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A 6-dimensional simply connected complex and symplectic manifold with no Kähler metric

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We construct a simply connected compact manifold which has complex and symplectic structures but does not admit Kähler metric, in the lowest possible dimension where this can happen, that is, dimension 6. Such a manifold is automatically formal and has even odd-degree Betti numbers but it does not satisfy the Lefschetz property for any symplectic form.

1. Introduction

A Kähler manifold (M, J, ω) is a smooth manifold M of dimension $2n$ endowed with an integrable almost complex structure J and a symplectic form ω such that $g(X,Y) = \omega(X, JY)$ defines a Riemannian metric, called *Kähler metric*. In order to check that a compact manifold does not carry any Kähler metric, one can use a collection of known topological obstructions to the existence of such a structure: theory of Kähler groups, evenness of odd-degree Betti numbers, Lefschetz property or the formality of the rational homotopy type (see [1, 7, 25]).

If M is a compact Kähler manifold, then it has a complex and a symplectic structure. However, the converse is not true. The first example of a compact manifold admitting complex and symplectic structures but no Kähler metric is the Kodaira-Thurston manifold [16, 23]. This 4-manifold is not simply connected (it is actually a nilmanifold) hence the fundamental group plays a key role in this property. The classification of complex and symplectic nilmanifolds of dimension 6 was given by Salamon in [22]. Generalizations to higher dimension $2n \geq 6$ of the Kodaira-Thurston manifold are the generalized Iwasawa manifolds considered in [6]. Such manifolds have complex and symplectic structures but carry no Kähler metric. Note that, in dimension 2, every oriented surface admits a Kähler metric.

If one restricts attention to manifolds with trivial fundamental group, then every complex manifold of real dimension 4 admits a Kähler structure. Indeed, by the Enriques-Kodaira classification [16], if M is a complex surface whose first Betti number b_1 is even (this holds in particular when $b_1 = 0$), then M is deformation equivalent to a Kähler surface (see also [2, Theorem 3.1, page 144] for a direct proof of this fact). We point out that Gompf [13] has constructed the first examples of simply connected compact symplectic but not complex 4-manifolds. Also Fintushel and Stern [12] have given a family of simply connected symplectic 4-manifolds not admitting complex structures (the latter was proved by Park [21]).

In dimensions higher than 4, we have the following results. The first examples of simply connected compact symplectic non-Kählerian manifolds were given in dimension 6 by Gompf in the aforementioned paper [13] and in dimension ≥ 10 by McDuff in [18] (these examples are not known to admit complex structures). Fine and Panov in [10] (see also [11]) have produced simply connected symplectic 6-manifolds with $c_1 = 0$ which do not have a compatible complex structure (but it is not known if they admit Kähler structures). Furthermore, Guan in [14] constructed the first family of simply connected, compact and holomorphic symplectic non-Kählerian manifolds of (real) dimension $4n \geq 8$. On the other hand, the first and third authors have proved [3] that the 8-dimensional manifold X constructed in [9] is an example of a simply connected, symplectic and complex manifold which does not admit a Kähler structure (since it is not formal). For higher dimensions $2n = 8 + 2k$, $k \geq 1$, one can take $X \times \mathbb{CP}^k$. This is simply connected, complex and symplectic but not Kähler. Thus, a natural question arises:

> *Does there exist a 6-dimensional simply connected, compact, symplectic and complex manifold which does not admit Kähler metrics?*

In this paper we answer this question in the affirmative by proving the following result:

Theorem 1.1. *There exists a* 6*-dimensional, simply connected, compact, symplectic and complex manifold which carries no Kähler metric.*

In order to construct such an example, we start with a 6-dimensional nilmanifold M admitting both a complex structure J and a symplectic structure ω. Then we quotient it by a finite group preserving J and ω to obtain a simply connected, 6-dimensional orbifold $\widehat{M}$ with an orbifold complex structure $\widehat{J}$ and an orbifold symplectic form $\widehat{\omega}$. By Hironaka Theorem [15], there is a complex resolution $(\widetilde{M_c}, \widetilde{J})$ of $(\widehat{M}, \widehat{J})$. As in [5], we resolve symplectically the singularities of $(\widehat{M}, \widehat{\omega})$ to obtain a smooth symplectic 6-manifold $(\widetilde{M_s}, \widetilde{\omega})$. However, in our situation, the singular locus of the orbifold $\widehat{M}$ does not consist only of a discrete set of points, in contrast with [5]. For a complex and symplectic orbifold, we provide conditions under which the complex and the symplectic resolution of singularities are diffeomorphic (Theorem 3.1). Using this we prove that the resolutions $\widetilde{M_c}$ and $\widetilde{M_s}$ are diffeomorphic. Thus, $\widetilde{M} = \widetilde{M_c}$ is not only a complex manifold but also a symplectic one.

To prove that $\widetilde{M}$ satisfies the conditions of Theorem 1.1, we show that $\widehat{M}$ is simply connected (Proposition 6.1), this resulting from the careful choice of the action of the finite group on M. Then, we have that $\widetilde{M}$ is also simply connected because any desingularization of a complex analytic variety with quotient singularities has the same fundamental group as the original variety [17, Theorem 7.8.1]. Since $\widetilde{M}$ is a 6-dimensional simply connected compact manifold, then $b_1(\widetilde{M}) = 0$, and $b_3(\widetilde{M})$ is even by Poincaré duality. Also $\widetilde{M}$ is automatically formal by [8, Theorem 3.2]. Therefore, to ensure that $\widetilde{M}$ does not carry any Kähler metric, we use the Lefschetz property; more precisely, we prove that the map $L_{[\Omega]} \colon H^2(\widetilde{M}) \to H^4(\widetilde{M})$ given by the cup product with $[\Omega]$ is not an isomorphism for any possible symplectic form Ω. Again the choice of nilmanifold M and finite group action makes possible to have a non-zero $[\beta] \in H^2(\widetilde{M})$ such that $[\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0$ for every $[\alpha_1], [\alpha_2] \in H^2(\widetilde{M})$, which gives the result.

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2. Orbifolds

Definition 2.1. A (smooth) n-dimensional orbifold is a Hausdorff, paracompact topological space X endowed with an atlas $\mathcal{A} = \{(U_p, \widetilde{U}_p, \Gamma_p, \varphi_p)\}$

of orbifold charts, that is $U_p \subset X$ is a neighbourhood of $p \in X$, $\widetilde{U}_p \subset \mathbb{R}^n$ an open set, $\Gamma_p \subset \mathrm{GL}(n,\mathbb{R})$ a finite group acting on $\widetilde{U}_p$, and $\varphi_p \colon \widetilde{U}_p \to U_p$ is a Γ_p-invariant map with $\varphi_p(0) = p$, inducing a homeomorphism $\widetilde{U}_p/\Gamma_p \cong U_p$.

The charts are compatible in the following sense: if $q \in U_q \cap U_p$, then there exist a connected neighbourhood $V \subset U_q \cap U_p$ and a diffeomorphism $f \colon \varphi_p^{-1}(V)_0 \to \varphi_q^{-1}(V)$, where $\varphi_p^{-1}(V)_0$ is the connected component of $\varphi_p^{-1}(V)$ containing q, such that $f(\sigma(x)) = \rho(\sigma)(f(x))$, for any x, and $\sigma \in \mathrm{Stab}_{\Gamma_p}(q)$, where $\rho \colon \mathrm{Stab}_{\Gamma_p}(q) \to \Gamma_q$ is a group isomorphism.

For each $p \in X$, let $n_p = \#\Gamma_p$ be the order of the orbifold point (if $n_p = 1$ the point is smooth, also called non-orbifold point). The singular locus of the orbifold is the set $S = \{p \in X \mid n_p > 1\}$. Therefore $M - S$ is a smooth n-dimensional manifold. The singular locus S is stratified: if we write $S_k = \{p \mid n_p = k\}$, and consider its closure $\overline{S_k}$, then $\overline{S_k}$ inherits the structure of an orbifold. In particular S_k is a smooth manifold, and the closure consists of some points of S_{kl}, $l \geq 2$.

We say that the orbifold is locally oriented if $\Gamma_p \subset \mathrm{GL}_+(n,\mathbb{R})$ for any $p \in X$. As Γ_p is finite, we can choose a metric on $\widetilde{U}_p$ such that $\Gamma_p \subset \mathrm{SO}(n)$. An element $\sigma \in \Gamma_p$ admits a basis in which it is written as

$$\sigma = \mathrm{diag}\left(\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \ldots, \begin{pmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{pmatrix}, 1, \ldots, 1 \right),$$

for $\theta_1, \ldots, \theta_r \in (0, 2\pi)$. In particular, the set of points fixed by σ is of codimension $2r$. Therefore the set of singular points $S \cap U_p$ is of codimension ≥ 2, and hence $X - S$ is connected (if X is connected). Also we say that the orbifold X is oriented if it is locally oriented and $X - S$ is oriented.

A natural example of orbifold appears when we take a smooth manifold M and a finite group Γ acting on M effectively. Then $\widehat{M} = M/\Gamma$ is an orbifold. If M is oriented and the action of Γ preserves the orientation, then $\widehat{M}$ is an oriented orbifold. Note that for every $\widehat{p} \in \widehat{M}$, the group $\Gamma_{\widehat{p}}$ is the stabilizer of $p \in M$, with $\widehat{p} = \widehat{\pi}(p)$ under the natural projection $\widehat{\pi} \colon M \to \widehat{M}$.

Definition 2.2. A complex orbifold is a $2n$-dimensional orbifold X whose orbifold charts have $\widetilde{U}_p \subset \mathbb{C}^n$, $\Gamma_p \subset \mathrm{GL}(n,\mathbb{C})$, and in the compatibility of charts the maps f are biholomorphisms. Note that X is automatically oriented.

If M is a complex manifold and Γ is a finite group acting effectively on M by biholomorphisms, then $\widehat{M} = M/\Gamma$ is a complex orbifold.

The complex structure of a complex orbifold X can be given by the orbifold $(1,1)$-tensor J with $J^2 = -\mathrm{id}$. This is given by tensors J_p on each $\widetilde{U}_p$ defining the complex structure, which are Γ_p-equivariant, for each $p \in X$, and which agree under the functions f defining the compatibility of charts.

Definition 2.3. A *complex resolution* of a complex orbifold (X, J) is a complex manifold $\widetilde{X}$ together with a holomorphic map $\pi\colon \widetilde{X} \to X$ which is a biholomorphism $\widetilde{X} - E \to X - S$, where $S \subset X$ is the singular locus and $E = \pi^{-1}(S)$ is the *exceptional locus*.

Let X be an orbifold. An orbifold k-form α consists of a collection of k-forms α_p on each $\widetilde{U}_p$ which are Γ_p-equivariant and that match under the compatibility maps between different charts.

Definition 2.4. A symplectic orbifold (X, ω) consists of a $2n$-dimensional oriented orbifold X and an orbifold 2-form ω such that $d\omega = 0$ and $\omega^n > 0$ everywhere.

If M is a symplectic manifold and Γ is a finite group acting effectively on M by symplectomorphisms, then $\widehat{M} = M/\Gamma$ is a symplectic orbifold.

Definition 2.5. A *symplectic resolution* of a symplectic orbifold (X, ω) consists of a smooth symplectic manifold $(\widetilde{X}, \widetilde{\omega})$ and a map $\pi\colon \widetilde{X} \to X$ such that:

- π is a diffeomorphism $\widetilde{X} - E \to X - S$, where $S \subset X$ is the singular locus and $E = \pi^{-1}(S)$ is the *exceptional locus*.
- $\widetilde{\omega}$ and $\pi^*\omega$ agree in the complement of a small neighbourhood of E.

3. Desingularization of orbifold points

In this section we suppose that X is an oriented orbifold whose singular locus S consists of a discrete set of points. Assume that X admits a complex structure J and a symplectic structure ω. Therefore we have a complex orbifold (X, J) and a symplectic orbifold (X, ω).

It is well-known that (X, J) admits a complex resolution $(\widetilde{X}_c, \widetilde{J})$ by Hironaka's desingularization [15]. Also, the symplectic orbifold (X, ω) admits a symplectic resolution $(\widetilde{X}_s, \widetilde{\omega})$ by Theorem 3.3 in [5]. We want to compare the two resolutions.

First, let us look at the complex resolution of (X, J). Consider $p \in S$, and let $U_p = \widetilde{U}_p/\Gamma_p$ be an orbifold neighbourhood. Recall that we denote

$\varphi_p\colon \widetilde{U}_p \to U_p$ the quotient map. By definition of complex orbifold, $\widetilde{U}_p \subset \mathbb{C}^n = \mathbb{R}^{2n}$ and $\Gamma_p \subset \mathrm{GL}(n,\mathbb{C})$. As Γ_p is a finite group, we can choose a Kähler metric invariant by Γ_p. With a linear change of variables, we can transform the Kähler metric into standard form. That is, we can suppose that there is an inclusion

$$\imath\colon \Gamma_p \hookrightarrow \mathrm{U}(n). \tag{3.1}$$

Shrinking $\widetilde{U}_p$ if necessary, we can assume that $\widetilde{U}_p = B_\epsilon(0)$, for some $\epsilon > 0$.

Consider now an algebraic resolution of the singularity of $Y = \mathbb{C}^n/\Gamma_p$, provided by [15]. Denote it $\pi\colon \widetilde{Y} \to Y$, and let $E = \pi^{-1}(p)$ be the exceptional locus. Write $B = B_\epsilon(0)/\Gamma_p$ and $\widetilde{B} = \pi^{-1}(B)$. The complex resolution is defined as the smooth manifold

$$\widetilde{X}_c = (X - \{p\}) \cup_\pi \widetilde{B},$$

where the identification uses the map $\pi\colon \tilde{B} - E \to B - \{p\} = U_p - \{p\}$. This has a natural complex structure since π is a biholomorphism.

Now we move to the construction of the symplectic resolution of (X, ω), as done in [5]. For $p \in S$, take an orbifold neighbourhood $U'_p = \widetilde{U}'_p/\Gamma'_p$, with $\varphi'_p\colon \widetilde{U}'_p \to U'_p$. By the equivariant Darboux theorem (see [20, Theorem 7.3.1]), there is a Γ'_p-equivariant symplectomorphism $(\widetilde{U}'_p, \omega_p) \cong (V, \omega_0)$, where $V \subset \mathbb{R}^{2n}$ is an open set, and ω_0 is the standard symplectic form (shrinking $\widetilde{U}'_p$ if necessary). So without loss of generality, we can assume that $\widetilde{U}'_p \subset (\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic form, and $\Gamma'_p \subset \mathrm{Sp}(2n, \mathbb{R})$. As Γ'_p is a finite group, and $\mathrm{U}(n) \subset \mathrm{Sp}(2n, \mathbb{R})$ is the maximal compact subgroup, we can choose a complex structure J on $\mathbb{R}^{2n}$ such that the pair (J, ω_0) determines a Kähler metric, which is invariant by Γ'_p. We perform a linear change of variables, which transforms the complex structure into standard form (so $\widetilde{U}'_p$ has the standard Kähler structure). Equivalently, we can suppose that there is an inclusion

$$\imath'\colon \Gamma'_p \hookrightarrow \mathrm{U}(n). \tag{3.2}$$

Shrinking $\widetilde{U}'_p$ if necessary, we can assume that $\widetilde{U}'_p = B_{\epsilon'}(0)$, for some $\epsilon' > 0$.

Consider an algebraic resolution of singularities of $Y' = \mathbb{C}^n/\Gamma'_p$, call it $\pi'\colon \widetilde{Y}' \to Y'$, and let $E' = (\pi')^{-1}(p)$ be the exceptional locus. Write $B' = B_{\epsilon'}(0)/\Gamma'_p$ and $\widetilde{B}' = (\pi')^{-1}(B')$. The symplectic resolution is defined as the smooth manifold

$$\widetilde{X}_s = (X - \{p\}) \cup_{\pi'} \widetilde{B}',$$

where $\widetilde{B}' - E'$ and $B' - \{p\} = U'_p - \{p\}$ are identified by π'. This has a symplectic structure that is constructed by gluing the symplectic structure of $X - \{p\}$ and the Kähler form of $\widetilde{B}'$ by a cut-off process, as done in Theorem 3.3 of [5].

Now we are going to compare $\widetilde{X}_c$ and $\widetilde{X}_s$. First note that for $p \in S$, we have $\Gamma_p \cong \Gamma'_p$. This follows from $\Gamma_p \cong \pi_1(B - \{p\})$ and $\Gamma'_p \cong \pi_1(B' - \{p\})$, and the fact that B, B' are homeomorphic. So we shall denote $\Gamma'_p = \Gamma_p$ henceforth. We have the following result.

Theorem 3.1. *If one can arrange that the inclusions $\imath$ and $\imath'$, given by* (3.1) *and* (3.2)*, respectively, are such that $\imath = \imath'$ for every singular point $p \in S$, then there is a diffeomorphism $\widetilde{X}_c \cong \widetilde{X}_s$, which is the identity outside a small neighbourhood of the exceptional loci. In particular, $\widetilde{X}_c$ admits both complex and symplectic structures.*

Proof. The key point is obviously that if $\imath = \imath'$, then $Y' = Y$, so we can take $\widetilde{Y}' = \widetilde{Y}$ and $\pi' = \pi$ in the constructions above.

We fix a point $p \in S$, and construct the required isomorphism in a neighbourhood of the exceptional locus over that point. Consider the map (reducing $\epsilon > 0$ if necessary)

$$f = (\varphi'_p)^{-1} \circ \varphi_p \colon B_\epsilon(0) = \widetilde{U}_p \to B_{\epsilon'}(0) = \widetilde{U}'_p;$$

f is Γ_p-equivariant and an open embedding (it might fail to be surjective) with $f(0) = 0$. We shall construct a map $F \colon B_\epsilon(0) \to B_{\epsilon'}(0)$ such that

- $F = \mathrm{id}$ in a small ball $B_{0.2\epsilon}(0)$,
- $F = f$ outside a slightly bigger ball $B_{0.9\epsilon}(0)$,
- F is a Γ_p-equivariant diffeomorphism onto its image.

This gives a diffeomorphism $F \colon \widetilde{X}_c \to \widetilde{X}_s$, defined by F on $B_\epsilon(0)/\Gamma_p - \{p\}$, extended by the identity on $\pi^{-1}(B_{0.2\epsilon}(0)/\Gamma_p)$, and also by the identity on $X - \pi^{-1}(B_{0.9\epsilon}(0)/\Gamma_p)$.

Write $f(x) = L(x) + R(x)$, where L is the linear part and $|R(x)| \leq C|x|^2$, for some constant $C > 0$. Both these maps are Γ_p-equivariant. Take a smooth, non-decreasing function $\rho_1 \colon [0, \epsilon] \to [0, 1]$ such that $\rho_1(t) = 0$ for $t \in [0, 0.8\epsilon]$ and $\rho_1(t) = 1$ for $t \in [0.9\epsilon, 1]$. Consider $g(x) = L(x) + \rho_1(|x|)R(x)$. Then,

$g(x) = L(x)$ for $|x| \leq 0.8\epsilon$, $g(x) = f(x)$ for $|x| \geq 0.9\epsilon$, and $g(x)$ is Γ_p-equivariant because $\Gamma_p \subset \mathrm{SO}(2n)$. Also

$$dg(x) - L = \rho_1'(|x|)R(x)d|x| + \rho_1(|x|)dR(x).$$

Using that $|\rho_1'(t)| \leq C/\epsilon$ and $|dR(x)| \leq C|x|$ (we denote by $C > 0$ uniform constants, that can vary from line to line) we have that $|dg(x) - L| \leq C|x|$. For $\epsilon > 0$ small enough, we have that g is a diffeomorphism onto its image.

Next, take the linear map $L\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. We can choose orthonormal (oriented) basis in both origin and target so that $L = \mathrm{diag}(\lambda_1, \dots, \lambda_{2n})$, where $\lambda_i > 0$ are real numbers (the first vector of the basis is a unitary vector e_1 such that $|L(e_1)|$ is maximized; then L maps $\langle e_1\rangle^\perp$ to $\langle L(e_1)\rangle^\perp$, and we proceed inductively). Consider the map

$$h(x) = \begin{cases} x, & |x| \leq 0.4\epsilon, \\ x + \rho_2\left(\left(\frac{|x|-0.4\epsilon}{0.3\epsilon}\right)^\alpha\right)(L(x) - x), & 0.4\epsilon \leq |x| \leq 0.7\epsilon, \\ g(x), & |x| \geq 0.7\epsilon, \end{cases}$$

where $\rho_2\colon [0,1] \to [0,1]$ is smooth non-decreasing with $\rho_2(t) = 0$ for $t \in [0, \frac{1}{3}]$, and $\rho_2(t) = 1$ for $t \in [\frac{2}{3}, 1]$. Here $\alpha > 0$ is a constant to be fixed soon.

Clearly h is Γ_p-equivariant, $h(x) = f(x)$ off $B_{0.9\epsilon}(0)$, and $h(x) = x$ in $B_{0.4\epsilon}(0)$ (but beware, we have chosen different coordinates on the origin $\mathbb{R}^{2n}$ and the target $\mathbb{R}^{2n}$, so h is not the identity in the ball). The map h is $\mathcal{C}^\infty$ because for $0.4\epsilon \leq |x| \leq 0.5\epsilon$ we have also $h(x) = x$. Let us see that h is a diffeomorphism onto its image. It only remains to see this for $0.5\epsilon \leq |x| \leq 0.7\epsilon$. Write $y = h(x)$, so in our coordinates $y_i = x_i + \rho_2(u)(\lambda_i - 1)x_i$, with $u = \left(\frac{|x|-0.4\epsilon}{0.3\epsilon}\right)^\alpha$. Then,

$$dy_i = (1 + (\lambda_i - 1)\rho_2(u))\, dx_i + (\lambda_i - 1)\rho_2'(u)\frac{\alpha}{0.3\epsilon}\left(\frac{|x| - 0.4\epsilon}{0.3\epsilon}\right)^{\alpha-1} x_i\gamma$$

with $\gamma = d|x| = \frac{1}{|x|}\sum x_j dx_j$. Write $\delta_i = (1 + (\lambda_i - 1)\rho_2(u))$, so δ_i takes values between 1 and λ_i. We compute

$$\begin{aligned} & dy_1 \wedge \dots \wedge dy_n \\ &= \delta_1 \dots \delta_n\, dx_1 \wedge \dots \wedge dx_n \\ &\quad + \sum \delta_1 \dots \hat{\delta}_i \dots \delta_n \frac{(\lambda_i - 1)\rho_2'(u)\alpha x_i}{0.3\epsilon} \\ &\qquad \times \left(\frac{|x| - 0.4\epsilon}{0.3\epsilon}\right)^{\alpha-1} dx_1 \wedge \dots \wedge \overset{(i)}{\gamma} \wedge \dots \wedge dx_n \end{aligned}$$

$$= \delta_1 \ldots \delta_n \left(1 + \alpha \sum \frac{(\lambda_i - 1)\rho_2'(u)(|x| - 0.4\epsilon)^{\alpha-1} x_i^2}{|x|\delta_i (0.3\epsilon)^\alpha}\right) dx_1 \wedge \ldots \wedge dx_n.$$

In the sum, the numerator is bounded above by $C(0.3\epsilon)^{\alpha+1}$ and the denominator is bounded below by $C^{-1}(0.3\epsilon)^{\alpha+1}$, for some uniform (independent of α) constant $C > 0$. Hence choosing $\alpha > 0$ small enough, we get that the above quantity does not vanish, hence h is a diffeomorphism onto its image.

After this step is done, recall that we have taken coordinates given by an orthonormal basis $\{e_i\}$ on the origin $\mathbb{R}^{2n}$, and by the orthonormal basis $\{L(e_i)/\lambda_i\}$ on the target $\mathbb{R}^{2n}$. Written with respect to the same coordinates, we have an orthogonal transformation $M\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ so that $h(x) = M$ on $B_{0.4\epsilon}(0)$. The final step is to change the isometry $M \in \mathrm{SO}(2n)$ by the identity. Take a smooth path M_t of matrices joining $M_0 = \mathrm{id}$ with $M_1 = M$. Take a smooth non-decreasing $\rho_3\colon [0,\epsilon] \to [0,1]$ with $\rho_3(t) = 0$ for $t \in [0, 0.2\epsilon]$, and $\rho_3(t) = 1$ for $t \in [0.3\epsilon, \epsilon]$. The map $F(x) = M_{\rho_3(|x|)}(x)$, $|x| \leq 0.4\epsilon$, and $F(x) = h(x)$ for $|x| \geq 0.4\epsilon$, is the required map. □

Remark 3.2. Let $F\colon (\widetilde{X}_c, \widetilde{J}) \to (\widetilde{X}_s, \widetilde{\omega})$ be the diffeomorphism provided by Theorem 3.1. Then if we denote $\widetilde{\omega}' = F^*\widetilde{\omega}$, we have that $\widetilde{X}_c$ admits a symplectic structure $\widetilde{\omega}'$ and a complex structure $\widetilde{J}$. These are not compatible in general, but they are compatible on a neighbourhood of the exceptional locus, and give a Kähler structure there.

Remark 3.3. The condition $\imath = \imath'$ in Theorem 3.1 is not vacuous. Consider for instance the unit ball $B = B(0,1) \subset \mathbb{C}^2$ with the standard complex structure and the symplectic form $\omega = -i\,(dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2)$. Let $\imath : \Gamma_p = \mathbb{Z}_m \hookrightarrow \mathrm{U}(2)$, $m > 2$, $\zeta = e^{2\pi i/m}$, with the action given by $\zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta z_2)$. Then $(B, \omega) \cong (B', \omega_0)$, with the symplectomorphism given by $w_1 = z_1$, $w_2 = \bar{z}_2$, and $\omega_0 = -i\,(dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2)$ the standard symplectic form. The inclusion $\imath' : \mathbb{Z}_m \hookrightarrow \mathrm{U}(2)$ is now given by the action $\zeta \cdot (w_1, w_2) = (\zeta w_1, \zeta^{m-1} w_2)$. Therefore $\imath \neq \imath'$, for $m > 2$.

4. A complex and symplectic 6-orbifold

Consider the complex Heisenberg group G, that is, the complex nilpotent Lie group of (complex) dimension 3 consisting of matrices of the form

$$\begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In terms of the natural (complex) coordinate functions (u_1, u_2, u_3) on G, we have that the complex 1-forms $\mu = du_1$, $\nu = du_2$ and $\theta = du_3 - u_2\, du_1$ are left invariant, and

$$d\mu = d\nu = 0, \quad d\theta = \mu \wedge \nu.$$

Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi i/6}$, and consider the discrete subgroup $\Gamma \subset G$ formed by the matrices in which $u_1, u_2, u_3 \in \Lambda$. We define the compact (parallelizable) nilmanifold

$$M = \Gamma\backslash G.$$

We can describe M as a principal torus bundle

$$T^2 = \mathbb{C}/\Lambda \hookrightarrow M \to T^4 = (\mathbb{C}/\Lambda)^2$$

by the projection $(u_1, u_2, u_3) \mapsto (u_1, u_2)$.

Consider the action of the finite group $\mathbb{Z}_6$ on G given by the generator

$$\begin{array}{rcl} \rho\colon G & \to & G \\ (u_1, u_2, u_3) & \mapsto & (\zeta^4\, u_1, \zeta\, u_2, \zeta^5\, u_3). \end{array}$$

This action satisfies that $\rho(p \cdot q) = \rho(p) \cdot \rho(q)$, for $p, q \in G$, where $\cdot$ denotes the natural group structure of G. Moreover, $\rho(\Gamma) = \Gamma$. Thus, ρ induces an action on the quotient $M = \Gamma\backslash G$. Denote by $\rho\colon M \to M$ the $\mathbb{Z}_6$-action. The action on 1-forms is given by

$$\rho^*\mu = \zeta^4\, \mu, \quad \rho^*\nu = \zeta\, \nu, \quad \rho^*\theta = \zeta^5\, \theta.$$

Proposition 4.1. *$\widehat{M} = M/\mathbb{Z}_6$ is a 6-orbifold admitting complex and symplectic structures.*

Proof. The nilmanifold M is a complex manifold whose complex structure J is the multiplication by i at each tangent space T_pM, $p \in M$. Then one can check that J commutes with the $\mathbb{Z}_6$-action ρ on M, that is, $(\rho_*)_p \circ J_p = J_{\rho(p)} \circ (\rho_*)_p$, for any point $p \in M$. Hence, J induces a complex structure on the quotient $\widehat{M} = M/\mathbb{Z}_6$.

Now we define the complex 2-form ω on M given by

$$\omega = -i\, \mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta}. \tag{4.1}$$

Clearly, ω is a real closed 2-form on M which satisfies $\omega^3 > 0$, that is, ω is a symplectic form on M. Moreover, ω is $\mathbb{Z}_6$-invariant. Indeed, $\rho^*\omega = -i\, \mu \wedge$

$\bar{\mu} + \zeta^6 \nu \wedge \theta + \zeta^{-6} \bar{\nu} \wedge \bar{\theta} = \omega$. Therefore $\widehat{M}$ is a symplectic 6-orbifold, with the symplectic form $\widehat{\omega}$ induced by ω. $\square$

We denote by

$$\widehat{\pi} \colon M \to \widehat{M}$$

the natural projection. The orbifold points of $\widehat{M}$ are the following:

1) The points $(\frac{1}{3}a(1+\zeta), \frac{1}{3}b(1+\zeta), \frac{1}{3}c(1+\zeta) + \frac{2}{9}ab(1+\zeta)^2) \in M$, with $a, b, c \in \{0, 1, 2\}$ and $(b, c) \neq (0, 0)$, are points of order 3; their isotropy group is $K = \{\mathrm{id}, \rho^2, \rho^4\}$. These points are mapped in pairs by $\mathbb{Z}_6$, so they define 12 orbifold points in $\widehat{M} = M/\mathbb{Z}_6$, with models $\mathbb{C}^3/K$.
2) The surfaces $S_{(p,q)} = \{(u_1, p, p\, u_1 + q) \mid u_1 \in \mathbb{C}/\Lambda\} \subset M$, where $p, q \in \{0, \frac{1}{2}, \frac{\zeta}{2}, \frac{1+\zeta}{2}\}$, $(p, q) \neq (0, 0)$. These are 15 tori, which consist of points of order 2, with isotropy $H = \{\mathrm{id}, \rho^3\}$. These surfaces are permuted by the group $\mathbb{Z}_6$, so they come in 5 groups of three tori each. Thus they define 5 tori in the orbifold $\widehat{M}$, formed by orbifold points of order 2.
3) The surface $S_0 = \{(u_1, 0, 0) \mid u_1 \in \mathbb{C}/\Lambda\} \subset M$ is a torus consisting generically of points of order 2, with isotropy H. Here $\rho \colon S_0 \to S_0$ and it is a map of order 3, with three fixed points $(\frac{1}{3}a(1+\zeta), 0, 0)$, $a = 0, 1, 2$. These points have isotropy $\mathbb{Z}_6$. The quotient $S_0/\langle\rho\rangle \subset \widehat{M}$ is homeomorphic to a sphere (with three orbifold points of order 3).

5. Resolution of the 6-orbifold

Now we want to desingularize the orbifold $\widehat{M}$. We shall treat each of the connected components of the singular locus determined before independently. Recall that $K = \{\mathrm{id}, \rho^2, \rho^4\} \cong \mathbb{Z}_3$ and $H = \{\mathrm{id}, \rho^3\} \cong \mathbb{Z}_2$. There is a natural isomorphism $\langle\rho\rangle = \mathbb{Z}_6 \cong K \times H$.

5.1. Resolution of the isolated orbifold points

We know that there are 12 isolated orbifold points in $\widehat{M}$. Let $\widehat{p} \in \widehat{M}$ be one of them. The preimage of $\widehat{p}$ under $\widehat{\pi}$ consists of two points, $\widehat{\pi}^{-1}(\widehat{p}) = \{p_1, p_2\}$. The isotropy group of p_1 is K. Consider a K-invariant neighbourhood U of p_1 in M. Then,

$$\widehat{U} = \widehat{\pi}(U) \cong U/K$$

is an orbifold neighbourhood of $\widehat{p}$ in $\widehat{M}$. This has complex and symplectic resolutions as in Section 3. In order to apply Theorem 3.1 we check that $\imath =$

$\imath'\colon K \to \mathrm{U}(3)$. For the complex resolution, we have $\imath(\zeta^2) = \mathrm{diag}(\zeta^2, \zeta^2, \zeta^4)$. For the symplectic resolution, the symplectic form (4.1) is, in our coordinates (u_1, u_2, u_3),

$$\omega = -i\, du_1 \wedge d\bar{u}_1 + du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3\,. \tag{5.1}$$

We have to do a change of variables to transform $K \subset \mathrm{Sp}(6, \mathbb{R})$ into a subgroup of $\mathrm{U}(3)$. This is obtained with

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= \frac{1}{\sqrt{2}}(u_2 - i\bar{u}_3) \\ v_3 &= \frac{1}{\sqrt{2}}(\bar{u}_2 - iu_3). \end{aligned}$$

This transforms (5.1) into

$$\omega = -i\, dv_1 \wedge d\bar{v}_1 - i\, dv_2 \wedge d\bar{v}_2 - i\, dv_3 \wedge d\bar{v}_3,$$

the standard Kähler form. In the new coordinates the K-action is given by $(v_1, v_2, v_3) \mapsto (\zeta^2 v_1, \zeta^2 v_2, \zeta^4 v_3)$, so $\imath'(\zeta^2) = \mathrm{diag}(\zeta^2, \zeta^2, \zeta^4)$, and $\imath = \imath'$.

5.2. Resolution of the singular sets $\widehat{\pi}(S_{(p,q)})$

Now we consider a connected component of the singular set which is homeomorphic to a 2-torus. There are 5 such components in $\widehat{M}$, all of them are images by $\widehat{\pi}$ of the sets $S_{(p,q)} = \{(u_1, p, p\,u_1 + q) \mid u_1 \in \mathbb{C}/\Lambda\}$, where $(p, q) \in I = \left(\{0, \frac{1}{2}, \frac{\zeta}{2}, \frac{1+\zeta}{s}\}\right)^2 - \{(0, 0)\}$.

Let us focus on one such component $\widehat{T} = \widehat{\pi}(T)$, $T \cong \mathbb{C}/\Lambda$. Then H fixes $S_{(p,q)}$, and its orbit under K is given by $S_{(p_i,q_i)}$, for three elements $(p_1, q_1) = (p, q), (p_2, q_2), (p_3, q_3) \in I$. Consider a neighbourhood U of $T \subset M$ via

$$\begin{array}{rcl} T \times B_\epsilon(0) & \to & U \\ (u_1, u_2, u_3) & \mapsto & (u_1, u_2 + p, u_3 + p\,u_1 + q), \end{array}$$

where $B_\epsilon(0) \subset \mathbb{C}^2$. The image is

$$\widehat{U} = \widehat{\pi}(U) \cong U/H \cong T \times (B_\epsilon(0)/H), \tag{5.2}$$

where $H \cong \mathbb{Z}_2$ acts as $(u_2, u_3) \mapsto (-u_2, -u_3)$.

We see that the complex structure on (5.2) is the product complex structure. Also, the symplectic structure $\omega = i\, du_1 \wedge d\bar{u}_1 + du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3$ is the product of the natural symplectic structure of $\mathbb{C}/\Lambda$ with an orbifold symplectic structure on $B_\epsilon(0)/H$. Using the construction of Section 3, we have a desingularization

$$\widetilde{Y} \to B_\epsilon(0)/H$$

which is a smooth manifold endowed with both a complex structure and a symplectic structure coinciding with the given ones outside a small neighbourhood of the exceptional locus E. The condition $\imath = \imath'$ of Theorem 3.1 is trivially satisfied, since $\imath(\rho^3) = \imath'(\rho^3) = -\mathrm{id}$. Multiplying by $T = \mathbb{C}/\Lambda$, we have that

$$\widetilde{U} = T \times \widetilde{Y}$$

is a smooth manifold endowed with a complex structure $\widetilde{J}$, and a symplectic structure $\widetilde{\omega}$, which coincide with those of $\widehat{U}$ outside a small neighbourhood of the exceptional locus $T \times E \subset \widetilde{U}$.

The complex and the symplectic resolutions of $\widehat{M}$ in a neighbourhood of $\widehat{T}$ are obtained by replacing $\widehat{U} \subset \widehat{M}$ with $\widetilde{U}$. The two resolutions are diffeomorphic by the considerations above.

5.3. Resolution of the singular set $\widehat{\pi}(S_0)$

Finally we consider the connected component of the singular set which is homeomorphic to a 2-sphere. This is $\widehat{S}_0 = \widehat{\pi}(S_0)$, where $S_0 = \{(u_1, 0, 0) \mid u_1 \in \mathbb{C}/\Lambda\}$. As before, a neighbourhood of S_0 in M is of the form

$$U_0 = (\mathbb{C}/\Lambda) \times B_\epsilon(0),$$

where $B_\epsilon(0) \subset \mathbb{C}^2$. The action of $H = \mathbb{Z}_2$ is trivial on $\mathbb{C}/\Lambda$ and as ± 1 on $\mathbb{C}^2$. The action of $K = \mathbb{Z}_3$ is of the form $\rho^2(u_1, u_2, u_3) = (\zeta^2 u_1, \zeta^2 u_2, \zeta^4 u_3)$.

Let us focus on $B_\epsilon(0)/H$. By the construction of Section 3, we have a complex desingularization $(\widetilde{Y}_c, \widetilde{J}) \to B_\epsilon(0)/H$. The holomorphic action of K on $B_\epsilon(0)$ induces an action on $(\widetilde{Y}_c, \widetilde{J})$. Also, there is a symplectic desingularization $(\widetilde{Y}_s, \widetilde{\omega}) \to B_\epsilon(0)/H$. The action of K on $B_\epsilon(0)$ induces an action on $(\widetilde{Y}_s, \widetilde{\omega})$. This follows by taking an orbifold chart of the singular point that is $(H \times K)$-equivariant, using the equivariant Darboux theorem.

By Theorem 3.1, there is a diffeomorphism $F\colon (\widetilde{Y}_c, \widetilde{J}) \to (\widetilde{Y}_s, \widetilde{\omega})$. Let us see that F can be taken to be K-equivariant. This follows by the arguments in the proof of Theorem 3.1 by using that $\imath\colon H \times K \to \mathrm{U}(2)$ and $\imath'\colon H \times K \to \mathrm{U}(2)$ are equal. For the complex case, $\imath$ is given by the representation

$(u_2, u_3) \mapsto (\zeta u_2, \zeta^5 u_3)$, so $\imath(\zeta) = \mathrm{diag}(\zeta, \zeta^5)$. For the symplectic case, we have to do a change of variables to transform $H \times K \subset \mathrm{Sp}(4, \mathbb{R})$ into a subgroup of U(2). This is given by

$$v_2 = \frac{1}{\sqrt{2}}(u_2 - i\bar{u}_3), \quad v_3 = \frac{1}{\sqrt{2}}(\bar{u}_2 - iu_3),$$

which transforms $\omega = du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3$ into the standard Kähler form $-i\, dv_2 \wedge d\bar{v}_2 - i\, dv_3 \wedge d\bar{v}_3$. As $(v_2, v_3) \mapsto (\zeta v_2, \zeta^5 v_3)$, we have that $\imath'(\zeta) = \mathrm{diag}(\zeta, \zeta^5)$. Hence $\imath = \imath'$.

This produces a desingularization $\widetilde{Y} \to B_\epsilon(0)/H$ with a symplectic and a complex structure, which match the given ones outside a small neighbourhood of the exceptional set $E \subset \widetilde{Y}$, which are compatible (they give a Kähler structure) in a smaller neighbourhood of E, by Remark 3.2, and which have an action of K preserving both the complex and symplectic structures. A desingularization of

$$U_0/H = (\mathbb{C}/\Lambda) \times (B_\epsilon(0)/H)$$

is given by substituting a neighbourhood of $\widehat{S}_0 = (\mathbb{C}/\Lambda) \times \{0\}$ by $(\mathbb{C}/\Lambda) \times \widetilde{Y}$. The fixed points of action of K in U_0/H lie on $\widehat{S}_0$, hence the fixed points of the action of K on the desingularization of U_0/H lie in the exceptional divisor. In this part of the manifold, we have a Kähler structure, so the symplectic and complex desingularization are the same.

This means that $(U_0/H)/K \cong U_0/(H \times K)$ admits a desingularization $\widetilde{V}$ with a complex and a symplectic structure. The resolution of $\widehat{M}$ in a neighbourhood of $\widehat{S}_0$ is obtained by substituting $\widehat{\pi}(U_0) = U_0/(H \times K) \subset \widehat{M}$ with $\widetilde{V}$.

All together, we get a smooth 6-manifold $\widetilde{M}$ with a complex structure and a symplectic structure, and with a map

$$\pi \colon \widetilde{M} \longrightarrow \widehat{M},$$

which is simultaneously a complex and a symplectic resolution.

6. Topological properties of $\widetilde{M}$

In this section, we are going to complete the proof of Theorem 1.1 by proving that $\widetilde{M}$ is simply-connected and that it does not admit a Kähler structure.

Proposition 6.1. *$\widetilde{M}$ is simply connected.*

Proof. By [17, Theorem 7.8.1], it is sufficient to prove that $\widehat{M}$ is simply connected.

We fix base points $p_0 = (0,0,0) \in M$ and $\widehat{p}_0 = \widehat{\pi}(p_0) \in \widehat{M}$. There is an epimorphism of fundamental groups

$$\Gamma = \pi_1(M, p_0) \twoheadrightarrow \pi_1(\widehat{M}, \widehat{p}_0),$$

since the $\mathbb{Z}_6$-action has a fixed point [4, Chapter II, Corollary 6.3]. Now the nilmanifold M is a principal 2-torus bundle over the 4-torus T^4, so we have an exact sequence

$$\mathbb{Z}^2 \hookrightarrow \Gamma \to \mathbb{Z}^4.$$

The group $\Gamma = \pi_1(M, p_0)$ is thus generated by the images of the fundamental groups of the surfaces $\Sigma_1 = \{(u_1, 0, 0)\}$, $\Sigma_2 = \{(0, u_2, 0)\}$ and $\Sigma_3 = \{(0, 0, u_3)\}$ in M. The image $\widehat{\pi}(\Sigma_1)$ is a 2-sphere, since $\widehat{\pi}\colon \Sigma_1 \to \widehat{\pi}(\Sigma_1)$ is a degree 3 map with three ramification points of order 3 (namely $(\frac{1}{2}a(1 + \zeta), 0, 0)$, with $a = 0, 1, 2$). The image of Σ_2 is also a 2-sphere, since $\widehat{\pi}\colon \Sigma_2 \to \widehat{\pi}(\Sigma_2)$ is a degree 6 map with one point of order 6, $(0,0,0)$, two of order 3, $(0, \frac{1}{2}b(1+\zeta), 0)$, $b = 1, 2$, and three of order 2 (namely $(0, p, 0)$, $p = \frac{1}{2}, \frac{\zeta}{2}, \frac{1+\zeta}{2}$). Analogously, $\widehat{\pi}(\Sigma_3)$ is a 2-sphere. This proves that $\pi_1(\widehat{M}, \widehat{p}_0) = \{1\}$.

Now we look at the resolution process. As mentioned before, the desingularisation process does not change the fundamental group [17, Theorem 7.8.1]. However, for simplicity, we give a direct proof of this result in the case at hand. Let $S \subset \widehat{M}$ be the singular locus and suppose $p \in S$ is an isolated orbifold point. The resolution replaces a neighbourhood $B = B_\epsilon(0)/\Gamma_p$ of p with a smooth manifold $\widetilde{B}$, such that $\pi\colon \widetilde{B} \to B$ is a complex resolution of singularities. The manifold $\widetilde{B}$ is simply connected by [24, Theorem 4.1]. A Seifert-Van Kampen argument gives that $\pi_1(\widehat{M})$ is the amalgamated sum of $\pi_1(\widehat{M} - \{p\})$ and $\pi_1(B)$ along $\pi_1(\partial B)$. Also $\pi_1(\widetilde{M})$ is the amalgamated sum of $\pi_1(\widetilde{M} - E)$ and $\pi_1(\widetilde{B})$ along $\pi_1(\partial B)$. As $\pi_1(B) = \pi_1(\widetilde{B}) = \{1\}$, we have that $\pi_1(\widehat{M}) = \pi_1(\widetilde{M})$.

Suppose now that we have a connected component S' of the singular locus S of positive dimension. Let $E' = \pi^{-1}(S')$ be the corresponding exceptional locus. The invariance of the fundamental group under resolution is proved along the same lines as before if we know that the map $\pi\colon E' \to S'$ induces an isomorphism $\pi_1(E') \to \pi_1(S')$. In our case, we have two possibilities: if $S' = \widehat{\pi}(S_{(p,q)}) \cong T^2$, then $E' = T^2 \times E$, where E is the exceptional divisor of the resolution $\widetilde{Y} \to B_\epsilon(0)/H$, which is clearly simply connected, and the result follows.

The second possibility is $S' = \widehat{\pi}(S_0)$. In this case, the exceptional divisor over S' is the exceptional divisor of the resolution of

$$((\mathbb{C}/\Lambda) \times (\mathbb{C}^2/H))/K.$$

The resolution of $\mathbb{C}^2/H$ is done by blowing-up $\mathbb{C}^2$ at the origin,

$$\widetilde{\mathbb{C}}^2 = \{(a, b, [u : v]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid av = bu\},$$

and then quotienting by $H = \{\pm \mathrm{id}\}$. Clearly, the fundamental groups of $(\mathbb{C}/\Lambda) \times (\mathbb{C}^2/H)$ and $(\mathbb{C}/\Lambda) \times (\widetilde{\mathbb{C}}^2/H)$ coincide. The action of K is given by $(a, b, [u : v]) \mapsto ((\zeta^2 a, \zeta^4 b), [u : \zeta^2 v])$, with fixed points $(0, 0, [1 : 0])$ and $(0, 0, [0 : 1])$ The fixed points of K on $((\mathbb{C}/\Lambda) \times (\widetilde{\mathbb{C}}^2/H)$ occur when K fixes both factors. Therefore, all fixed points are isolated, and the second resolution does not alter the fundamental group. □

In order to prove that $\widetilde{M}$ does not admit a Kähler structure, we are going to check that it does not satisfy the Lefschetz condition *for any symplectic form*. For this, it is necessary to understand the cohomology $H^*(\widetilde{M})$.

We start by computing the cohomology of $\widehat{M}$. By Nomizu theorem [19], the cohomology of the nilmanifold M is:

$$\begin{aligned}
H^0(M,\mathbb{C}) &= \langle 1 \rangle,\\
H^1(M,\mathbb{C}) &= \langle [\mu], [\bar\mu], [\nu], [\bar\nu] \rangle,\\
H^2(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu], [\mu\wedge\bar\nu], [\bar\mu\wedge\nu], [\nu\wedge\bar\nu], [\mu\wedge\theta], [\bar\mu\wedge\bar\theta], [\nu\wedge\theta], [\bar\nu\wedge\bar\theta] \rangle,\\
H^3(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\theta], [\mu\wedge\bar\mu\wedge\bar\theta], [\nu\wedge\bar\nu\wedge\theta], [\nu\wedge\bar\nu\wedge\bar\theta], [\mu\wedge\nu\wedge\theta],\\
&\qquad [\bar\mu\wedge\bar\nu\wedge\bar\theta], [\mu\wedge\bar\nu\wedge\theta], [\mu\wedge\bar\nu\wedge\bar\theta], [\bar\mu\wedge\nu\wedge\theta], [\bar\mu\wedge\nu\wedge\bar\theta] \rangle,\\
H^4(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\nu\wedge\theta], [\mu\wedge\bar\mu\wedge\bar\nu\wedge\bar\theta], [\bar\mu\wedge\nu\wedge\bar\nu\wedge\bar\theta], [\mu\wedge\nu\wedge\bar\nu\wedge\theta],\\
&\qquad [\mu\wedge\bar\mu\wedge\theta\wedge\bar\theta], [\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta], [\mu\wedge\bar\nu\wedge\theta\wedge\bar\theta], [\bar\mu\wedge\nu\wedge\theta\wedge\bar\theta] \rangle,\\
H^5(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\nu\wedge\theta\wedge\bar\theta], [\mu\wedge\bar\mu\wedge\bar\nu\wedge\theta\wedge\bar\theta], [\mu\wedge\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta],\\
&\qquad [\bar\mu\wedge\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta] \rangle,\\
H^6(M,\mathbb{C}) &= \langle [\mu\wedge\bar\mu\wedge\nu\wedge\bar\nu\wedge\theta\wedge\bar\theta] \rangle.
\end{aligned}$$

The cohomology of $\widehat{M}$ is $H^*(\widehat{M},\mathbb{C}) = H^*(M,\mathbb{C})^{\mathbb{Z}_6}$:

$$\begin{aligned}
H^0(\widehat{M},\mathbb{C}) &= \langle 1 \rangle,\\
H^1(\widehat{M},\mathbb{C}) &= 0,\\
H^2(\widehat{M},\mathbb{C}) &= \langle [\mu\wedge\bar\mu], [\nu\wedge\bar\nu], [\nu\wedge\theta], [\bar\nu\wedge\bar\theta] \rangle,
\end{aligned}$$

$H^3(\widehat{M}, \mathbb{C}) = 0,$
$H^4(\widehat{M}, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \theta], [\mu \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\theta}], [\mu \wedge \bar{\mu} \wedge \theta \wedge \bar{\theta}], [\nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle,$
$H^5(\widehat{M}, \mathbb{C}) = 0,$
$H^6(\widehat{M}, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle.$

Proposition 6.2. *$\widetilde{M}$ does not admit a Kähler structure since it does not satisfy the Lefschetz property for any symplectic form on $\widetilde{M}$.*

Proof. Let Ω be a symplectic form on $\widetilde{M}$. The Lefschetz map $L_{[\Omega]} \colon H^2(\widetilde{M}) \to H^4(\widetilde{M})$ is given by the cup product with $[\Omega]$. We show that there is a class $[\beta] \in H^2(\widetilde{M})$ which is in the kernel of $L_{[\Omega]}$. We prove this by checking that $[\Omega] \wedge [\beta] \wedge [\alpha] = 0$, for any 2-form $[\alpha] \in H^2(\widetilde{M})$.

We need to determine the cohomology $H^2(\widetilde{M})$. For this, the first step is to construct a map $H^2(\widehat{M}) \to H^2(\widetilde{M})$. Let $h \colon M \to M$ be a map which:

- is the identity outside small neighbourhoods of each point with non-trivial isotropy,
- contracts a neighbourhood of each of the 24 isolated points whose isotropy is K onto the corresponding point,
- contracts a neighbourhood of each $S_{(p,q)}$ onto $S_{(p,q)}$ (fixing $S_{(p,q)}$ pointwise),
- in a neighbourhood of S_0, is the composition of a contraction onto S_0 with a map that contracts neighbourhoods (in S_0) of the 3 fixed points to the points, and
- is $\mathbb{Z}_6$-equivariant.

h induces a map $\widehat{h} \colon \widehat{M} \to \widehat{M}$. Note that for any closed form $\alpha \in \Omega^*(\widehat{M})$, $\widehat{h}^*(\alpha) \in \Omega^*(\widehat{M})$ is cohomologous to α and can be lifted to a form $\pi^* \widehat{h}^*(\alpha) \in \Omega^*(\widetilde{M})$, where $\pi \colon \widetilde{M} \to \widehat{M}$ is the resolution map. This induces a well-defined map

$$\Psi = \pi^* \circ \widehat{h}^* \colon H^*(\widehat{M}) \to H^*(\widetilde{M}).$$

Now consider $U = \widehat{M} - S$, where $S \subset \widehat{M}$ is the singular locus and $V \subset \widehat{M}$ is a small neighbourhood of S. Let also $\widetilde{U} = \pi^{-1}(U)$ and $\widetilde{V} = \pi^{-1}(V) \subset \widetilde{M}$.

Using compactly supported de Rham cohomology, we have a diagram

$$\begin{array}{ccccccc} H^2_c(U)\oplus H^2_c(V) & \to & H^2_c(\widehat{M}) & \to & H^3_c(U\cap V) & \to & H^3_c(U)\oplus H^3_c(V) \\ \downarrow = \quad \Psi\downarrow & & \downarrow\Psi & & \downarrow\cong & & \downarrow = \quad \Psi\downarrow \\ H^2_c(\widetilde{U})\oplus H^2_c(\widetilde{V}) & \to & H^2_c(\widetilde{M}) & \to & H^3_c(\widetilde{U}\cap \widetilde{V}) & \to & H^3_c(\widetilde{U})\oplus H^3_c(\widetilde{V}) \end{array}$$

Since V retracts onto a set of dimension 2, $H^3(V)=0$. By Poincaré duality, $H^3_c(V)=0$ as well. Now a simple diagram chasing proves that $H^2(\widetilde{M})=H^2_c(\widetilde{M})$ is generated by $H^2(\widehat{M})=H^2_c(\widehat{M})$ and $H^2_c(\widetilde{V})$.

Consider the closed form $\nu\wedge\bar\nu\in\Omega^2(\widehat{M})$. Since $\nu\wedge\bar\nu|_{S_{(p,q)}}=0$ for any surface $S_{(p,q)}$ and $\nu\wedge\bar\nu|_{S_0}=0$ as well, the 2-cohomology class

$$[\beta]=\Psi([\nu\wedge\bar\nu])$$

vanishes on $\widetilde{V}$. Clearly $[\beta]\wedge[\alpha_1]\wedge[\alpha_2]=0$ if either $[\alpha_1],[\alpha_2]\in H^2_c(\widetilde{V})$. Moreover, one can check that $[\beta]\wedge[\alpha_1]\wedge[\alpha_2]=0$, for $[\alpha_1],[\alpha_2]\in H^2(\widehat{M})$, which completes the proof. □

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