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Canonical bases for the equivariant cohomology and K-theory rings of symplectic toric manifolds

M. Pabiniak and S. Sabatini

Let M be a symplectic toric manifold acted on by a torus $\mathbb{T}$. In this work we exhibit an explicit basis for the equivariant K-theory ring $\mathcal{K}_{\mathbb{T}}(M)$ which is canonically associated to a generic component of the moment map. We provide a combinatorial algorithm for computing the restrictions of the elements of this basis to the fixed point set; these, in turn, determine the ring structure of $\mathcal{K}_{\mathbb{T}}(M)$. The construction is based on the notion of local index at a fixed point, similar to that introduced by Guillemin and Kogan in [GK].

We apply the same techniques to exhibit an explicit basis for the equivariant cohomology ring $H_{\mathbb{T}}(M;\mathbb{Z})$ which is canonically associated to a generic component of the moment map. Moreover we prove that the elements of this basis coincide with some well-known sets of classes: the equivariant Poincaré duals to certain smooth flow up submanifolds, and also the canonical classes introduced by Goldin and Tolman in [GT], which exist whenever the moment map is index increasing.

1. Introduction

Let (M^{2n}, ω) be a compact symplectic toric manifold of dimension $2n$, i.e. a compact symplectic manifold equipped with an effective Hamiltonian action of an n-dimensional torus $\mathbb{T}$ with Lie algebra $\mathfrak{t}$, and let $\psi\colon M \to \mathfrak{t}^*$ be a moment map for the action. Moreover assume that the fixed point set $M^{\mathbb{T}}$ is discrete. Let $\mathbb{H}_{\mathbb{T}}(M)$ denote either the equivariant cohomology ring with $\mathbb{Z}$ coefficients or the equivariant K-theory ring of M. The inclusion $i\colon M^{\mathbb{T}} \hookrightarrow M$ is $\mathbb{T}$-equivariant, hence it induces a map

$$i^*\colon \mathbb{H}_{\mathbb{T}}(M) \to \mathbb{H}_{\mathbb{T}}(M^{\mathbb{T}})\,.$$

Since $M^{\mathbb{T}}$ is discrete the map i^* is always injective (cf. [Ki] for a proof in the equivariant cohomology setting and [GK] for the equivariant K-theory ring). Therefore $\mathbb{H}_{\mathbb{T}}(M)$ can be viewed as a subring of $\mathbb{H}_{\mathbb{T}}(M^{\mathbb{T}})$.

The main goal of this paper is to construct an explicit basis of $\mathbb{H}_{\mathbb{T}}(M)$ canonically associated to a generic component μ of the moment map ψ (see page 1124). The restrictions of the basis elements to the fixed points, i.e. their images in $\mathbb{H}_{\mathbb{T}}(M^{\mathbb{T}})$, determine the equivariant structure constants associated to this basis, hence the ring structure of $\mathbb{H}_{\mathbb{T}}(M)$.

A result of Kirwan guarantees that there always exists a basis for $\mathbb{H}_{\mathbb{T}}(M)$ associated to a generic component of the moment map, the elements of this basis being called *Kirwan classes* (see [Ki] for the equivariant cohomology setting, and Proposition 2.2 for a generalization of this idea to equivariant K-theory). This basis is, however, not unique. Several authors have added different conditions that would ensure this basis to be unique, i.e. to be *canonically* associated to a generic component of the moment map. For example, Guillemin and Zara [GZ02, GZ03] study this problem for the equivariant cohomology ring of *GKM spaces* (see Section 2.2). The elements of the basis they introduce are called equivariant Thom classes, and should be thought of as the "equivariant Poincaré duals" to the closures of the unstable manifolds $W^u(p)$ of a generic component μ of the moment map with respect to an invariant metric. When such closures are smooth, these equivariant Poincaré duals can be computed explicitly in terms of the $\mathbb{T}$-representations on the normal bundle of $\overline{W^u(p)}$. However in general these closures are not smooth, and in this case the restrictions of the equivariant Thom classes to the fixed point set are analyzed by means of the combinatorics of the GKM graph. In [GT] Goldin and Tolman study a similar problem on Hamiltonian $\mathbb{T}$-spaces, and introduce basis elements for the equivariant cohomology ring which are canonically associated to a generic component μ of the moment map. However in both cases, the existence and uniqueness of such classes is proved only under the assumption that μ is *index increasing* (see Definition 4.2). This is satisfied for example when the stable and unstable manifolds of μ with respect to an invariant metric meet transversally. Later work of Zara [Z] provides a basis of $H_{\mathbb{T}}(M;\mathbb{Q})$ also in the non-index increasing case, however the same procedure would not in general produce a basis of $H_{\mathbb{T}}(M;\mathbb{Z})$.

In [GK] Guillemin and Kogan introduce equivariant K-theory classes which are a basis of the equivariant K-theory ring $\mathcal{K}_{\mathbb{T}}(M)$ of a Hamiltonian $\mathbb{T}$-space (as a module over the equivariant K-theory of a point), not necessarily endowed with an index-increasing component of the moment map. However no explicit connection is given between such basis and the “natural” basis given by the K-theoretical equivariant Poincaré duals to the closures of the unstable manifolds (in the case in which there exists an invariant metric for which these are all smooth). To express the extra conditions imposed on Kirwan classes, Guillemin and Kogan introduce the *local index map*, that associates to each class in $\mathcal{K}_{\mathbb{T}}(M)$ and each fixed point $q \in M^{\mathbb{T}}$ an element of $\mathcal{K}_{\mathbb{T}}(\mathrm{pt})$. Note that $\mathcal{K}_{\mathbb{T}}(\mathrm{pt})$ can be identified with $R(\mathbb{T})$, the representation ring of $\mathbb{T}$.

Inspired by this idea, we present a slightly different definition of local index of an equivariant K-theory class τ at a fixed point q

$$\mathrm{Ind}_q\colon \mathcal{K}_{\mathbb{T}}(M) \to R(\mathbb{T}),$$

and give an explicit combinatorial recipe for computing it (see Definition 3.2 in Section 3). Using this notion we introduce a basis for the equivariant K-theory ring of a symplectic toric manifold M which is canonically associated to a generic component of the moment map, both in the index increasing and non-index increasing case. The main result of the paper is the following. Let F_p be the *flow-up* manifold at $p \in M^{\mathbb{T}}$ (see page 1131, Section 3), corresponding to the closure of the unstable manifold at p.

Theorem 1.1. *Let $(M,\omega,\mathbb{T},\psi)$ be a symplectic toric manifold of dimension $2n$, together with a choice of a generic component of the moment map $\mu\colon M \to \mathbb{R}$. Let $\mathcal{K}_{\mathbb{T}}(M)$ be the equivariant K-theory ring of M. Then for each $p \in M^{\mathbb{T}}$ there exists a unique Kirwan class $\tau_p \in \mathcal{K}_{\mathbb{T}}(M)$, called the* **i-canonical class at the fixed point** *p (see Definition 3.5), satisfying*

1) *$\mathrm{Ind}_q(\tau_p) = 1$ for all points $q \in F_p \cap M^{\mathbb{T}}$;*

2) *$Ind_q(\tau_p) = 0$ for all points $q \notin F_p \cap M^{\mathbb{T}}$.*

Moreover, the set $\{\tau_p\}_{p \in M^{\mathbb{T}}}$ is a basis for $\mathcal{K}_{\mathbb{T}}(M)$ as a module over $R(\mathbb{T})$.

Being a Kirwan class means that $\tau_p(p) = \mathsf{e}^-_{\mathbb{T}}(p)$, the equivariant (K-theoretical) Euler class of the negative normal bundle N_p^- of μ at p, and that $\tau_p(q) = 0$ for all $q \in M^{\mathbb{T}}$ with $\mu(q) < \mu(p)$ (see Proposition 2.2).

We prove the claims of the above theorem in two separate Propositions: uniqueness is proved in Proposition 3.7, existence in Proposition 4.1.

Moreover, we show that in the index increasing case these classes are indeed the equivariant Poincaré duals to the flow-up manifolds F_p (see Lemma 4.3 and Prop. 4.5).

Note that we require our classes τ_p to have local index 1 on F_p, not only at p as in [GK]. As a consequence, the trivial bundle $\mathbf{1} \in \mathcal{K}_{\mathbb{T}}(M)$, endowed with the trivial action on the fiber, is an element of our basis of i-canonical classes, while it is a non-trivial $R(\mathbb{T})$-linear combination of the elements of the basis exhibited in [GK]. Moreover, when M is a complex projective space endowed with the standard toric action, the basis of i-canonical classes consists of powers of the (equivariant) prequantization line bundle (see Example 4.6). Another important advantage of our approach is that the local index, and thus also the i-canonical classes, is easy to calculate directly from the combinatorics underlying the symplectic toric manifold, as we demonstrate by various examples. We give explicit formulas for the elements of this basis when the component of the moment map is index increasing, and inductive formulas otherwise.

The definition of local index can also be translated to the equivariant cohomology setting (Section 5), thus allowing us to define a canonical basis for the equivariant cohomology ring of M.

Theorem 1.2. *Let $(M, \omega, \mathbb{T}, \psi)$ be a symplectic toric manifold of dimension $2n$, together with a choice of a generic component of the moment map $\mu\colon M \to \mathbb{R}$. Let $H^*_{\mathbb{T}}(M;\mathbb{Z})$ be the equivariant cohomology ring of M with integer coefficients. Then for each $p \in M^{\mathbb{T}}$ there exists a unique Kirwan class $\tau_p \in H^*_{\mathbb{T}}(M;\mathbb{Z})$, called the* **i-canonical class at the fixed point** *p (see Definition 5.5), satisfying*

1) *$Ind_p(\tau_p) = 1$;*

2) *$Ind_q(\tau_p) = 0$ for all points $q \in M^{\mathbb{T}} \setminus \{p\}$.*

The set $\{\tau_p\}_{p \in M^{\mathbb{T}}}$ is a basis for $H^*_{\mathbb{T}}(M;\mathbb{Z})$ as a module over $H^*_{\mathbb{T}}(pt;\mathbb{Z})$.

Indeed, in Proposition 5.17 we prove that the i-canonical classes in equivariant cohomology coincide with the equivariant Poincaré duals to the flow-up manifolds F_p. Moreover, when the chosen component of the moment map is index increasing, they also coincide with the canonical classes introduced by Goldin and Tolman in [GT] (see Proposition 5.10).

Note that the definition of i-canonical classes in equivariant cohomology is not a direct translation of the definition in K-theory. Here we want the local index of τ_p to vanish at all fixed points other than p. The reason for this difference is that we want the class $\mathbf{1} \in H^*_{\mathbb{T}}(M;\mathbb{Z})$ to be one of the elements of the basis of $H^*_{\mathbb{T}}(M;\mathbb{Z})$.

Organization. Section 2 contains the background material and some preliminary results. The definition and properties of the local index are in Section 3. In Section 4 we construct i-canonical classes in equivariant K-theory, thus proving their existence. Section 5 proves similar results in the equivariant cohomology setting. We finish the paper with an appendix about an explicit description of the Kirwan map.

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2. Background

2.1. Hamiltonian spaces

Let (M,ω) be a compact symplectic manifold of dimension $2n$, and $\mathbb{T}$ a compact real torus of dimension d with Lie algebra $\mathfrak{t}$. Suppose that $\mathbb{T}$ acts on (M,ω) in a Hamiltonian fashion, i.e. there exists a $\mathbb{T}$-invariant map $\psi\colon M \to \mathfrak{t}^*$, called **moment map**, satisfying

$$\iota_{\xi^\#}\omega = -d\langle \psi, \xi\rangle \ , \tag{2.1}$$

where $\xi^\#$ denotes the vector field on M associated with the flow of symplectomorphisms generated by $\xi \in \mathfrak{t}$, and $\langle \cdot,\cdot\rangle$ the dual pairing between $\mathfrak{t}^*$

and $\mathfrak{t}$. Unless otherwise stated, we assume the action to be effective, and the fixed point set $M^{\mathbb{T}}$ of the action to be discrete. We refer to $(M, \omega, \mathbb{T}, \psi)$ as a **Hamiltonian $\mathbb{T}$-space**. Recall that when $\dim(\mathbb{T}) = \frac{\dim(M)}{2}$, the Hamiltonian $\mathbb{T}$-space $(M, \omega, \mathbb{T}, \psi)$ is called a **symplectic toric manifold**. Before specializing to the case of symplectic toric manifolds, we first introduce some notions that are be used throughout this note and do not depend on the action being toric.

Let $\mathcal{K}_{\mathbb{T}}(M)$ denote the equivariant K-theory ring of M, i.e. the abelian group associated to the semigroup of isomorphism classes of complex $\mathbb{T}$-vector bundles over M, endowed with the direct sum operation $\oplus$ and the tensor product $\otimes$. Thus, if M is a point, $\mathcal{K}_{\mathbb{T}}(\mathrm{pt})$ is the representation ring of the torus $\mathbb{T}$, henceforth denoted by $R(\mathbb{T})$. Observe that, if $\ell^* \subset \mathfrak{t}^*$ denotes the weight lattice of $\mathfrak{t}^*$ and $\ell^* = \mathbb{Z}\langle x_1, \dots, x_d \rangle$, then $R(\mathbb{T})$ can be identified with the ring of finite sums $\left\{ \sum_{j \in J} n_j e^{2\pi \mathrm{i} w_j} \text{ s.t. } |J| < \infty, n_j \in \mathbb{Z} \text{ and } w_j \in \ell^* \right\}$, or equivalently

$$R(\mathbb{T}) = \mathbb{Z}[e^{2\pi \mathrm{i} x_1}, \dots, e^{2\pi \mathrm{i} x_d}, e^{2\pi \mathrm{i}(-x_1 - x_2 - \dots - x_d)}] \tag{2.2}$$

i.e. $R(\mathbb{T})$ is identified with the character ring of $\mathbb{T}$. The unique map $\pi \colon M \to \{\mathrm{pt}\}$ induces a map $\pi^* \colon R(\mathbb{T}) \to \mathcal{K}_{\mathbb{T}}(M)$ which gives $\mathcal{K}_{\mathbb{T}}(M)$ the structure of an $R(\mathbb{T})$-module.

Observe that the inclusion $i \colon M^{\mathbb{T}} \hookrightarrow M$, which is clearly $\mathbb{T}$-equivariant, gives rise to a map in equivariant K-theory:

$$i^* \colon \mathcal{K}_{\mathbb{T}}(M) \to \mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}}) \, .$$

Since we assume $M^{\mathbb{T}}$ to be discrete we have $\mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}}) = \bigoplus_{p \in M^{\mathbb{T}}} R(\mathbb{T})$, which can be regarded as the ring of maps that assign to each fixed point $p \in M^{\mathbb{T}}$ a representation in $R(\mathbb{T})$. In [GK, Corollary 2.2], the authors prove that for (compact) Hamiltonian $\mathbb{T}$-spaces with discrete fixed point set $M^{\mathbb{T}}$, the above restriction map i^* is **injective** (this result is quoted here in Theorem 2.3; see also [HL, Theorem 2.5] for the case in which M is not necessarily compact). Thus $\mathcal{K}_{\mathbb{T}}(M)$ can be regarded as a subring of $\mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}})$, which is a much easier object to deal with. Henceforth we identify $\mathcal{K}_{\mathbb{T}}(M)$ with $i^*(\mathcal{K}_{\mathbb{T}}(M))$:

$$\mathcal{K}_{\mathbb{T}}(M) \cong i^*(\mathcal{K}_{\mathbb{T}}(M)) \subset \mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}}) \cong \bigoplus_{p \in M^{\mathbb{T}}} R(\mathbb{T}).$$

Let $p \in M^{\mathbb{T}}$ and $i_p^* \colon \mathcal{K}_{\mathbb{T}}(M) \to \mathcal{K}_{\mathbb{T}}(\{p\})$ the map induced by the inclusion $i_p \colon \{p\} \hookrightarrow M$. For every $\tau \in \mathcal{K}_{\mathbb{T}}(M)$ we denote by $\tau(p) \in R(\mathbb{T})$ the value

$i_p^*(\tau)$, and define the **support** of τ to be

$$\operatorname{supp}(\tau) = \{q \in M^{\mathbb{T}} \mid \tau(q) \neq 0\} \subset M^{\mathbb{T}}.$$

Observe that injectivity of i^* implies that $\tau = 0 \in \mathcal{K}_{\mathbb{T}}(M)$ if and only if $\tau(p) = 0$ for all $p \in M^{\mathbb{T}}$, or equivalently if and only if $\operatorname{supp}(\tau) = \emptyset$.

Let $J\colon TM \to TM$ be a $\mathbb{T}$-invariant almost complex structure compatible with ω, i.e. $\omega(J\cdot,\cdot)$ defines an (invariant) inner product on M. If p is a fixed point of the action, the $\mathbb{T}$-action on M induces a representation on $T_pM \simeq \mathbb{C}^n$, called the **isotropy representation of $\mathbb{T}$ at p** , which is given by

$$\exp(\xi)\cdot(z_1,\dots,z_n) = (e^{2\pi \mathrm{i} w_1(\xi)}z_1,\dots,e^{2\pi \mathrm{i} w_n(\xi)}z_n) \text{ for every } \xi \in \mathfrak{t}\,. \tag{2.3}$$

Here the w_i's are well-defined nonzero elements of ℓ^* and are called the **weights (of the isotropy representation of $\mathbb{T}$) at** p. The weights are nonzero because we are assuming $M^{\mathbb{T}}$ to be discrete, and the isotropy action commutes with the $\mathbb{T}$-action on M around p. We denote the set of these weights, counted with multiplicities, by W_p, and the set of all isotropy weights, counted with multiplicities, by $W = \coprod_{p \in M^{\mathbb{T}}} W_p$.

Take a vector $\overline{\xi} \in \mathfrak{t}$ such that $w(\overline{\xi}) \neq 0$ for all $w \in W$, and consider the circle subgroup generated by $\overline{\xi}$, $S^1 = \{\exp(t\overline{\xi}) \mid t \in \mathbb{R}\} \subset \mathbb{T}$. It is well-known (cf. [F]) that the $\overline{\xi}$-component of the moment map $\mu = \psi^{\overline{\xi}}\colon M \to \mathbb{R}$, defined as $\mu(q) = \langle \psi(q), \overline{\xi}\rangle$, is a $\mathbb{T}$-invariant Morse function whose critical set coincides with the fixed point set $M^{\mathbb{T}}$. It is easy to check that equation (2.1) implies that the isotropy weights in the representation on the negative (resp. positive) normal bundle of μ at p, denoted by N_p^- (resp. N_p^+), coincide with the **positive (resp. negative) weights**, i.e. with those w's in W_p such that $w(\overline{\xi}) > 0$ (resp. $w(\overline{\xi}) < 0$). We denote this (multi)set by W_p^+ (resp. W_p^-). Observe that, with this convention, all the weights at the minimum p_0 of μ are negative, so that $W_{p_0} = W_{p_0}^-$.

Let λ_p denote the number of positive weights at p for every $p \in M^{\mathbb{T}}$; then the Morse index of μ at p is precisely $2\lambda_p$.

Definition 2.1. Given $p \in M^{\mathbb{T}}$, the **equivariant (K-theoretical) Euler class** of the negative normal bundle N_p^- of μ at p, denoted by $\mathsf{e}_{\mathbb{T}}^-(p)$, is an element of $\mathcal{K}_{\mathbb{T}}(\{p\})$ defined as

$$\mathsf{e}_{\mathbb{T}}^-(p) = \prod_{w_j \in W_p^+} (1 - e^{2\pi \mathrm{i} w_j})$$

This class plays a key role in the construction of i-canonical classes.

We say that μ **separates fixed points** if $\mu(p) \neq \mu(q)$ for every $p, q \in M^{\mathbb{T}}$ with $p \neq q$. Observe that, since we only deal with symplectic toric manifolds, the moment map for the $\mathbb{T}$-action is always injective when restricted to the fixed point set. Thus, for an open dense subset of $\overline{\xi} \in \mathfrak{t}$, the corresponding μ separates fixed points. We call $\overline{\xi} \in \mathfrak{t}$ **generic** [1] if $w(\overline{\xi}) \neq 0$ for all $w \in W$ and the corresponding μ separates fixed points. Such μ is called a **generic component of the moment map**. Henceforth, we order the fixed points of the action $p_0, \dots, p_N$ in such a way that $\mu(p_0) < \mu(p_1) < \cdots < \mu(p_N)$ and denote this ordering by

$$p_0 \prec p_1 \prec \cdots \prec p_N \,. \tag{2.4}$$

The following proposition is not new, but since it plays a key role in our work, we include the proof for the readers' convenience.

Proposition 2.2. *Let $(M, \omega, \mathbb{T}, \psi)$ be a Hamiltonian $\mathbb{T}$-space, and let $\mu : M \to \mathbb{R}$ be a generic component of the moment map. Then for every $p \in M^{\mathbb{T}}$ there exists a class $\nu_p \in \mathcal{K}_{\mathbb{T}}(M)$, called a* **Kirwan class** *at p, such that*

(i) *$\nu_p(p) = \mathsf{e}_{\mathbb{T}}^{-}(p)$;*

(ii) *$\nu_p(q) = 0$ for every $q \in M^{\mathbb{T}}$ such that $q \prec p$, i.e. $\mu(q) < \mu(p)$.*

Moreover the set $\{\nu_p\}_{p \in M^{\mathbb{T}}}$ is a basis for $\mathcal{K}_{\mathbb{T}}(M)$ as an $R(\mathbb{T})$-module.

Before giving the proof of this Proposition, we recall here a few important facts about the equivariant K-theory ring of Hamiltonian $\mathbb{T}$-spaces. For every $p \in M^{\mathbb{T}}$ and a small $\varepsilon > 0$, let

$$M_p^{\pm} = \{q \in M \mid \mu(q) \leq \mu(p) \pm \varepsilon\}.$$

We don't include ε in the notation as for all sufficiently small ε's homotopy type of the above set is the same. Let D^{λ_p} be a $2\lambda_p$-dimensional disc centered at 0 in the subspace $\mathbb{C}^{\lambda_p} \subset \mathbb{C}^{\lambda_p} \oplus \mathbb{C}^{n-\lambda_p} \simeq T_pM$ corresponding to the λ_p complex coordinates on which the isotropy action (2.3) has positive weights. Note that D^{λ_p} is $\mathbb{T}$-invariant with respect to the isotropy action. By a standard Morse-theoretic argument, for $\varepsilon > 0$ sufficiently small, there exists an equivariant homotopy equivalence between (M_p^+, M_p^-) and $(D^{2\lambda_p}, \partial D^{2\lambda_p})$,

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[1] Note that in literature it is common to call $\overline{\xi}$ generic if $w(\overline{\xi}) \neq 0$ for all $w \in W$.

thus implying that $\mathcal{K}_{\mathbb{T}}(M_p^+, M_p^-) \simeq \mathcal{K}_{\mathbb{T}}(D^{2\lambda_p}, \partial D^{2\lambda_p})$. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{K}_{\mathbb{T}}(M_p^+, M_p^-) \arrow[r, "\alpha_p"] & \mathcal{K}_{\mathbb{T}}(M_p^+) \arrow[d, "\iota_p^*"'] \\
\mathcal{K}_{\mathbb{T}}(\{p\}) \arrow[u, "\mathcal{T}_p"'] \arrow[r] & \mathcal{K}_{\mathbb{T}}(\{p\})
\end{tikzcd} \tag{2.5}$$

where $\mathcal{T}_p$ is the Thom isomorphism, α_p the map in the long exact sequence of the pair (M_p^+, M_p^-) and ι_p^* the restriction map. We have that $\iota_p^* \circ \alpha_p \circ \mathcal{T}_p$ is just the multiplication by $\mathsf{e}_{\mathbb{T}}^-(p)$, which is not a zero divisor in $\mathcal{K}_{\mathbb{T}}(\{p\})$, thus implying that α_p is injective for every $p \in M^{\mathbb{T}}$. This is the main ingredient of the following Theorem (whose proof is omitted here, but the reader can refer to [GK, Lemma 2.1 and Corollary 2.2]):

Theorem 2.3. *For every $p \in M^{\mathbb{T}}$ and $\varepsilon > 0$ (used to define $M_p^{\pm}$) sufficiently small, the K-theory long exact sequence of the pair (M_p^+, M_p^-) splits into short exact sequences*

$$0 \longrightarrow \mathcal{K}_{\mathbb{T}}(M_p^+, M_p^-) \xrightarrow{\alpha_p} \mathcal{K}_{\mathbb{T}}(M_p^+) \xrightarrow{\beta_p} \mathcal{K}_{\mathbb{T}}(M_p^-) \longrightarrow 0 \tag{2.6}$$

Moreover, the following map

$$\mathcal{K}_{\mathbb{T}}(M_p^{\pm}) \to \mathcal{K}_{\mathbb{T}}(M_p^{\pm} \cap M^{\mathbb{T}}) \tag{2.7}$$

is injective for every $p \in M^{\mathbb{T}}$, hence so is

$$i^* \colon \mathcal{K}_{\mathbb{T}}(M) \to \mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}}). \tag{2.8}$$

Finally, the map

$$\mathcal{K}_{\mathbb{T}}(M) \to \mathcal{K}_{\mathbb{T}}(M_p^{\pm}) \tag{2.9}$$

is surjective for every $p \in M^{\mathbb{T}}$.

Proof of Proposition 2.2. Consider any K-theory class ν in $\mathcal{K}_{\mathbb{T}}(M_p^+)$. Recall that we denote $\iota_p^*(\nu)$ by $\nu(p)$. By the exactness of (2.6) and the analysis of the diagram (2.5) done before, we obtain that ν is in $\ker(\beta_p)$ if and only if it satisfies $\nu(p) = f\, \mathsf{e}_{\mathbb{T}}^-(p)$ for some $f \in R(\mathbb{T}) = \mathcal{K}_{\mathbb{T}}(\{p\})$ and $\nu(q) = 0$ for all $q \in M_p^- \cap M^{\mathbb{T}}$. By Theorem 2.3 the restriction map $\mathcal{K}_{\mathbb{T}}(M_p^+) \to \mathcal{K}_{\mathbb{T}}(M_p^+ \cap M^{\mathbb{T}})$ is injective, so specifying f uniquely determines the class ν. By taking

$f = 1$ and extending the class ν to M, which can be achieved by using the surjectivity of (2.9), we obtain a class $\nu_p \in \mathcal{K}_{\mathbb{T}}(M)$ satisfying properties (i) and (ii) in Proposition 2.2, henceforth called a Kirwan class.

Consider a collection of Kirwan classes $\{\nu_p\}_{p\in M^{\mathbb{T}}}$. We first need to prove that they generate $\mathcal{K}_{\mathbb{T}}(M)$ as an $R(\mathbb{T})$-module. Let $\gamma \in \mathcal{K}_{\mathbb{T}}(M)$, and let q_0 be the first fixed point (in the $\prec$ order) where $\gamma(q_0) \neq 0$. Since the restriction of γ to $M_{q_0}^-$ is zero, from what we observed before, and by property (i) of ν_{q_0}, we have $\gamma(q_0) = f_0\, \mathsf{e}_{\mathbb{T}}^-(q_0) = f_0\, \nu_{q_0}(q_0)$, for some $f_0 \in R(\mathbb{T})$. Thus the class $\gamma - f_0\nu_{q_0}$ is zero at q_0, and by property (ii) of ν_{q_0}, the first fixed point q_1 where it doesn't vanish satisfies $q_0 \prec q_1$. By repeating this argument we can construct a class $\gamma - \sum_{i=1}^m f_i\nu_{q_i}$, with $f_i \in R(\mathbb{T})$ for all $i = 1, \ldots, m$, whose restriction to the fixed point set vanishes identically. By the injectivity of (2.8) it follows that $\gamma - \sum_{i=1}^m f_i\nu_{q_i} = 0$, and hence $\{\nu_p\}_{p\in M^{\mathbb{T}}}$ is a set of generators of $\mathcal{K}_{\mathbb{T}}(M)$ as an $R(\mathbb{T})$-module.

Now suppose that $\delta = \sum_{j=0}^s c_j\nu_{p_{i_j}} = 0$, where $c_j \in R(\mathbb{T})$ and $c_j \neq 0$ for every $j = 0, \ldots, s$, and assume that $i_1 < i_2 < \cdots < i_s$. Observe that $\nu_{p_{i_1}}$ is the only class that does not vanish at p_{i_1}, and hence $\delta(p_{i_1}) = c_1\nu_{p_{i_1}}(p_{i_1}) = c_1\mathsf{e}_{\mathbb{T}}^-(p_{i_1}) \neq 0$, which gives a contradiction. We conclude that the set $\{\nu_p\}_{p\in M^{\mathbb{T}}}$ is a basis for $\mathcal{K}_{\mathbb{T}}(M)$ as an $R(\mathbb{T})$-module. ☐

Finally we recall the following Lemma, whose proof follows, *mutatis mutandis*, from that of [GS, Lemma 2.4].

Lemma 2.4. *With the same hypotheses of Proposition 2.2, let* $\{\nu_p\}_{p\in M^{\mathbb{T}}}$ *be a basis of* $\mathcal{K}_{\mathbb{T}}(M)$ *consisting of Kirwan classes. Consider the equivariant structure constants* $\{a_{p,q}^s\}_{p,q,s\in M^{\mathbb{T}}} \subset R(\mathbb{T})$ *associated to the basis* $\{\nu_p\}_{p\in M^{\mathbb{T}}}$*, namely*

$$\nu_p\nu_q = \sum_{s\in M^{\mathbb{T}}} a_{p,q}^s\nu_s.$$

Then there exists an explicit algorithm that computes these equivariant structure constants from the restrictions, $\{i^*(\nu_p)\}_{p\in M^{\mathbb{T}}}$*, of the basis elements to the fixed point set.*

Observe that Kirwan classes are never unique, unless M is a point. Indeed, if ν_p is a Kirwan class at p, then the class

$$\nu_p + \sum_{\{q\in M^{\mathbb{T}} \mid \mu(q)>\mu(p)\}} a_q\nu_q$$

also is, for any set of $a_q \in R(\mathbb{T})$.

In the next sections we introduce "special" Kirwan classes, i.e. Kirwan classes satisfying some extra assumptions that ensure their uniqueness, and compute their restrictions to the fixed point set.

2.2. The equivariant K-theory ring of GKM spaces

Let $(M,\omega,\mathbb{T},\psi)$ be a Hamiltonian $\mathbb{T}$-space (so $M^{\mathbb{T}}$ is discrete) and assume that $\dim(\mathbb{T}) \geq 2$. Then the $\mathbb{T}$-action is called **GKM (Goresky-Kottwitz-MacPherson [GKM])**, or equivalently $(M,\omega,\mathbb{T},\psi)$ is called a **GKM space**, if for every codimension one subtorus $K \subset \mathbb{T}$, the submanifold fixed by K has dimension at most 2. It can be checked that this condition is equivalent to requiring that for every fixed point $p \in M^{\mathbb{T}}$, the weights of the isotropy action at p, $w_1,\dots,w_n \in \ell^* \subset \mathfrak{t}^*$, are pairwise linearly independent. Let $\mathfrak{k}_i \subset \mathfrak{t}$ be $\ker(w_i)$, and $K_i = \exp(\mathfrak{k}_i) \subset \mathbb{T}$. From the definition of GKM space it follows that for each $i = 1,\dots,n$, the connected component of M^{K_i} containing p is a 2-sphere, called **isotropy sphere**. The circle group $\mathbb{T}/K_i$ acts effectively on such sphere, and this action has two fixed points, one of them being p. The combinatorics of the arrangement of isotropy spheres, together with the information on their stabilizers, is encoded in a labeled graph, called **GKM graph** $\Gamma = (V,E)$:

- The vertex set V coincides with the fixed point set $M^{\mathbb{T}}$.
- Given distinct $p,q \in V$, there exists a directed edge $e = \overrightarrow{p\,q}$ from p to q if and only if there exists a 2-sphere S^2 fixed by some codimension one subgroup $K \subset \mathbb{T}$ such that the fixed points of the action of the quotient circle $\mathbb{T}/K$ on S^2 are precisely p and q; we refer to this sphere as the **sphere associated to (the edge)** e.
- Every edge $e = \overrightarrow{p\,q} \in E$ is labeled by a weight $w(e) \in \ell^*$, defined as the weight of the isotropy $\mathbb{T}$-action on T_qS^2, where S^2 is the sphere associated to $e = \overrightarrow{p\,q}$.

Every time p is connected to q by an edge $e = \overrightarrow{p\,q}$ (with weight $w(e)$), then by definition also q is connected to p by an edge $-e = \overrightarrow{q\,p}$ (with weight $w(-e) = -w(e)$). In order to avoid having two edges representing geometrically the same sphere, we choose one of these edges by picking an *orientation* on the edge set E in the following way. Pick a generic $\overline{\xi}$ and let $\mu\colon M \to \mathbb{R}$ be the $\overline{\xi}$-component of the moment map ψ, as defined before. Each isotropy sphere is a symplectic submanifold with an effective Hamiltonian action of a circle with two fixed points p and q. Since $\overline{\xi}$ is generic we have $w(\overline{\xi}) \neq 0$, for all $w \in W$. This implies that $\mu(p) \neq \mu(q)$, and so for each isotropy sphere we choose the directed edge $e = \overrightarrow{p\,q}$ such that $\mu(p) < \mu(q)$. We refer to this graph as the **oriented GKM graph** (associated to $(M, \omega, \mathbb{T}, \psi, \overline{\xi})$) and denote it by $\Gamma^o = (V, E^o)$. We also define an **increasing path** γ from p to q in the oriented GKM graph Γ^o, where $p, q \in V$, to be an ordered sequence of edges in E^o of the form $(\overrightarrow{p\,p_1}, \overrightarrow{p_1\,p_2}, \dots, \overrightarrow{p_j\,q})$. Observe that if $\mu(p) \geq \mu(q)$, (i.e. $p \succeq q$), then the set of increasing paths from p to q is empty.

For a GKM space $(M, \omega, \mathbb{T}, \psi, \overline{\xi})$ with oriented GKM graph $\Gamma^o = (V, E^o)$, the weights in the negative normal bundle at p, namely those in W_p^+, coincide with the weights associated to the edges $e_j \in E^o$ of the form $\overrightarrow{q_j\,p}$. Thus the equivariant (K-theoretical) Euler class of the negative bundle at $p \in M^{\mathbb{T}}$ in Definition 2.1 can be expressed as

$$\mathsf{e}_{\mathbb{T}}^-(p) = \prod_{e_j} (1 - e^{2\pi \mathrm{i} w(e_j)})$$

where the product is over all the λ_p edges $e_j \in E^o$ ending at p, i.e. $e_j = \overrightarrow{q_j\,p} \in E^o$, for some $q_j \in V$.

In analogy with the equivariant cohomology ring, the GKM graph determines which elements in $\mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}})$ come from classes in $\mathcal{K}_{\mathbb{T}}(M)$. This is proved by Knutson and Rosu in the Appendix of [R].

Theorem 2.5 (Knutson, Rosu '03). *Let $(M, \omega, \mathbb{T}, \psi)$ be a GKM space, and $\Gamma = (V, E)$ the associated GKM graph. Then $\tau \in \mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}}) \simeq \bigoplus_{p \in M^{\mathbb{T}}} R(\mathbb{T})$ is an element of $\mathcal{K}_{\mathbb{T}}(M)$ if and only if for every $e = \overrightarrow{p\,q} \in E$*

$$\tau(p) - \tau(q) = \alpha\,(1 - e^{2\pi \mathrm{i} w(e)}) \quad \textit{for some} \quad \alpha \in R(\mathbb{T})\,. \tag{2.10}$$

Observe that the elements $\tau \in \mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}})$ satisfying (2.10) indeed form a ring. Moreover condition (2.10) is equivalent to requiring

$$\tau(p) - \tau(q) = \tilde{\alpha}\,(1 - e^{-2\pi \mathrm{i} w(e)}) \quad \text{for some} \quad \tilde{\alpha} \in R(\mathbb{T}),$$

thus it is sufficient to check condition (2.10) on the edges of the oriented GKM graph.

2.3. Symplectic toric manifolds as GKM spaces

In the following sections we focus on symplectic toric manifolds, i.e. Hamiltonian $\mathbb{T}$-spaces $(M, \omega, \mathbb{T}, \psi)$ with $\dim(\mathbb{T}) = \dfrac{\dim(M)}{2}$. In this case, for every $p \in M^{\mathbb{T}}$ the isotropy weights at p form a $\mathbb{Z}$-basis of ℓ^*, the weight lattice of $\mathfrak{t}^*$, hence they are pairwise linearly independent. Thus symplectic toric manifolds are a special class of GKM spaces. Moreover, their oriented GKM graph can be recovered from the one-skeleton of the image of the moment map $\psi(M)$:

- The vertices of the GKM graph are the vertices of the polytope.
- The oriented edges of the GKM graph are precisely the edges of the polytope, oriented by using μ.
- The weight labeling the edge $e = \overrightarrow{pq}$ is precisely the primitive element $w \in \ell^*$ such that $\psi(q) - \psi(p) = m\, w$, for some $m \in \mathbb{R}_{>0}$.

3. The local index

Let $(M, \omega, \mathbb{T}, \psi)$ be a symplectic toric manifold. In this section, for every $q \in M^{\mathbb{T}}$, we construct a map

$$\mathrm{Ind}_q \colon \mathcal{K}_{\mathbb{T}}(M) \to R(\mathbb{T})$$

called the local index map at q. The procedure to construct $\mathrm{Ind}_q(\tau)$, for $\tau \in \mathcal{K}_{\mathbb{T}}(M)$, is summarised as follows:

1) Construct a smooth, toric $\mathbb{CP}^{\lambda_q}$ via symplectic cutting;
2) Define an equivariant K-theory class $\widetilde{\tau}_q$ on $\mathbb{CP}^{\lambda_q}$ that only depends on $\tau(q)$ and on the positive weights at q (with respect to the circle generated by $\overline{\xi}$);
3) Compute its index on $\mathbb{CP}^{\lambda_q}$.

We now describe the procedure in details.

First of all, recall that the **equivariant K-theory push-forward map** $\mathrm{Ind} \colon \mathcal{K}_{\mathbb{T}}(M) \to R(\mathbb{T})$, also called the **(equivariant) index homomorphism**, can be computed by using the Atiyah-Segal formula [AS] in the following way:

$$\mathrm{Ind}(\tau) = \sum_{p \in M^{\mathbb{T}}} \frac{\tau(p)}{\prod\limits_{w_j \in W_p} (1 - e^{2\pi \mathrm{i} w_j})}\,. \tag{3.1}$$

Note that this is not in general a homomorphism of rings, but only a homomorphism of $R(\mathbb{T})$-modules. As we will see later (Proposition 3.4), the local index map is not a homomorphism of $R(\mathbb{T})$-modules.

For the trivial bundle $\mathbf{1} \in \mathcal{K}_{\mathbb{T}}(M)$ one has:

$$\operatorname{Ind}(\mathbf{1}) = \sum_{p \in M^{\mathbb{T}}} \frac{1}{\prod\limits_{w_j \in W_p} (1 - e^{2\pi \mathrm{i} w_j})}.$$

For any generic $\overline{\xi} \in \mathfrak{t}$, consider the restriction map $r_{\overline{\xi}} \colon \mathcal{K}_{\mathbb{T}}(\mathrm{pt}) \to \mathcal{K}_{S^1}(\mathrm{pt})$, where $S^1 = \exp(t\,\overline{\xi})$, and let $\mu \colon M \to \mathbb{R}$ be the associated $\overline{\xi}$-component of the moment map. Using Corollary 2.7 in [H] for almost complex manifolds (equation (2.8)'), we have that $r_{\overline{\xi}}(\operatorname{Ind}(\mathbf{1}))$ is the number of fixed points with no negative weights w.r.t. μ, which is always 1 as M is connected and the action Hamiltonian. Since $\overline{\xi}$ was generic, we obtain

$$\operatorname{Ind}(\mathbf{1}) = 1. \tag{3.2}$$

(1) Constructing $\mathbb{CP}^{\lambda_q}$.

Let $w_1, \dots, w_{\lambda_q}$ be the weights in the negative normal bundle of μ at $q \in M^{\mathbb{T}}$. Note that $w_j(\overline{\xi}) > 0$ for $j = 1, \dots, \lambda_q$. Define $\mathcal{H}_q$ to be λ_q-dimensional affine subspace of $\mathbb{R}^n$ given by $\psi(q) + \mathbb{R}\langle w_1, \dots, w_{\lambda_q}\rangle$. It is easy to see that $\mathcal{H}_q \cap \psi(M)$ is a λ_q-dimensional face of the polytope $\psi(M)$. Define

$$H_q := \psi^{-1}(\mathcal{H}_q \cap \psi(M)).$$

As we explain below, H_q is a *smooth, symplectic toric* submanifold of M.

The action of $\mathbb{T}$ on H_q is clearly not effective. In fact the subtorus

$$\mathbb{T}_q^0 = \exp(\{\xi \in \mathfrak{t} \mid w_i(\xi) = 0,\ i = 1, \dots, \lambda_q\}) \subset \mathbb{T}$$

acts trivially on an open neighborhood of q in H_q, and hence it acts trivially on H_q. Thus H_q is acted on by the quotient torus $\mathbb{T}_q = \mathbb{T}/\mathbb{T}_q^0$, whose dual Lie algebra $Lie(\mathbb{T}_q)^*$ can be identified with $\mathbb{R}\langle w_1, \dots, w_{\lambda_q}\rangle$. Let ℓ_q^* denote the weight lattice of $Lie(\mathbb{T}_q)^*$, which is given by $\mathbb{R}\langle w_1, \dots, w_{\lambda_q}\rangle \cap \ell^*$. The action of $\mathbb{T}_q$ on H_q is effective, as we now show. Every point $s \in H_q$ fixed by $\mathbb{T}_q$ is also a $\mathbb{T}$-fixed point in M, and the weights $w'_1, \dots, w'_{\lambda_q}$ of the isotropy $\mathbb{T}_q$ action on T_sH_q correspond to those weights $w'_1, \dots, w'_n$ of the isotropy action of $\mathbb{T}$ on T_sM which belong to $\mathbb{R}\langle w_1, \dots, w_{\lambda_q}\rangle$. Moreover, since the $\mathbb{T}$-action on M is toric, we have that $\mathbb{Z}\langle w'_1, \dots, w'_n\rangle = \ell^*$, which implies that $\mathbb{Z}\langle w'_1, \dots, w'_{\lambda_q}\rangle = \ell_q^*$. Thus the $\mathbb{T}_q$ action is effective, and the polytope $\mathcal{H}_q \cap \psi(M) \subset \mathbb{R}\langle w_1, \dots, w_{\lambda_q}\rangle$ is Delzant with respect to the lattice ℓ_q^*. We conclude that $H_q \subset M$, endowed with the restriction of the symplectic form ω and the action of $\mathbb{T}_q$, is a symplectic toric manifold itself.

Moreover, observe that H_q is the "**flow down**" submanifold of q, i.e. the closure of the stable manifold of the gradient flow μ_t of μ (taken with respect to a $\mathbb{T}$-invariant Kähler metric on M). Indeed, the $\mathbb{R}$-action associated with the gradient flow μ_t commutes with the action of the complexified torus $\mathbb{T}_{\mathbb{C}} = \mathbb{T} \otimes \mathbb{C}$ (see for example [A, page 8]), and therefore the stable manifold at q, i.e. $W^s(q) = \{x \in M;\ \lim_{t \to +\infty} \mu_t(x) = q\}$, is invariant under the $\mathbb{T}_{\mathbb{C}}$-action. From the local model of the $\mathbb{T}$-action around q we deduce that H_q and $W^s(q)$ agree around q. This (together with $\mathbb{T}_{\mathbb{C}}$-invariance) also implies that the action of $\mathbb{T}_q^0$ on $W^s(q)$ is trivial, and thus $W^s(q)$ must be contained in a connected component of the fixed point set of $\mathbb{T}_q^0$ containing q, which is exactly H_q. As H_q is closed in M, we deduce that $\overline{W^s(q)} \subset H_q$. Moreover, the invariance implies that $W^s(q)$ must be a union of $\mathbb{T}_{\mathbb{C}}$-orbits. The set $\psi^{-1}(Int\,(\mathcal{H}_q \cap \psi(M)))$ is a $\mathbb{T}_{\mathbb{C}}$-orbit whose intersection with $W^s(q)$ is non-trivial. Thus the whole orbit $\psi^{-1}(Int\,(\mathcal{H}_q \cap \psi(M)))$ must be contained in $W^s(q)$ implying that the closure of this orbit, which is exactly H_q, is contained in the closure of $W^s(q)$. Combining this with the inclusion $\overline{W^s(q)} \subset H_q$ proved above, we deduce that $\overline{W^s(q)} = H_q$.

Similarly, define $\mathcal{F}_q$ to be the $(n - \lambda_q)$-dimensional affine space given by $\psi(q) + \mathbb{R}\langle w_{\lambda_q+1}, \ldots, w_{\lambda_n}\rangle$ and F_q to be

$$F_q = \psi^{-1}(\mathcal{F}_q \cap \psi(M)).$$

Note that F_q is also a closed $\mathbb{T}$-invariant symplectic submanifold of M, which can be thought as the "**flow-up**" submanifold at q, namely the closure of the unstable manifold at q with respect to μ_t. Therefore, *mutatis mutandis*, what is claimed above for H_q also holds for F_q.

Let S^2 be the 2-sphere endowed with the standard symplectic form and a symplectic toric S^1 action rotating the sphere S^2, with speed 1, keeping the north and the south poles fixed. Consider a moment map, $h\colon S^2 \to \mathbb{R}$, such that $0 = h(S) < h(N) = 1$, where N and S denote the north and the south poles respectively. Let $H_q \times S^2$ be the symplectic manifold with symplectic form given by the sum of the pull-backs of the symplectic forms on H_q and S^2. This manifold is endowed with the (non effective) Hamiltonian action of $\mathbb{T} \times S^1$, i.e. $(t, t') * (s, s') = (t * s, t' * s')$ for every $(t, t') \in \mathbb{T} \times S^1$ and $(s, s') \in H_q \times S^2$. Denote by $s_0 = q, s_1, \ldots, s_j$ the $\mathbb{T}$-fixed points in H_q. The $\mathbb{T} \times S^1$ fixed points in $H_q \times S^2$ are given by

$$H_q^{\mathbb{T}} \times \{S\} = \{q_0 = (s_0, S) = (q, S), \ldots, q_j = (s_j, S)\}$$

and

$$H_q^{\mathbb{T}} \times \{N\} = \{q_0' = (s_0, N), \dots, q_j' = (s_j, N)\},$$

for all $i = 0, \dots, j$.

Note that the splitting $\mathbb{T} \times S^1$ allows us to regard $\mathfrak{t}^*$ as a subspace of $Lie(\mathbb{T} \times S^1)^*$. With abuse of notation we consider the weights of the $\mathbb{T}$-action as elements in $Lie(\mathbb{T} \times S^1)^*$. Let $-w_0 \in Lie(\mathbb{T} \times S^1)^*$ be the weight of the isotropy $\mathbb{T} \times S^1$ action at $T_q(\{q_0\} \times S^2) \subset T_{q_0}(H_q \times S^2)$ (see Figure 3.1).

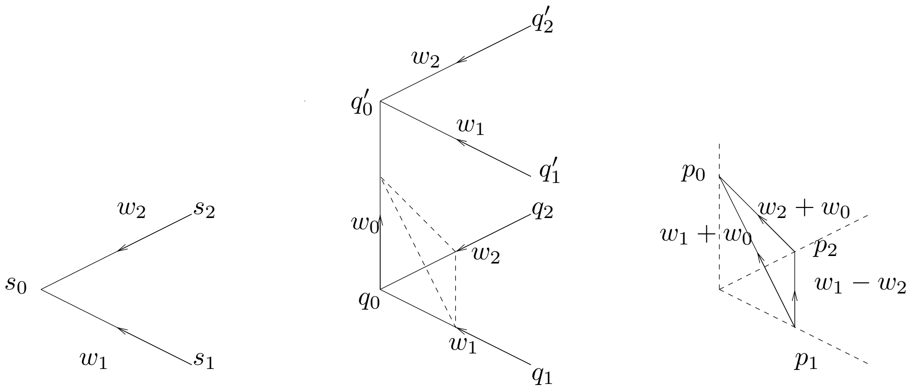

Figure 3.1: Local pictures for H_q, $H_q \times S^2$ and $\widetilde{H}_q^{\epsilon}$.

Around $q_0 = (q, S)$, the moment map ψ' of the Hamiltonian $\mathbb{T} \times S^1$ action on $H_q \times S^2$ can be written as

$$\psi'(z_1, \dots, z_{\lambda_q}, z) = -w_0 \frac{1}{2}|z|^2 + w_1 \frac{1}{2}|z_1|^2 + \cdots + w_{\lambda_q} \frac{1}{2}|z_{\lambda_q}|^2 + \psi(q) \ ,$$

where $(z_1, \dots, z_{\lambda_q}, z)$ are complex coordinates on $H_q \times S^2$ around q_0. Since the $\mathbb{T}$-action on M is toric, the weights of the isotropy $\mathbb{T}$-action at q, given by $w_1, \dots, w_n$, form a $\mathbb{Z}$-basis of ℓ^*, the weight lattice of $\mathfrak{t}^*$; hence $w_0, w_1, \dots, w_n$ is a $\mathbb{Z}$-basis of the dual lattice of $\mathbb{T} \times S^1$. Let ξ_q be the element of $Lie(\mathbb{T} \times S^1)$ such that $w_0(\xi_q) = -1$, $w_i(\xi_q) = 1$ for all $i = 1, \dots, \lambda_q$, and $w_i(\xi_q) = 0$ for all $i = \lambda_q + 1, \dots, n$. Then $C_q = \exp(\mathbb{R}\langle \xi_q \rangle) \subset \mathbb{T} \times S^1$ is a circle acting effectively on $H_q \times S^2$ with moment map φ which around q_0 is given by

$$\varphi(z_1, \dots, z_{\lambda_q}, z) = \frac{1}{2}|z|^2 + \frac{1}{2}|z_1|^2 + \cdots + \frac{1}{2}|z_{\lambda_q}|^2 + \psi(q)(\xi_q) \, .$$

Let $\widetilde{H}_q^\epsilon$ be the symplectic reduction of $H_q \times S^2$ at $\epsilon + \psi(q)(\xi_q)$ with respect to the action of C_q, for $\epsilon > 0$ sufficiently small, i.e.

$$\widetilde{H}_q^\epsilon = \varphi^{-1}(\epsilon + \psi(q)(\xi_q))/C_q.$$

Observe that $\widetilde{H}_q^\epsilon$ is symplectomorphic to the complex projective space $\mathbb{CP}^{\lambda_q}$, endowed with the (Hamiltonian) residual action of $\widetilde{\mathbb{T}}_q = (\mathbb{T} \times S^1)/C_q$. Denote its fixed points by $p_0, p_1, \ldots, p_{\lambda_q}$ with

$$\begin{aligned} p_0 &= \left(\varphi^{-1}(\epsilon + \psi(q)(\xi_q)) \cap (\{q\} \times S^2)\right)/C_q, \\ p_j &= \left(\varphi^{-1}(\epsilon + \psi(q)(\xi_q)) \cap (\psi')^{-1}(\overrightarrow{q_j\, q_0})\right)/C_q \end{aligned}$$

(see Figure 3.1). Moreover, the tuples of isotropy weights of the $\widetilde{\mathbb{T}}_q$-action at the fixed points of $\widetilde{H}_q^\epsilon$ are given by:

$$W_{p_0} = \{w_0 + w_1, \ldots, w_0 + w_{\lambda_q}\}$$

and

$$W_{p_i} = \{-(w_i + w_0), w_1 - w_i, \ldots, w_{i-1} - w_i, w_{i+1} - w_i, \ldots, w_{\lambda_q} - w_i\}$$

for all $i = 1, \ldots, \lambda_q$. Note that all the elements in W_{p_i} vanish on ξ_q, hence they belong to the lattice of $Lie(\widetilde{\mathbb{T}}_q)^*$.

(2) Defining the class $\widetilde{\tau}_q$ on $\mathbb{CP}^{\lambda_q}$.

The definition of such class is divided into two steps, **(i)** and **(ii)**.

(i) *Mapping $\tau \in \mathcal{K}_{\mathbb{T}}(M)$ to $\mathcal{K}_{\mathbb{T}\times S^1}(H_q \times S^2)$:*
Consider the following commuting diagram

$$\begin{tikzcd}
\mathcal{K}_{\mathbb{T}}(M) \arrow[r, "r_1"] \arrow[d] \arrow[drr] & \mathcal{K}_{\mathbb{T}}(H_q) \arrow[r, "r_2"] \arrow[d, dotted, "\widetilde{i}^*\circ\widetilde{r}"] & \mathcal{K}_{\mathbb{T}\times S^1}(H_q \times S^2) \arrow[d, "\widetilde{i}^*"'] \\
\mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}}) \arrow[r] & \mathcal{K}_{\mathbb{T}}(H_q^{\mathbb{T}}) \arrow[r] & \mathcal{K}_{\mathbb{T}\times S^1}((H_q \times S^2)^{\mathbb{T}\times S^1})
\end{tikzcd}$$

where the maps involved are defined as follows. The map r_1 is the restriction induced by the $\mathbb{T}$-equivariant inclusion $H_q \hookrightarrow M$. To define the map $r_2 \colon \mathcal{K}_{\mathbb{T}}(H_q) \to \mathcal{K}_{\mathbb{T}\times S^1}(H_q \times S^2)$ note that, as S^1 acts trivially on H_q and $\mathbb{T}$

acts trivially on S^2, we have a canonical identification

$$\mathcal{K}_{\mathbb{T}\times S^1}(H_q \times S^2) \cong \mathcal{K}_{\mathbb{T}}(H_q) \otimes \mathcal{K}_{S^1}(S^2).$$

Define r_2 as tensoring with $\mathbf{1} \in \mathcal{K}_{S^1}(S^2)$, i.e. $r_2(r_1(\tau))$ is the class $r_1(\tau) \otimes \mathbf{1}$ regarded as an element of $\mathcal{K}_{\mathbb{T}\times S^1}(H_q \times S^2)$ under the above identification. The vertical arrows are the restrictions of the equivariant K-theory rings to those of the fixed point sets; in particular $\widetilde{i}\colon (H_q \times S^2)^{\mathbb{T}\times S^1} \hookrightarrow H_q \times S^2$. Let

$$\widetilde{r} := r_2 \circ r_1.$$

In Step **(i)** we map $\tau \in \mathcal{K}_{\mathbb{T}}(M)$ to $\widetilde{r}(\tau) \in \mathcal{K}_{\mathbb{T}\times S^1}(H_q \times S^2)$. In practice, we work with $\widetilde{i}^* \circ \widetilde{r}(\tau) \in \mathcal{K}_{\mathbb{T}\times S^1}\big((H_q \times S^2)^{\mathbb{T}\times S^1}\big)$. An explicit procedure is given as follows:

- determine the restrictions of $\tau \in \mathcal{K}_{\mathbb{T}}(M)$ to $H_q^{\mathbb{T}} = \{s_0 = q, \dots, s_j\}$;
- then $\widetilde{i}^* \circ \widetilde{r}(\tau)$ is $\tau(s_l)$ at q_l and q_l', for all $l = 0, \dots, j$.

Note that these restrictions live in $R(\mathbb{T})$, which we regard as a subring of $R(\mathbb{T}\times S^1)$ using the identifications

$$\begin{aligned} R(\mathbb{T}) &= \mathbb{Z}[e^{2\pi \mathrm{i} w_1}, \dots, e^{2\pi \mathrm{i} w_n}, e^{2\pi \mathrm{i}(-w_1 - \dots - w_n)}] \quad \text{and} \\ R(\mathbb{T}\times S^1) &= \mathbb{Z}[e^{2\pi \mathrm{i} w_0}, e^{2\pi \mathrm{i} w_1}, \dots, e^{2\pi \mathrm{i} w_n}, e^{2\pi \mathrm{i}(-w_1 - \dots - w_n - w_0)}] \quad \text{(see (2.2))}. \end{aligned}$$

(ii) *Defining* $\widetilde{\tau}_q$ *as the image of* $\widetilde{r}(\tau) \in \mathcal{K}_{\mathbb{T}\times S^1}(H_q \times S^2)$ *under a map*

$$\kappa\colon \mathcal{K}_{\mathbb{T}\times S^1}(H_q \times S^2) \to \mathcal{K}_{\widetilde{\mathbb{T}}_q}(\widetilde{H}_q^{\epsilon}) \tag{3.3}$$

described below.

In practice, given $\widetilde{r}(\tau) \in \mathcal{K}_{\mathbb{T}\times S^1}(H_q \times S^2)$ we define the class $\widetilde{\tau}_q := \kappa(\widetilde{r}(\tau))$ by specifying its restrictions to each of the $\lambda_q + 1$ fixed points $p_0, p_1, \dots, p_{\lambda_q}$ of $\widetilde{H}_q^{\epsilon} \simeq \mathbb{CP}^{\lambda_q}$ using the recipe below. The value $\widetilde{r}(\tau)(q_0)$ (equal to the value $\tau(s_0)$) is an element of $R(\mathbb{T})$, which we regard as a subring of $R(\mathbb{T}\times S^1)$. Since the weights $w_1, \dots, w_n$ form a $\mathbb{Z}$-basis of ℓ^*, we can express $\tau(q_0)$ as $f(e^{2\pi \mathrm{i} w_1}, \dots, e^{2\pi \mathrm{i} w_n}, e^{2\pi \mathrm{i}(-w_1 - \dots - w_n)})$, where $f(X_1, \dots, X_{n+1})$ is a polynomial with integer coefficients. Let

$$f_0 = f\big(e^{2\pi \mathrm{i}(w_0 + w_1)}, \dots, e^{2\pi \mathrm{i}(w_0 + w_{\lambda_q})}, e^{2\pi \mathrm{i} w_{\lambda_q + 1}}, \dots, e^{2\pi \mathrm{i} w_n}, e^{2\pi \mathrm{i}(-w_1 - \dots - w_n - \lambda_q w_0)}\big)$$

and

$$f_j = f\big(e^{2\pi \mathrm{i}(w_1 - w_j)}, \dots, e^{2\pi \mathrm{i}(w_{\lambda_q} - w_j)}, e^{2\pi \mathrm{i} w_{\lambda_q + 1}}, \dots, e^{2\pi \mathrm{i} w_n}, e^{2\pi \mathrm{i}(-w_1 - \dots - w_n + \lambda_q w_j)}\big),$$

for all $j = 1, \dots, \lambda_q$ (i.e. the j-th argument of f_j is $e^0 = 1$). Define the restriction of $\widetilde{\tau}_q$ to the fixed point set to be f_j at the point p_j, for all $j = 0, \dots, \lambda_q$. Observe that this element of $\mathcal{K}_{\widetilde{\mathbb{T}}_q}((\widetilde{H}_q^\epsilon)^{\widetilde{\mathbb{T}}_q})$ does indeed represent a class in $\mathcal{K}_{\widetilde{\mathbb{T}}_q}(\widetilde{H}_q^\epsilon)$. By Theorem 2.5 it is sufficient to check that $f_0 - f_j \equiv 0 \mod (1 - e^{2\pi \mathrm{i}(w_0 + w_j)})$ and $f_i - f_j \equiv 0 \mod (1 - e^{2\pi \mathrm{i}(w_i - w_j)})$, i.e. one needs to check that f_0 is equal to f_j when setting $w_0 + w_j = 0$ and f_i is equal to f_j when setting $w_i - w_j = 0$. These follow easily from the definition of the f_i's.

Remark 3.1. It is worth observing (though we are not going to use it) that the above map κ is the Kirwan map relating the K-theory rings of a manifold and of its symplectic reduction. We devote the Appendix to a careful description of the Kirwan map.

(3) Computing the index of $\widetilde{\tau}_q$.

Let $\alpha_q \colon R(\widetilde{\mathbb{T}}_q) = \mathcal{K}_{\widetilde{\mathbb{T}}_q}(\mathrm{pt}) \to \mathcal{K}_{\mathbb{T}}(\mathrm{pt}) = R(\mathbb{T})$ be the homomorphism sending w_0 to 0 and w_j to itself, for all $j \neq 0$.

Definition 3.2. The **local index** of τ at q, denoted by $\mathrm{Ind}_q(\tau)$, is the element of $R(\mathbb{T})$ defined as

$$\mathrm{Ind}_q(\tau) := \alpha_q \circ \mathit{Ind}(\widetilde{\tau}_q)\,,$$

where $\mathit{Ind} \colon \mathcal{K}_{\widetilde{\mathbb{T}}_q}(\widetilde{H}_q^\epsilon) \to R(\widetilde{\mathbb{T}}_q)$ is the index homomorphism. The **local index map** at $q \in M^{\mathbb{T}}$ is the map

$$\mathrm{Ind}_q \colon \mathcal{K}_{\mathbb{T}}(M) \to R(\mathbb{T}) \tag{3.4}$$

that assigns $\mathrm{Ind}_q(\tau)$ to τ.

Note that the only information needed for computing $\mathrm{Ind}_q(\tau)$ are the weights of $\mathbb{T}$-action on T_qH_q and the value of τ at q. Moreover the computation is relatively easy thanks to the combinatorial recipe for computing the restriction of $(\kappa \circ \widetilde{r})(\tau) \in \mathcal{K}_{\widetilde{\mathbb{T}}_q}(\widetilde{H}_q^\epsilon)$ to the $\widetilde{\mathbb{T}}_q$ fixed point set given in **(ii)**, and thanks to the Atiyah-Segal formula (3.1).

Example 3.3. As an example, we calculate the local index of an equivaraint K-theory class of the Hirzebruch surface. Let τ denote the class presented at the second picture in Figure 4.2 on page 1148 and q denote the fixed point where its value is $(1 - e^{2\pi i(x-y)})e^{2\pi i y}$. Observe that the weights at q are y and $x-y$. Following the notation from the definition of the local index, we have

$$\tau(q) = f(e^{2\pi i y}, e^{2\pi i(x-y)}, e^{2\pi i(-x)}) = (1-e^{2\pi i(x-y)})e^{2\pi i y},$$
$$\kappa(\tau)(p_0)=f_0=f(e^{2\pi i(y+w_0)}, e^{2\pi i(x-y)}, e^{2\pi i(-x-w_0)})=(1-e^{2\pi i(x-y)})e^{2\pi i(y+w_0)},$$
$$\kappa(\tau)(p_1) = f_1 = f(1, e^{2\pi i(x-y)}, e^{2\pi i(-x+y)}) = 1-e^{2\pi i(x-y)}.$$

The values of the class $\widetilde{r}(\tau)$ at the fixed points of $H_q \times S^2$ close to (q, S) are presented on the left side of Figure 3.2, while the picture on the right represents the values of the class $\kappa(\tau)$ (c.f. the Appendix).

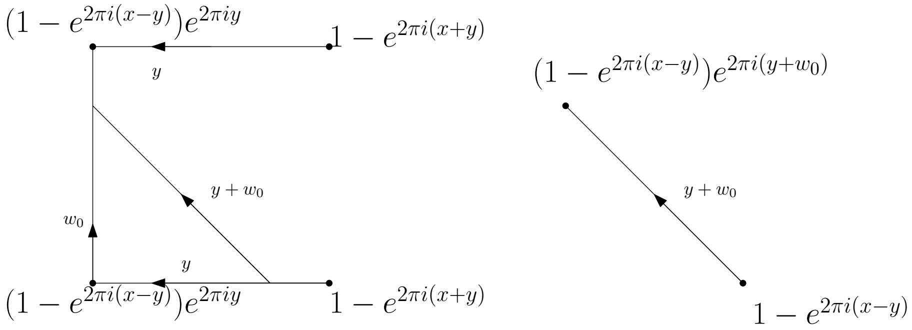

Figure 3.2: The values of the class $\widetilde{r}(\tau)$ at the fixed points of $H_q \times S^2$ close to (q, S) and the values of the class $\kappa(\tau)$ at the fixed points of H_q^ϵ.

The local index of τ at q is equal to

$$\begin{aligned}\alpha_q(\mathrm{Ind}(\widetilde{\tau}_q)) &= \alpha_q\left(\frac{(1-e^{2\pi i(x-y)})e^{2\pi i(y+w_0)}}{1-e^{2\pi i(y+w_0)}} + \frac{1-e^{2\pi i(x-y)}}{1-e^{-2\pi i(y+w_0)}}\right)\\ &= \alpha_q\left(-\frac{1-e^{2\pi i(x-y)}}{1-e^{-2\pi i(y+w_0)}} + \frac{1-e^{2\pi i(x-y)}}{1-e^{-2\pi i(y+w_0)}}\right) = \alpha_q(0) = 0.\end{aligned}$$

In the next proposition we prove some properties of the local index. For the construction of our canonical classes, only properties (i), (ii) and (iii) are needed, whereas properties (iv) and (v) are used in Subsection 3.1 to compare our results with those of Guillemin and Kogan [GK].

Proposition 3.4. *The local index satisfies the following properties:*

(i) *(Additivity) for any* $\tau, \tau' \in \mathcal{K}_{\mathbb{T}}(M)$, *and any* $q \in M^{\mathbb{T}}$ *have*

$$Ind_q(\tau + \tau') = Ind_q(\tau) + Ind_q(\tau').$$

(ii) *If* $\tau(q) = \alpha\, \mathsf{e}^-_{\mathbb{T}}(q)$ *for some* $\alpha \in R(\mathbb{T})$ *then*

$$Ind_q(\tau) = \alpha = \tau(q)/\mathsf{e}^-_{\mathbb{T}}(q).$$

In particular $Ind_q(\tau) = 0$ *if* $\tau(q) = 0$.

(iii) *If* $\tau'(q) = 0$ *then* $Ind_q(\tau + \alpha\,\tau') = Ind_q(\tau)$, *for every* $\alpha \in R(\mathbb{T})$.

(iv) *The local index map* $Ind_q \colon \mathcal{K}_{\mathbb{T}}(M) \to R(\mathbb{T})$ *is not a morphism of* $R(\mathbb{T})$*-modules.*

(v) *The local index map* $Ind_q \colon \mathcal{K}_{\mathbb{T}}(M) \to R(\mathbb{T})$ *is a morphism of* $R(\mathbb{T}/C^0_q)$*-modules, where* C^0_q *is the image of the circle* $C_q \subset \mathbb{T} \times S^1$ *used in the reduction under the map* $\mathbb{T} \times S^1 \to \mathbb{T}$.

Note that the condition in (ii) is satisfied for instance whenever $\tau(q') = 0$ for all $q' \prec q$, see the proof of Proposition 2.2.

Proof. Property (i) follows from additivity of the index homomorphism. To prove (ii), present α as $f(e^{2\pi \mathsf{i} w_1}, \dots, e^{2\pi \mathsf{i} w_n}, e^{2\pi \mathsf{i}(-w_1-\cdots-w_n)})$, where $f(X_1, \dots, X_{n+1})$ is a polynomial with integer coefficients; here $w_1, \dots, w_n$ are the weights of $\mathbb{T}$-action on T_qM, and $w_1, \dots, w_{\lambda_q}$ are the weights of $\mathbb{T}$-action on T_qH_q, i.e. the weights in W_q^+ (cf. page 1134). Let f_0 be the polynomial obtained from f by substituting w_j with $w_j + w_0$ for $j = 1, \dots, \lambda_q$. Then the values of $\kappa(\widetilde{r}(f\tau))$ at the fixed points $p_0, \dots, p_{\lambda_q}$ of $\widetilde{H}^\epsilon_q$ are (respectively)

$$f_0 \prod_{j=1}^{\lambda_q} (1 - e^{2\pi \mathsf{i}(w_j + w_0)}), 0, \dots, 0.$$

Therefore, from (3.1) it follows that the index of $\kappa(\widetilde{r}(f\tau))$ is f_0, and the local index $\mathrm{Ind}_q(f\tau)$ is the image of f_0 under the homomorphism $\mathcal{K}_{\widetilde{\mathbb{T}}_q}(\mathrm{pt}) \to \mathcal{K}_{\mathbb{T}}(\mathrm{pt})$ sending w_0 to 0, so it is f. Property (iii) follows from the first two.

In order to prove (iv) it is sufficient to find $\alpha \in R(\mathbb{T})$ and $\tau \in \mathcal{K}_{\mathbb{T}}(M)$ such that $\alpha \cdot \mathrm{Ind}_q(\tau) \neq \mathrm{Ind}_q(\alpha \cdot \tau)$. Take τ to be $\mathbf{1}$, the trivial line bundle over M with trivial action on the fibers, and α to be $e^{2\pi \mathsf{i}(-w_1-\cdots-w_n)}$. Then, following the procedure on page 1134, the value of $\kappa(\widetilde{r}(\mathbf{1}))$ at the fixed points $p_0, p_1 \dots, p_{\lambda_q}$ is always one, and equation (3.2) implies that $e^{2\pi \mathrm{i}(-w_1-\dots-w_n)}\mathrm{Ind}_q(\mathbf{1}) = e^{2\pi \mathrm{i}(-w_1-\dots-w_n)}$. However we prove that

$$\mathrm{Ind}_q(e^{2\pi \mathrm{i}(-w_1-\dots-w_n)}) = 0.$$

The value of $\kappa(\widetilde{r}(e^{2\pi \mathrm{i}(-w_1-\dots-w_n)}))$ at the fixed point p_j is $e^{2\pi \mathrm{i} a_j}$, where $a_0 = -w_1 - \dots - w_n - \lambda_q w_0$ and $a_j = -w_1 - \dots - w_n + \lambda_q w_j$ for all $j = 1, \dots, \lambda_q$. We prove that there exists an equivariant cohomology class $a \in H^2_{\mathbb{T}}(\mathbb{CP}^{\lambda_q}; \mathbb{Z})$ such that its restriction to the fixed point p_j is exactly a_j, for all $j = 0, \dots, \lambda_q$. Let $\eta \in H^2_{\mathbb{T}}(\mathbb{CP}^{\lambda_q}; \mathbb{Z})$ be the equivariant Poincaré dual to the $\mathbb{CP}^{\lambda_q - 1} \subset \mathbb{CP}^{\lambda_q}$ corresponding to the $(\lambda_q - 1)$-dimensional sub-simplex spanned by $p_1, \dots, p_{\lambda_q}$. Hence the value of η at p_0 is zero, and at p_j is $-(w_0 + w_j)$, for all $j = 1, \dots, \lambda_q$. It is easy to check that $\widetilde{\mathrm{c}}_1 = (\lambda_q + 1)\eta + \beta$ for some $\beta \in H^2_{\mathbb{T}}(pt; \mathbb{Z})$, where $\widetilde{\mathrm{c}}_1$ is the first equivariant Chern class of $\mathbb{CP}^{\lambda_q}$. Then the class $a = (-\lambda_q)\eta + a_0$ is in $H^2_{\mathbb{T}}(\mathbb{CP}^{\lambda_q}; \mathbb{Z})$, and its value at p_j is exactly a_j, for all $j = 0, \dots, \lambda_q$. Consider a generic $\overline{\xi} \in \mathfrak{t}$ (see 1124) generating a circle subgroup $S^1 \subset \mathbb{T}$ and let $r_{\overline{\xi}} \colon R(\mathbb{T}) \to R(S^1)$ be the restriction map. Theorem 1.1 and Corollary 1.2 in [HY] imply that there exists an equivariant line bundle, denoted by $e^{2\pi \mathrm{i} a}$, whose equivariant first Chern class is exactly a. From [S, Prop. 3.9] it follows that $r_{\overline{\xi}}(\mathrm{Ind}(e^{2\pi \mathrm{i} a})) = 0$ for all such $\overline{\xi}$'s. (Observe that the first Chern class of this line bundle is $-\frac{c_1(T\mathbb{CP}^{\lambda_q})}{\lambda_q + 1}$.) Since $\overline{\xi}$ is generic, this implies that $\mathrm{Ind}(e^{2\pi \mathrm{i} a}) = 0$, and hence $\mathrm{Ind}_q(e^{2\pi \mathrm{i}(-w_1-\dots-w_n)}) = 0$.

Property (v) can be proved as follows. Observe that C_q^0 is the circle subgroup of $\mathbb{T}$ generated by $\xi_q^0 \in Lie(\mathbb{T})$ such that ξ_q^0 is 1 when evaluated at the λ_q positive weights at q and is 0 when evaluated at the $n - \lambda_q$ negative weights at q. Every element of $R(\mathbb{T}/C_q^0)$ is the evaluation at $e^{2\pi \mathrm{i} w_1}, \dots, e^{2\pi \mathrm{i} w_n}, e^{2\pi \mathrm{i}(-w_1-\dots-w_n)}$ of a polynomial g in $n+1$ variables with integer coefficients (cf. 1134) satisfying

$$g(X_1, \dots, X_{n+1}) = g(aX_1, \dots, aX_{\lambda_q}, X_{\lambda_q+1}, \dots, X_{n+1}) \quad \text{for all } a \in \mathbb{C}.$$

In order to pass from a computation of the local index of a class τ at q to the computation of the local index of $g\tau$ at q, one needs to multiply the polynomials f_0 and f_j from page 1134 respectively by

$$g_0 = g(e^{2\pi \mathrm{i}(w_0+w_1)}, \dots, e^{2\pi \mathrm{i}(w_0+w_{\lambda_q})}, e^{2\pi \mathrm{i} w_{\lambda_q+1}}, \dots, e^{2\pi \mathrm{i} w_n}, e^{2\pi \mathrm{i}(-w_1-\dots-w_n)})$$

and

$$g_j = g(e^{2\pi \mathrm{i}(w_1-w_j)}, \dots, e^{2\pi \mathrm{i}(w_{\lambda_q}-w_j)}, e^{2\pi \mathrm{i} w_{\lambda_q+1}}, \dots, e^{2\pi \mathrm{i} w_n}, e^{2\pi \mathrm{i}(-w_1-\dots-w_n)})$$

for all $j = 1, \dots, \lambda_q$. The invariance of g gives that

$$g_0 = g_1 = \cdots = g_{\lambda_q} = g(e^{2\pi \mathrm{i} w_1}, \dots, e^{2\pi \mathrm{i} w_n}, e^{2\pi \mathrm{i}(-w_1 - \cdots - w_n)}) .$$

Thus the local index gets multiplied by the above polynomial. □

Recall that by F_p we denote the flow-up manifold at $p \in M^{\mathbb{T}}$, as defined on page 1131. We can now define *i-canonical classes.*

Definition 3.5. Let $(M, \omega, \mathbb{T}, \psi)$ be a symplectic toric manifold of dimension $2n$, together with a choice of a generic component of the moment map $\mu = \psi^{\bar{\xi}} \colon M \to \mathbb{R}$. Then for each $p \in M^{\mathbb{T}}$, a Kirwan class $\tau_p \in \mathcal{K}_{\mathbb{T}}(M)$ satisfying the following properties:

1) $\mathrm{Ind}_q(\tau_p) = 1$ for all points $q \in F_p \cap M^{\mathbb{T}}$;
2) $\mathrm{Ind}_q(\tau_p) = 0$ for all points $q \notin F_p \cap M^{\mathbb{T}}$;

is called an **i-canonical class** at (the fixed point) p.

Remark 3.6. Note that the equivariant K-theory class of the trivial bundle $\mathbf{1}$ is an i-canonical class. Indeed, if $p_{min} \in M^{\mathbb{T}}$ is the fixed point at which μ attains its minimum, then $F_{p_{min}}$ is the whole manifold M and for any $q \in M^{\mathbb{T}}$ we have that $\mathrm{Ind}_q(\mathbf{1}) = \mathrm{Ind}(\mathbf{1}_{\mathbb{CP}^{\lambda_q}}) = 1$ (see (3.2)).

Observe that as the i-canonical classes are a special set of Kirwan classes, we have that:

- if for each $p \in M^{\mathbb{T}}$ there exists an i-canonical class τ_p, then the set $\{\tau_p\}_{p \in M^{\mathbb{T}}}$ is a basis for $\mathcal{K}_{\mathbb{T}}(M)$ as a module over $R(\mathbb{T})$;
- $\tau_p(p) = \mathsf{e}^{-}_{\mathbb{T}}(p)$;
- $\tau_p(q) = 0$ for all $q \in M^{\mathbb{T}} \setminus \{p\}$ with $q \prec p$.

We will see later that a stronger condition is true: $\tau_p(q) = 0$ for all $q \in M^{\mathbb{T}} \setminus V_p^+$, where V_p^+ is given in Definition 4.7.

As we remarked in Section 2.1, Kirwan classes are never unique, unless M is a point. Conditions 1) and 2) in Definition 3.5 guarantee that if i-canonical classes exist then they are unique, hence their name: they are *canonically associated* to $(M, \omega, \mathbb{T}, \psi, \bar{\xi})$, and the "*i*" refers to "index", as they are defined using the notion of local index.

Proposition 3.7 (Uniqueness of i-canonical classes). *If an i-canonical class τ_p exists then it is unique.*

Proof. Suppose that there exist two K-theory classes τ_p and τ_p' satisfying all conditions of Definition 3.5, for some $p \in M^{\mathbb{T}}$. Then the class $\eta = \tau_p - \tau_p'$ is supported on points living above p with respect to the order $\prec$ defined in (2.4), and its local index is zero at all fixed points. If η were nonzero then, by injectivity of (2.8), there would exist a fixed point $q \succ p$ with $\eta(q) \neq 0$. Take minimal such q (minimal with respect to $\prec$). Then, if $q_1, \ldots, q_{\lambda_q}$ are the fixed points in H_q connected to q through an edge of the (oriented) GKM graph Γ, we have that $q_j \prec q$, implying $\eta(q_j) = 0$ for $j = 1, \ldots, \lambda_q$. So Theorem 2.5 implies that $\eta(q)$ must be a (nonzero) multiple of $\mathsf{e}_{\mathbb{T}}^{-}(q) = \prod_{w_j \in W_q^+}(1 - e^{2\pi \mathrm{i} w_j})$. But condition (ii) of Proposition 3.4 would then imply that $\mathrm{Ind}_q(\eta) \neq 0$. This contradiction proves that η must be the zero class and hence $\tau_p = \tau_p'$. $\square$

Section 4 is devoted to proving that, indeed, for symplectic toric manifolds i-canonical classes always exist.

3.1. Comparison with Guillemin-Kogan results [GK]

The definition of local index (Definition 3.2) is inspired by, though different from, the one in [GK]. In our definition of local index, we make an explicit choice of the circle C_q used for obtaining $\widetilde{H}_q^{\epsilon}$ through symplectic reduction. The choice is different for each $q \in M^{\mathbb{T}}$. With this choice the reduced space is always smooth and one can explicitly calculate the local index using the algorithm we provide. In the work of Guillemin and Kogan, the reduction at each fixed point is done with respect to the same fixed circle. Therefore the reduced space is often an orbifold, making explicit computations much harder. It is important to notice that the local indices I_q^i at $q \in M^{\mathbb{T}}$ defined in [GK], associated to reductions with respect to two different circles C_i, $i = 1, 2$, are not the same. Indeed $I_q^i\colon R(\mathbb{T}) \to R(\mathbb{T})$ is only an $R(\mathbb{T}/C_i)$, and not an $R(\mathbb{T})$, -module homomorphism; thus different choices of C_i's cannot result in the same local index map (see Remark 5.1 and bottom of page 371 in [GK]). Our local index is also only an $R(\mathbb{T}/C_q^0)$, and not an $R(\mathbb{T})$, -module homomorphism (see Proposition 3.4 (iv) and (v)). Moreover, in [GK, Section 7] the authors define a new variant of the local index, which they call $\widetilde{I}_q$, where the circle used in the reduction can vary at each fixed point. If at each fixed point the choice of their circle agrees with ours, then the local indices Ind_q and $\widetilde{I}_q$ from [GK, Section 7] are the same. In this case it is easy to see that our i-canonical classes $\{\tau_p\}_{p \in M^{\mathbb{T}}}$ and their classes $\{\tau_p^{GK}\}_{p \in M^{\mathbb{T}}}$ are related by the following formula:

$$\tau_p = \sum_{q \in F_p \cap M^{\mathbb{T}}} \tau_q^{GK} \,. \tag{3.5}$$

4. Construction

The main goal of this section is to prove the following

Proposition 4.1 (Existence of i-canonical classes). *Let $(M, \omega, \mathbb{T}, \psi)$ be a symplectic toric manifold of dimension $2n$, together with a choice of a generic component of the moment map $\mu = \psi^{\overline{\xi}} \colon M \to \mathbb{R}$. Then for each $p \in M^{\mathbb{T}}$, there exists an i-canonical class τ_p.*

We prove this by explicitly exhibiting a set of K-theory classes satisfying properties 1) and 2) in Definition 3.5. The proof of Proposition 4.1 is divided into two parts: the index increasing case (Proposition 4.5) and non-index increasing case (Subsection 4.2).

Definition 4.2. Let $(M, \omega, \mathbb{T}, \psi, \overline{\xi})$ be a GKM space with oriented GKM graph $\Gamma^o = (V, E^o)$. Then the space is called **index increasing** if for every edge $e = \overrightarrow{pq} \in E^o$ we have $\lambda_p < \lambda_q$, and **non-index increasing** otherwise.

The Hirzebruch surface in Figure 4.2 is an example of non-index increasing GKM space.

For toric manifolds there is a natural algorithm for constructing a basis of the equivariant K-theory ring consisting of special Kirwan classes which are equivariant Poincaré duals to the flow up submanifolds. As we will prove in Section 4.1, in the index increasing case these equivariant K-theory classes are indeed i-canonical classes (see Proposition 4.5). In the non-index increasing case we will need to modify them to make them "canonical", as explained in Section 4.2. Below we recall this algorithm.

Let $(M^{2n}, \omega, \mathbb{T}, \psi)$ be a symplectic toric manifold, and let $\mu = \psi^{\overline{\xi}} \colon M \to \mathbb{R}$ be a generic component of the moment map. Let $\Gamma = (V, E)$ (resp. $\Gamma^o = (V, E^o)$) be the associated GKM graph (resp. oriented GKM graph). This GKM space is not necessarily index increasing. We recall that for every $p \in M^{\mathbb{T}}$ the flow-up at p, denoted by F_p, is a $\mathbb{T}$-invariant submanifold of M (see the discussion on page 1131, Section 3). Thus the normal bundle to F_p, which we denote by $N(F_p)$, is $\mathbb{T}$-invariant, and for each $q \in M^{\mathbb{T}} \cap F_p$ the set of weights of the $\mathbb{T}$-representation on $N(F_p)|_q$ is given by $\{w(\overrightarrow{rq})\}$, where $r \in M^{\mathbb{T}} \setminus F_p$ and $\overrightarrow{rq} \in E$; note that such an edge does not necessarily belong to E^o. We have that the (K-theoretical) equivariant Euler class of the normal bundle $N(F_p)|_q$ is

$$\mathsf{e}_{\mathbb{T}}\big(N(F_p)|_q\big) = \prod_{\substack{r \in M^{\mathbb{T}} \setminus F_p \\ \overrightarrow{r\,q} \in E}} (1 - e^{2\pi \mathrm{i}\, w(\overrightarrow{r\,q})}). \tag{4.1}$$

In particular we have that $\mathsf{e}_{\mathbb{T}}\big(N(F_p)|_p\big) = \mathsf{e}_{\mathbb{T}}^{-}(p)$.

Lemma 4.3. *Let $(M^{2n}, \omega, \mathbb{T}, \psi)$ be a symplectic toric manifold, and let $\mu = \psi^{\xi} \colon M \to \mathbb{R}$ be a generic component of the moment map. For each $p \in M^{\mathbb{T}}$ define $\eta_p \in \mathcal{K}_{\mathbb{T}}(M^{\mathbb{T}})$ to be*

$$\eta_p(q) = \begin{cases} 0 & \text{for } q \in M^{\mathbb{T}} \setminus F_p \\ \mathsf{e}_{\mathbb{T}}\big(N(F_p)|_q\big) & \text{for } q \in F_p \cap M^{\mathbb{T}}. \end{cases}$$

Then η_p is an element of $\mathcal{K}_{\mathbb{T}}(M)$, and it is a Kirwan class in the sense of Proposition 2.2.

The element $\eta_p \in \mathcal{K}_{\mathbb{T}}(M)$ defined in the above lemma is called the **equivariant (K-theoretical) Poincaré dual** to the flow-up manifold F_p.

Proof. The proof of this Lemma is quite standard, but we include it here for completeness. In order to prove that η_p is indeed an element of $\mathcal{K}_{\mathbb{T}}(M)$, we need to verify condition (2.10) in Theorem 2.5, for every edge e of the GKM graph Γ.

Let $\overrightarrow{r\,s} \in E$. If neither r nor s belong to F_p, then by definition of η_p we have $\eta_p(r) = \eta_p(s) = 0$, so (2.10) holds. If $r \in F_p$ but $s \notin F_p$, then by definition of $\eta_p(r)$ and (4.1), $\eta_p(r) = Q\,(1 - e^{2\pi \mathrm{i}\, w(\overrightarrow{s\,r})})$, for some $Q \in R(\mathbb{T})$, and (2.10) holds. Similarly if $s \in F_p$ but $r \notin F_p$. Finally, suppose that both r and s belong to F_p. Consider the subtorus $\mathbb{T}' = \exp\{\xi \in \mathit{Lie}(\mathbb{T}) \mid w(\overrightarrow{r\,s})(\xi) = 0\} \subset \mathbb{T}$ fixing the sphere S^2 associated to the edge $\overrightarrow{r\,s}$. Since $\mathbb{T}'$ acts trivially on S^2 the representations of $\mathbb{T}'$ on $N(S^2)|_r$ and $N(S^2)|_s$ agree. This implies that there exists an isomorphism $\varphi \colon W_r \to W_s$ such that for every $w \in W_r$

$$\varphi(w) - w = n_w\, w(\overrightarrow{r\,s}) \tag{4.2}$$

for some $n_w \in \mathbb{Z}$. Note that S^2 is also an invariant submanifold in F_p. Denote by $W_r^{F_p}$ (resp. $W_s^{F_p}$) the set of weights of the $\mathbb{T}$-representation on the tangent space of F_p at r (resp. at s), and observe that the isomorphism φ restricts to an isomorphism from $W_r^{F_p}$ to $W_s^{F_p}$ satisfying (4.2), and hence to an isomorphism from $W_r \setminus W_r^{F_p}$ to $W_s \setminus W_s^{F_p}$ satisfying (4.2). Observe that $e^{2\pi \mathsf{i}\, w} - e^{2\pi \mathsf{i}\, \varphi(w)} = e^{2\pi \mathsf{i}\, w}(1 - e^{2\pi \mathsf{i}\, n_w\, w(\overrightarrow{r\,s})}) = Q'\,(1 - e^{2\pi \mathsf{i}\, w(\overrightarrow{r\,s})}))$, for some $Q' \in R(\mathbb{T})$, and that the weights of the $\mathbb{T}$-representation on $N(F_p)|_r$ (resp. $N(F_p)|_s$) are precisely those in $W_r \setminus W_r^{F_p}$ (resp. $W_s \setminus W_s^{F_p}$). It follows that

$$\mathsf{e}_{\mathbb{T}}\big(N(F_p)|_r\big) - \mathsf{e}_{\mathbb{T}}\big(N(F_p)|_s\big) = Q''(1 - e^{2\pi \mathsf{i}\, w(\overrightarrow{r\,s})}))$$

for some $Q'' \in R(\mathbb{T})$. By Theorem 2.5 we can conclude that η_p is an element of $\mathcal{K}_{\mathbb{T}}(M)$.

In order to finish the proof we need to show that η_p satisfies properties (i) and (ii) of Proposition 2.2. The first property follows from observing that, by definition of F_p, the negative normal bundle of μ at p coincides with the normal bundle $N(F_p)$ at p. The second one follows from observing that p is a minimum of μ on F_p and that F_p is connected. Therefore every fixed point q in $F_p \setminus \{p\}$ satisfies $\mu(q) > \mu(p)$. Hence any q' with $\mu(q') < \mu(p)$ must be in $M^{\mathbb{T}} \setminus F_p$, and thus $\eta_p(q') = 0$. This concludes the proof. ☐

Remark 4.4. In the above proof we show that the normal bundle of F_p at p coincides with the negative normal bundle of μ at p, so $\mathsf{e}_{\mathbb{T}}(N(F_p)|_p) = \mathsf{e}_{\mathbb{T}}^{-}(p)$. Therefore Propostion 3.4 (ii) implies that

$$\mathrm{Ind}_p(\eta_p) = 1\,.$$

4.1. Toric manifolds: the index increasing case

In this Subsection we analyze symplectic toric manifolds which are also index increasing GKM spaces. In this case Proposition 4.1 follows from the following

Proposition 4.5. *Let $(M^{2n}, \omega, \mathbb{T}, \psi)$ be a symplectic toric manifold, and let $\mu = \psi^{\overline{\xi}} \colon M \to \mathbb{R}$ be a generic component of the moment map. Let $\Gamma^o = (V, E^o)$ be the associated oriented GKM graph, and assume it is index increasing. Then for each $p \in M^{\mathbb{T}}$, the Kirwan classes η_p defined in Lemma 4.3 are the i-canonical classes τ_p.*

In the proof of this proposition we use some standard facts about index increasing GKM spaces which, for the sake of completeness, are proved in Subsection 5.2.

Proof of Proposition 4.5. We need to show that

$$\mathrm{Ind}_q(\eta_p) = \begin{cases} 0 & \textit{for } q \in M^{\mathbb{T}} \setminus F_p \\ 1 & \textit{for } q \in F_p \cap M^{\mathbb{T}}. \end{cases}$$

Consider any two fixed points p, q. Suppose first that $q \notin F_p$. Then $\eta_p(q) = 0$, and therefore the local index at q is 0 (see Propostion 3.4 (ii)). Now suppose that $q \in F_p$. By Lemma 5.9 we have $\lambda_q > \lambda_p$. In the GKM graph of M the vertex q is connected to λ_q vertices $q_1, \ldots, q_{\lambda_q}$ by edges terminating at q, and to $n - \lambda_q$ vertices, $q_{\lambda_q+1}, \ldots, q_n$ by edges starting at q. As the moment map is index increasing, from Corollary 5.16 it follows that the vertices $q_{\lambda_q+1}, \ldots, q_n$ are also in F_p, as q is. Therefore λ_p of the points $q_1, \ldots, q_{\lambda_q}$ are not in F_p (see also Corollary 5.16). To simplify the notation assume that $q_1, \ldots, q_{\lambda_p}$ are not in F_p. The value of η_p at these points is 0. Let $w_1, \ldots, w_{\lambda_q}$ be the weights of $\mathbb{T}$-action on the tangent spaces at q of the spheres corresponding to the edges $\overrightarrow{q_1\, q}, \ldots, \overrightarrow{q_{\lambda_q}\, q}$. Then, by (4.1), the value of η_p at q is given by

$$\eta_p(q) = \prod_{j=1}^{\lambda_p} (1 - e^{2\pi \mathrm{i} w_j}).$$

To calculate the local index of η_p at q we look at the class $\kappa(\widetilde{r}(\eta_p))$ in $\mathcal{K}_{\widetilde{\mathbb{T}}_q}(\widetilde{H}_q^\epsilon)$. Using the algorithm and notation from Section 3 we find the values of this class at the fixed points of $\widetilde{H}_q^\epsilon$:

$$\kappa(\widetilde{r}(\eta_p))(p_l) = \prod_{j=1}^{\lambda_p} (1 - e^{2\pi \mathrm{i}(w_j - w_l)}) = 0, \qquad l = 1, \ldots, \lambda_p,$$

$$\kappa(\widetilde{r}(\eta_p))(p_l) = \prod_{j=1}^{\lambda_p} (1 - e^{2\pi \mathrm{i}(w_j - w_l)}) \neq 0, \qquad l = \lambda_p + 1, \ldots, \lambda_q,$$

$$\kappa(\widetilde{r}(\eta_p))(p_0) = \prod_{j=1}^{\lambda_p} (1 - e^{2\pi \mathrm{i}(w_j + w_0)}) \neq 0.$$

Note that at a point p_l where $\kappa(\widetilde{r}(\eta_p))(p_l)$ is nonzero, this value is exactly equal to the product of terms $(1 - e^{2\pi \mathrm{i} w})$ taken over the weights w on the edges connecting p_l to fixed points in $(\widetilde{H}_q^\epsilon) \cong \mathbb{CP}^{\lambda_q}$ where the value of $\kappa(\widetilde{r}(\eta_p))$ is 0. This observation, and Atiyah-Segal formula (3.1), imply that the index of $\kappa(\widetilde{r}(\eta_p))$ is equal to the index of a class $\mathbf{1}$ on $\mathbb{CP}^{\lambda_q - \lambda_p} \subset \mathbb{CP}^{\lambda_q}$ containing the fixed points of $\mathbb{CP}^{\lambda_q}$ where $\kappa(\widetilde{r}(\eta_p))$ is nonzero. By (3.2) we have that $\mathrm{Ind}(\mathbf{1}_{\mathbb{CP}^{\lambda_q-\lambda_p}})=1$. In other words, $\mathrm{Ind}(\kappa(\eta_p))$ is equal to

$$\frac{\prod_{j=1}^{\lambda_p}(1-e^{2\pi\mathsf{i}(w_j+w_0)})}{\prod_{j=1}^{\lambda_q}(1-e^{2\pi\mathsf{i}(w_j+w_0)})}+0+\sum_{l=\lambda_p+1}^{\lambda_q}\frac{\prod_{j=1}^{\lambda_p}(1-e^{2\pi\mathsf{i}(w_j-w_l)})}{(1-e^{2\pi\mathsf{i}(-w_l-w_0)})\prod_{j=1,j\neq l}^{\lambda_q}(1-e^{2\pi\mathsf{i}(w_j-w_l)})}=$$

$$\frac{1}{\prod_{j=\lambda_p+1}^{\lambda_q}(1-e^{2\pi\mathsf{i}(w_j+w_0)})}+\sum_{l=\lambda_p+1}^{\lambda_q}\frac{1}{(1-e^{2\pi\mathsf{i}(-w_l-w_0)})\prod_{j=\lambda_p+1,j\neq l}^{\lambda_q}(1-e^{2\pi\mathsf{i}(w_j-w_l)})}$$

$$=\mathrm{Ind}(\mathbf{1}_{\mathbb{CP}^{\lambda_q-\lambda_p}})=1.$$

This shows that the classes defined naturally as the equivariant Poincaré duals to the flow-up manifolds F_p are i-canonical classes. □

Example 4.6 (The complex projective space $\mathbb{CP}^n$). Consider the complex projective space $(\mathbb{CP}^n,\omega)$ where ω denotes the Fubini-Study symplectic form rescaled so that $[\omega]$ is integral and primitive, i.e. $[\omega]$ is a generator of $H^2(\mathbb{CP}^n;\mathbb{Z})=\mathbb{Z}$, regarded as a lattice in $H^2(\mathbb{CP}^n;\mathbb{R})$. Endow $(\mathbb{CP}^n,\omega)$ with the standard toric action of an n-dimensional torus $\mathbb{T}$ and moment map ψ; as before, let $\mathfrak{t}$ be the Lie algebra of $\mathbb{T}$ and $\ell\subset\mathfrak{t}$ the integral lattice. Since the action is Hamiltonian, the symplectic form extends to an equivariant form in the Cartan complex, called equivariant symplectic form, given by $\omega+\psi$. We can choose the moment map so that $\psi(q)\in\ell^*$ for every $q\in(\mathbb{CP}^n)^{\mathbb{T}}$. Then the above equivariant form represents a well-defined element $[\omega+\psi]$ in $H^2_{\mathbb{T}}(\mathbb{CP}^n;\mathbb{Z})$ (regarded as a lattice in $H^2_{\mathbb{T}}(\mathbb{CP}^n;\mathbb{R})$). At the level of the GKM graph (V,E) associated to $(\mathbb{CP}^n,\omega,\mathbb{T},\psi)$, the condition of $[\omega]$ being integral and primitive translates into saying that

$$\psi(q)-\psi(p)=w(\overrightarrow{p\,q})\quad\text{for every}\quad\overrightarrow{p\,q}\in E. \tag{4.3}$$

Indeed, this is equivalent to saying that the symplectic volume of the sphere $\psi^{-1}(\overrightarrow{p\,q})$ is 1, for every $\overrightarrow{p\,q}\in E$. Note that for every pair of fixed points $p,q\in(\mathbb{CP}^n)^{\mathbb{T}}$ there exists an edge $\overrightarrow{p\,q}\in E$.

Consider the equivariant K-theory class represented by the equivariant line bundle $\mathbb{L}^{S^1}$ satisfying $\mathrm{c}_1^{S^1}(\mathbb{L}^{S^1})=-[\omega+\psi]$, where $\mathrm{c}_1^{S^1}(\mathbb{L}^{S^1})$ denotes the equivariant first Chern class of $\mathbb{L}^{S^1}$. Such bundle exists by Theorem 1.1 and Corollary 1.2 proved by Hattori and Yoshida in [HY]. Note that $\mathbb{L}^{S^1}(s)=$

$e^{-2\pi \mathrm{i}\psi(s)}$ for every $s \in (\mathbb{CP}^n)^{\mathbb{T}}$. Let $\overline{\xi} \in \mathfrak{t}$ be a generic vector, and consider the ordering induced by $\mu = \psi^{\overline{\xi}}$ on the fixed points. Note that for every choice of generic vector $\overline{\xi}$, the oriented GKM graph associated to $(\mathbb{CP}^n, \omega, \mathbb{T}, \psi, \overline{\xi})$ is index increasing.

For every $p \in M^{\mathbb{T}}$ consider the class

$$\tau_p = \prod_{\substack{q \in M^{\mathbb{T}} \\ q \prec p}} \left(1 - e^{2\pi \mathrm{i}\psi(q)} \mathbb{L}^{S^1}\right).$$

Observe that $q \prec p$ if and only if $q \notin F_p$ (see also Proposition 5.15). Therefore, for each $s \in (\mathbb{CP}^n)^{\mathbb{T}}$, we have that

$$\tau_p(s) = \begin{cases} 0 & \text{if} \quad s \notin F_p \\ \displaystyle\prod_{\substack{q \in M^{\mathbb{T}} \\ q \prec p}} \left(1 - e^{2\pi \mathrm{i}(\psi(q) - \psi(s))}\right) = \prod_{\substack{q \in M^{\mathbb{T}} \\ q \notin F_p}} \left(1 - e^{2\pi \mathrm{i} w(\overrightarrow{qs})}\right) & \text{if} \quad s \in F_p. \end{cases}$$

Thus τ_p coincides with the equivariant (K-theoretical) Poincaré dual to F_p which, by Proposition 4.5, is the i-canonical class at p.

The i-canonical classes for $\mathbb{CP}^2$ are shown in Figure 4.1.

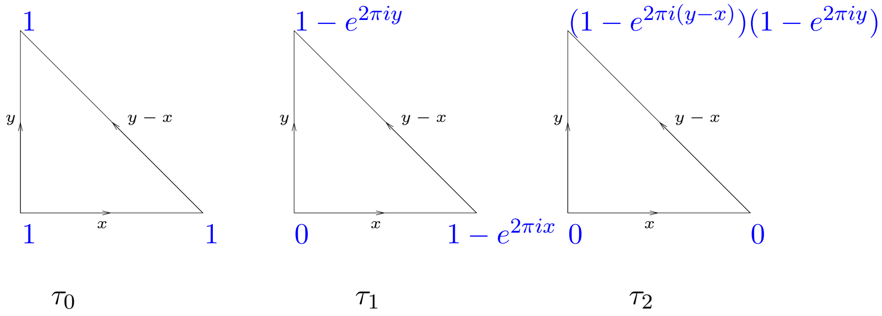

Figure 4.1: The basis of i-canonical classes $\{\tau_p\}$ for $\mathbb{CP}^2$.

4.2. Toric manifolds: the non-index increasing case

If the moment map is not index increasing, to each fixed point $p \in M^{\mathbb{T}}$ we can still associate a K-theory class η_p, as in Lemma 4.3. The value of η_p on points $q \in M^{\mathbb{T}} \setminus F_p$ is zero and therefore the local index $\mathrm{Ind}_q(\eta_p)$ is also zero (see Propostion 3.4). However it may no longer be true that $\mathrm{Ind}_q(\eta_p) = 1$ for $q \in F_p \cap M^{\mathbb{T}} =: F_p^{\mathbb{T}}$. Below we present an algorithm to construct i-canonical classes τ_p out of the equivariant Poincaré duals η_p as in Lemma 4.3. To simplify notation, we use p to denote both $p \in M^{\mathbb{T}}$ and $\psi(p)$, as ψ is injective on the fixed point set.

Definition 4.7. For any $p \in M^{\mathbb{T}}$ we define V_p^+ to be the set of fixed points which can be joined to p through an increasing path in E^o, i.e. $q \in V_p^+$ if and only if there exists a sequence of edges $\gamma = (r_0 = p, r_1, \dots, r_m = q)$ such that $\overrightarrow{r_i r_{i+1}} \in E^o$ for every $i = 0, \dots, m-1$.

This definition implies that $\mu(q) > \mu(p)$, i.e. $p \prec q$, for all $q \in V_p^+ \setminus \{p\}$. Also observe that (by Lemma 5.8) $F_p^{\mathbb{T}} \subset V_p^+$ and thus for all $q \in F_p^{\mathbb{T}}$ we have that $F_q^{\mathbb{T}} \subseteq V_q^+ \subset V_p^+$. (In the index increasing case a stronger statement is true: $F_q \subseteq F_p$ for all $q \in F_p^{\mathbb{T}}$, see Corollary 5.16).

Proof of Proposition 4.1 - the non-index increasing case. Fix $p \in M^{\mathbb{T}}$ and let $V_p^+ = \{q_0 = p, q_1, \dots, q_k\}$ be ordered so that $q_j \prec q_l$ for $0 \le j < l \le k$. As the restriction of μ to F_{q_l} attains its minimum at q_l, we have that

$$j < l \Rightarrow q_j \notin F_{q_l}, \text{ and therefore } \eta_{q_l}(q_j) = 0.$$

We inductively construct auxiliary classes $a_1, \dots, a_k$ satisfying

- $\mathrm{Ind}_{q_j}(a_l) = 1$ if $j \le l$ and $q_j \in F_p^{\mathbb{T}}$;
- $\mathrm{Ind}_{q_j}(a_l) = 0$ if $j \le l$ and $q_j \in V_p^+ \setminus F_p^{\mathbb{T}}$;
- $\mathrm{Ind}_q(a_l) = 0$ if $q \notin V_p^+$.

Then we will show that a_k is the i-canonical class τ_p. In the following we make use of the fact that $F_p^{\mathbb{T}} \subset V_p^+$ (see Lemma 5.8).

Define

$$a_1 := \begin{cases} \eta_p + (1 - \mathrm{Ind}_{q_1}(\eta_p))\ \eta_{q_1} & \text{if } q_1 \in F_p^{\mathbb{T}}, \\ \eta_p - \mathrm{Ind}_{q_1}(\eta_p)\ \eta_{q_1} & \text{if } q_1 \in V_p^+ \setminus F_p^{\mathbb{T}}. \end{cases}$$

As $\eta_{q_1}(p) = 0$, we have $a_1(p) = \eta_p(p) = \mathsf{e}_{\mathbb{T}}^-(p)$, and hence $\mathrm{Ind}_p(a_1) = \mathrm{Ind}_p(\eta_p) = 1$ (see Propostion 3.4 and Remark 4.4). Also, observe that if $\eta_{q_1}(s) \neq 0$ for some $s \in M^{\mathbb{T}}$ then $s \in F_{q_1}^{\mathbb{T}} \subset V_{q_1}^+ \subset V_p^+$, where the first inclusion follows from Lemma 5.8, and the second is obvious. Thus the class a_1 restricts to zero on $M^{\mathbb{T}} \setminus V_p^+$ and $\mathrm{Ind}_s(a_1) = 0$ if $s \notin V_p^+$. Moreover, by Proposition 3.4 and Remark 4.4 we can conclude that

$$\mathrm{Ind}_{q_1}(a_1) := \begin{cases} \mathrm{Ind}_{q_1}(\eta_p) + (1 - \mathrm{Ind}_{q_1}(\eta_p)) \cdot 1 = 1 & \text{if } q_1 \in F_p^{\mathbb{T}}, \\ \mathrm{Ind}_{q_1}(\eta_p) - \mathrm{Ind}_{q_1}(\eta_p) \cdot 1 = 0 & \text{if } q_1 \in V_p^+ \setminus F_p^{\mathbb{T}}. \end{cases}$$

Then we proceed inductively and define

$$a_j = \begin{cases} a_{j-1} + (1 - \mathrm{Ind}_{q_j}(a_{j-1}))\eta_{q_j} & \text{if } q_j \in F_p^{\mathbb{T}}, \\ a_{j-1} - \mathrm{Ind}_{q_j}(a_{j-1})\eta_{q_j} & \text{if } q_j \in V_p^+ \setminus F_p^{\mathbb{T}}. \end{cases}$$

As the fixed points are ordered with $\prec$, the restrictions of η_{q_j} to fixed points $q_0, q_1, \ldots, q_{j-1}$ are zero. Thus the local index of a_j at q_l is the same as of a_{j-1}, for all $l = 0, \ldots, j-1$. Similarly as before, Remark 4.4 and Proposition 3.4 prove that $\mathrm{Ind}_{q_j}(a_j) = 1$ if $q_j \in F_p^{\mathbb{T}}$ and is zero on $V_p^+ \setminus F_p^{\mathbb{T}}$. Moreover a_j restricts to zero on $M^{\mathbb{T}} \setminus V_p^+$ and $\mathrm{Ind}_s(a_j) = 0$ if $s \notin V_p^+$.

The algorithm ends when we exhaust all the points in $V_p^+ = \{q_0, q_1, \ldots, q_k\}$ and we define the class τ_p to be a_k. Thus $\mathrm{Ind}_q(\tau_p) = 1$ if $q \in F_p^{\mathbb{T}}$ and $\mathrm{Ind}_q(\tau_p) = 0$ if $q \notin F_p^{\mathbb{T}}$ as needed. What is left to prove is that the classes τ_p are Kirwan classes in the sense of Proposition 2.2. This follows immediately from observing that $\tau_p = \eta_p + \sum_{q_l \in V_p^+ \setminus \{p\}} \alpha_l \, \eta_{q_l}$, where $\alpha_l \in R(\mathbb{T})$ for every l, and from the η_{q_l}'s also being Kirwan classes. $\square$

Figure 4.2 presents the basis of i-canonical classes for the Hirzebruch surface.

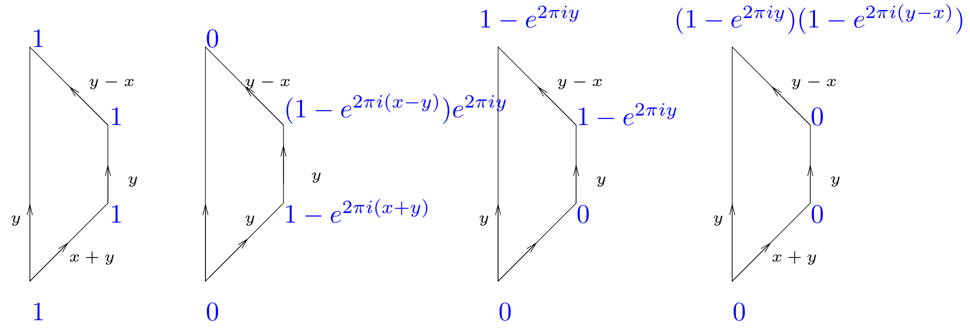

Figure 4.2: The basis of i-canonical classes $\{\tau_p\}$ for the Hirzebruch surface.

Remark 4.8. Note that the collection of i-canonical classes $\{\tau_p\}_{p\in M^{\mathbb{T}}}$ is obtained from the basis of Kirwan classes $\{\eta_p\}_{p\in M^{\mathbb{T}}}$ by applying a lower triangular matrix with 1's on diagonal. This gives an alternative proof that they form a basis of $\mathcal{K}_{\mathbb{T}}(M)$.

5. Applications to equivariant cohomology

As it was already observed by Guillemin and Kogan, it is possible to use the local index map to define "canonical classes" forming a basis of the equivariant cohomology ring for every Hamiltonian $\mathbb{T}$-space, in particular for symplectic toric manifolds (see [GK, Remark 1.4]). This idea was already used by Zara [Z] to construct a basis for the equivariant cohomology ring with *rational* coefficients. We follow the idea of Guillemin and Kogan, but use a different definition of local index to ensure *integrality* of the basis elements produced with this method. Moreover we relate our basis to the bases already commonly used in equivariant cohomology (with integer coefficients). Namely, the basis thus obtained coincides with the basis of the equivariant Poincaré duals $\{\eta_p\}$ defined below. In addition, when the generic component of the moment map is the index increasing, Goldin and Tolman [GT] introuduced another basis for the equivariant cohomology (see Definition 5.3). In this case we show that the three sets of bases - the i-canonical classes, the equivariant Poincaré duals $\{\eta_p\}$, and the Goldin-Tolman [GT] canonical classes - are the same.

Our definition of i-canonical classes for equivariant cohomology is slightly different from the definition given in K-theory. Namely, here we require the local index of a class associated to a fixed point p to vanish on all $M^{\mathbb{T}} \setminus \{p\}$. The reason for this change is that we would like the class $\mathbf{1}_M$ to be an element of our i-canonical basis.

Recall that in the Borel description, the equivariant cohomology ring $H^*_{\mathbb{T}}(M;\mathbb{Z})$ is defined to be the ordinary cohomology ring of the space $M \times_{\mathbb{T}} E\mathbb{T}$, where $E\mathbb{T}$ is a contractible space on which $\mathbb{T}$ acts freely. If $\mathbb{T}$ is a d-dimensional torus, and $x_1,\dots,x_d$ a $\mathbb{Z}$-basis of the dual lattice of $\mathbb{T}$, then $H^*_{\mathbb{T}}(\mathrm{pt};\mathbb{Z}) = H^*((\mathbb{CP}^\infty)^d;\mathbb{Z}) = \mathbb{Z}[x_1,\dots,x_d]$. The unique map $M \to \mathrm{pt}$ gives $H^*_{\mathbb{T}}(M;\mathbb{Z})$ the structure of an $H^*_{\mathbb{T}}(\mathrm{pt};\mathbb{Z})$-module.

Let $(M,\omega,\mathbb{T},\psi)$ be a GKM space (see Section 2.2) with GKM graph (V,E). For an edge $\overrightarrow{pq} \in E$, let $\mathbb{T}' = \mathbb{T}'_{\overrightarrow{pq}} = \exp\{\xi \in \mathfrak{t} \mid w(\overrightarrow{pq})(\xi) = 0\}$ be the subtorus fixing the sphere $S^2_{\overrightarrow{pq}} = \psi^{-1}(\overrightarrow{pq})$ corresponding to the edge $\overrightarrow{pq}$. Let $\mathbb{S}(\mathfrak{t}^*)$ be the symmetric algebra of $\mathfrak{t}^*$ and $\pi_{\overrightarrow{pq}}\colon \mathbb{S}(\mathfrak{t}^*) \to \mathbb{S}((\mathfrak{t}')^*)$ be the homomorphism induced by the inclusion of $\mathfrak{t}' = Lie(\mathbb{T}')$ into $\mathfrak{t}$. Then for any class $\tau \in H^*_{\mathbb{T}}(M;\mathbb{Z})$ we must have

$$\pi_{\overrightarrow{p\,q}}(\tau(q)) = \pi_{\overrightarrow{p\,q}}(\tau(p)) \quad \text{for every} \quad \overrightarrow{p\,q} \in E. \tag{5.1}$$

This condition is necessary, but not sufficient to guarantee that a class $\tau \in H^*_{\mathbb{T}}(M^{\mathbb{T}};\mathbb{Z})$ is in $H^*_{\mathbb{T}}(M;\mathbb{Z})$. However, if working with rational coefficients, a theorem of Goresky-Kottwitz-MacPherson ([GKM]) implies that each τ in $H^*_{\mathbb{T}}(M^{\mathbb{T}};\mathbb{Q})$ satisfying (5.1) is in $H^*_{\mathbb{T}}(M;\mathbb{Q})$. (Compare with Theorem 2.5.)

5.1. Choosing a basis

We start by recalling the choices of bases already used in the literature: the basis consisting of equivariant Poincaré duals to flow up manifolds and the Goldin-Tolman canonical classes in the index increasing case.

Let $(M,\omega,\mathbb{T},\psi)$ be a Hamiltonian $\mathbb{T}$-space, and let $\mu\colon M \to \mathbb{R}$ be a generic component of the moment map. For a fixed point p, the **equivariant (cohomological) Euler class** of the negative normal bundle N_p^- of μ at p is the element $\Lambda_p^- \in H^{2\lambda_p}(\{p\};\mathbb{Z})$ given by

$$\Lambda_p^- = \prod_{w_j \in W_p^+} w_j \,. \tag{5.2}$$

The following Proposition is due to Kirwan.

Proposition 5.1 (Kirwan). *Let $(M,\omega,\mathbb{T},\psi)$ be a Hamiltonian $\mathbb{T}$-space, and let $\mu\colon M \to \mathbb{R}$ be a generic component of the moment map. Then for every $p \in M^{\mathbb{T}}$ there exists a class $\nu_p \in H^{2\lambda_p}_{\mathbb{T}}(M;\mathbb{Z})$ such that*

(i) $\nu_p(p) = \Lambda_p^-$;

(ii) $\nu_p(q) = 0$ *for every $q \in M^{\mathbb{T}}$ such that $q \prec p$ (i.e. $\mu(q) < \mu(p)$).*

Moreover the set $\{\nu_p\}_{p\in M^{\mathbb{T}}}$ is a basis for $H^*_{\mathbb{T}}(M;\mathbb{Z})$ as a module over $H^*_{\mathbb{T}}(pt;\mathbb{Z})$.

An equivariant cohomology class satisfying properties (i) and (ii) above is called a **Kirwan class** (at the fixed point p).

Note that Proposition 2.2 is a generalization of the above original result of Kirwan from the equivariant cohomology setting to the K-theory setting. Due to this similarity we omit the proof of Proposition 5.1, which is based on the fact that Λ_p^- is not a zero divisor in $H_{\mathbb{T}}(\mathrm{pt};\mathbb{Z})$.

Henceforth $(M,\omega,\mathbb{T},\psi,\overline{\xi})$ denotes a symplectic toric manifold endowed with a choice of generic $\overline{\xi}\in\mathfrak{t}$. In analogy with Section 4 for equivariant K-theory, we define the equivariant (cohomological) Poincaré duals to the flow-up manifolds F_p. Let F_p be the (smooth) flow-up manifold at the fixed point p (see page 1131), and $N(F_p)$ the normal bundle of F_p in M. Then the equivariant (cohomological) Euler class of the normal bundle $N(F_p)$ of F_p in M at a fixed point q is an element in $H^{2\lambda_p}(\{q\};\mathbb{Z})$ given by

$$\chi_{\mathbb{T}}\big(N(F_p)|_q\big)=\prod_{\substack{r\in M^{\mathbb{T}}\setminus F_p\\ \overrightarrow{r\,q}\in E}} w(\overrightarrow{r\,q}). \tag{5.3}$$

In particular, since $N(F_p)|_p=N_p^-$, we have that $\chi_{\mathbb{T}}\big(N(F_p)|_p\big)=\Lambda_p^-$.

Definition 5.2. Let $(M,\omega,\mathbb{T},\psi)$ be a symplectic toric manifold, and let $\mu=\psi^{\overline{\xi}}\colon M\to\mathbb{R}$ be a generic component of the moment map. For each $p\in M^{\mathbb{T}}$ define the **equivariant Poincaré dual** to the flow-up manifold F_p to be the class $\eta_p\in H_{\mathbb{T}}^{2\lambda_p}(M;\mathbb{Z})$ whose restriction to the fixed points is given by

$$\eta_p(q)=\begin{cases}0 & \textit{for } q\in M^{\mathbb{T}}\setminus F_p\\ \chi_{\mathbb{T}}\big(N(F_p)|_q\big) & \textit{for } q\in F_p\cap M^{\mathbb{T}}.\end{cases}$$

It is easy to check that η_p is a Kirwan class in the sense of Proposition 5.1 for every $p\in M^{\mathbb{T}}$, and thus $\{\eta_p\}_{p\in M^{\mathbb{T}}}$ forms a basis of $H_{\mathbb{T}}^*(M;\mathbb{Z})$ as a module over $H_{\mathbb{T}}^*(\mathrm{pt};\mathbb{Z})$.

Goldin and Tolman [GT] define another basis for the equivariant cohomology ring of a symplectic manifold endowed with a Hamiltonian torus action.

Definition 5.3. Let $(M,\omega,\mathbb{T},\psi)$ be a Hamiltonian $\mathbb{T}$-space, and let $\mu\colon M\to\mathbb{R}$ be a generic component of the moment map. A cohomology class $\zeta_p\in H_{\mathbb{T}}^{2\lambda_p}(M;\mathbb{Z})$ is a canonical class in the sense of Goldin and Tolman (henceforth referred to as **GT-canonical class**) at $p\in M^{\mathbb{T}}$ if

(i) $\zeta_p(p)=\Lambda_p^-$;

(ii) $\zeta_p(q)=0$ for all $q\in M^{\mathbb{T}}\setminus\{p\}$ such that $\lambda_q\leq\lambda_p$.

GT-canonical classes do not always exist; however, if they exist, they are uniquely associated to the chosen component of the moment map μ (see [GT, Lemma 2.7]). Moreover GT-canonical classes are Kirwan classes in the sense of Proposition 5.1 (cf. [GT, Lemma 2.8]), hence if they exist for every fixed point p, they form a basis of $H^*_{\mathbb{T}}(M;\mathbb{Z})$ as a module over $H^*_{\mathbb{T}}(\mathrm{pt};\mathbb{Z})$.

If the $\mathbb{T}$ action above is GKM, and if the chosen component of the moment map μ is index increasing, then for each fixed point p the GT-canonical class ζ_p exists. Conversely, if for all $p \in M^{\mathbb{T}}$ the GT-canonical class exist, then μ must be index increasing (see Theorem 4.1 and Remark 4.2 in [GT]).

Below we propose a different choice of basis for the equivariant cohomology ring, making use of our definition of local index translated to the equivariant cohomology case.

Similarly to what we did in Section 3, for a fixed point $q \in M^{\mathbb{T}}$ we define the map $\widetilde{r}\colon H^*_{\mathbb{T}}(M;\mathbb{Z}) \to H^*_{\mathbb{T}\times S^1}(H_q \times S^2;\mathbb{Z})$ and denote by κ the Kirwan map $\kappa\colon H^*_{\mathbb{T}\times S^1}(H_q \times S^2;\mathbb{Z}) \to H^*_{\widetilde{\mathbb{T}}_q}(\widetilde{H}^\epsilon_q;\mathbb{Z})$ which is surjective (for the surjectivity of κ over the integers see [TW, Proposition 7.3]). The index homomorphism in equivariant cohomology, corresponding to the equivariant K-theory push-forward map, is simply given by integration. By the Atiyah-Bott-Berline-Vergne Localization Formula [AB, BV] this integral

$$\mathrm{Ind}\colon H^*_{\widetilde{\mathbb{T}}_q}(\widetilde{H}^\epsilon_q;\mathbb{Z}) \to H^{*-2\lambda_q}_{\widetilde{\mathbb{T}}_q}(\mathrm{pt};\mathbb{Z})$$

is the following sum of rational functions:

$$\mathrm{Ind}(\alpha) = \int_{\widetilde{H}^\epsilon_q} \alpha = \sum_{q_j \in (\widetilde{H}^\epsilon_q)^{\mathbb{T}_q}} \frac{\alpha(p)}{\prod\limits_{w_l \in W^\epsilon_{q_j}} w_l},$$

where $W^\epsilon_{q_j}$ is the set of isotropy weights at q_j in $\widetilde{H}^\epsilon_q$.

Observe that if $\alpha \in H^{2\lambda}_{\widetilde{\mathbb{T}}_q}(\widetilde{H}^\epsilon_q;\mathbb{Z})$ and $\lambda < \lambda_q$ then $\mathrm{Ind}(\alpha) = 0$.

For a class $\tau \in H^*_{\mathbb{T}}(M;\mathbb{Z})$ let the **local index of τ at** q, denoted by $\mathrm{Ind}_q(\tau)$, be the image of the above $\mathrm{Ind}(\kappa \circ \widetilde{r}(\tau)) \in H^{*-2\lambda_q}_{\widetilde{\mathbb{T}}_q}(\mathrm{pt};\mathbb{Z})$ under the natural homomorphism $H^*_{\widetilde{\mathbb{T}}_q}(\mathrm{pt};\mathbb{Z}) \to H^*_{\mathbb{T}}(\mathrm{pt};\mathbb{Z})$ (compare with Definition 3.2).

Remark 5.4. An analogous of Proposition 3.4 holds for the local index in equivariant cohomology (the only difference is that $R(\mathbb{T})$ from the K-theory setting needs to be replaced by $H^*_{\mathbb{T}}(\mathrm{pt};\mathbb{Z})$). Moreover, by comparing the degrees, one sees that if $\tau \in H^{2\lambda}_{\mathbb{T}}(M;\mathbb{Z})$ then $\mathrm{Ind}_q(\tau) = 0$ for all $q \in M^{\mathbb{T}}$ with $\lambda < \lambda_q$, and that for all $q \in M^{\mathbb{T}}$ with $\lambda \geq \lambda_q$ the degree of $\mathrm{Ind}_q(\tau)\eta_q$ is equal to the degree of τ (here η_q is the equivariant Poincaré dual to F_q).

Definition 5.5. Let $(M, \omega, \mathbb{T}, \psi)$ be a symplectic toric manifold of dimension $2n$, together with a choice of a generic component of the moment map $\mu = \psi^{\xi}: M \to \mathbb{R}$. Then for each $p \in M^{\mathbb{T}}$, a Kirwan class $\tau_p \in H^*_{\mathbb{T}}(M; \mathbb{Z})$ satisfying

1) $\operatorname{Ind}_p(\tau_p) = 1$,
2) $\operatorname{Ind}_q(\tau_p) = 0$ for all points $q \in M^{\mathbb{T}} \setminus \{p\}$,

is called an **i-canonical class** at (the fixed point) p.

Remark 5.6. Note that as the above classes are Kirwan classes, they satisfy $\tau_p(p) = \Lambda_p^-$, $\tau_p(q) = 0$ for $q \prec p$, and they form a basis of $H^*_{\mathbb{T}}(M; \mathbb{Z})$ as an $H^*_{\mathbb{T}}(\mathrm{pt}; \mathbb{Z})$-module. (We will see later that in fact τ_p is zero at all the points $q \in M^{\mathbb{T}} \setminus V_p^+$.) By repeating the argument of Proposition 3.7 one can show that if i-canonical classes exist they are unique.

In what follows we prove the existence of i-canonical classes, compare them with the equivariant Poincaré duals to the flow-up manifolds, and with the GT-canonical classes in the index increasing case. Before doing so we prove some technical lemmas.

5.2. Technical Lemmas

We recall that, by the local normal form for the moment map μ, a fixed point with all negative isotropy weights is a local minimum of μ. Moreover, by [A], a local minimum is a global minimum, and the connectedness of the fibers of μ implies that the subset where μ achieves its minimum is connected. Hence, if the action has only isolated fixed points, there exists a unique fixed point where μ achieves its minimum (see [GS, Lemma 5.1]), and it is the only point with all negative weights.

Lemma 5.7. *Let $(X, \omega, \mathbb{T}, \psi)$ be a symplectic toric manifold together with a choice of a generic component $\mu: M \to \mathbb{R}$ of the moment map inducing an orientation on the associated GKM graph (V, E). Let p be the vertex corresponding to the fixed point where μ attains its minimum. Then for every vertex $q \in V$ there exists an increasing path $\overrightarrow{p\,p_k}, \overrightarrow{p_k\,p_{k-1}} \ldots, \overrightarrow{p_1\,q}$ from p to q.*

Proof. The unique point p where μ achieves its minimum is the unique fixed point with only negative weights (i.e. it corresponds to the only vertex with no incoming edges in the oriented GKM graph, that is $W_p^+ = \emptyset$). Take any vertex $q \in V$. If $q = p$ we are done. Otherwise there exists a vertex p_1 and an edge $\overrightarrow{p_1\, q} \in E$ with $w(\overrightarrow{p_1\, q}) \in W_q^+$, hence $\overrightarrow{p_1\, q} \in E^o$. If $p_1 = p$ we are done. Otherwise we continue this process and construct a path $\overrightarrow{p_l\, p_{l-1}}, \dots, \overrightarrow{p_1\, q}$ in the oriented GKM graph. As the GKM graph is finite and connected this procedure must end at p. $\square$

We recall that given a symplectic toric manifold $(M, \omega, \mathbb{T}, \psi, \overline{\xi})$ with oriented GKM graph (V, E^o) (which is not necessarily index increasing) and given $p \in M^{\mathbb{T}}$, V_p^+ is defined to be the set of vertices which can be joined to p through an increasing path in E^o (see Definition 4.7). The previous lemma, applied to $X = F_p$, implies the following.

Lemma 5.8. *Given a symplectic toric manifold $(M, \omega, \mathbb{T}, \psi, \overline{\xi})$, for every $p \in M^{\mathbb{T}}$ we have that*

$$F_p^{\mathbb{T}} \subset V_p^+$$

In Proposition 5.15 we prove that the opposite inclusion also holds provided that $(M, \omega, \mathbb{T}, \psi, \overline{\xi})$ is index increasing.

<u>Convention:</u> In the rest of subsection 5.2, $(M, \omega, \mathbb{T}, \psi, \overline{\xi})$ denotes a compact symplectic toric manifold together a choice of generic $\overline{\xi} \in \mathfrak{t}$ such that the corresponding moment map component $\mu = \psi^{\overline{\xi}} \colon M \to \mathbb{R}$ is <u>*index increasing.*</u> The associated oriented GKM graph is denoted by $\Gamma^o = (V, E^o)$. These hypotheses and notation apply to all of the following lemmas, propositions and corollaries.

Lemma 5.9. *Let $p \in M^{\mathbb{T}}$ and $(F_p, \omega_{|F_p})$ the flow up at p (see definition in Section 3). Then for any $q \in F_p^{\mathbb{T}} \setminus \{p\}$ we have that $\lambda_q > \lambda_p$.*

Proof. Take any $q \in F_p^{\mathbb{T}} \setminus \{p\}$. By Lemma 5.8, $q \in V_p^+$ and thus there exists an oriented path from p to q such that μ increases on each edge. The index increasing assumption gives that $\lambda_q > \lambda_p$. $\square$

Proposition 5.10. *Let $\eta_p \in H_{\mathbb{T}}^{2\lambda_p}(M; \mathbb{Z})$ be the equivariant Poincaré dual to F_p (see Definition 5.2) and $\zeta_p \in H_{\mathbb{T}}^{2\lambda_p}(M; \mathbb{Z})$ the GT-canonical class at $p \in M^{\mathbb{T}}$. Then $\eta_p = \zeta_p$.*

Proof. Notice that our index increasing assumption implies that GT-canonical classes exist at each $p \in M^{\mathbb{T}}$. As the conditions in Definition 5.3 define GT-canonical classes uniquely, it is sufficient to prove that $\eta_p(p) = \Lambda_p^-$

and that for all $q \in M^{\mathbb{T}} \setminus \{p\}$ with $\lambda_q \leq \lambda_p$ we have $\eta_p(q) = 0$. The first assertion follows from observing that the normal bundle of F_p at p coincides with the negative normal bundle of μ at p. To prove the second one we use Lemma 5.9, which implies that for any $q \in M^{\mathbb{T}} \setminus \{p\}$ with $\lambda_q \leq \lambda_p$ we must have $q \in M^{\mathbb{T}} \setminus F_p$, and at these points $\eta_p(q) = 0$ by definition. $\square$

Remark 5.11. There is an alternative way of proving that $\eta_p = \zeta_p$ in the index increasing case. If the class $\eta_p - \zeta_p$ were nonzero, it would be nonzero at some fixed point q. As both η_p and ζ_p vanish on points in $M^{\mathbb{T}} \setminus \{p\}$ of index smaller or equal to the index of p (see Lemma 5.9), the class $\eta_p - \zeta_p$ can only be nonzero at fixed points q with $\lambda_q > \lambda_p$. Since GT-canonical classes form a basis of $H^*_{\mathbb{T}}(M, \mathbb{Z})$, from the previous observation it follows that $\eta_p - \zeta_p = \sum_{q;\lambda_q > \lambda_p} c_q \zeta_q$, where $c_q \in H^*_{\mathbb{T}}(\mathrm{pt}; \mathbb{Z})$, with $c_s \neq 0$ for some s. By comparing the degrees of the classes on the right and on the left hand side of the above equation we obtain a contradiction.

Lemma 5.12. *Let $q \in M^{\mathbb{T}}$ and $(H_q, \omega_{|H_q})$ be the flow down at q (see the definition on page 1131). Let p be a fixed point connected to q by an oriented edge $\overrightarrow{pq}$ and such that $\lambda_p + 1 = \lambda_q$. Denote by $p_1, \ldots, p_{\lambda_p}$ the fixed points connected to p by oriented edges $\overrightarrow{p_j\, p}$. Then $p_1, \ldots, p_{\lambda_p} \in H_q$.*

Proof. Note that the GKM graph for H_q is a graph of valency λ_q. Therefore exactly λ_q of the points connected to p must be in H_q. Take any $r \neq q$ connected to p by an edge $\overrightarrow{pr}$. There are $n - 1 - \lambda_p = n - \lambda_q$ such points. Then $\lambda_r > \lambda_p = \lambda_q - 1$, thus $\lambda_r \geq \lambda_q$. Lemma 5.9 applied to $-\mu$ and H_q (instead of μ and F_q) gives that $r \notin H_q$. Therefore these $n - \lambda_q$ points connected to p are not in H_q. It follows that the remaining λ_q points, q and the $\lambda_q - 1 = \lambda_p$ points $p_1, \ldots, p_{\lambda_p}$ directly below p, must be in H_q. $\square$

To continue we need to recall some definitions from [GT]. Given a weight $w \in \ell^*$ in the weight lattice of $\mathfrak{t}^*$, and a generic $\xi \in \mathfrak{t}$, the projection which sends $X \in \mathfrak{t}^*$ to $X - \frac{X(\xi)}{w(\xi)} w \in \xi^{\perp} \subset \mathfrak{t}^*$ can be extended to be an endomorphism ρ_w of $\mathbb{S}(\mathfrak{t}^*)$, the symmetric algebra of $\mathfrak{t}^*$. Given an edge $\overrightarrow{r_1\, r_2} \in E$ of the GKM graph of M, the (pairwise) independence of the set of weights at each fixed point implies that $\rho_{w(\overrightarrow{r_1 r_2})}(\Lambda^-_{r_1}) \neq 0$ and $\rho_{w(\overrightarrow{r_1, r_2})}\left(\frac{\Lambda^-_{r_2}}{w(\overrightarrow{r_1 r_2})}\right) \neq 0$. Therefore the following nonzero elements of the field of fractions of $\mathbb{S}(\mathfrak{t}^*)$ are well defined for all $\overrightarrow{r_1 r_2} \in E$

$$\Theta(r_1, r_2) = \frac{\rho_{w(\overrightarrow{r_1 r_2})}(\Lambda^-_{r_1})}{\rho_{w(\overrightarrow{r_1 r_2})}\left(\dfrac{\Lambda^-_{r_2}}{w(\overrightarrow{r_1 r_2})}\right)}.$$

In [GT, Theorem 1.6] Goldin and Tolman prove that $\Theta(r_1,r_2)\in\mathbb{Z}\setminus\{0\}$ for all edges $\overrightarrow{r_1r_2}\in E$ with $\lambda_{r_2}=\lambda_{r_1}+1$. (Note that the assumption that the difference of the indices is 1, though not explicitly stated in their theorem, is implied and required.) Moreover, they prove the following formula for computing the restriction of a GT-canonical class to a fixed point. Let (V,E_{can}) be the subgraph of the GKM graph $\Gamma=(V,E)$ where $E_{can}=\{e=\overrightarrow{r_1\,r_2}\in E\mid \lambda_{r_2}=\lambda_{r_1}+1\}$. Since the oriented GKM graph is index increasing, this implies that $\mu(r_1)<\mu(r_2)$ for every $\overrightarrow{r_1\,r_2}\in E_{can}$, i.e. E_{can} is a subset of E^o. Then for every $p,q\in M^{\mathbb{T}}$ we have that

$$\zeta_p(q)=\Lambda_q^-\sum_{\gamma\in\sum_p^q}\prod_{i=1}^{|\gamma|}\frac{\psi(r_i)-\psi(r_{i-1})}{\psi(q)-\psi(r_{i-1})}\frac{\Theta(r_{i-1},r_i)}{w(\overrightarrow{r_{i-1}\,r_i})},\tag{5.4}$$

where $\sum_p^q$ is the set of paths from p to q in (V,E_{can}) and $\gamma\in\sum_p^q$ is given by the sequence of vertices $\gamma=(r_0,\dots,r_{|\gamma|})$; here $|\gamma|$ denotes the length of γ, i.e. the number of edges composing it.

Lemma 5.13. *For all $e=\overrightarrow{r_1r_2}\in E_{can}$ we have that $\Theta(r_1,r_2)=1$.*

Note that the above lemma may not hold for GKM manifolds which are not toric (see [GT, Example 5.2])

Proof. Recall that W_r^+ denotes the set of weights of the edges ending at r. Observe that $\rho_{w(\overrightarrow{r_1r_2})}$ sends $w(\overrightarrow{r_1r_2})$ to 0, so it is enough to prove that there exists a bijection $\theta\colon W_{r_1}^+\to W_{r_2}^+\setminus\{w(\overrightarrow{r_1r_2})\}$ such that $\theta(w)-w=m\cdot w(\overrightarrow{r_1r_2})$ for some $m\in\mathbb{Z}$ that depends on w. This is equivalent to proving that the weights of $\mathbb{T}$ representation on $T_{r_2}H_{r_2}$ and $T_{r_1}H_{r_2}$ agree modulo $w(\overrightarrow{r_1r_2})$. The last fact follows from observing that the subtorus

$$\mathbb{T}'=\exp(\{\eta\in\mathfrak{t}\mid w(\overrightarrow{r_1r_2})(\eta)=0\})$$

fixes the sphere $S^2_{\overrightarrow{r_1r_2}}=\psi^{-1}(\overrightarrow{r_1r_2})$, an embedded $\mathbb{T}$ invariant submanifold of H_{r_2} (which, in turn, is a smooth $\mathbb{T}$-invariant submanifold of M). Thus the representations of the torus $\mathbb{T}'$ on the normal bundle of $S^2_{\overrightarrow{r_1r_2}}$ in H_{r_2} need to agree at r_1 and r_2. □

The next lemma proves that for each $q\in V_p^+$ there exists a path γ whose edges belong to E_{can}.

Lemma 5.14. *Let $p\in M^{\mathbb{T}}$ and let Σ_p^q be defined as before. Then $q\in V_p^+$ if and only if $\sum_p^q\neq\emptyset$.*

Proof. If $\sum_p^q \neq \emptyset$ then clearly $q \in V_p^+$. Vice versa, suppose that $q \in V_p^+$. Assume first that there exists a path from p to q composed by one edge $\overrightarrow{p\,q} \in E^o$. Note that the GT-canonical class ζ_p does not vanish when restricted to q. Indeed, $\zeta_p(q) = 0$ and condition (5.1) would imply that $w(\overrightarrow{p\,q})$ divides $\zeta_p(p) = \Lambda_p^-$, which contradicts the assumption about linear independence of weights at p. By (5.4) we conclude that $\sum_p^q \neq \emptyset$. If the path from p to q is composed by edges $\overrightarrow{p\,r_1} = \overrightarrow{r_0\,r_1}, \dots, \overrightarrow{r_{m-1}\,r_m} = \overrightarrow{r_{m-1}\,q}$, each of them in E^o, then the preceding argument implies that the sets of paths $\sum_{r_0}^{r_1}, \dots, \sum_{r_{m-1}}^{r_m}$ are all nonempty, and so $\sum_p^q$ is nonempty as well. ☐

Proposition 5.15. *For any $p \in M^{\mathbb{T}}$, let ζ_p be the GT-canonical class. Then for any $q \in M^{\mathbb{T}}$ have*

$$\zeta_p(q) \neq 0 \Leftrightarrow q \in V_p^+.$$

Together with Proposition 5.10 this implies

$$F_p^{\mathbb{T}} = V_p^+.$$

Proof. If $\zeta_p(q) \neq 0$ then by (5.4) the set $\sum_p^q$ is nonempty, hence $q \in V_p^+$. Now consider $q \in V_p^+$. By Lemma 5.14 we have that $\sum_p^q$ is not empty. We use formula (5.4) quoted from [GT] to analyze $\zeta_p(q)$. Using Lemma 5.13 observe that each summand in (5.4) for $\zeta_p(q)$ is positive (in the sense that it gives a positive number when evaluated on $\bar{\xi}$), therefore $\zeta_p(q) \neq 0$. ☐

Corollary 5.16. *If $q \in F_p^{\mathbb{T}}$ and $\overrightarrow{q\,q_0} \in E^o$ then $q_0 \in F_p^{\mathbb{T}}$. As a consequence, the normal bundle of F_p at q (denoted by $N(F_p)|_q$) is a subbundle of the negative normal bundle of μ at q (denoted by N_q^-).*

Proof. By definition of V_p^+ it follows that if $q \in V_p^+$ then $V_q^+ \subseteq V_p^+$. Together with Proposition 5.15 this gives that $F_q^{\mathbb{T}} \subseteq F_p^{\mathbb{T}}$. From the definition of F_q it is straightforward to see that if $\overrightarrow{q\,q_0} \in E^o$ then $q_0 \in F_q^{\mathbb{T}}$, and the first claim follows. As a consequence we have that $N(F_p)|_q$ splits as a direct sum of line bundles $\mathbb{L}_i$, each of them being the tangent bundle at q of the sphere associated to the edge $\overrightarrow{q_i\,q} \in E^o$, for some $q_i \notin F_p$. This implies the second claim. ☐

5.3. The proof of Theorem 1.2

We are now ready to prove Theorem 1.2. It follows immediately from the Proposition below.

Proposition 5.17. *Let $(M,\omega,\mathbb{T},\psi)$ be a symplectic toric manifold, together with a choice of a generic component of the moment map $\mu=\psi^{\overline{\xi}}\colon M\to\mathbb{R}$, not necessarily index increasing. Then for each $p\in M^{\mathbb{T}}$ the i-canonical class τ_p exists, and is equal to the equivariant Poincaré dual η_p to flow-up submanifold F_p.*

If the moment map $\mu=\psi^{\overline{\xi}}\colon M\to\mathbb{R}$ is index increasing (thus GT-canonical classes exist) then the above proposition, together with Proposition 5.10, implies that for each $p\in M^{\mathbb{T}}$ the following equivariant cohomology classes are the same:

- the Poincaré dual η_p to flow-up submanifold F_p;
- the GT-canonical class ζ_p;
- the i-canonical class τ_p.

Proof. Fix $p\in M^{\mathbb{T}}$. We show that the equivariant Poincaré dual η_p to the flow-up submanifold F_p satisfies: $\mathrm{Ind}_p(\eta_p)=1$ and $\mathrm{Ind}_q(\eta_p)=0$ for each $q\in M^{\mathbb{T}}\setminus\{p\}$. Since η_p is also a Kirwan class, this proves that η_p is the i-canonical class at p. As $\eta_p(p)=\Lambda_p^-$, Remark 5.4 implies that $\mathrm{Ind}_p(\eta_p)=1$. For $q\in M^{\mathbb{T}}\setminus F_p$ we have $\eta_p(q)=0$, thus $\mathrm{Ind}_q(\eta_p)=0$. Consider a point $q\in F_p\setminus\{p\}$. Let $q_1,\dots,q_{\lambda_q},\dots,q_n\in M^{\mathbb{T}}$ be the fixed points connected to q by an edge in the GKM graph of M, and let $w_1,\dots,w_{\lambda_q},\dots,w_n$ denote the weights on the corresponding oriented edges, with $\Lambda_q^-=\prod_{j=1}^{\lambda_q}w_j$. By definition of equivariant Poincaré dual we have that

$$\eta_p(q)=\prod_{\substack{1\le j\le n\\ q_j\notin F_p}}w_j=\prod_{\substack{1\le j\le \lambda_q\\ q_j\notin F_p}}w_j\cdot\prod_{\substack{\lambda_q+1\le j\le n\\ q_j\notin F_p}}w_j\,.$$

To calculate the index $\mathrm{Ind}_q(\eta_p)$ we use the algorithm and the notation of Section 3. Observe that

$$f_0=\prod_{\substack{1\le j\le \lambda_q\\ q_j\notin F_p}}(w_j+w_0)\cdot\prod_{\substack{\lambda_q+1\le j\le n\\ q_j\notin F_p}}w_j,$$

$$f_i=\begin{cases}0 & \text{if } q_i\notin F_p\\ \displaystyle\prod_{\substack{1\le j\le \lambda_q\\ q_j\notin F_p}}(w_j-w_i)\cdot\prod_{\substack{\lambda_q+1\le j\le n\\ q_j\notin F_p}}w_j & \text{if } q_i\in F_p\end{cases}$$

Therefore the local index $\mathrm{Ind}_q(\eta_p)$ is equal to

$$\mathrm{Ind}_q(\eta_p) = \frac{\displaystyle\prod_{\substack{1\leq j\leq\lambda_q\\ q_j\notin F_p}}(w_j+w_0)\cdot\prod_{\substack{\lambda_q+1\leq j\leq n,\\ q_j\notin F_p}} w_j}{\displaystyle\prod_{1\leq j\leq\lambda_q}(w_j+w_0)} + \sum_{\substack{1\leq i\leq n\\ q_i\in F_p}} \frac{\displaystyle\prod_{\substack{1\leq j\leq\lambda_q\\ q_j\notin F_p}}(w_j-w_i)\cdot\prod_{\substack{\lambda_q+1\leq j\leq n\\ q_j\notin F_p}} w_j}{\displaystyle(-w_i-w_0)\prod_{\substack{1\leq j\leq\lambda_q\\ j\neq i}}(w_j-w_i)}$$

$$= \left(\prod_{\substack{\lambda_q+1\leq j\leq n\\ q_j\notin F_p}} w_j\right)\left(\frac{1}{\displaystyle\prod_{\substack{1\leq j\leq\lambda_q\\ q_j\in F_p}}(w_j+w_0)} + \sum_{\substack{1\leq i\leq n\\ q_i\in F_p}}\frac{1}{\displaystyle(-w_i-w_0)\prod_{\substack{1\leq j\leq\lambda_q\\ j\neq i,\, q_j\in F_p}}(w_j-w_i)}\right)$$

$$= \left(\prod_{\substack{\lambda_q+1\leq j\leq n\\ q_j\notin F_p}} w_j\right)\mathrm{Ind}(\mathbf{1}_{\mathbb{CP}^s}),$$

where s is the number of weights $w_1,\ldots,w_{\lambda_q}$ appearing in the representation of $\mathbb{T}$ on T_qF_p (in the index increasing case, s would be $\lambda_q-\lambda_p$). Note that $s\neq 0$ because if $s=0$ then q would be the minimum of μ on F_p (c.f. Lemma 5.7) which contradicts our assumption that $q\neq p$. As $\mathrm{Ind}(\mathbf{1}_{\mathbb{CP}^s})=0$ for $s>0$ it follows that $\mathrm{Ind}_q(\eta_p)=0$. ☐

Appendix A. An explicit description of the Kirwan map

The map κ used in the definition of local index is in fact the surjective Kirwan map relating the equivariant K-theory or cohomology ring of a manifold X with that of the reduced spaces. Indeed, below we describe a combinatorial algorithm for calculating the Kirwan map, and the reader can compare it with the combinatorial algorithm for calculating the local index in Section 3. For simplicity we only deal with the equivariant cohomology setting.

Suppose that $X=X^{2d}$ is a $2d$-dimensional symplectic manifold equipped with an effective Hamiltonian toric action of torus $\mathbb{T}=T^d$. Let $\psi'\colon X\to Lie(T^d)^*\cong\mathbb{R}^d$ be a choice of moment map. Choose a subtorus $T^k\hookrightarrow\mathbb{T}$, $k<d$, and consider the induced action of T^k on X. Let $\pi\colon Lie(T^d)^*\to Lie(T^k)^*$ be the map induced by the inclusion $T^k\hookrightarrow T^d$. Then $\varphi=\pi\circ\psi'\colon X\to\mathbb{R}^k$ is a moment map for this action. Take any regular value a of the function φ. Then $X_{red}:=\varphi^{-1}(a)/T^k$ is a symplectic toric orbifold.

If the T^k action on $\varphi^{-1}(a)$ is free then $\varphi^{-1}(a)/T^k$ is a manifold. Moreover it is equipped with a Hamiltonian action of the residual torus $\mathbb{K} := T^d/T^k$. By a theorem of Kirwan the following map (called Kirwan map)

$$\kappa\colon H^*_{\mathbb{T}}(X;\mathbb{Z}) \twoheadrightarrow H^*_{\mathbb{T}}(\varphi^{-1}(a);\mathbb{Z}) \simeq H^*_{\mathbb{K}}(X_{red};\mathbb{Z})$$

is surjective (see [TW, Prop. 7.3]). We describe κ explicitly in the situation that appears in our algorithm for calculating the local index (where $X = H_q \times S^2$ and $X_{red} = \widetilde{H}^{\epsilon}_q$), namely when:

- $k = 1$ so $T^k \cong S^1$, henceforth denoted $\mathbb{S}^1$ to avoid confusion,
- $\mathbb{S}^1$ acts freely on $\varphi^{-1}(a)$, hence X_{red} is a manifold and
- the level of the reduction a is close to the maximum of φ, i.e. the hyperplane $\pi^{-1}(a)$ cuts the moment map image of X close to the fixed point q_0 of X where φ attains its maximum.

The moment map image of X_{red} is the intersection of $\psi'(X)$ with the affine hyperplane $\pi^{-1}(a)$ in $Lie(T^d)^*$. The fixed points of X_{red} correspond to the points of intersection of this affine hyperplane with the edges of the 1-skeleton of the moment polytope of X; the set of these edges is denoted by E. An example is presented in Figure A1. The weights of the $T^d = T^2$ action at p_0 are w_1 and w_2. The affine hyperplane is perpendicular to the vector $v = w_1 + w_2$.

Recall the description of the kernel of the Kirwan map from the work of Goldin [G] and Tolman-Weitsman [TW], and observe that in our situation any class in $H^*_{\mathbb{T}}(X;\mathbb{Z})$ which has value 0 when restricted to q_0 is in the kernel. Therefore

$$\kappa(\alpha) = \kappa(\alpha(q_0)\cdot \mathbf{1})$$

for any α in $H^*_{\mathbb{T}}(X;\mathbb{Z})$. This reduces our problem to analyzing only the classes of the form $f\cdot\mathbf{1}$, with $f \in H^*_{\mathbb{T}}(\mathrm{pt};\mathbb{Z})$.

We describe $\kappa(f\cdot\mathbf{1})$ by calculating its restrictions to the fixed points $X^{\mathbb{K}}_{red}$. Let $p_i \in X^{\mathbb{K}}_{red}$ be any fixed point. Denote by $q_i, q_0 \in X^{\mathbb{T}}$ the fixed points in X connected by an edge $\overrightarrow{q_iq_0} \in E$ such that p_i is the intersection of the edge $\overrightarrow{q_iq_0}$ with the affine hyperplane $\pi^{-1}(a)$. Denote by S^2_i the sphere in X corresponding to the edge $\overrightarrow{q_iq_0}$ and by H_i the subtorus of $\mathbb{T}$ fixing S^2_i, i.e. $H_i = \exp(\{\xi \mid w(\overrightarrow{q_iq_0})(\xi) = 0\})$. Note that $S^2_i \cap \varphi^{-1}(a)$ is a circle, denoted by $\mathcal{C}_i$, equipped with a free $\mathbb{S}^1$-action (the restriction of the free $\mathbb{S}^1$ action on $\varphi^{-1}(a)$ to $S^2_i \cap \varphi^{-1}(a)$) and that

$$\mathcal{C}_i/\mathbb{S}^1 = \{p_i\}.$$

To find the weights of the $\mathbb{K}$ action on $T_{p_i}X_{red}$ we proceed as in [GH, Example 3]. Observe that H_i is complementary to $\mathbb{S}^1$ in $\mathbb{T}$, so $H_i \cong \mathbb{T}/\mathbb{S}^1 = \mathbb{K}$ (this follows from the assumption that the reduced space is a manifold). Therefore the $\mathbb{K}$ action on $T_{p_i}X_{red}$ is isomorphic to the H_i action on this space. The weights of the $\mathbb{K}$ action on $T_{p_i}X_{red}$ are obtained by projecting the $\mathbb{T}$ weights at q_0 to $\mathfrak{h}_i^* = Lie(H_i)^*$. (Using q_i instead of q_0 gives the same result as the $\mathbb{T}$ weights at q_i differ from those at q_0 by a multiple of $w(\overrightarrow{q_iq_0})$, so the difference vanishes after applying the projection to $\mathfrak{h}_i^*$). Note also that the weights of the $\mathbb{K}$ action on $T_{p_i}X_{red}$, together with $w(\overrightarrow{q_iq_0})$, form a $\mathbb{Z}$-basis of the lattice ℓ^*.

We need to find the image of $f \cdot \mathbf{1} \in H_{\mathbb{T}}^*(X;\mathbb{Z})$ under the composition

$$H_{\mathbb{T}}^*(X;\mathbb{Z}) \to H_{\mathbb{T}}^*(\mathcal{C}_i;\mathbb{Z}) \overset{\cong}{\to} H_{\mathbb{K}}^*(\{p_i\};\mathbb{Z}) = H_{H_i}^*(\mathrm{pt};\mathbb{Z}),$$

where the first map is induced by the inclusion $\mathcal{C}_i \hookrightarrow X$ and sends $f \cdot \mathbf{1} \in H_{\mathbb{T}}^*(X;\mathbb{Z})$ to $f \cdot \mathbf{1} \in H_{\mathbb{T}}^*(\mathcal{C}_i;\mathbb{Z})$. As for the second map, on classes of the form $f \cdot \mathbf{1}$, with $f \in H_{\mathbb{T}}^*(\mathrm{pt};\mathbb{Z})$, it acts exactly as the map $H_{\mathbb{T}}^*(\mathrm{pt};\mathbb{Z}) \to H_{H_i}^*(\mathrm{pt};\mathbb{Z})$ we used above to find the weights. Note that the value $\kappa(f\cdot\mathbf{1})(p_i)$ is in the $\mathbb{Z}$-span of the weights of the $\mathbb{K}$ action on $T_{p_i}X_{red}$, as it should be.

In conclusion, the procedure for finding the value of the restriction of $\kappa(\alpha)$ to p_i is the following:

- Present the value $\alpha(q_0) \in H_{\mathbb{T}}^*(\mathrm{pt};\mathbb{Z})$ in the basis consisting of the $\mathbb{K}$ weights at $T_{p_i}X_{red}$ and of the weight $w(\overrightarrow{q_iq_0})$.
- Map such value to $H_{\mathbb{K}}^*(\mathrm{pt};\mathbb{Z}) \cong H_{H_i}^*(\mathrm{pt};\mathbb{Z})$ by sending the weight $w(\overrightarrow{q_iq_0})$ to 0.

The result is $\kappa(\alpha)$ restricted to the point p_i. Note that starting from $\alpha(q_i)$ instead of $\alpha(q_0)$ gives exactly the same result because $\alpha(q_0)$ and $\alpha(q_i)$ differ by a multiple of the weight $w(\overrightarrow{q_iq_0})$ (see (5.1)). By repeating the same argument for each fixed point of X_{red} we obtain the image of $\kappa(\alpha)$ in $H_{\mathbb{K}}^*(X_{red}^{\mathbb{K}})$.

In the example in Figure A1, $\pi^{-1}(a)$ is generated by the vector $(1,-1)$. At p_1 we get a $\mathbb{Z}$-basis $\{w_1 - w_2 = (1,-1),\ w_1 = (1,0)\}$, while at p_2 we get a $\mathbb{Z}$-basis $\{w_2 - w_1 = (-1,1),\ w_2 = (0,1)\}$. We find the image of the class α presented in black on the right of Figure A1 (with $\alpha_j = \alpha(q_j)$). The torus $\mathbb{K}$ is 1 dimensional, so the dual of its Lie algebra can be identified with $\mathbb{R}[x]$, where $x = w_1 - w_2$. At the point p_1 our procedure applied to α_0 gives

$$w_1 + w_2 = -(w_1 - w_2) + 2w_1 \to -(w_1 - w_2) + 0 = -x.$$

(Observe that if we use α_1 instead of α_0 we indeed get the same result: $4w_1 + w_2 = -(w_1 - w_2) + 5w_1 \to -(w_1 - w_2) + 0 = -x$.) At the point p_2 we get

$$w_1 + w_2 = (w_1 - w_2) + 2w_2 \to (w_1 - w_2) + 0 = x.$$

Therefore $\kappa(\alpha)$ (presented in blue) restricted to p_1 gives $-x$, and restricted to p_2 gives x.

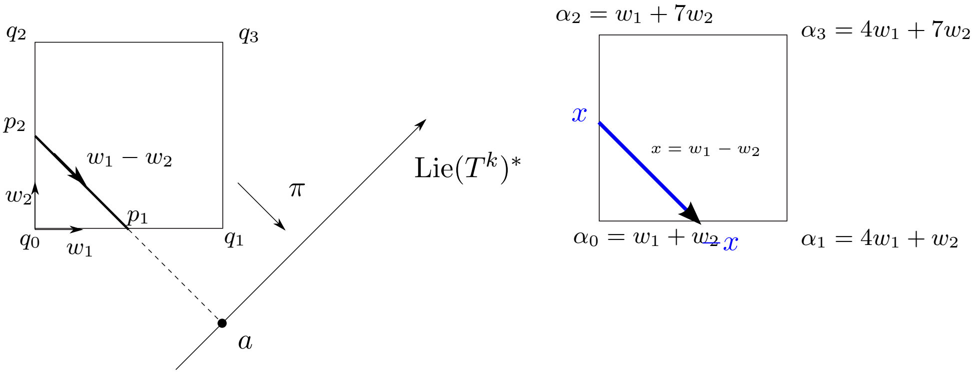

Figure A1: An example of a computation of the Kirwan map.

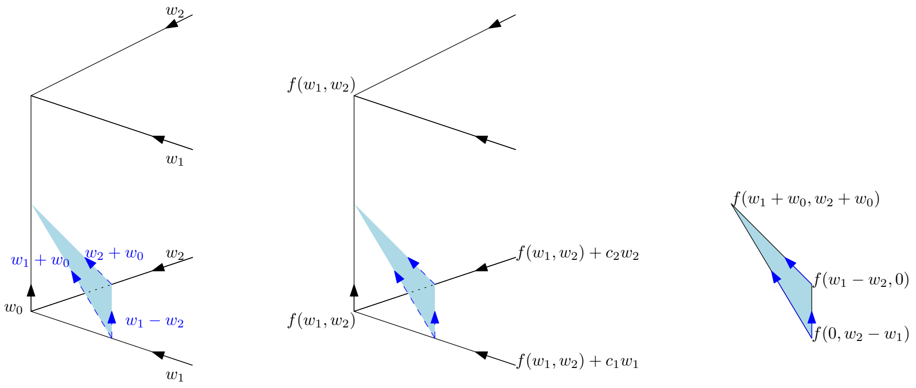

Figure A2: A calculation of the Kirwan map in the situation encountered in the definition of local index.

Our second example of computation of the Kirwan map goes back to the situation presented in Figure 3.1, i.e. the situation we encountered while calculating the local index. Given a class $\tau \in H^*_{\mathbb{T}}(H_q;\mathbb{Z})$ we want to calculate the class $\kappa(\widetilde{r}(\tau))$ in $H^*_{\widetilde{\mathbb{T}}_q}((\widetilde{H}^\epsilon_q)^{\widetilde{\mathbb{T}}_q};\mathbb{Z})$. Figure A2 consists of three pictures: the weights around the fixed point q_0, the values of $\widetilde{r}(\tau)$ at the fixed points of $H_q \times S^2$ in the neighborhood of q_0, and the values of $\kappa(\widetilde{r}(\tau))$ at the fixed points of $\widetilde{H}^\epsilon_q$. The values of $\kappa(\widetilde{r}(\tau))$ at p_0, p_1, p_2, respectively, are calculated in the following way:

$$\begin{aligned}
&f(w_1,w_2) = f((w_1+w_0)-w_0,(w_2+w_0)-w_0) \to f(w_1+w_0,w_2+w_0),\\
&f(w_1,w_2) = f(w_1,(w_2-w_1)+w_1) \to f(0,w_2-w_1),\\
&f(w_1,w_2) = f((w_1-w_2)+w_2,w_2) \to f(w_1-w_2,0).
\end{aligned}$$

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Mathematisches Institut, Universität zu Köln
Weyertal 86-90, D-50931 Köln, Germany
E-mail address: `pabiniak@math.uni-koeln.de`
E-mail address: `sabatini@math.uni-koeln.de`

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