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Quasi-isometry type of the metric space derived from the kernel of the Calabi homomorphism

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We prove that the set of symmetrized conjugacy classes of the kernel of the Calabi homomorphism on the group of area-preserving diffeomorphisms of the 2-disk is not quasi-isometric to the half line.

1. Introduction

Suppose that G is a simple group and $K \subseteq G$ is a subset. Here, we assume that K contains non-trivial elements of G. Since the group G is simple, any non-trivial element g of G can be written as a product of conjugates of elements of $K \cup K^{-1}$. We define for each $g \in G$ the number $q_K(g)$ by the minimal number of conjugates of elements of $K \cup K^{-1}$ whose product is equal to g. Here, for the identity element e, we define $q_K(e) = 0$. The function $q_K \colon G \to \mathbb{Z}_{\geq 0}$ is obviously invariant under conjugations and defines a conjugation-invariant norm on G. Such a conjugation-invariant norm is called a *conjugation-generated norm*. In this paper, we mainly consider the case K consists of a single non-trivial element.

Elements f and g of a group G are *symmetrized conjugate* to each other if f is conjugate to g or g^{-1}. It is easy to see that symmetrized conjugacy is an equivalence relation. We denote by $[g]$ the symmetrized conjugacy class represented by $g \in G$. We define $\mathcal{M}(G)$ to be the set of non-trivial symmetrized conjugacy classes of elements of G. In [17], Tsuboi introduced a metric d on $\mathcal{M}(G)$ defined by

$$d([f],[g]) = \log \max\{q_{\{g\}}(f), q_{\{f\}}(g)\}.$$

In fact, it is easy to see that the inequality

$$q_{\{f\}}(h) \leq q_{\{f\}}(g) q_{\{g\}}(h)$$

holds for any $f, g, h \in G$ and thus the function $d\colon \mathcal{M}(G) \times \mathcal{M}(G) \to \mathbb{R}_{\geq 0}$ satisfies the triangle inequality. We are interested in this metric space $\mathcal{M}(G)$, which is an invariant of simple group.

In [12], Kodama studied the metric space $(\mathcal{M}(G), d)$ for the case G is the infinite alternating group A_∞ and proved the following.

Theorem 1.1 (Kodama [12]). *The metric space* $(\mathcal{M}(A_\infty), d)$ *is quasi-isometric to the half line.*

We define the 2-disk D^2 and the standard area form Ω on D^2 to be

$$D^2 = \{(x, y) \in \mathbb{R}; x^2 + y^2 \leq 1\} \text{ and } \Omega = \frac{1}{\pi} dx \wedge dy$$

respectively. Let $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ be the group of C^∞-diffeomorphisms of the 2-disk D^2, which preserve Ω and are the identity on a neighborhood of the boundary. It is classically known that the group $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ admits a homomorphism

$$\mathrm{Cal}\colon \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2) \to \mathbb{R}$$

called the Calabi homomorphism. The Calabi homomorphism gives an abelianization of $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and its kernel KerCal is simple [1]. In this paper, we study the metric space $(\mathcal{M}(G), d)$ for the case $G = \mathrm{KerCal}$ and prove the following theorem.

Theorem 1.2. *For any non-trivial element* $f \in \mathrm{KerCal}$, *there exist a sequence* $\{f_n\}_{n\geq 0}$ *contained in* KerCal *with* $f_0 = f$, *an element* $g \in \mathrm{KerCal}$ *and positive constants* C_1, C_2, C_3 *which satisfy the following.*

(i) $d([f_n], [f_m]) \geq C_1|n - m|$,

(ii) $d([f_n], [f_{n+1}]) \leq C_2$,

(iii) $d([f_n], [g^m]) \geq \log m + C_3$.

As a corollary, we obtain the following statement answering to a problem raised by Tsuboi [18, Problem4.4].

Theorem 1.3. *The metric space* $(\mathcal{M}(\mathrm{KerCal}), d)$ *is not quasi-isometric to the half line.*

2. Quasi-morphisms

In this section, we prepare a notion of quasi-morphism, which is a useful tool to evaluate a lower bound for a conjugation-generated norm q_K and prove Proposition 2.2. On quasi-morphisms and conjugation-generated norms, see [7] for more details.

Let G be a group. A *quasi-morphism* on G is a function $\phi\colon G \to \mathbb{R}$ such that there exists a constant $C \geq 0$ satisfying $|\phi(gh) - \phi(g) - \phi(h)| \leq C$ for any $g, h \in G$. The real number

$$D(\phi) = \sup_{g,h\in G} |\phi(gh) - \phi(g) - \phi(h)|$$

is called the *defect* of ϕ. A quasi-morphism ϕ on G is *homogeneous* if $\phi(g^p) = p\phi(g)$ for any $g \in G$ and any $p \in \mathbb{Z}$. For any quasi-morphism ϕ on an arbitrary group G, there exists a unique homogeneous quasi-morphism $\tilde{\phi}$ on G such that $\tilde{\phi} - \phi$ is a bounded function on G and $\tilde{\phi}$ is explicitly written as

$$\tilde{\phi}(g) = \lim_{p\to\infty} \frac{1}{p}\phi(g^p).$$

We denote by $Q(G)$ the $\mathbb{R}$-vector space consisting of homogeneous quasi-morphisms on G. Note that homogeneous quasi-morphisms are invariant under conjugations.

2.1. Conjugation-invariant norms and quasi-morphisms

Let K be a subset of G. We define the vector subspace $Q(G, K)$ of $Q(G)$ by

$$Q(G, K) = \{\phi \in Q(G); \phi \text{ is bounded on } K\}.$$

Note that this definition is different from that given in [7]. Suppose that $g \in G$ is written as

$$g = f_1 \cdots f_n,$$

where $f_1, \ldots, f_n$ are conjugates of elements of $K \cup K^{-1}$. Then for each $\phi \in Q(G, K)$ the inequation

$$|\phi(g) - \phi(f_1) - \cdots - \phi(f_n)| \leq (n-1)(D(\phi))$$

holds. If we set $C_K = \sup_{h\in K} |\phi(h)|$, then we have

$$\frac{|\phi(g)|}{D(\phi) + C_K} \leq n.$$

This means that

$$\frac{|\phi(g)|}{D(\phi)+C_K} \leq q_K(g).$$

Denoting by $[K]$ the set of symmetrized conjugacy classes represented by the elements of K, we have the following lemma on the metric d of $\mathcal{M}(G)$.

Lemma 2.1. *Let $\phi \in Q(G,K)$ and $g \in G$ such that $\phi(g) \neq 0$. Then*

$$\log \frac{|\phi(g)|}{D(\phi)+C_K} \leq d([g],[K]).$$

In particular,

$$\log n + \log \frac{|\phi(g)|}{D(\phi)+C_K} \leq d([g^n],[K]) \text{ for any } n.$$

A simple group G is *uniformly simple* if the metric space $(\mathcal{M}(G), d)$ is bounded. This is equivalent to saying that $(\mathcal{M}(G), d)$ is quasi-isometric to a point. Since $Q(G,K) = Q(G)$ for any bounded set K, if the group G admits a non-trivial quasi-morphism then $(\mathcal{M}(G), d)$ is unbounded by Lemma 2.1 and thus G is not uniformly simple.

2.2. Gambaudo-Ghys' construction of quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$

It is known that the vector space $Q(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2))$ is infinite-dimensional [8][9][10]. To prove Theorem 1.2, we use quasi-morphisms on KerCal obtained by Brandenbursky generalizing Gambaudo-Ghys' construction [4].

Let $X_n(D^2)$ be the n-fold configuration space of D^2. Fix a base point $x^0 = (x^0_1, \ldots, x^0_n) \in X_n(D^2)$. For any $g \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and almost every $x = (x_1, \ldots, x_n) \in X_n(D^2)$, we set a loop $l(g;x)\colon [0,1] \to X_n(D^2)$ by

$$l(g;x)(t) = \begin{cases} \{(1-3t)x^0_i + 3tx_i\} & \left(0 \leq t \leq \frac{1}{3}\right) \\ \{g_{3t-1}(x_i)\} & \left(\frac{1}{3} \leq t \leq \frac{2}{3}\right) \\ \{(3-3t)g(x_i) + (3t-2)x^0_i\} & \left(\frac{2}{3} \leq t \leq 1\right), \end{cases}$$

where $\{g_t\}_{t\in[0,1]}$ is a path in $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ such that g_0 is the identity and $g_1 = g$. Of course for some $x \in X_n(D^2)$ the loop $l(g;x)$ may not be defined.

However, for almost every x the loop $l(g;x)$ is well-defined. We define the pure braid $\gamma(g;x)$ to be the homotopy class relative to the base point x^0 represented by the loop $l(g;x)$. Since the group of diffeomorphisms of D^2 is contractible [16] and is homotopy equivalence to $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ [15], the pure braid $\gamma(g;x)$ is independent of the choice of the path $\{g_t\}$. Let $P_n(D^2)$ be the pure braid group on n-strands. For a homogeneous quasi-morphism ϕ on $P_n(D^2)$, if we consider the function

$$g \mapsto \int_{x\in X_n(D^2)} \phi(\gamma(g;x))\Omega^n,$$

then this function is well-defined [4][6] and is further a quasi-morphism on $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ since the diffeomorphism g preserves Ω. Thus we have the linear map $\Gamma_n\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ defined by

$$\Gamma_n(\phi)(g) = \lim_{p\to\infty} \frac{1}{p} \int_{x\in X_n(D^2)} \phi(\gamma(g^p;x))\Omega^n.$$

Let $B_n(D^2)$ be the braid group on n strands and $i\colon P_n(D^2) \to B_n(D^2)$ the natural inclusion. Then the linear map $Q(i)\colon Q(B_n(D^2)) \to Q(P_n(D^2))$ is induced. For $n \geq 3$, the vector space $Q(B_n(D^2))$ is infinite-dimensional [3] and the composition $\Gamma_n \circ Q(i)\colon Q(B_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ of the linear maps is injective [10]. Hence the image $\mathrm{Im}\Gamma_n \subset Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ is also infinite dimensional.

For $r > 1$, we denote the small disk $\{(x,y)\in\mathbb{R}; x^2+y^2 \leq r^{-2}\}$ of radius $1/r$ by $D(r^{-1})$. Let $\varphi_r\colon D^2 \to D(r^{-1})$ be the C^∞-diffeomorphism defined by

$$\varphi_r(x,y) = \left(\frac{x}{r}, \frac{y}{r}\right).$$

We define the homomorphism $s_r\colon \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2) \to \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ by

$$s_r(f)(x,y) = \begin{cases} \varphi_r \circ f \circ \varphi_r^{-1}(x,y) & \text{if } (x,y) \in D(r^{-1}) \\ (x,y) & \text{if } (x,y) \notin D(r^{-1}). \end{cases}$$

Note that if f is in KerCal, then $s_r(f)$ is also.

Let $\sigma_1 \in B_3(D^2)$ be the braid on 3 strands as indicated in Figure 1. The following proposition is essentially introduced in [6, Lemma 3.11].

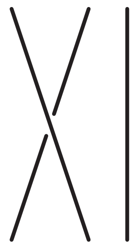

Figure 1: the braid σ_1.

Proposition 2.2. *If $\phi \in Q(B_3)$ satisfies $\phi(\sigma_1) = 0$, then*

$$\Gamma_3 \circ Q(i)(\phi)(s_r(f)) = \frac{1}{r^6}\Gamma_3 \circ Q(i)(\phi)(f)$$

for any $f \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and any $r > 1$.

Proof. Let $x = (x_1, x_2, x_3)$ be in $X_3(D^2)$. For any $f \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and any $r > 1$, if two or three of x_1, x_2, x_3 are not in $D(r^{-1})$, then the pure braid $\gamma(s_r(f); x)$ is trivial. Hence we have

$$\begin{aligned}\int_{x \in X_3(D^2)} \phi(\gamma(s_r(f); x))\Omega^3 = &\int_{x_1,x_2,x_3 \in D(r^{-1})} \phi(\gamma(s_r(f); x))\Omega^3 \\ &+ 3\int_{x_1,x_2 \in D(r^{-1}), x_3 \notin D(r^{-1})} \phi(\gamma(s_r(f); x))\Omega^3\end{aligned}$$

for any $\phi \in Q(B_3(D^2))$.

If $x_1, x_2 \in D(r^{-1})$ and $x_3 \notin D(r^{-1})$, then the pure braid $\gamma(s_r(f); x)$ is a conjugate of a power of σ_1 and hence $\phi(\gamma(s_r(f); x)) = 0$. Since

$$\int_{x_1,x_2,x_3 \in D(r^{-1})} \phi(\gamma(s_r(f); x))\Omega^3 = \frac{1}{r^6}\int_{x \in X_3(D^2)} \phi(\gamma(f; x))\Omega^3,$$

we have the desired equality. $\square$

3. Proof of the main theorem

In this section, we prove the main theorem. Before starting the proof, we show the following lemma as a preliminary step.

Lemma 3.1. *For any $f \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and $r > 1$, the following holds.*

(I) $d([s_r^m(f)],[s_r^n(f)]) \geq (2\log r)|m-n|$.

(II) $d([s_r^n(f)],[s_r^{n+1}(f)]) \leq d([f],[s_r(f)])$.

Proof. Assume that $m<n$. Since the area of the support of $s_r^m(f)$ is just $r^{2(n-m)}$ times of that of $s_r^n(f)$, we have $q_{\{s_r^n(f)\}}(s_r^m(f)) \geq r^{2(n-m)}$. This implies (I).

Suppose that $s_r(f)$ is written as a product

$$s_r(f) = (h_1 f^{\varepsilon_1} h_1^{-1}) \cdots (h_k f^{\varepsilon_k} h_k^{-1}),$$

where each ε_i is 1 or -1. Since the map

$$s_r\colon \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2) \to \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$$

is a homomorphism, we have

$$s_r^{n+1}(f) = (s_r^n(h_1)s_r^n(f)^{\varepsilon_1}s_r^n(h_1)^{-1}) \cdots (s_r^n(h_k)s_r^n(f)^{\varepsilon_k}s_r^n(h_k)^{-1})$$

and thus $q_{\{s_r^n(f)\}}(s_r^{n+1}(f)) \leq q_{\{f\}}(s_r(f))$. The inequality $q_{\{s_r^{n+1}(f)\}}(s_r^n(f)) \leq q_{\{s_r(f)\}}(f)$ similarly follows. Hence we have (II). □

Proof of Theorem 1.2. Fix $f \in \mathrm{KerCal}$ and $r>1$. If we set $f_n = s_r^n(f)$, then the properties (i) and (ii) immediately follow from Lemma 3.1.

Since the vector space $Q(B_n(D^2))$ is infinite-dimensional for $n \geq 3$ [3], considering the linear combination it is guaranteed that there exists a non-trivial homogeneous quasi-morphism ϕ on B_3 satisfying $\phi(\sigma_1)=0$. Since the composition of the linear maps $\Gamma_n \circ Q(i)\colon Q(B_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ is injective for $n \geq 3$ [10], its image $\Gamma_3 \circ Q(i)(\phi)$ is also non-trivial. If we denote it by ϕ', then $|\phi'(f_n)| \leq |\phi'(f)|$ by Proposition 2.2 and thus ϕ' is in $Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2),\{f_n; n\geq 0\})$. Moreover, choose $g \in \mathrm{KerCal}$ such that $\phi'(g) \neq 0$. Then we have by Lemma 2.1

$$\log m + \log \frac{|\phi'(g)|}{D(\phi') + |\phi'(f)|} \leq d([g^m],[f_n; n \geq 0]) \text{ for any } m \in \mathbb{N},$$

which is the property (iii). □

Proof of Theorem 1.3. If the metric spaces $\mathcal{M}(\mathrm{KerCal})$ and $\mathbb{R}_{\geq 0}$ are quasi-isometric, then there exists a quasi-isometric embedding $\Phi\colon \mathcal{M}(\mathrm{KerCal}) \to \mathbb{R}_{\geq 0}$. By the property (iii), we have $\Phi([f]) < \Phi([g^m])$ for sufficiently large $m \in \mathbb{N}$. By the property (i), there exists $n \in \mathbb{N}$ such that $\Phi([g^m]) < \Phi([f_n])$. If we set $n_m = \min\{n \in \mathbb{N}; \Phi([g^m]) < \Phi([f_n])\}$, then $\Phi([f_{n_m}]) - \Phi([g^m])$ is bounded independently on m by the property (ii). However this contradicts the property (iii) since we can make n_m arbitrarily large by taking larger m. □

Remark 3.2. Let M be a closed C^∞-manifold and fix a symplectic form ω of M. Then the group $\mathrm{Ham}^\infty(M)$ of Hamiltonian diffeomorphisms of M is a simple group [1].

Let U be a closed ball in M. Taking the subgroup $\mathrm{Ham}^\infty(U)$ of $\mathrm{Ham}^\infty(M)$, consisting of diffeomorphisms supported by U, as in the case of D^2 we can consider the shrinking homomorphism $s_r\colon \mathrm{Ham}^\infty(U) \to \mathrm{Ham}^\infty(U)$ and construct a sequence $\{f_n\}$ in $\mathrm{Ham}^\infty(M)$ which satisfies the properties (i) and (ii) in Theorem 1.2. Hence if there exists a quasi-morphisms on $\mathrm{Ham}^\infty(M)$ whose restriction in $\mathrm{Ham}^\infty(U)$ have the property as Proposition 2.2, then Theorem 1.2 holds for $\mathrm{Ham}^\infty(M)$ and Theorem 1.3 for $\mathcal{M}(\mathrm{Ham}^\infty(M))$.

When M is a closed surface, we can construct quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(M)_0$ by Gambaudo-Ghys' way [5] and verify by an argument similar to the case of D^2 that there exists a quasi-morphism ϕ on $\mathrm{Ham}^\infty(M)$ satisfying $\phi(s_r(f)) = r^{-6}\phi(f)$ for any $f \in \mathrm{Ham}^\infty(U)$.

When M is the one point blow up of a closed symplectic 4-manifold (X, ω_X) such that ω_X and the first Chern class $c_1(X)$ vanish on $\pi_2(X)$, then $\mathrm{Ham}^\infty(M)$ admits a non-trivial quasi-morphism μ, which is called a Calabi quasi-morphism [8][14]. If we take U sufficiently small, then μ satisfies $\mu(s_r(f)) = r^{-8}\mu(f)$ for any $f \in \mathrm{Ham}^\infty(U)$.

Remark 3.3. Let $\mathrm{Ham}^\infty_C(D^{2n})$ and $\mathrm{Ham}^\infty_C(\mathbb{R}^{2n})$ be the groups of Hamiltonian diffeomorphisms of D^{2n} and $\mathbb{R}^{2n}$ respectively with respect to the standard symplectic form ω. These groups admits the Calabi homomorphisms $\mathrm{Cal}\colon \mathrm{Ham}^\infty_C(D^{2n}) \to \mathbb{R}$ and $\mathrm{Cal}_\mathbb{R}\colon \mathrm{Ham}^\infty_C(\mathbb{R}^{2n}) \to \mathbb{R}$ and their kernels KerCal and $\mathrm{KerCal}_\mathbb{R}$ are simple [1]. The group $\mathrm{Ham}^\infty_C(D^{2n})$ admits a quasi-morphism τ, which is constructed by Barge and Ghys [2]. The quasi-morphism $\tau \in Q(\mathrm{Ham}^\infty_C(D^{2n}))$ satisfies $\tau(s_r(f)) = r^{-2n}(f)$.

Although the group $\mathrm{KerCal}_\mathbb{R}$ does not admit non-trivial quasi-morphisms [13], Kawasaki constructed a homogeneous conjugation invariant function on $\mathrm{KerCal}_\mathbb{R}$, which is called a partial quasi-morphism [11]. If we denote it by μ, then the equation $\mu(s_r(f)) = r^{-2n}\mu(f)$ is satisfied.

Therefore a statement similar to Lemma 2.1 hold for τ and μ. Hence Theorem 1.2 holds for KerCal and $\mathrm{KerCal}_\mathbb{R}$ and Theorem 1.3 for $\mathcal{M}(\mathrm{KerCal})$ and $\mathcal{M}(\mathrm{KerCal}_\mathbb{R})$.

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