvalues and that they are independent, by proving the complete integrability of the collective hamiltonians defined by the chain (28).

Theorem 9.1. *The collective hamiltonians $F(\mathfrak{u}(n), \ldots, \mathfrak{u}(1))$ define a completely integrable model on $M_\phi = SO(2n)/U(n)$. The Nijenhuis tensor N_ϕ*

(20) is of maximal rank and its ordered eigenvalues are

$$(36) \qquad \lambda_i^{(k)} = 1 - 2\tilde{\lambda}_i^{(k)}, \quad 1 \le i \le n + 1 - k \le n$$

where $i\tilde{\lambda}_i^{(k)}$ are the eigenvalues of the moment map $\mu_{\mathfrak{u}(n+1-k)}$. The bihamiltonian polytope is the $n(n-1)/2$-dimensional $\mathcal{C}(N_\phi) \subset \mathbb{R}^{n(n+1)/2}$ where $(\lambda_i^{(k)}) \in \mathcal{C}(N_\phi)$ if, for $n = 2N$,

$$-1 \le \lambda_{2N}^{(1)} = \lambda_{2N-1}^{(1)} \le \cdots \le \lambda_2^{(1)} = \lambda_1^{(1)} \le 3, \quad \lambda_i^{(k)} \le \lambda_i^{(k+1)} \le \lambda_{i+1}^{(k)},$$

and, for $n = 2N + 1$,

$$-1 = \lambda_{2N+1}^{(1)} \le \lambda_{2N}^{(1)} = \lambda_{2N-1}^{(1)} \le \cdots \le \lambda_2^{(1)} = \lambda_1^{(1)} \le 3, \quad \lambda_i^{(k)} \le \lambda_i^{(k+1)} \le \lambda_{i+1}^{(k)}.$$

Proof. If we write $X \in M_\phi$ in a block form with $X_{ij} \in M_n(\mathbb{R})$,

$$X = g\rho_\phi g^{-1} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad g \in SO(2n),$$

the moment map for the $\mathfrak{u}(n)$ action is

$$\mu_{\mathfrak{u}(n)}(X) = X_{11} + X_{22} + i(X_{12} - X_{21}),$$

so that if $\tilde{\lambda} = \mathrm{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$, a generic matrix in $\mu_{\mathfrak{u}(n)}^{-1}(i\tilde{\lambda})$ can be written as

$$(37) \qquad Z = \begin{pmatrix} A & B + \tilde{\lambda}/2 \\ B - \tilde{\lambda}/2 & -A \end{pmatrix},$$

with A, B antisymmetric satisfying

$$(38) \qquad [A, B] = [A, \tilde{\lambda}] = [B, \tilde{\lambda}] = 0, \qquad A^2 + B^2 = \frac{1}{4}(\tilde{\lambda}^2 - 1).$$

as a consequence of $X^2 = \rho_\phi^2 = -1/4$.

By using the Weyl group of $U(n)$ we can take $\tilde{\lambda}_{i-1} \le \tilde{\lambda}_i$ so that

$$\tilde{\lambda} = \mathrm{diag}\left(\tilde{\lambda}_1 1_{m_1}, \ldots, \tilde{\lambda}_s 1_{m_s}\right),$$

i.e. $\tilde{\lambda}_i$ has multiplicity m_i, $\sum m_i = n$. The condition (38) implies that A, B are block diagonal $A = \mathrm{diag}\,(A_1, \ldots, A_s)$, $A_i \in M_{m_i}(\mathbb{R})$, and the same goes for B. If m_i is odd, then $A_i^2 + B_i^2 = 1/4(\tilde{\lambda}_i^2 - 1)1_{m_i}$ implies $\tilde{\lambda}_i = \pm 1$. We

exclude first the possibility of $\tilde{\lambda}_i = -1$. Let $A_t = tA$, $B_t = tB$ and $\tilde{\lambda}_{it} = \text{sgn}\,\tilde{\lambda}_i \cdot (1 - t^2 + t^2\tilde{\lambda}_i^2)^{1/2}$ (take $\text{sgn}\,(0) = \pm 1$ does not matter), then Z_t as given in (37) is a family of $\mathfrak{sp}(n)$ matrices. Since M_ϕ is the orbit of $SO(2n)$, then the Pfaffian $pf(Z_t)$ is equal to $pf(\rho_\phi) = (1/2)^n$; by evaluating it in $t = 0$ we get

$$\text{sgn}\prod \tilde{\lambda}_i^{m_i} = +1.$$

Hence $\tilde{\lambda}_i = -1$ with m_i odd is excluded.

Thus one must always have even m_i, except possibly the last m_s odd when $\tilde{\lambda}_s = 1$. Thus for $n = 2N$ all m_i are even while for $n = 2N + 1$, all but the last m_i are even. We have then the Gelfand-Tsetlin pattern

$$-1 \le \tilde{\lambda}_1 = \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_{n-1} = \tilde{\lambda}_n \le 1, \quad n = 2N$$
$$-1 \le \tilde{\lambda}_1 = \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_{n-2} = \tilde{\lambda}_{n-1} \le \tilde{\lambda}_n = 1, \quad n = 2N + 1$$

To solve for A, B, we focus on the dense open subset where maximal amount of eigenvalues are distinct, thus $m_i = 2$ for $n = 2N$ case, and $m_i = 2$, $i = 1, \ldots, N$, $m_{N+1} = 1$ for $n = 2N + 1$. Then for both cases, one writes $A_i = a_i\sigma$, $B_i = b_i\sigma$, with σ denoting the 2×2 antisymmetric matrix, and

$$(39) \qquad a_i^2 + b_i^2 = \frac{1}{4}(1 - \tilde{\lambda}_i^2), \quad i = 1, \ldots, N.$$

To prove the complete integrability we use again Proposition 2.1 showing that for almost all $\tilde{\lambda}$ the action of the stabilizer subgroup $U(n)_{i\tilde{\lambda}}$ on $\mu_{\mathfrak{u}(n)}^{-1}(i\tilde{\lambda})$ is transitive. From (39), the orbits corresponding to fixed $\tilde{\lambda}$ are a product of N-circles. It is a direct check that the action of $U(n)_{i\tilde{\lambda}}$ rotates $a_i + ib_i \to e^{2i\theta}(a_i + ib_i)$. The action is clearly transitive. The same logic applies to n odd. Finally to get (36) one applies Proposition 6.1. $\qquad\square$

Remark 9.1. It is now straightforward to check that the number of independent eigenvalues described in the above theorem is the correct one. In the even case, e.g. for $n = 4$, after renaming the independent eigenvalues the above pattern gives

$$\begin{bmatrix} x_1 & & x_1 & & x_2 & & x_2 \\ & x_1 & & x_3 & & x_2 & \\ & & x_4 & & x_5 & & \\ & & & x_6 & & & \end{bmatrix},$$

and $\dim SO(8)/U(4) = 28 - 16 = 12$. The counting for the general even case $n = 2N$, when $\dim M_\phi = 2N(2N - 1)$, goes as

$$N + (N - 1) + \big(2N - 2\ + 2N - 3\ + \cdots + 1\big) = N(2N - 1).$$

In the odd case, we have for $n = 5$

$$\begin{bmatrix} 1 & x_1 & x_1 & x_2 & x_2 \\ & x_3 & x_1 & x_4 & x_2 \\ & & x_5 & x_6 & x_7 \\ & & & x_8 & x_9 \\ & & & & x_{10} \end{bmatrix},$$

and $\dim SO(10)/U(5) = 45 - 25 = 20$. The general counting for $n = 2N + 1$, when $\dim M_\phi = 2N(2N + 1)$, goes as

$$N + N + \big(2N - 1\ + 2N - 2\ + \cdots + 1\big) = N(2N + 1).$$

10. $M_\phi = SO(n + 2)/SO(n) \times SO(2)$

Let us consider $M_\phi = SO(n + 2)/SO(n) \times SO(2)$; in Section 6.3 we showed that the moment map of the subalgebra

$$\mathfrak{g}_k = \mathfrak{so}(n + 2 - 2k) \oplus \underbrace{\mathfrak{so}(2) \oplus \cdots \mathfrak{so}(2)}_{k}$$

in the representation $(S, i/2, \ldots, i/2)$ solves the master equation (21) so that every eigenvalue defines a Nijenhuis eigenvalue by (22). We show in this section that by varying k we get all Nijenhuis eigenvalues and that they are independent.

We again make contact with the collective hamiltonians defined by (30). This is equivalently described as the space of collective hamiltonians of the reduced chain

$$(40) \quad \mathfrak{so}(2N + 1) \supset \mathfrak{so}(2N - 1) \oplus \mathfrak{so}(2) \supset \cdots \supset \mathfrak{so}(3) \oplus \mathfrak{so}(2) \supset \mathfrak{so}(2) \supset 0,$$
$$\mathfrak{so}(2N) \supset \mathfrak{so}(2N - 2) \oplus \mathfrak{so}(2) \supset \cdots \supset \mathfrak{so}(4) \oplus \mathfrak{so}(2) \supset \mathfrak{so}(2) \oplus \mathfrak{so}(2) \supset 0$$

where the k-th subalgebra of the chain $\mathfrak{g}'_k = \mathfrak{so}(n + 2 - 2k) \oplus \mathfrak{so}(2)$ is the subalgebra of $\mathfrak{so}(n + 2 - 2(k - 1)) \subset \mathfrak{g}'_{k-1}$ corresponding to the non–compact root α_1. If $n + 2 = 2N + 1$ the last step is then $\mathfrak{g}'_N = \mathfrak{so}(2)$, if $n + 2 = 2N$ the last step is then $\mathfrak{g}'_{N-1} = \mathfrak{so}(2) \oplus \mathfrak{so}(2)$. We stress the fact that the difference between the chain (30) and (40) is relevant only for the determination

of the Nijenhuis eigenvalues and not for the definition of the collective hamiltonians.

Theorem 10.1. *The collective hamiltonians $F(\mathfrak{so}(n) \oplus \mathfrak{so}(2), \mathfrak{so}(n-2) \oplus \mathfrak{so}(2), \ldots)$ define a completely integrable model on $M_\phi = SO(n+2)/SO(n) \times SO(2)$. Let $n+2 = 2N$ or $2N+1$. The Nijenhuis tensor (20) is of maximal rank and its eigenvalues are*

$$
(41) \qquad \lambda_{\pm}^{(k)} = \pm|a_k| - \sum_{j=1}^{k} b_j + 1, \ \ k = 1, \ldots N-1, \quad \text{and}
$$

$$
\lambda^{(N)} = 1 - \sum_{j=1}^{N} b_j \quad \text{if } n+2 = 2N+1
$$

where $\pm i a_k$ are the eigenvalues of the moment map $\mu_{\mathfrak{so}(n+2-2k)}$ for $\mathfrak{so}(n+2-2k) \subset \mathfrak{g}'_k$ and $b_k = pf(\mu_{\mathfrak{so}(2)})$ with $\mathfrak{so}(2) \subset \mathfrak{g}'_k$.

The bihamiltonian polytope is then described as $\mathcal{C}(N_\phi) \subset \mathbb{R}^n$, where $(a_k, b_k) \in \mathcal{C}(N_\phi)$ if

$$
(42) \qquad 0 \leq |a_k| \leq |a_{k-1}|, \qquad |b_k| \leq |a_{k-1}| - |a_k|, \quad k = 1, \ldots, N,
$$

$a_0 = 1, a_N = 0$ and, if $n+2 = 2N$, $b_N = 0$.

Proof. Even though the Nijenhuis eigenvalues must be computed from the spin representations, integrability of collective hamiltonians will depend on the properties of the moment maps of (40) in the fundamental representation. We are going first to characterize the coadjoint orbits contained in the image of the moment map of the subalgebras appearing in (40). Let us parametrize $g \in SO(n+2)$ as

$$
(43) \qquad g = \begin{pmatrix} \cdots & \vec{\xi} & \vec{\eta} \\ \cdots & \vec{x} & \vec{y} \end{pmatrix}, \quad \vec{\xi}, \vec{\eta} \in M_{n,1}(\mathbb{R}), \quad \vec{x}, \vec{y} \in M_{2,1}(\mathbb{R}).
$$

Since ρ is written in block diagonal form as $\mathrm{diag}(0_n, \sigma)$ we get

$$
\mu = g \rho g^t = \begin{pmatrix} (\vec{\xi}, \vec{\eta}) \sigma (\vec{\xi}, \vec{\eta})^t & (\vec{\xi}, \vec{\eta}) \sigma (\vec{x}, \vec{y})^t \\ (\vec{x}, \vec{y}) \sigma (\vec{\xi}, \vec{\eta})^t & (\vec{x}, \vec{y}) \sigma (\vec{x}, \vec{y})^t \end{pmatrix} = \begin{pmatrix} h \, \mathrm{diag}(0_{n-2}, a\sigma) h^t & A \\ -A^t & b\sigma \end{pmatrix}
$$

where $b = \det(\vec{x}, \vec{y})$, $a \in \mathbb{R}$ and $h \in SO(n)$. The last equality is just the standard form of a rank 2 antisymmetric matrix. The reduction to $\mathfrak{h}_\phi$ removes the off-diagonal blocks.

The $SO(n) \times SO(2)$ orbits contained in $\mu_{\mathfrak{h}_\phi}(M_\phi)$ are then the orbits $\mathcal{O}_{ab}$ through $\alpha_{ab} \equiv \mathrm{diag}(0_{n-2}, a\sigma, b\sigma)$ parametrized by a, b. Let $\mathfrak{h}_{\alpha_{ab}} \subset \mathfrak{h}_\phi$ the stabilizer subalgebra of α_{ab}. If $a \neq 0$ then $\mathcal{O}_{ab}$ is isomorphic to the compact hermitian symmetric space of $SO(n)$ and $h_{\alpha_{ab}} = \mathfrak{so}(n-2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2)$. A generic point in $\mu_{\mathfrak{h}_\phi}^{-1}(\alpha_{ab})$ is of the form

$$(44) \qquad\qquad P = \begin{pmatrix} 0_{n-2} & 0 & 0 \\ 0 & a\sigma & X \\ 0 & -X^t & b\sigma \end{pmatrix}$$

where $X = (\vec{u}, \vec{v})\sigma(\vec{x}, \vec{y})^t \in M_2(\mathbb{R})$, with $\vec{u}, \vec{v}, \vec{x}, \vec{y} \in \mathbb{R}^2$ satisfying $\det(\vec{u}, \vec{v}) = a$, $\det(\vec{x}, \vec{y}) = b$ and

$$g_4 = \begin{pmatrix} \vec{u} & \vec{v} \\ \vec{x} & \vec{y} \end{pmatrix} \quad g_4 g_4^t = 1.$$

The action of $(g_{n-2}, h, k) \in SO(n-2) \times SO(2) \times SO(2)$ integrating $\mathfrak{h}_{\alpha_{ab}}$ is given by $X \to hXk^t$. By combining left $SO(2)$ action on X and re-parametrization of $(\vec{x}, \vec{y})$ we can choose $(\vec{u}, \vec{v}) = \mathrm{diag}(u, v)$; indeed we can choose $h, k \in SO(2)$ such that

$$X = (\vec{u}, \vec{v})\sigma(\vec{x}, \vec{y})^t = hh^t(\vec{u}, \vec{v})k\sigma k^t(\vec{x}, \vec{y})^t = h\,\mathrm{diag}(u, v)\sigma(\vec{p}, \vec{q})^t.$$

Orthogonality of g_4 then means

$$u^2 + |\vec{p}|^2 = 1 = v^2 + |\vec{q}|^2, \quad \vec{p}^t \vec{q} = 0.$$

Let $\vec{p} = (p_1, p_2) \neq 0$, then $\vec{q} = c(-p_2, p_1)$ for some c. Since $b = \det(\vec{p}, \vec{q}) = c|\vec{p}|^2$ and $a = uv$, we get

$$(45) \qquad\qquad 1 + a^2 - b^2 = u^2 + \frac{a^2}{u^2}.$$

The condition that there are real solutions for u^2, together with the upper bound of $|a|$, gives the range of (a, b)

$$(46) \qquad\qquad b^2 \leq (1 - |a|)^2, \quad |a| \leq 1.$$

The space of solutions to the above equation is just the space of those $\vec{p} \in \mathbb{R}^2$ with $|\vec{p}|^2 = 1 - u^2$ and the right $SO(2)$ action on X is transitive on this circle. We conclude that the action of $\mathfrak{h}_{\alpha_{ab}}$ on $\mu_{\mathfrak{h}_\phi}^{-1}(\alpha_{ab})$ is transitive.

If $a = 0$ then $\mathfrak{h}_{\alpha_{0b}} = \mathfrak{h}_\phi$; moreover $\vec{\xi}, \vec{\eta}$ appearing in (43) are collinear and it can be shown that $|b| \leq 1$, extending (46) to the case $a = 0$.

The orbits of the subgroups appearing in the two chains of (40) will have the same pattern, compact hermitian symmetric spaces or points. We get two new variables (a_k, b_k) for each step, until we get to $\mathfrak{g}'_{N-1}$, which is the last step for the even case. In the odd case there is one more reduction to $\mathfrak{g}'_N = \mathfrak{so}(2)$ that gives us one more b_N variable. In both cases we get

$$n = \frac{1}{2} \dim SO(n+2) - \frac{1}{2} \dim(SO(n) \times SO(2))$$

variables, which is consistent. In order to establish the range of these variables let $\mathcal{O}_{a_k b_k} \subset p_{k-1,k}(\mathcal{O}_{a_{k-1} b_{k-1}})$, where $p_{k-1,k} : \mathfrak{g}'_{k-1} \to \mathfrak{g}'_k$ is the dual of the inclusion map, denote the adjoint orbit of $\alpha_{a_k, b_k} = \mathrm{diag}(0_{n-2k}, a_k \sigma, b_k \sigma) \in \mathfrak{g}'_k$. Then, since $\mathcal{O}_{a_{k-1} b_{k-1}}$ is isomorphic to the $SO(n+2-2(k-1))$ orbit of $a_{k-1} \rho_{k-1}$, where ρ_{k-1} is the normalized generator of the non–compact root, we repeat the above considerations and conclude that $(a = a_k/a_{k-1}, b = b_k/a_{k-1})$ satisfy inequalities (46) and so (42). Moreover, we showed above that the action of the stabilizer subgroup of $\alpha_{a_k, b_k} \in \mathfrak{g}'_k$ is transitive on $p_{k-1,k}^{-1}(\alpha_{a_k, b_k})$. By applying Proposition 2.1 we prove the complete integrability.

Finally we have to compute the eigenvalues of the moment map of $\mathfrak{g}_k$ in the representation $(S, i/2, \dots, i/2)$ in terms of a_k, b_k. Since the weights of the spin representation are $(\pm 1/2, \dots, \pm 1/2)$ they are easily computed as

$$\pm \frac{i}{2} a_k + \frac{i}{2} \sum_{j=1}^{k} b_j.$$

By using Proposition 6.1, these lead to the pointwise eigenvalues of N_ϕ^*

$$(47) \qquad \pm a_i - \sum_{j=1}^{i} b_j + 1 \quad i = 1, \dots, n.$$

$\square$

Remark 10.2. Note that the spin representation for n even is reducible, but it does not have any effect on the proof. Also for n even, the last reduction $\mathfrak{so}(4)^* \to \mathfrak{so}(2)^* \oplus \mathfrak{so}(2)^*$ does not take place through removing the root α_1 as the earlier steps, but this again has no effect on the validity of the proof.

Remark 10.3. Let us identify the hamiltonians of the action of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{so}(n+2)$. Indeed, the b_k are the hamiltonians of the $(n+2-$

k)-th $\mathfrak{so}(2)$, $k \leq N$ when $n + 2 = 2N + 1$ and $k \leq N - 1$ for $n + 2 = 2N$. In the even case, the missing generator is given by the last a_{N-1}. These variables are of course global smooth functions.

Remark 10.4. The value of a appearing in (44) can always be assumed to be non-negative, except in the even case in the last step $\mathfrak{so}(4)^* \rightarrow \mathfrak{so}(2)^* \oplus \mathfrak{so}(2)^*$. Indeed, conjugating P by a rotation of π along, say, the $(n-2)$, $(n-1)$ direction flips $a \rightarrow -a$. If we think to the definition of the action variables described at the end of Section 2.3, then $|a_k|$ is obtained by projecting $\mu_{\mathfrak{so}(n+2-2k)}$ to the positive Weyl chamber.

   In the last even step it is then convenient not to introduce the absolute value in the definition of the Nijenhuis eigenvalue, since a_{N-1} and then $\lambda_{\pm}^{(N-1)}$ are smooth global functions while the absolute value would introduce a singularity.

11. Conclusions

In this paper we proved that the PN structures defined on compact hermitian symmetric spaces are of maximal rank, or equivalently that they define a completely integrable model that admits a bihamiltonian description. In the case of Grassmannians we recover the well known Gelfand-Tsetlin integrable model, so that our result can be phrased by saying that we show that Gelfand-Tsetlin variables are in involution also with respect to the Bruhat-Poisson structure. In the other cases, the results are new also on the symplectic side. From our point of view, it is natural to look for the information about the Poisson pencil that are contained in these models. We collect here some observations that we plan to develop in the future.

1) *Poisson connections.* The Poisson vector bundles discussed in Section 5 are a device for organizing the computations needed for the diagonalization of the Nijenhuis tensor. It would be interesting to understand more deeply their role, in particular in connection with the Gelfand-Tsetlin chain of subgroups; the role of the representations of the Nijenhuis algebroid in the integrable model defined by the canonical hierarchy should also be investigated.

2) *Thimm method and Bruhat-Poisson structure.* We used the Thimm method for proving maximality of the rank of the Nijenhuis tensor. This well known construction in symplectic geometry, which relies on the properties of the moment map of Hamiltonian G–spaces, is compatible here with the Bruhat-Poisson structure. For instance the subgroups appearing in the

Gelfand-Tsetlin chain are all Poisson subgroups. It could be interesting to study the nature of this compatibility, for instance to have an alternative more direct proof that the collective Hamiltonians form an abelian subalgebra with respect to the Bruhat-Poisson bracket.

3) *Geometry of the Poisson pencil and log symplectic structures.* The description of the spectrum of the Nijenhuis tensor N_ϕ gives information on the geometry of the pencil $\pi_t = \pi_0 + t\Omega^{-1}$, where π_0 is the Bruhat-Poisson structures and Ω is the KKS symplectic form. We collect here few basic observations.

The knowledge of eigenvalues allows us to reconstruct the strata of symplectic leaves of a given dimension. In fact, the corank of π_t at a given point is the multiplicity of the eigenvalue $-t$ so that the symplectic foliation can be analyzed by means of the hyperplanes $\mathcal{C}^{(k)}(t)$ of $\mathcal{C}(N_\phi)$ defined as the set of points where the $k-th$ eigenvalue is equal to $-t$. For instance we can conclude that on the complement of the preimage of $\mathcal{C}(t) = \bigcup_k \mathcal{C}^{(k)}(t)$ π_t is nondegenerate; in particular π_t is the inverse of a symplectic form for all t bigger than the radius of the smallest ball containing $\mathcal{C}(N_\phi)$. This behaviour is a clear hint of a *log symplectic structure*, that we plan to discuss in a separate paper. In particular, we plan to investigate the relation with the framework of *tropical moment map* introduced in [7] and the very recent [12].

Moreover, as described at the end of Subsection 2.2, for each t the modular vector field of π_t with respect to the symplectic volume form is given by the symplectic vector field $\Omega_{kks}^{-1}d\mathrm{Tr}N_\phi$. This vector field is not hamiltonian in general for π_t, but it is easy to see that $\log\det(N_\phi + t)$ gives a local hamiltonian, which is well defined provided that no Nijenhuis eigenvalue is equal to $-t$.

4) *Lifting to the symplectic groupoid.* In [2] the Poisson Nijenhuis structure on $\mathbb{C}P_n$ was used to quantize the symplectic groupoid of the Bruhat-Poisson structure. As briefly summarized in the Introduction, the procedure requires the integration to a groupoid cocycle of the Poisson vector field $\Omega_{kks}^{-1}d\lambda$ associated to every Nijenhuis eigenvalue λ. This gives an integrable model on the symplectic groupoid compatible with the multiplication. In this construction, it is crucial that the eigenvalues are smooth global functions. In general we know that the Nijenhuis eigenvalues are globally continuous functions but their differential becomes singular on the boundary of the Weyl chamber of each of the subalgebras appearing in (1). So the singularity locus can be read from our construction and this analysis will be done in a separate paper. In general, it is an interesting problem to put this peculiar

procedure of integration of cocycles under the light of the more canonical integration of Poisson Nijenhuis structures developed in [17].

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