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On the Lee classes of locally conformally symplectic complex surfaces

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We prove that the deRham cohomology classes of Lee forms of locally conformally symplectic structures taming the complex structure of a compact complex surface S with first Betti number equal to 1 is either a non-empty open subset of $H^1_{dR}(S, \mathbb{R})$, or a single point. In the latter case, we show that S must be biholomorphic to a blow-up of an Inoue–Bombieri surface. Similarly, the deRham cohomology classes of Lee forms of locally conformally Kähler structures of a compact complex surface S with first Betti number equal to 1 is either a non-empty open subset of $H^1_{dR}(S, \mathbb{R})$, a single point or the empty set. We give a characterization of Enoki surfaces in terms of the existence of a special foliation, and obtain a vanishing result for the Lichnerowicz–Novikov cohomology groups on the class VII compact complex surfaces with infinite cyclic fundamental group.

1. Introduction

This paper is a sequel to our previous work [2] in which we have established the following result

Theorem 1.1. *[2] Any compact complex surface $S = (M, J)$ admits a non-degenerate 2-form ω which tames the complex structure J, i.e. its $(1, 1)$-part*

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$\omega^{1,1}$ *with respect to* J *is positive-definite, and satisfies*

$$d\omega = \alpha \wedge \omega \tag{1}$$

for some closed 1*-form* α*.*

A non-degenerate 2-forms ω satisfying (1) is called a *locally conformally symplectic* (LCS) structure and the corresponding closed 1-form α is referred to as the *Lee form* of ω. The notion of an LCS structure is conformally invariant in the sense that if ω is an LCS structure with Lee form α, then $\tilde{\omega} = e^f \omega$ is an LCS structure with Lee form $\tilde{\alpha} = \alpha + df$. Thus, the deRham class $[\alpha] \in H^1_{dR}(S, \mathbb{R})$ is a natural invariant of (the conformal class of) an LCS structure, which we shall call the *Lee class* of ω. We thus consider the set of all Lee classes of LCS structures taming the complex structure of S

$$\mathcal{T}(S) := \{[\alpha] \in H^1_{dR}(S, \mathbb{R}) : \exists\, \omega \in \mathcal{E}^2(S, \mathbb{R}) \text{ s. t. } \omega^{1,1} > 0,\ d\omega = \alpha \wedge \omega\}.$$

Theorem 1.1 then states that $\mathcal{T}(S) \neq \emptyset$. This result is new only when the first Betti number $b_1(S)$ is odd, in which case $0 \notin \mathcal{T}(S)$ (see e.g. [2, Prop. 3.5]). Conversely, when $b_1(S)$ is even, it follows from [13, Lemme II.3] and [16, p. 185] that $\mathcal{T}(S) = \{0\}$.

A further motivation for studying LCS structures comes from the following recent result by Eliashberg–Murphy [10] (see also [7, Thm. 2.15 & Rem. 2.16]):

Theorem 1.2. *[10] Let* (M, J) *be a compact almost complex manifold and* $a \neq 0 \in H^1_{dR}(M, \mathbb{R})$ *a non-trivial deRham class. Then for any* $C > 1$ *sufficiently large and any closed* 1*-form* $\alpha \in Ca$*, there exists an LCS structure with Lee form* α*, compatible with the orientation on* M *induced by* J*.*

The LCS forms ω constructed in [10] are in fact *exact* in the sense that

$$\omega = d_\alpha \beta := d\beta - \alpha \wedge \beta$$

for a 1-form β on M. On a compact complex surface containing a rational curve, such LCS forms cannot tame the underlying complex structure, i.e. generically they are different from the LCS structures provided by Theorem 1.1.

The main result of this paper is the following structure theorem for the set $\mathcal{T}(S)$.

Theorem 1.3. *Let S be a compact complex surface with first Betti number equal to 1. Then, either $\mathcal{T}(S)$ is a non-empty open subset in $H^1_{dR}(S,\mathbb{R})$, or else $\mathcal{T}(S)$ is a single point and S is a blow-up of an Inoue–Bombieri surface [17].*

The proof of this result relies, at one hand, on a characterization of the case when $\mathcal{T}(S)$ is not open in terms of the "Kähler rank" theory developed in [8] (see Theorem 4.1, Remark 4.6 and Lemma 4.7 below), and, at the other hand, on the characterization of the blow-ups of Inoue–Bombieri surfaces obtained by Brunella in [6].

As observed recently in [27], Theorem 1.3 shows that on the Inoue–Bombieri complex surfaces (and their blow-ups) the existence result for LCS structures provided by Theorem 1.1 is complementary to the one provided by Theorem 1.2, see Corollary 4.9 below.

Theorem 1.3 is to be compared with recent results of R. Goto [15] about the deformations of Lee classes of locally conformally Kähler structures. As a matter of fact, combining [15, Thm. 2.3] with Theorem 4.1 in this paper, we obtain the following

Theorem 1.4. *Let S be a compact complex surface with first Betti number equal to 1 and $\mathcal{C}(S)\subset\mathcal{T}(S)$ the set of Lee classes of locally conformally Kähler structures on S, i.e.*

$$\mathcal{C}(S)=\{[\alpha]\in H^1_{dR}(S,\mathbb{R}) : \exists\ \omega\in\mathcal{E}^{1,1}(S,\mathbb{R}),\ \omega>0,\ d\omega=\alpha\wedge\omega\}.$$

Then $\mathcal{C}(S)$ is either empty, a single point or a non-empty open subset in $H^1_{dR}(S,\mathbb{R})$.

Examples of either type do exist, due to [4, 30], see [2, Thm. 1.4].

The proofs of Theorems 1.3 and 1.4 use a vanishing result for the first cohomology group $H^1_{d_L}(S,L)$ associated to the sheaf of parallel sections of the flat real line bundle L corresponding to a deRham class $a\in\mathcal{T}(S)$ (see Theorem 4.1), which holds true for all surfaces with $b_1(S)=1$ except the blow-ups of Inoue–Bombieri surfaces, according to [6, Thm. 1], Theorem 4.1, and Lemma 4.7 below. One is thus naturally led to ask whether or not the assumption $a\in\mathcal{T}(S)$ can be removed from this statement. In the final section of the paper, we recollect some observations regarding this and some related questions, and establish the following vanishing result.

Theorem 1.5. *Let S be a compact complex surface with first Betti number equal to* 1 *and fundamental group isomorphic to $\mathbb{Z}$. Then, for any real flat bundle L associated to a class $a \neq 0 \in H^1_{dR}(S,\mathbb{R}) \overset{\exp}{\cong} H^1(S,\mathbb{R}^*_+)$,*

$$\dim_{\mathbb{R}} H^k_{d_L}(S,L) = 0,\ k \neq 2, \quad \dim_{\mathbb{R}} H^2_{d_L}(S,L) = \dim_{\mathbb{R}} H^2(S,\mathbb{R}).$$

2. Preliminaries

Let $X = (M,J)$ be a compact complex manifold of complex dimension n, and α a closed 1-form on X, representing de Rham class $a = [\alpha] \in H^1_{dR}(X,\mathbb{R})$. We denote by $L_\alpha = X \times \mathbb{R}$ the topologically trivial real line bundle over X, endowed with the flat connection $\nabla^\alpha s := ds + \alpha \otimes s$, where s is a smooth function on X, also viewed as a smooth section of L_α. Similarly, ∇^α induces a holomorphic structure on the complex bundle $\mathcal{L}_\alpha := L_\alpha \otimes \mathbb{C}$ such that parallel sections are holomorphic. Writing $\alpha_{|U_i} = df_i$ on an open covering $\mathfrak{U} = (U_i)$ of X, $\{(U_i, e^{-f_i})\}$ defines a parallel (respectively holomorphic) trivialization of L_α (resp. of $\mathcal{L}_\alpha$) with transition functions $e^{f_i - f_j}$ on $U_i \cap U_j$. With respect to this trivialization, $s_0 = (U_i, e^{f_i})$ is a nowhere vanishing smooth section of L_α. This construction fits in into the sequence of natural morphisms

$$H^1_{dR}(X,\mathbb{R}) \overset{\exp}{\cong} H^1(X,\mathbb{R}^*_+) \longrightarrow H^1(X,\mathbb{C}^*) \longrightarrow \mathrm{Pic}(X), \tag{2}$$

where $\mathbb{R}^*_+$ denotes the sheaf of locally constant positive real functions, and $\mathrm{Pic}(X) = H^1(X,\mathcal{O}^*)$ is the group of isomorphism classes of holomorphic line bundles. Indeed, L_α represents the isomorphism class $\exp(a) \in H^1(X,\mathbb{R}^*_+)$ given by (2) whereas $\mathcal{L}_\alpha$ represents its image in $\mathrm{Pic}^0(X)$, where $\mathrm{Pic}^0(X)$ denotes the subgroup of $H^1(X,\mathcal{O}^*)$ of isomorphism classes of holomorphic line bundle with zero first Chern class.

In what follows, we shall tacitly identify L_α and $L_{\tilde\alpha}$ (resp. $\mathcal{L}_\alpha$ and $\mathcal{L}_{\tilde\alpha}$) for any two $\alpha, \tilde\alpha \in a$, and denote (with a slight abuse of notation) by L_a (resp. $\mathcal{L}_a$) a flat line bundle obtained by some choice of $\alpha \in a$; we shall refer to L_a (resp. $\mathcal{L}_a$) as the flat real (resp. the flat holomorphic) line bundle corresponding to $a \in H^1_{dR}(X,\mathbb{R})$. Similarly, we shall implicitly identify a flat real line bundle (resp. a holomorphic line bundle) with the class it represents in $H^1(X,\mathbb{R}^*_+)$ (resp. in $\mathrm{Pic}(X)$).

We denote by $\mathcal{E}^k(X,\mathbb{R})$, resp. $\mathcal{E}^{p,q}(X,\mathbb{C})$ the space of smooth real k-forms on X, resp. of smooth complex-valued (p,q)-forms on X. The α-twisted differential

$$d_\alpha := d - \alpha \wedge \cdot$$

defines the Lichnerowicz–Novikov complex

$$\cdots \xrightarrow{d_\alpha} \mathcal{E}^{k-1}(X,\mathbb{R}) \xrightarrow{d_\alpha} \mathcal{E}^{k}(X,\mathbb{R}) \xrightarrow{d_\alpha} \cdots \tag{3}$$

which is isomorphic to the de Rham complex of differential forms with values in L^*

$$\cdots \xrightarrow{d_{L^*}} \mathcal{E}^{k-1}(X,L^*) \xrightarrow{d_{L^*}} \mathcal{E}^{k}(X,L^*) \xrightarrow{d_{L^*}} \cdots \tag{4}$$

associated to the sheaf of locally constant sections of L^*. This can be viewed by writing $\alpha_{|U_i} = df_i$ on an open covering $\mathfrak{U} = (U_i)$ of X: then, for any d_α-closed smooth form ω on X, $\omega_{i|U_i} := e^{-f_i}\omega$ gives rise to a smooth section in $\mathcal{E}^k(X,L^*)$, satisfying $d\omega_i = 0$ on U_i, i.e. $\{U_i,\omega_i)$ defines a d_{L^*}-closed form with values in L^*. In particular, we have an isomorphism between the cohomology groups

$$H^k_\alpha(X,\mathbb{R}) \cong H^k_{d_{L^*}}(X,L^*), \tag{5}$$

associated to the complexes (3) and (4), respectively.

Considering complex-valued forms, one can similarly introduce the operators

$$d_{L^*} = \partial_{\mathcal{L}^*} + \bar\partial_{\mathcal{L}^*}, \quad \text{and} \quad d_\alpha = \partial_\alpha + \bar\partial_\alpha$$

with

$$\partial_\alpha = \partial - \alpha^{1,0}\wedge \quad \text{and} \quad \bar\partial_\alpha = \bar\partial - \alpha^{0,1}\wedge,$$

acting respectively on $\mathcal{E}^{p,q}(X,\mathcal{L}^*)$ and $\mathcal{E}^{p,q}(X,\mathbb{C})$. These give rise to the isomorphisms

$$H^{p,q}_{\bar\partial_\alpha}(X,\mathbb{C}) \cong H^{p,q}(X,\mathcal{L}^*) \cong H^q(X,\Omega^p\otimes\mathcal{L}^*), \tag{6}$$

where $H^{p,q}_{\bar\partial_\alpha}(X,\mathbb{C}) = \mathrm{Ker}(\bar\partial_\alpha)/\mathrm{Im}(\bar\partial_\alpha)$ whereas $H^{p,q}(X,\mathcal{L}^*) \cong H^q(X,\Omega^p\otimes\mathcal{L}^*)$ is the usual Dolbeault cohomology group of X with values in the flat holomorphic line bundle $\mathcal{L}^*$, and Ω^p stands for the holomorphic vector bundle of $(p,0)$ forms on X.

For any $(k-1)$-form ϕ and $(2n-k)$-form ψ on X, we have

$$d(\phi\wedge\psi) = (d_\alpha\phi)\wedge\psi + (-1)^{k-1}\phi\wedge(d_{-\alpha}\psi).$$

Integrating the above formula over the closed manifold X leads to a natural pairing between $H^k_\alpha(X,\mathbb{R})$ and $H^{2n-k}_{-\alpha}(X,\mathbb{R})$.

For any Riemannian metric g on X, the L^2 adjoint of d_α is

$$d_\alpha^* = - * d_{-\alpha} *,$$

where $*$ is the Hodge operator with respect to g. (We have used that X is oriented and even dimensional.) It follows that the corresponding twisted Laplace operator $\Delta_\alpha^g = d_\alpha d_\alpha^* + d_\alpha^* d_\alpha$ has the same index as the usual Laplacian Δ^g and satisfies

$$*\Delta_\alpha^g = \Delta_{-\alpha}^g * .$$

Hodge theory (see e.g. [1, 28]) and (5) then imply that the spaces $H_\alpha^k(X,\mathbb{R})$ are finite dimensional and the pairing between $H_\alpha^k(X,\mathbb{R})$ and $H_{-\alpha}^{2n-k}(X,\mathbb{R})$ defined above is a perfect pairing, i.e.

$$H_\alpha^k(X,\mathbb{R}) \cong \big(H_{-\alpha}^{2n-k}(X,\mathbb{R})\big)^*, \quad H_{d_L}^k(X,L) \cong \big(H_{d_{L^*}}^{2n-k}(X,L^*)\big)^*, \tag{7}$$

where the upper $*$ denotes the dual vector space. The index theorem also implies (as observed in [12])

$$\sum_{k=0}^{2n}(-1)^k \dim_{\mathbb{R}} H_\alpha^k(X,\mathbb{R}) = e(X), \tag{8}$$

where $e(M)$ is the Euler characteristic of M. Notice that if $[\alpha] \neq 0 \in H_{dR}^1(X,\mathbb{R})$, then

$$H_\alpha^0(X,\mathbb{R}) = H_\alpha^{2n}(X,\mathbb{R}) = \{0\}, \tag{9}$$

Indeed, suppose $d_\alpha f = 0$ for some smooth non-zero function f. This means that f satisfies the linear system $df = f\alpha$, so f cannot vanish on X, showing that $\alpha = d\log|f|$, a contradiction. Thus, $H_\alpha^0(X,\mathbb{R}) = \{0\}$ and $H_{-\alpha}^0(X,\mathbb{R}) \cong (H_\alpha^{2n}(X,\mathbb{R}))^* = \{0\}$.

We shall use the following elementary fact

Lemma 2.1. *Suppose M is a compact manifold and $p : \tilde{M} \to M$ a finite cover. For any flat real line bundle $L = L_\alpha$ denote $\tilde{L} = L_{\tilde{\alpha}}$ the corresponding pullback to $\tilde{M}$, where $\tilde{\alpha} = p^*\alpha$. Then the natural pull-back map $p^* : H_\alpha^k(M,\mathbb{R}) \to H_{\tilde{\alpha}}^k(\tilde{M},\mathbb{R})$ is injective.*

Proof. We need to show that if β is d_α-closed k-form on M such that $\tilde{\beta} := p^*(\beta) = d_{\tilde{\alpha}}\tilde{\gamma}$ on $\tilde{M}$, then β is d_α-exact on M. Denote by Γ the finite group of diffeomorphisms of $\tilde{M}$ such that $M = \tilde{M}/\Gamma$. As both $\tilde{\alpha}$ and $\tilde{\beta}$ are invariant under the action of Γ of $\tilde{M}$, the average of $\tilde{\gamma}$ over Γ is a Γ-invariant $(k-1)$-form γ on $\tilde{M}$, satisfying $\tilde{\beta} = d_{\tilde{\alpha}}\gamma$. As γ descends to M (being Γ-invariant), we also have $\beta = d_\alpha \gamma$ on M. □

Similarly to (7), we have isomorphisms

$$H^{p,q}_{\bar{\partial}_\alpha}(X, \mathbb{C}) \cong (H^{n-p,n-q}_{\bar{\partial}_{-\alpha}}(X, \mathbb{C}))^*, \quad H^{p,q}(X, \mathcal{L}) \cong \left(H^{n-p,n-q}(X, \mathcal{L}^*)\right)^*$$

where the second identification is the usual Serre duality and the first follows from the second via (6).

We shall now specialize to the case when $X = S$ is a compact complex surface. For a flat real line bundle $L := L_\alpha$ and $\mathcal{L} = L_\alpha \otimes \mathbb{C}$ the corresponding flat holomorphic line bundle, we shall denote by

- $b_k(S, L) := \dim_{\mathbb{R}} H^k_{d_L}(S, L) = \dim_{\mathbb{R}} H^k_{-\alpha}(S, \mathbb{R})$,
- $h^{p,q}(S, \mathcal{L}) := \dim_{\mathbb{C}} H^{p,q}(S, \mathcal{L}) = \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}_{-\alpha}}(S, \mathbb{C})$,

the corresponding dimensions. We then have

Lemma 2.2. *Let S be a compact complex surface, $L = L_\alpha$ a flat real line bundle corresponding to a closed 1-form α and $\mathcal{L} = L_\alpha \otimes \mathbb{C}$ the corresponding flat holomorphic line bundle. Let $B_x : \hat{S} \to S$ be the blow-down map from the complex surface $\hat{S}$ obtained from S by blowing up a point $x \in S$ and denote by $\hat{\alpha} = B_x^*(\alpha)$, $\hat{L} = L_{\hat{\alpha}}$ and $\hat{\mathcal{L}} = \mathcal{L}_{\hat{\alpha}}$ the corresponding objects on $\hat{S}$, obtained by the natural pull-back map. Then,*

(a) $b_1(\hat{S}, \hat{L}) = b_1(S, L)$, $b_3(\hat{S}, \hat{L}) = b_3(S, L)$, $b_2(\hat{S}, \hat{L}) = b_2(S, L) + 1$;

(b) $h^{k,0}(\hat{S}, \hat{\mathcal{L}}) = h^{k,0}(S, \mathcal{L}), k = 0, 1, 2$.

Proof. (a) Notice that as $e(\hat{S}) = e(S) + 1$ (see e.g. [3]), the last equality in (a) follows from the first two and (8)-(9). Also, using the duality $H^3_\alpha(S, \mathbb{R}) \cong (H^1_{-\alpha}(S, \mathbb{R}))^*$ (see (7)), it is enough to show that for each $[\alpha] \in H^1_{dR}(S, \mathbb{R})$, $\dim_{\mathbb{R}} H^1_\alpha(\hat{S}, \mathbb{R}) = \dim_{\mathbb{R}} H^1_\alpha(S, \mathbb{R})$. As the dimension of $H^1_\alpha(S, \mathbb{R})$ does not depend on the choice of $\alpha \in a$, we can choose α such that it identically vanishes on a open ball U centred at x. We are going to prove that the natural pull-back map $B_x^* : H^1_\alpha(S, \mathbb{R}) \to H^1_{\hat{\alpha}}(\hat{S}, \mathbb{R})$ is then an isomorphism.

We shall first prove that B_x^* is surjective. With our choice for α, any $d_{\hat{\alpha}}$-closed 1-form $\hat{\varphi}$ on $\hat{S}$ is closed over $\hat{U} = B_x^{-1}(U)$. As

$$H^1_{dR}(\hat{U},\mathbb{R}) \cong H^1_{dR}(\mathbb{C}P^1,\mathbb{R}) = \{0\},$$

we can write $\hat{\varphi}_{|\hat{U}} = d(\hat{\xi}_{|\hat{U}})$. Multiplying $\hat{\xi}_{|\hat{U}}$ by the pull-back via B_x of a bump function centred at x and supported in U, we can assume $\hat{\xi}$ is globally defined on $\hat{S}$ and $\hat{\phi} = \hat{\varphi} - d_{\hat{\alpha}}\hat{\xi}$ is another form representing $[\hat{\varphi}] \in H^1_{\hat{\alpha}}(\hat{S})$ which vanishes identically on a neighbourhood of E. Then, the diffeomorphism $(B_x^{-1}) : S \setminus \{x\} \to \hat{S} \setminus E$ allows us to define a smooth 1-form $\phi = (B_x^{-1})^*(\hat{\phi})$ on S with $d_\alpha \phi = 0$ and $B_x^*(\phi) = \hat{\phi}$.

We now prove that $B_x^* : H^1_\alpha(S,\mathbb{R}) \to H^1_{\hat{\alpha}}(\hat{S},\mathbb{R})$ is injective. Suppose φ is a d_α-closed 1-form on S, such that $\hat{\varphi} = B_x^*(\varphi) = d_{\hat{\alpha}}\hat{\xi}$. As $H^1_{dR}(U,\mathbb{R}) = \{0\}$, we can modify φ with a d_α-exact 1-form (as we did above with $\hat{\varphi}$) and assume without loss that $\varphi_{|U} \equiv 0$. It follows that the function $\hat{\xi}$ satisfies $d\hat{\xi}_{|\hat{U}} \equiv 0$, i.e. $\hat{\xi}$ is a smooth function on $\hat{S}$ which is constant on $\hat{U}$ and, therefore, is the pull back to $\hat{S}$ of a smooth function ξ on S (which is constant on U). It follows that $\varphi = d_\alpha \xi$.

(b) Again, we assume without loss that the closed 1-form α identically vanishes on a open ball U centred at x. Clearly, we have an injective pull-back map $B_x^* : H^{k,0}_{\bar{\partial}_\alpha}(S,\mathbb{C}) \to H^{k,0}_{\bar{\partial}_{\hat{\alpha}}}(\hat{S},\mathbb{C})$ so we need to establish its surjectivity. Suppose $\hat{\beta}$ is a $(k,0)$-form on $\hat{S}$ satisfying $\bar{\partial}_{\hat{\alpha}}\hat{\beta} = 0$. Pulling back $\hat{\beta}$ by the biholomorphism $B_x^{-1} : S \setminus \{x\} \to \hat{S} \setminus E$ defines a $(k,0)$-form β on $S \setminus \{x\}$, which satisfies $\bar{\partial}_\alpha \beta = 0$. As α vanishes on U, the $(k,0)$-form β is holomorphic on $U \setminus \{x\}$, and therefore extends over x by Hartogs' extension theorem. By construction, $\hat{\beta} = B_x^*(\beta)$ on $\hat{S} \setminus E$, hence everywhere by continuity. □

3. Complex surfaces with $b_1 = 1$

From now on, S will denote a compact complex surface whose first Betti number $b_1(S) = 1$. Kodaira [19] has shown that for such a surface either $H^0(S, K_S^m) = \{0\}$ for all $m \geq 1$, where $K_S = \Omega^2$ stands for the canonical bundle of S, or there exists $m_0 \geq 1$ such that $K_S^{m_0} \cong \mathcal{O}$ is trivial. In the first case, the surface is said to belong to the class VII (we follow the terminology of [3]) whereas in the latter case, Kodaira proved that the minimal model S_0 of S must be a *secondary Kodaira surface*, see [3, 19]. The classification of compact complex surfaces in the class VII is still open, but the special case when the minimal model S_0 of S satisfies $b_2(S_0) = 0$ has been settled by [5, 21, 29]: S_0 must then be either a Hopf surface [18] or an Inoue–Bombieri surface [17]. The class of the minimal complex surfaces $S_0 \in$ VII for which

$b_2(S_0) > 0$ is commonly denoted by VII_0^+. We summarize the situation in the following

Theorem 3.1. *[5, 19, 21, 29] Any compact complex surface S with first Betti number $b_1(S) = 1$ is obtained by blowing up a minimal complex surface S_0 of one of the following types*

- *a secondary Kodaira surface;*
- *a Hopf surface;*
- *an Inoue–Bombieri surface;*
- *a minimal complex surface in the class VII_0^+.*

We shall use the following key vanishing result (see e.g. [2, Lemma 2.13] for a proof).

Lemma 3.2. *[2, 23] Let S be a compact complex surface whose minimal model belongs to the class VII_0^+. Then, for any non-trivial holomorphic line bundle $\mathcal{L} \in \mathrm{Pic}^0(S)$*

$$H^2(S, \mathcal{L}) \cong H^0(S, K_S \otimes \mathcal{L}^*) = \{0\}.$$

4. The space of Lee classes

4.1. A characterization of the case when $\mathcal{T}(S)$ is a single point

In this section we are going to establish the following result.

Theorem 4.1. *Let S be a compact complex surface with $b_1(S) = 1$ and $\mathcal{T}(S) \subset H^1_{dR}(S, \mathbb{R})$ the set of Lee classes of LCS forms taming the complex structure on S. Then the following conditions are equivalent.*

(1) *$\exists\ 0 \neq a \in H^1_{dR}(S, \mathbb{R})$ and $\alpha \in a$ such that*

$$d_{-\alpha} d^c_{-\alpha}(1) = dJ\alpha + \alpha \wedge J\alpha = 0;$$

(2) *$\mathcal{T}(S) = \{a\}$;*

(3) *$\mathcal{T}(S) \subset H^1_{dR}(S, \mathbb{R})$ is not open;*

(4) *$\exists\ a \in \mathcal{T}(S)$ such that $H^1_{d_{L_a}}(S, L_a) \neq \{0\}$.*

We first need a variation of a result in [15]:

Proposition 4.2. *Let M be a compact $2n$-dimensional manifold which admits an LCS structure with Lee class $0 \neq a \in H^1_{dR}(M, \mathbb{R})$. Let $L = L_a$ be the corresponding flat real line bundle and L^* its dual. If $H^3_{d_{L^*}}(M, L^*) = \{0\}$, then for any $b \in H^1_{dR}(M, \mathbb{R})$ there exists $\varepsilon > 0$ such that for $|t| < \varepsilon$, M admits an LCS structure with Lee class in $(a + tb) \in H^1_{dR}(M, \mathbb{R})$. The statement holds true for the Lee classes of LCS structures taming a given almost complex structure J on M.*

Proof. Let α, β be representatives of the de Rham classes a and b, respectively, and ω_0 a smooth 2-form on M satisfying $d_\alpha \omega_0 = 0$ and $\omega_0^n \neq 0$. As the latter condition is open, it is enough to construct a $\mathcal{C}^\infty$ family $\omega(t)$ of 2-forms, satisfying $d_{(\alpha+t\beta)}\omega(t) = 0$ for $|t| < \varepsilon$. (The case when ω_0 tames J is handled similarly, by noting that $\omega^{1,1} > 0$ is an open condition.) To this end, we take $\omega(t)$ be a formal power series

$$\omega(t) = \sum_{i=0}^{\infty} \omega_i t^i,$$

where ω_i are smooth 2-forms on M (and ω_0 is the 2-form chosen above). Using $d_\alpha \omega_0 = 0$, the condition $d_{(\alpha+t\beta)}\big(\omega(t)\big) = 0$ reads as

$$d_\alpha \omega_{i+1} = \beta \wedge \omega_i, \quad i = 0, \dots \tag{10}$$

which can be used to build ω_i by induction: Indeed, the right hand side is d_α-closed as

$$d_\alpha(\beta \wedge \omega_i) = -\beta \wedge (d_\alpha \omega_i) = -\beta \wedge (\beta \wedge \omega_{i-1}) = 0.$$

As $H^3_\alpha(M, \mathbb{R}) \cong H^3_{d_{L^*}}(M, L^*) = \{0\}$ by the hypothesis, one can solve (10) inductively as follows. Let g be a Riemannian metric on M and d_α^* the formal L^2 adjoint of d_α (with respect to g) acting on p-forms. The vanishing of $H^3_\alpha(M, \mathbb{R})$ ensures that the Laplacian $\Delta^g_\alpha = d_\alpha d_\alpha^* + d_\alpha^* d_\alpha$ is invertible on $\mathcal{E}^3(M, \mathbb{R})$, with inverse $\mathbb{G}_\alpha$. Note that, as $\beta \wedge \omega_i$ is a d_α-closed 3-form, $d_\alpha\Big(\mathbb{G}_\alpha(\beta \wedge \omega_i)\Big) = 0$. Letting

$$\omega_{i+1} := d_\alpha^*(\mathbb{G}_\alpha(\beta \wedge \omega_i),$$

we have

$$\begin{aligned} d_\alpha \omega_{i+1} &= \Delta^g_\alpha\Big(\mathbb{G}_\alpha(\beta \wedge \omega_i)\Big) - d_\alpha^* d_\alpha\Big(\mathbb{G}_\alpha(\beta \wedge \omega_i)\Big) \\ &= \beta \wedge \omega_i. \end{aligned}$$

The convergence of the series in $\mathcal{C}^k(M)$ follows by standard Schauder estimates for the Laplacian, whereas the smoothness of the solutions follows from standard regularity theory for elliptic PDE's. Indeed, notice that $\omega(t)$ satisfies the PDE

$$(d^*_\alpha d_{\alpha+t\beta} + d_{\alpha+t\beta} d^*_\alpha)(\omega(t)) = d_{\alpha+t\beta} d^*_\alpha \omega_0,$$

in which the rhs is $\mathcal{C}^\infty$. The above PDE is elliptic for t small enough (as it is so for $t=0$). □

Proof of Theorem 4.1. '(1) ⇒ (2)': By Theorem 1.1, $\mathcal{T}(S) \neq \emptyset$. Let $\tilde{a} \in \mathcal{T}(S)$. As $b_1(S)=1$, $\tilde{a}$ can be represented by the closed 1-form $t\alpha$ for some real constant $t \neq 0$. Let ω be a $d_{t\alpha}$-closed 2-form with $\omega^{1,1} > 0$. Using that $d_{-\alpha} d^c_{-\alpha}(1) = 0$, we have

$$d_{-t\alpha} d^c_{-t\alpha}(1) = t dJ\alpha + t^2 \alpha \wedge J\alpha = t(t-1)\alpha \wedge J\alpha.$$

It follows that

$$\begin{aligned} 0 = \int_M d^c_{-t\alpha}(1) \wedge d_{t\alpha}\omega = \int_M d_{-t\alpha} d^c_{-t\alpha}(1) \wedge \omega \\ = t(t-1) \int_M \alpha \wedge J\alpha \wedge \omega \\ = t(t-1) \int_M \alpha \wedge J\alpha \wedge \omega^{1,1}. \end{aligned}$$

As $\int_M \alpha \wedge J\alpha \wedge \omega^{1,1} > 0$, it follows that $t=1$, i.e. $\mathcal{T}(S) = \{[\alpha]\}$.

'(2) ⇒ (3)' is obvious.

'(3) ⇒ (4)' follows from Proposition 4.2 and the facts that $b_1(S)=1$ and $H^3_{d_{L^*}}(S, L^*) \cong H^1_{d_L}(S, L)$, see (7).

'(4) ⇒ (1)': We shall consider four cases, according to the type of the minimal model S_0 in Theorem 3.1

Cases 1 and 2: S_0 is either a secondary Kodaira surface or a Hopf surface. These cases are impossible because of the following

Lemma 4.3. *If the minimal model of S is a secondary Kodaira surface or a Hopf surface, then for any $a \neq 0 \in H^1_{dR}(S,\mathbb{R})$, $H^1_{d_{L_a}}(S, L_a) = \{0\}$.*

Proof. Belgun [4] has shown that any secondary Kodaira surface S_0 admits a Vaisman locally conformally Kähler structure ω_0, i.e. an LCS structure ω_0 with $\omega_0 = \omega_0^{1,1} > 0$ and such that the Lee form α_0 is parallel with respect to $g_0(\cdot,\cdot)=\omega_0(\cdot,J\cdot)$. By [20], $H^1_{t\alpha_0}(S_0,\mathbb{R})=\{0\}$ for any $t\neq 0$. As $b_1(S_0)=1$, it follows that for any $\check{a}\neq 0\in H^1_{dR}(S_0,\mathbb{R})$, $H^1_{d_{L_{\check{a}}}}(S_0,L_{\check{a}})=\{0\}$. As the blow-down map $b: S\to S_0$ induces an isomorphism between $H^1_{dR}(S_0,\mathbb{R})$ and $H^1_{dR}(S,\mathbb{R})$, any non-zero class $a\in H^1_{dR}(S,\mathbb{R})$ is the pull-back of a class $\check{a}\neq 0\in H^1_{dR}(S_0,\mathbb{R})$, thus by Lemma 2.2, $H^1_{d_{L_a}}(S,L_a)=\{0\}$.

Hopf surfaces are classified in [18], and they are either primary or secondary. Gauduchon–Ornea [14] showed that any primary Hopf surfaces S_0 is a small deformation of a primary Hopf surface admitting Vaisman locally conformally metrics, so that the same argument as in the case of a secondary Kodaira surface proves the claim. If S_0 is a secondary Hopf surface, it is covered by a primary one. Using Lemma 2.1 and Lemma 2.2, we conclude again that $H^1_{d_{L_a}}(S,L_a)=\{0\}$ for each non-trivial flat real line bundle $L=L_a$. □

Case 3: S_0 is an Inoue–Bombieri surface. In this case our claim follows from

Lemma 4.4. *If the minimal model of S is an Inoue–Bombieri surface, then $\mathcal{T}(S)=\{a\}$, $H^1_{d_{L_a}}(S,L_a)\neq 0$ and there exits $\alpha\in a$ satisfying the identity in Theorem 4.1* (1).

Proof. The Inoue–Bombieri surfaces S_0 are classified in [17] and appear as three types of quotients of $\mathbb{H}\times\mathbb{C}$, where

$$\mathbb{H}=\{w=w_1+iw_2,: w_2>0\}$$

denotes the upper half-plane. An inspection upon the explicit forms of the desk transformations on $\mathbb{H}\times\mathbb{C}$ (see [17]) shows that in each case the $(1,0)$-form $-idw/w_2$ is invariant and therefore descends to S_0. Let $\alpha := dw_2/w_2 = \mathrm{Re}(-idw/w_2)$ denote the corresponding real 1-form on S_0. We thus have that $J\alpha=\mathrm{Im}(-idw/w_2)=-dw_1/w_2$ satisfies

$$dJ\alpha = -dw_1\wedge dw_2/w_2^2 = -\alpha\wedge J\alpha. \tag{11}$$

Pulling back α from S_0 to S by the blow-down map, we have a non-zero 1-form α on S satisfying (11). As α does not vanish on S_0, we have that $a:=[\alpha]\neq 0$ on S_0, and hence also on S (see Lemma 2.2). By '(1) ⇒ (2)', we have $\mathcal{T}(S)=\{[\alpha]\}$. By Proposition 4.2, $H^3_\alpha(S,\mathbb{R})\neq\{0\}$ so that, by (7) and (5), $\dim_{\mathbb{R}} H^3_\alpha(S,\mathbb{R})=\dim_{\mathbb{R}} H^1_{-\alpha}(S,\mathbb{R})=\dim_{\mathbb{R}} H^1_{d_{L_\alpha}}(S,L_\alpha)\neq 0$. □

Remark 4.5. An alternative way to show $H^1_{d_{L_\alpha}}(S,L_\alpha)\neq\{0\}$ appears in [27].

Case 4: S_0 belongs to the class VII_0^+. Let $[\alpha] \in \mathcal{T}(S)$ and $0 \neq [\beta] \in H^1_{-\alpha}(S, \mathbb{R})$. As $d_{-\alpha}\beta = 0$, $\bar{\partial}_{-\alpha}\beta^{0,1} = 0$ where $\beta^{0,1} = \frac{1}{2}(\beta - iJ\beta)$ is the $(0,1)$-part of β. As $H^{0,1}_{\bar{\partial}_{-\alpha}}(S, \mathbb{C}) \cong H^1(S, \mathcal{L}_\alpha)$ and $H^2(S, \mathcal{L}_\alpha) = \{0\}$ (see Lemma 3.2), it follows from the Riemann–Roch formula and [2, Lemmas 2.4 & 2.11] that

$$\dim_{\mathbb{C}} H^{0,1}_{\bar{\partial}_{-\alpha}}(S, \mathbb{C}) = \dim_{\mathbb{C}} H^0(S, \mathcal{L}_\alpha) = 0.$$

Thus, $\beta^{0,1} = \bar{\partial}_{-\alpha}(h + if)$ for some smooth real-valued functions f, h on S. Equivalently, $\beta = d_{-\alpha}h + d^c_{-\alpha}f$. Since $[\beta] \neq 0 \in H^1_{-\alpha}(S, \mathbb{R})$, f is not identically zero whereas $d_{-\alpha}\beta = 0$ implies

$$d_{-\alpha}d^c_{-\alpha}f = dJdf + \alpha \wedge Jdf - J\alpha \wedge df + f(dJ\alpha + \alpha \wedge J\alpha) = 0. \tag{12}$$

Let ω be a $d_{-\alpha}$-closed 2-form with $(1,1)$-part $F = \omega^{1,1} > 0$. We consider the hermitian metric g on S whose fundamental 2-form is F. By [2, Lemmas 2.4 & 2.5], we have

$$\delta^g(\theta^g - \alpha) + g(\theta^g - \alpha, \alpha) = 0, \tag{13}$$

where we recall $\theta^g = J\delta^g F$. Taking contraction with F in (12) yields

$$M(f) := \Delta^g f + g(df, \theta^g - 2\alpha) + f\big(\delta^g\alpha + g(\alpha, \theta^g - \alpha)\big) = 0.$$

The adjoint operator $M^*(f)$ of $M(f)$ with respect to the L^2-product induced by g is

$$\begin{aligned} M^*(f) &= \Delta^g f - g(df, \theta^g - 2\alpha) + f\Big(\delta^g(\theta^g - \alpha) + g(\alpha, \theta^g - \alpha)\Big) \\ &= \Delta^g f - g(df, \theta^g - 2\alpha), \end{aligned}$$

where for the last equality we have used the property (13) of the hermitian metric g. By Hopf maximum principle, the kernel of M^* consist of the constant functions. It follows (see [2, App. A]) that the principal eigenvalues satisfy $\lambda_0(M^*) = \lambda_0(M) = 0$ so that $M(f) = 0$ admits a unique up to scale non-zero solution f which never vanishes on M. Letting

$$\tilde{\alpha} := \frac{d_{-\alpha}f}{f} = \alpha + d\log|f|,$$

it is straightforward to check that (12) is equivalent to

$$dJ\tilde{\alpha} + \tilde{\alpha} \wedge J\tilde{\alpha} = 0.$$

Noting that $[\tilde{\alpha}] = [\alpha] \neq 0$ (as $[\alpha] \in \mathcal{T}(S)$ by assumption, see [2, Prop. 3.5]), this concludes the proof of '(4) $\Rightarrow$ (1)' in Case 4 too. □

Remark 4.6. The condition (1) of Theorem 4.1 above gives a direct link to the theory of complex surfaces of Kähler rank one, see [8, 16]. Indeed, the condition (1) of Theorem 4.1 implies that the $(1,1)$-form $\alpha \wedge J\alpha$ is closed and positive in the sense of [8], thus by [8, Cor. 4.3], forcing the Kähler rank of S be equal to 1. We shall use a further ramification of the theory of [8] in the proof of Theorem 1.3 below, see in particular Lemma 4.7.

4.2. Proof of Theorem 1.3

By Theorem 4.1, we only need to show that if the condition (1) of Theorem 4.1 holds true, then S must be a blow-up of an Inoue–Bombieri surface.

As $b_1(S) = 1$, the torsion-free part $H_1(S,\mathbb{Z})^f$ of $H_1(S,\mathbb{Z})$ is $\mathbb{Z}$, so that S admits an infinite cyclic cover $\tilde{S}$ whose fundamental group is the kernel of the morphism $\pi_1(S) \to \big(\pi_1(S)/[\pi_1(S),\pi_1(S)]\big)^f$. Denote by γ the deck transformation on $\tilde{S}$, such that $S = \tilde{S}/\langle\gamma\rangle$. We then have the following reinterpretation of condition (1) of Theorem 4.1 in terms of the theory developed in [8].

Lemma 4.7. *Let S be a compact complex surface with $b_1(S) = 1$ and $\tilde{S}$ the infinite cyclic cover of S with $S = \tilde{S}/\langle\gamma\rangle$. Then, the condition* (1) *of Theorem 4.1 is equivalent to the existence of a positive pluriharmonic function $\tilde{h}$ on $\tilde{S}$, which is automorphic in the sense that $\tilde{h} \circ \gamma = C\tilde{h}$ for a positive constant C.*

Proof. Suppose first that α is a closed 1-form on S satisfying the condition (1) of Theorem 4.1. Let $\tilde{\alpha}$ be the pull back to $\tilde{S}$ of α. By the very construction of $\tilde{S}$, any closed 1-form on S pulls back to an exact 1-form on $\tilde{S}$. We thus can write $\tilde{\alpha} = d\tilde{f}$ for some smooth function $\tilde{f}$ on $\tilde{S}$. Using that $\tilde{\alpha}$ is invariant under the action of γ, it follows that $\tilde{f}$ satisfies

$$\tilde{f} \circ \gamma - \tilde{f} = c,$$

for some real constant c. It is easily seen that the relation $d_{-\tilde{\alpha}} d^c_{-\tilde{\alpha}}(1) = 0$ is equivalent to $dd^c(e^{\tilde{f}}) = 0$, i.e. $\tilde{h} := e^{\tilde{f}}$ is a positive pluriharmonic function on $\tilde{S}$ satisfying $\tilde{h} \circ \gamma = e^c \tilde{h}$.

Conversely, if $\tilde{h}$ is a positive pluriharmonic function on $\tilde{S}$ satisfying $\tilde{h} \circ \gamma = e^c\tilde{h}$, letting $\tilde{\alpha} := d\log\tilde{h}$ defines a closed, γ-invariant 1-form on $\tilde{S}$ which satisfies $d_{-\tilde{\alpha}}d^c_{-\tilde{\alpha}}(1) = 0$. It follows that $\tilde{\alpha}$ is the pull-back of a (non-zero) closed 1-form α on S, satisfying $d_{-\alpha}d^c_{-\alpha}(1) = 0$. As we noticed in the proof of Lemma 4.4, such a form α defines a nontrivial class $[\alpha] \neq 0 \in H^1_{dR}(S, \mathbb{R})$. □

Theorem 1.3 is thus a direct consequence of Theorem 4.1 and [6, Thm 1]. □

Combining Theorem 4.1 with some observations from [2] leads to

Corollary 4.8. *Let S be a compact complex surface with $b_1(S) = 1$ and S_0 its minimal model. Then,*

(a) *If S_0 is a secondary Kodaira surface or a Hopf surface, then $\mathcal{T}(S) = (-\infty, 0)$ where $H^1_{dR}(S, \mathbb{R}) \cong \mathbb{R}$ is oriented by the sign of the degree with respect to any Gauduchon metric on S of the flat holomorphic line bundle $\mathcal{L}_a$ associated to $a \in H^1_{dR}(S)$, see [2].*

(b) *If S_0 is an Inoue–Bombieri surface, then $\mathcal{T}(S) = \{a\}$.*

(c) *If S_0 belongs to VII_0^+, then $\mathcal{T}(S) \subset \mathbb{R}$ is a non-empty and open subset of $(-\infty, 0)$.*

Proof. (a) The pull-back by the blow-down map $b : S \to S_0$ defines an isomorphism $b^* : H^1_{dR}(S_0, \mathbb{R}) \to H^1_{dR}(S, \mathbb{R})$ (compare with Lemma 2.2(a)) which by [30, 33] (see also [2, Prop. 3.4]) embeds $\mathcal{T}(S_0)$ into $\mathcal{T}(S)$. According to [2, Prop. 4.3], $\mathcal{T}(S_0) \subset \mathcal{T}(S) \subset (-\infty, 0)$, so it is enough to show that $\mathcal{T}(S_0) = (-\infty, 0)$. In the case when S_0 is a Hopf surface this is established in [2, Prop. 5.1] and it follows from [31, Thm. 5.1], by using that any that Hopf surfaces can be obtained as a small deformations of a Vaisman Hopf surface (see [14]). If S_0 is a secondary Kodaira surface, it admits a Vaisman locally conformally Kähler metric by [4]. Thus, $\mathcal{T}(S_0) = (-\infty, 0)$ again follows from [31, Thm. 5.1] (or [2, Lemma 3.7] by noting that Vaisman metrics are pluricanonical).

The statement (b) is established in Lemma 4.4.

(c) See Theorem 4.1. □

As another application of Theorem 4.1, one can consider *exact* LCS structures, i.e. LCS structures for which $\omega = d_\alpha \eta = d\eta - \alpha \wedge \eta$ for some 1-form η. Theorem 1.2 provides the existence of exact LCS structures with arbitrary large Lee classes on any compact almost complex $2n$-manifold with non-zero $b_1(M)$. As an exact LCS structure does not admit symplectically embedded spheres (this is because by making a conformal modification of ω we can assume that $\alpha = 0$ on a tubular neighbourhood of the sphere) they cannot tame the complex structure of a complex surface with a rational curve, in particular of a *non-minimal* complex surface or a minimal surface in the class VII_0^+ which contains a cycle of rational curves. Similarly, as observed by A. Otiman [27]

Corollary 4.9. *[27] The Inoue–Bombieri surfaces admit no exact LCS structure taming its almost complex structure.*

Proof. If $\omega = d_\alpha \eta$ is an exact LCS structure which tames J, so is then $\tilde{\omega} = d_{\tilde{\alpha}}\eta$ for each closed 1-form $\tilde{\alpha}$ which is $\mathcal{C}^\infty$ close to α. It follows that α is an interior point of $\mathcal{T}(S)$, a contradiction. □

By contrast, on a secondary Kodaira surface and on a Hopf surface, for any $[\alpha] \in \mathcal{T}(S) = \mathcal{C}(S) = (-\infty, 0)$ (see the proof of Corollary 4.8) one can find a locally conformally Kähler structure with potential, i.e. of the form $\omega = d_\alpha d_\alpha^c f$ for some smooth function f, see [26]. This follows from the facts that, up to a finite covering, these surfaces can be obtained as small deformations of surfaces admitting a Vaisman locally conformally Kähler metric [4, 14, 32] and that the fundamental $(1,1)$-form ω of a Vaisman locally conformally Kähler structure can be written, up to scale, as $\omega = d_\alpha d_\alpha^c(1)$, see [26] or [2, (14)], noting that the positive-definiteness of $d_\alpha d_\alpha^c(1)$ is an open condition under small deformation of the complex structure [26]. Thus, on these minimal complex surfaces, any $[\alpha] \in \mathcal{T}(S)$ can be realized as the Lee class of an exact LCS structure which tames the complex structure.

4.3. Proof of Theorem 1.4

Let S be a compact complex surface with $b_1(S) = 1$ and S_0 its minimal model. Identifying $H^1_{dR}(S_0, \mathbb{R}) \cong H^1_{dR}(S, \mathbb{R}) \cong \mathbb{R}$ via the blow-down map $b : S \to S_0$, we have by [30, 33], $\mathcal{C}(S) = \mathcal{C}(S_0) \subset \mathbb{R}$. Thus, we can assume without loss that $S = S_0$ is minimal. We consider the following three cases, see Theorem 3.1.

Case 1: $S \in \mathrm{VII}_0^+$. We shall use a result of Goto [15, Thm. 2.3] (compare with Proposition 4.2 above).

Theorem 4.10. *[15] Let X be a compact complex manifold endowed with a locally conformally Kähler form ω with Lee form α. Suppose that $H^3(X, L_\alpha^*) = \{0\}$ and that for every $\bar{\partial}_\alpha$-closed $(0,2)$-form ψ there exists a $(0,1)$-form γ such that $\partial_\alpha \psi = \partial_\alpha \bar{\partial}_\alpha \gamma$. Then, for any closed 1-form β, there exists an $\varepsilon > 0$ such that for any $0 < |t| < \varepsilon$, X has a locally conformally Kähler form ω_t with Lee form $\alpha + t\beta$.*

As $S \in \mathrm{VII}_0^+$, we know by Lemma 3.2 that $H^{0,2}_{\bar\partial_\alpha}(S,\mathbb{C}) \cong H^2(S,\mathcal{L}^*_\alpha) = \{0\}$, which shows that the second necessary condition of Theorem 4.10 for the class $[\alpha] \in \mathcal{C}(S)$ to be an interior point is aways satisfied. In other words, $[\alpha] \in \mathcal{C}(S)$ is an interior point for $\mathcal{C}(S)$ provided that $H^3_{d_{L^*}}(S,L^*) \cong H^1_{d_L}(S,L) = \{0\}$ where $L = L_\alpha$ (see (7)). By Theorem 4.1, the latter condition fails if and only if $\mathcal{C}(S) \subset \mathcal{T}(S) = \{[\alpha]\}$, showing that then $\mathcal{C}(S)$ is either empty or a single point.

Case 2: S is a Hopf surface or a secondary Kodaira surface. Then $\mathcal{C}(S) = \mathcal{T}(S) = (-\infty, 0) \subset \mathbb{R}$ by the arguments in the proof of Corollary 4.8(a).

Case 3: S is an Inoue–Bombieri surface. In this case, according to Lemma 4.4, $\mathcal{C}(S) \subset \mathcal{T}(\mathcal{S}) = \{a\}$ is either empty or a point. Both cases do appear, as noticed in [4]. □

5. Vanishing results for twisted cohomologies on class VII surfaces.

5.1. Flat versus topologically trivial holomorphic line bundles

Most of the theory developed in Section 2 for a *real* closed 1-form α generalizes *mutatis mutandis* to the case when α is a closed complex-valued 1-form: We can associate to such an α the deRham complex $d_\alpha : \mathcal{E}^k(X,\mathbb{C}) \mapsto \mathcal{E}^{k+1}(X,\mathbb{C})$ with cohomology groups $H^k_\alpha(X,\mathbb{C})$ and the Dolbeault complexes with respect to the operator $\bar\partial_\alpha = \bar\partial - \alpha^{0,1}\wedge$, which give rise to cohomology groups $H^{k,0}_{\bar\partial_\alpha}(X,\mathbb{C})$. Furthermore, α induces a holomorphic structure on $\mathcal{L}_\alpha = \mathbb{C} \times X$, given by $\bar\partial_\alpha s = (ds)^{0,1} + \alpha^{0,1} \otimes s$ and we have the identification

$$H^0(X, \Omega^k \otimes \mathcal{L}_\alpha) \cong H^{k,0}_{\bar\partial_{-\alpha}}(X,\mathbb{C}).$$

The conclusion of Lemma 2.2(b) holds true as well in this more general context.

Recall that equivalence classes of flat complex line bundles are classified by elements of $H^1(X,\mathbb{C}^*)$. By the short exact sequences

$$\{0\} \to \mathbb{Z} \hookrightarrow \mathbb{C} \overset{\exp 2\pi i\cdot}{\longrightarrow} \mathbb{C}^* \to \{1\}$$
$$\{0\} \to \mathbb{Z} \hookrightarrow \mathcal{O} \overset{\exp 2\pi i\cdot}{\longrightarrow} \mathcal{O}^* \to \{1\}$$

we obtain the commutative digram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathbb{C}) & \longrightarrow & H^1(X,\mathbb{C}^*) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) & \longrightarrow \cdots \\ & \| & & \downarrow j & & \downarrow k & & \| & \\ 0 \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}) & \longrightarrow & H^1(X,\mathcal{O}^*) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) & \longrightarrow \cdots \end{array}$$

The first line shows that for any *topologically trivial* flat holomorphic line bundle $\mathcal{L} \in H^1(X,\mathbb{C}^*)$, there exists a closed complex-valued α with $\mathcal{L} = \mathcal{L}_\alpha$. Furthermore, if j is injective, then k is injective too whereas if j is an isomorphism, then k is an isomorphism between the group $H^1_0(X,\mathbb{C}^*)$ of equivalent classes of topologically trivial flat holomorphic line bundles and the group $\mathrm{Pic}^0(X)$ of equivalence classes of topologically trivial holomorphic line bundles.

In this section, we shall apply this construction in the special case of a compact complex surface S with first Betti number $b_1(S) = 1$. It is well-known (see e.g. [19, (14)]) that on a compact complex surface S, the morphism j in the above diagram is always surjective and is an isomorphism iff $H^{1,0}(S,\mathbb{C}) = \{0\}$. As the latter property holds true for a complex surface with $b_1(S) = 1$ (see [19, Thm. 3]), we have the following well-known (see e.g. [22])

Lemma 5.1. *On a compact complex surface S with $b_1(S) = 1$,*

$$H^1_0(S,\mathbb{C}^*) \cong \mathrm{Pic}^0(S).$$

In particular, for any holomorphic line bundle $\mathcal{L} \in \mathrm{Pic}^0(S)$ there exists a closed complex-valued 1-form α such that $\mathcal{L} = \mathcal{L}_\alpha$.

5.2. A characterization of Enoki surfaces

An important class of examples of minimal complex surfaces in the class VII_0^+, called *Enoki surfaces*, was introduced and studied by Enoki in [11]. We shall use here the following characterization of such surfaces

Theorem 5.2. *[11]. A minimal compact complex surface in the class VII_0^+ is an Enoki surface if and only if there exits a non-trivial holomorphic line bundle $\mathcal{L} \in \mathrm{Pic}^0(S)$ which admits a meromorphic section f. In this case, the divisor defined by (f) is mD for some $m \in \mathbb{Z}$, where D is the unique cycle of rational curves of S.*

By virtue of Lemma 2.2(b) and the remarks at the beginning of Section 5.1, it follows that

Corollary 5.3. *A compact complex surface S whose minimal model is in class VII_0^+ is a blow-up of an Enoki surface if and only if S admits a non-trivial holomorphic line bundle $\mathcal{L} \in \mathrm{Pic}^0(S)$ with $H^0(S, \mathcal{L}) \neq \{0\}$.*

We recall the following general observation

Lemma 5.4. *Let S be a compact complex surface whose minimal model is in the class VII_0^+. Then, for any topologically trivial holomorphic line bundle $\mathcal{L}$*

$$\dim_{\mathbb{C}} H^0(S, \Omega^1 \otimes \mathcal{L}) \leq 1.$$

Proof. By Lemma 2.2(b) and the remarks at the beginning of Section 5.1 (and since the fundamental group does not change under blow-down), we can assume without loss of generality that $S = S_0$ is a *minimal* complex surface in the class VII_0^+. Let $\beta_i \in H^0(S, \Omega^1 \otimes \mathcal{L})$, $i = 1, 2$, be two non-trivial holomorphic 1-forms with values in $\mathcal{L}$. It is clear (for instance by thinking of β_i as smooth $(1,0)$-forms satisfying $\bar{\partial}_{-\alpha}\beta_i = 0$, see (6) and Section 5.1, and Lemma 5.1) that $\beta_1 \wedge \beta_2 \in H^0(S, \Omega^2 \otimes \mathcal{L}^2) = H^0(S, K_S \otimes \mathcal{L}^2)$. By Lemma 3.2, $\beta_1 \wedge \beta_2 \equiv 0$. Letting $A \subset S$ be the vanishing locus of β_1, we thus have on $S \setminus A$,

$$\beta_2 = f\beta_1,$$

where f is a holomorphic function defined on $S \setminus A$. We claim that f extends as a meromorphic function over A, i.e. on S.

Let $D_{\max}$ be the *maximal divisor* of S (see e.g. [23]). As A is an analytic subset of S, it is composed of curves contained in $D_{\max}$, and of isolated points. By Hartogs' extension theorem, f extends holomorphically over the isolated points of A, so we consider a point $p \in A$ which belongs to an irreducible component D_0 of $D_{\max}$ with $D_0 \subset A$. Let U be an open neighbourhood of p over which both vector bundles Ω^1 and $\mathcal{L}$ trivialize. Since $\mathbb{C}\{z_1, z_2\}$ is a factorial ring, we can write (with respect to holomorphic coordinates $z = (z_1, z_2)$ on U)

$$\beta_i = \mu_i(z)(a_i(z)dz_1 + b_i(z)dz_2), i = 1, 2,$$

where $\mu_i(z), a_i(z), b_i(z)$ are holomorphic functions such that the codimension of the vanishing locus $Z(a_i, b_i)$ of a_i and b_i is 2. Thus, $Z(a_1, b_1)$ consists of isolated points in U. Avoiding theses points, at least one of the coefficients

a_1, b_1 does not vanish at p, say $a_1(p) \neq 0$. We then have (in a neighbourhood of p)

$$\beta_1 \wedge \beta_2 = \mu_1(z)\mu_2(z)\big(a_1(z)b_2(z) - a_2(z)b_1(z)\big)dz_1 \wedge dz_2 \equiv 0$$

hence $a_1b_2 - a_2b_1 \equiv 0$, i.e. $\beta_2 = \frac{\mu_2 a_2}{\mu_1 a_1}\beta_1$ where $\frac{\mu_2 a_2}{\mu_1 a_1}$ is a meromorphic function on U which extends f over p. It thus follows that f extends meromorphically over U minus the isolated points $Z(a_1, b_1)$, hence on U (by Levi's extension theorem). Thus, f extends meromorphically on S. Since any meromorphic function on S is constant (see [19]), we conclude that

$$\dim_{\mathbb{C}} H^0(S, \Omega^1 \otimes \mathcal{L}) = 1.$$

□

Another feature of the Enoki surfaces is given by the following

Lemma 5.5. *Let S be an Enoki surface. Then, for any non-trivial holomorphic line bundle $\mathcal{L} \in \mathrm{Pic}^0(S)$*

$$\dim_{\mathbb{C}} H^0(S, \Omega^1 \otimes \mathcal{L}) = \dim_{\mathbb{C}} H^0(S, \mathcal{L}) \leq 1.$$

Moreover, the equality $\dim_{\mathbb{C}} H^0(S, \Omega^1 \otimes \mathcal{L}) = \dim_{\mathbb{C}} H^0(S, \mathcal{L}) = 1$ holds if and only if $\mathcal{L} = m[D],\ m \in \mathbb{N}^*$, where D is the cycle of rational curves of S

Proof. Since there is no non-trivial meromorphic functions, $\dim_{\mathbb{C}} H^0(S, \mathcal{L}) \leq 1$ for any line bundle $\mathcal{L} \in \mathrm{Pic}(S)$. In [11], the Enoki surfaces are obtained as compactifications of affine line bundles by a cycle $D = \sum_{i=0}^{n-1} C_i$ of $n = b_2(S)$ rational curves. Theorem 5.2 then shows that $\dim_{\mathbb{C}} H^0(S, \mathcal{L}) = 1$ if and only if $\mathcal{L} = m[D]$ with $m \in \mathbb{N}^*$.

By Lemma 5.4, we also have $\dim_{\mathbb{C}} H^0(S, \Omega^1 \otimes \mathcal{L}) \leq 1$ for any $\mathcal{L} \in \mathrm{Pic}^0(S)$. Enoki surfaces can be also described (see [9, Thm. 1.19]) by a polynomial germ of the form

$$F(z_1, z_2) = \left(z_1 z_2^n t^n + \sum_{i=0}^{n-1} \alpha_i t^{i+1} z_2^{i+1}, t z_2\right), \quad 0 < |t| < 1. \tag{14}$$

In terms of (14), the maximal divisor D has local equation $z_2 = 0$. It follows from (14) that $\frac{dz_2}{z_2}$ is a meromorphic $(1,0)$-form on S, which has a pole of order 1 along D, or equivalently, $\beta_0 := dz_2$ is a holomorphic $(1,0)$-form with values in $\mathcal{L}_0 = [D]$ on S, showing that $\dim_{\mathbb{C}} H^0(S, \Omega^1 \otimes \mathcal{L}_0^m) = \dim_{\mathbb{C}} H^0(S, \mathcal{L}_0^m) = 1$ for any $m \in \mathbb{N}^*$.

Let $\mathcal{L} \in \mathrm{Pic}^0(S)$ be such that $\dim_{\mathbb{C}} H^0(S, \Omega^1 \otimes \mathcal{L}) = 1$, and

$$\beta \neq 0 \in H^0(S, \Omega^1 \otimes \mathcal{L}).$$

A similar argument as the one used in the proof of Lemma 5.4 shows that there is a meromorphic section f of $\mathcal{L} \otimes \mathcal{L}_0^{-1}$, such that $\beta = f\beta_0$. It thus follows from Theorem 5.2 that $(f) = pD$, $p \in \mathbb{Z}$. Since β is a holomorphic form (and β_0 does not vanish along D) we conclude that $p \in \mathbb{N}$, so that $\mathcal{L} = [pD] \otimes [D] = [mD]$ with $m = p + 1 \in \mathbb{N}^\star$, i.e. $\dim_{\mathbb{C}} H^0(S, \mathcal{L}) = 1$. □

Our main objective, which will occupy the remainder of the section, is establishing the following partial converse of Lemma 5.5, which characterizes Enoki surfaces by the existence of a special type of singular holomorphic foliation:

Theorem 5.6. *Let S be a compact complex surface whose minimal model S_0 is in class VII_0^+, and whose fundamental group is isomorphic to $\mathbb{Z}$. Then, for any non-trivial holomorphic line bundle $\mathcal{L} \in \mathrm{Pic}^0(S)$*

$$\dim_{\mathbb{C}} H^0(S, \Omega^1 \otimes \mathcal{L}) = \dim_{\mathbb{C}} H^0(S, \mathcal{L}) \leq 1$$

with $\dim_{\mathbb{C}} H^0(S, \Omega^1 \otimes \mathcal{L}) = 1$ if and only if S_0 is an Enoki surface.

Recall that, by Lemma 5.1, on a class VII surface S any $\mathcal{L} \in \mathrm{Pic}^0(S)$ can be written as $\mathcal{L} = \mathcal{L}_\alpha$ for some closed complex-valued 1-form α. We then have

Lemma 5.7. *Let $\mathcal{L} = \mathcal{L}_\alpha \in \mathrm{Pic}^0(S)$ be a non-trivial holomorphic line bundle on a complex surface S with minimal model in the class VII_0^+. Then, the following isomorphisms hold true.*

(a) *If $H^0(S, \mathcal{L}) = 0$, then $H^0(S, \Omega^1 \otimes \mathcal{L}) \cong H^1_{-\alpha}(S, \mathbb{C})$.*

(b) *If $H^0(S, \mathcal{L}) \neq 0$, then $H^0(S, \mathcal{L}) \overset{d}{\cong} H^0(S, \Omega^1 \otimes \mathcal{L})$.*

Proof. (a) In view of the identification (6) (see also Section 5.1), we are going to construct an isomorphism $s : H^{1,0}_{\bar{\partial}_{-\alpha}}(S, \mathbb{C}) \to H^1_{-\alpha}(S, \mathbb{C})$. Let β be a $(1,0)$-form satisfying $\bar{\partial}_{-\alpha}\beta = 0$. Then, the $(2,0)$-form $\partial_{-\alpha}\beta$ satisfies $\bar{\partial}_{-\alpha}(\partial_{-\alpha}\beta) = 0$ and, therefore, $\partial_{-\alpha}\beta = 0$ since $H^{2,0}_{\bar{\partial}_{-\alpha}}(S, \mathbb{C}) \cong H^0(S, \Omega^2 \otimes \mathcal{L}) = H^0(S, K_S \otimes \mathcal{L}) = \{0\}$ by Lemma 3.2. As $d_{-\alpha}\beta = (\partial_{-\alpha} + \bar{\partial}_{-\alpha})(\beta) = 0$, we thus have a natural map

$$s: \begin{array}{ccc} H^{1,0}_{\bar{\partial}_{-\alpha}}(S,\mathbb{C}) & \to & H^1_{-\alpha}(S,\mathbb{C}) \\ \beta & \mapsto & [\beta]. \end{array}$$

It is easy to see that s is injective when $H^0(S,\mathcal{L}) = \{0\}$. Indeed, let β be a $\bar{\partial}_{-\alpha}$-closed $(1,0)$-form such that $[\beta] = 0$ in $H^1_{-\alpha}(S,\mathbb{C})$. This means that there exists a complex-valued smooth function φ on S with $\beta = d_{-\alpha}\varphi$. Considering bi-degree, it follows that $\bar{\partial}_{-\alpha}\varphi = 0$, i.e. φ defines a section in $H^0_{\bar{\partial}_{-\alpha}}(S,\mathbb{C}) \cong H^0(S,\mathcal{L})$, thus $\varphi = 0$ and $\beta = 0$.

To prove the surjectivity of s, let θ be a complex-valued $d_{-\alpha}$-closed 1-form on S with $[\theta] \neq 0 \in H^1_{-\alpha}(S,\mathbb{C})$. Then the $(0,1)$-part $\theta^{0,1}$ of θ satisfies $\bar{\partial}_{-\alpha}\theta^{0,1} = 0$, i.e. $\theta^{0,1}$ defines a class in $H^{0,1}_{\bar{\partial}_{-\alpha}}(S,\mathcal{L}) \cong H^1(S,\mathcal{L})$. Using $H^0(S,\mathcal{L}) = \{0\}$, Lemma 3.2 and Riemann–Roch, we have $H^1(S,\mathcal{L}) = \{0\}$ which shows that $\theta^{0,1} = \bar{\partial}_{-\alpha}\varphi$ for some complex-valued smooth function φ. Thus, the 1-form $\tilde{\theta} := \theta - d_{-\alpha}\varphi$ is another representative of $[\theta]$, which is of type $(1,0)$. It thus follows that $\tilde{\theta} \in H^{1,0}_{\bar{\partial}_{-\alpha}}(S,\mathbb{C})$ and $s(\tilde{\theta}) = [\tilde{\theta}] = [\theta]$.

(b) Using the identifications (6), it is enough to show that the natural morphism

$$d_{-\alpha} : H^0_{\bar{\partial}_{-\alpha}}(S,\mathbb{C}) \to H^{1,0}_{\bar{\partial}_{-\alpha}}(S,\mathbb{C})$$

is an isomorphism. For any smooth complex-valued function φ satisfying $\bar{\partial}_{-\alpha}\varphi = 0$, the operator $d_{-\alpha}$ associates the $(1,0)$ form $\beta := d_{-\alpha}\varphi$. As β is $d_{-\alpha}$-closed, it also satisfies $\bar{\partial}_{-\alpha}\beta = 0$. The map

$$d_{-\alpha} : H^0_{\bar{\partial}_{-\alpha}}(S,\mathbb{C}) \to H^{1,0}_{\bar{\partial}_{-\alpha}}(S,\mathbb{C})$$

is injective as $H^0_{-\alpha}(S,\mathbb{C}) = \{0\}$ for $[\alpha] \neq 0 \in H^1_{dR}(S,\mathbb{C})$, see (9). It is surjective because $H^{1,0}_{\bar{\partial}_{-\alpha}}(S,\mathbb{C}) \cong H^0(S,\Omega^1 \otimes \mathcal{L})$ must be 1-dimensional by Lemma 5.4. $\square$

Remark 5.8. The isomorphisms in Lemma 5.7 can be alternatively derived from the following exact sequences of sheaves

$$0 \to \mathbb{C}(\mathcal{L}) \to \mathcal{O}(\mathcal{L}) \xrightarrow{d} d\mathcal{O}(\mathcal{L}) \to 0, \tag{15}$$

$$0 \to d\mathcal{O}(\mathcal{L}) \to \Omega^1 \otimes \mathcal{L} \xrightarrow{d} \Omega^2 \otimes \mathcal{L} \to 0, \tag{16}$$

where, for a flat holomorphic line bundle $\mathcal{L} \in H^1(S,\mathbb{C}^*)$, $\mathbb{C}(\mathcal{L})$ denotes the sheaf of local parallel sections of $\mathcal{L}$, and d is the deRham differential defined on smooth forms with values in $\mathcal{L}$ by using the flat connection on $\mathcal{L}$. By the long exact sequence of cohomologies associated (16) and Lemma 3.2 we deduce an isomorphism

$$i : H^0(S, d\mathcal{O}(\mathcal{L})) \cong H^0(S, \Omega^1 \otimes \mathcal{L}).$$

The two isomorphisms appearing in Lemma 5.7 are then the natural maps $s = i \circ \delta$ and d in the long cohomology sequence associated to (15)

$$\begin{array}{ccccccccc} 0 \longrightarrow H^0(S, \mathbb{C}(\mathcal{L})) & \longrightarrow & H^0(S, \mathcal{L}) & \xrightarrow{d} & H^0(S, d\mathcal{O}(\mathcal{L})) & \xrightarrow{\delta} & H^1(S, \mathbb{C}(\mathcal{L})) & \longrightarrow & H^1(S, \mathcal{L}) \\ & & & & {\scriptstyle i}\Big\| & & & & \\ & & & & H^0(S, \Omega^1 \otimes \mathcal{L}) & & & & \end{array}$$

One then deduces (a) from the vanishing of $H^0(S, \mathcal{L})$ and $H^1(S, \mathcal{L})$ (using Lemma 3.2 and Riemann-Roch) and the fact that if $\mathcal{L} = \mathcal{L}_\alpha$ for a closed complex-valued 1-form α, the deRham-Weil theorem gives

$$H^k(S, \mathbb{C}(\mathcal{L})) \cong H^k_{-\alpha}(S, \mathbb{C}).$$

Similarly, using that $H^0(S, \mathbb{C}(\mathcal{L})) = \{0\}$ for a non-trivial holomorphic line bundle $\mathcal{L}$ (as a parallel section of $\mathcal{L}$ is either identically zero or never vanishes), (b) follows from the injectivity of d and Lemma 5.4. $\square$

Proof of Theorem 5.6. Using Lemma 2.2(b), we can assume without loss of generality that $S = S_0$ is a *minimal* complex surface, i.e. it is in the class VII_0^+. As we have already observed, then S does not admit non-constant meromorphic functions, thus for any $\mathcal{L} \in \mathrm{Pic}^0(S)$ we have $\dim_{\mathbb{C}} H^0(S, \mathcal{L}) = 0, 1$.

Case 1: There exists a non-trivial $\mathcal{L} \in \mathrm{Pic}^0(S)$, such that $\dim_{\mathbb{C}} H^0(S, \mathcal{L}) = 1$. By Theorem 5.2, S is an Enoki surface so that Theorem 5.6 follows from Lemma 5.5.

Case 2: For any non-trivial $\mathcal{L} \in \mathrm{Pic}^0(S)$, $\dim_{\mathbb{C}} H^0(S, \mathcal{L}) = 0$. We thus need to show that in this case $H^0(S, \Omega^1 \otimes \mathcal{L}) = \{0\}$ for any non-trivial $\mathcal{L} \in \mathrm{Pic}^0(S)$.

Suppose for contradiction that $H^0(S, \Omega^1 \otimes \mathcal{L}) \neq \{0\}$. Let α be a closed complex valued 1-form α such that $\mathcal{L} = \mathcal{L}_\alpha$, so we have $H^0(S, \Omega^1 \otimes \mathcal{L}) \cong H^{1,0}_{\bar\partial_{-\alpha}}(S, \mathbb{C})$, see Sect. 2. Let $\beta \neq 0$ be a $(1,0)$-form on S such that $\bar\partial_{-\alpha}\beta = 0$. By Lemma 5.7, β satisfies $d_{-\alpha}\beta = 0$. Let $p : \tilde S \to S$ be the universal covering space of S, $\tilde\alpha = p^*\alpha$, $\tilde\beta = p^*\beta$ and $\tilde\Phi : \tilde S \to \tilde S$ a biholomorphism such that $S = \tilde S/\Gamma$ where $\Gamma \cong \mathbb{Z}$ is the infinite cyclic fundamental group of S generated by $\tilde{\Phi}$. The forms $\tilde{\alpha}$ and $\tilde{\beta}$ are invariant by the action of $\tilde{\Phi}$ as they are pull-backs of forms on S. As $\tilde{\alpha}$ is closed on $\tilde{S}$ (and $\tilde{S}$ is simply connected), there exists a complex-valued function $\tilde{f}$ on $\tilde{S}$, such that $\tilde{\alpha} = d\tilde{f}$. The invariance of $\tilde{\alpha}$ under $\tilde{\Phi}$ then means $d(\tilde{f} \circ \tilde{\Phi} - \tilde{f}) = 0$, so there exists a constant $C \in \mathbb{C}$ such that

$$\tilde{f} \circ \tilde{\Phi} - \tilde{f} = C. \tag{17}$$

Notice that the constant C cannot be zero as then f would be Γ-invariant and would descend to S to define a primitive of α which contradicts the assumption that $\mathcal{L} = \mathcal{L}_\alpha \neq \mathcal{O}$.

As $d(e^{\tilde{f}}\tilde{\beta}) = e^{\tilde{f}}(d_{-\tilde{\alpha}}\tilde{\beta}) = 0$, the $(1,0)$-form $e^{\tilde{f}}\tilde{\beta}$ is closed and therefore exact on $\tilde{S}$. Thus, there exists a complex-valued smooth function $\tilde{g}$ on $\tilde{S}$, such that $e^{\tilde{f}}\tilde{\beta} = d\tilde{g}$. Considering bi-degree, $\bar{\partial}\tilde{g} = 0$ i.e. $\tilde{g}$ is holomorphic. Using that $\tilde{\beta}$ is $\tilde{\Phi}$ invariant and (17), it follows that

$$\tilde{\Phi}^* d\tilde{g} = e^C d\tilde{g},$$

i.e. there exists a constant $K \in \mathbb{C}$ such that

$$\tilde{g} \circ \tilde{\Phi} = e^C \tilde{g} + K.$$

Setting $\tilde{h} := \tilde{g} + \frac{K}{e^C - 1}$, we obtain a non-zero holomorphic function on $\tilde{S}$ satisfying $\tilde{\Phi}^* \tilde{h} = e^C \tilde{h}$. Thus, $h := e^{-\tilde{f}}\tilde{h}$ is a smooth complex-valued function on $\tilde{S}$ which is Γ-invariant and satisfies $\bar{\partial}_{-\tilde{\alpha}} h = 0$. It follows that h descends to S to define a non-zero section of $H^0(S, \mathcal{L})$, a contradiction. □

Remark 5.9. There are no known examples of class VII_0^+ surfaces whose fundamental group is not isomorphic to $\mathbb{Z}$. In general, as the first Betti number of any class VII_0^+ surface S is equal to 1, S admits a unique infinite cyclic cover $\tilde{S}$. The arguments in the proof of Theorem 5.6 extend under the (a priori weaker) assumption $H^1(\tilde{S}, \mathbb{R}) = \{0\}$. □

5.3. Proof of Theorem 1.5

We recall the following vanishing result obtained in [20].

Lemma 5.10. *[20] Let S be a compact complex surface diffeomorphic to $(S^1 \times S^3)\sharp n\overline{\mathbb{C}P^2}$, $n \in \mathbb{N}^*$, and $L = L_\alpha$, $[\alpha] \neq 0 \in H^1_{dR}(S, \mathbb{R})$ a non-trivial flat real line bundle. Then for $k \neq 2$, $H^k_{d_L}(S, L) = 0$.*

Proof. As $H^k_{d_L}(S,L)$ depend only upon the smooth structure of S, we can assume without loss that S is a complex surface obtained by a diagonal Hopf surface S_0 by blowing up n points. By (9) and Lemma 2.2, it is enough to consider the case $n=0$, i.e. $S \cong S^1 \times S^3$. It is well-known (see e.g. [32]) that this smooth manifold admits a complex structure and a compatible Vaisman product metric with a parallel Lee form α_0. Applying [20, Thm 4.5], we conclude that the cohomology $H^k_{t\alpha_0}(S,\mathbb{R})$ vanishes for each $t \neq 0$. As $b_1(S) = 1$, it follows that there exists $t \neq 0$ such that $H^k_{d_L}(S,L) \cong H^k_{t\alpha_0}(S,\mathbb{R}) = \{0\}$. □

We then have

Lemma 5.11. *Let S be a compact complex surface whose minimal model is in the class* VII_0^+*. If there exists a non-trivial flat real line bundle* $L = L_\alpha, [\alpha] \neq 0 \in H^1_{dR}(S,\mathbb{R})$*, such that* $H^1_{d_L}(S,L) \neq \{0\}$*, then* $H^0(S,\Omega^1 \otimes \mathcal{L}) \neq \{0\}$.

Proof. Let $\mathcal{L} = L_\alpha \otimes \mathbb{C}$ be the corresponding flat holomorphic line bundle. We first show that $H^0(S,\mathcal{L}) = \{0\}$. Indeed, if $H^0(S,\mathcal{L}) \neq \{0\}$, then by Corollary 5.3, S must be obtained by blowing up an Enoki surface, and thus S must be diffeomorphic to $(S^1 \times S^3)\sharp n\overline{\mathbb{C}P^2}, n \in \mathbb{N}^*$ (see e.g. [25]). This contradicts $H^1_{d_L}(S,L) \neq \{0\}$ (according to Lemma 5.10).

Thus, $H^0(S,\mathcal{L}) = \{0\}$ and by Lemma 5.7 and (5) we have

$$\begin{aligned}\dim_{\mathbb{R}} H^1_{d_L}(S,L) &= \dim_{\mathbb{R}} H^1_{-\alpha}(S,\mathbb{R}) \\ &= \dim_{\mathbb{C}} H^1_{-\alpha}(S,\mathbb{C}) \\ &= \dim_{\mathbb{C}} H^0(S,\Omega^1 \otimes \mathcal{L}).\end{aligned}$$

□

We can now prove Theorem 1.5 (which in turn generalizes Lemma 5.10): As noticed in [12], by (7), (9), and (8), it is enough to show $H^1_{d_L}(S,L) = \{0\}$. By Lemma 2.2, we can assume that S is minimal whereas by Theorem 3.1, Lemma 4.3, and the fact that the fundamental group of Inoue–Bombieri surfaces is not isomorphic to $\mathbb{Z}$, we can also assume that S is in the class VII_0^+.

If $H^1_{d_L}(S,L) \neq \{0\}$, by Lemma 5.11 we will have $H^0(S,\Omega^1 \otimes \mathcal{L}) \neq \{0\}$ whereas Theorem 5.6 implies that S must be an Enoki surface, and therefore S must be diffeomorphic to $(S^1 \times S^3)\sharp\, n\overline{\mathbb{C}P^2}$ (see e.g. [25]). According to Lemma 5.10, this contradicts the assumption $H^1_{d_L}(S,L) \neq \{0\}$. □

References

[1] D. Angella and H. Kasuya, *Hodge theory for twisted differentials*, Complex Manifolds **1** (2014), 64–85.

[2] V. Apostolov and G. Dloussky, *Locally Conformally Symplectic Structures on Compact Non-Kähler Complex Surfaces*, Int. Math. Res. Not. IMRN **9** (2016), 2717–2747.

[3] W. Barth, K. Hulek, C. Peters, and A. Van de Ven, Compact Complex Surfaces, Springer, Heidelberg, second Edition, 2004.

[4] F. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), 1–40.

[5] F. Bogomolov, *Classification of surfaces of class* VII_0 *with* $b_2 = 0$, Math. USSR Izv **10** (1976), 255–269.

[6] M. Brunella, *A characterization of Inoue surfaces*, Comment. Math. Helv. **88** (2013), 859–874.

[7] B. Chantraine and E. Murphy, *Conformal symplectic geometry of cotangent bundles*, `arXiv:1606.00861`, to appear in J. Sympl. Geom.

[8] I. Chiose and M. Toma, *On compact complex surfaces of Kähler rank one*, Amer. J. Math. **135** (2013), 851–860.

[9] G. Dloussky and F. Kohler *Classification of singular germs of mappings and deformations of compact complex surfaces of class* VII_0, Annales Polonici Mathematici **LXX** (1998), 49–83.

[10] Y. Eliashberg and E. Murphy, *Making cobordism symplectic*, `arXiv:1504.06312`.

[11] I. Enoki, *Surfaces of class* VII_0 *with curves*, Tôhuku Math. J. **33** (1981), 453–492.

[12] A. Fujiki and M. Pontecorvo, *Bihermitian metrics on Kato surfaces*, `arXiv:1607.00192`.

[13] P. Gauduchon, *La* 1*-forme de torsion d'une variété hermitienne*, Math. Ann. **267** (1984), 495–518.

[14] P. Gauduchon and L. Ornea, *Locally conformally Kähler metrics on Hopf surfaces*, Ann. Inst. Fourier (Grenoble), **48** (1998), 1107–1127.

[15] R. Goto, *On the stability of locally conformal Kähler structures*, J. Math. Soc. Japan **66** (2014), 1375–1401.

[16] R. Harvey and H. Blaine Lawson, Jr., *An intrinsic characterization of Kähler manifolds*, Invent. Math. **74** (1983), 169–198.

[17] M. Inoue, *On surfaces of class* VII_0, Invent. Math. **24** (1974), 269–310.

[18] M. Kato, *Topology of Hopf surfaces*, J. Math. Soc. Japan **27** (1975), 222–238. *Erratum*, J. Math. Soc. Japan **41** (1989), 173–174.

[19] K. Kodaira, *On the structure of complex analytic surfaces I, II, III*, Am. J. Math **86** (1966), 751–798; Am. J. Math. **88** (1966), 682–721; Am. J. Math. **90** (1968), 55–83.

[20] M. de Léon, B. López, J. C. Marrero, and E. Padrón, *On the computation of the Lichnerowicz–Jacobi cohomology*, J. Geom. Phys. **44** (2003), 507–522.

[21] J. Li and S. T. Yau, *Hermitian Yang–Mills connections on non-Kähler manifolds*, Math. aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys. **1** (1987), 560–573.

[22] M. Lübke and A. Teleman, The Kobayashi–Hitchin Correspondence, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[23] I. Nakamura, *On surfaces of class* VII_0 *with curves*, Invent. Math. **78** (1984), 393–343.

[24] I. Nakamura, *Classification of non-Kähler complex surfaces* (Japanese), translated in Sugaku Expositions **2** (1989) 209–229. Sugaku **36** (1984), no. 2, 110–124.

[25] I. Nakamura, *On surfaces of class* VII_0 *with curves, II*, Tohoku J. Math. **42** (1990), 475–516.

[26] L. Ornea and M. Verbitsky, *LCK rank of locally conformally Kahler manifolds with potential*, J. Geom. Phys. **107** (2016), 92–98.

[27] A. Otiman, *Morse-Novikov cohomology of locally conformally Kähler surfaces*, Math. Zeit. **289** (2018), no. 1-2, 605–628.

[28] C. T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 5–95.

[29] A. Teleman, *Projectively flat surfaces and Bogomolov's theorem on class* VII_0*-surfaces*, Int. J. Math. **5** (1994), 253–264.

[30] F. Tricerri, *Some examples of locally conformal Kähler manifolds*, Rend. Sem. Mat. Univers. Politecn. Torino **40** (1982), 81–92.

[31] K. Tsukada, *Holomorphic forms and holomorphic vector fields on compact generalized Hopf manifolds*, Compositio Math. **93** (1994), 1–22.

[32] I. Vaisman, *Generalized Hopf manifolds*, Geom. Dedicata **13** (1982), 231–255.

[33] V. Vuletescu, *Blowing-up points on locally conformally Kähler manifolds*, Bull. Math. Soc. Sci. Math. Roumanie **52** (2009), 387–390.

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