

Hamiltonian loops on the symplectic one-point blow up

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We lift a loop of Hamiltonian diffeomorphisms on a symplectic manifold to loop of Hamiltonian diffeomorphisms on the symplectic one-point blow up of the symplectic manifold. Then we use Weinstein's morphism to show that the lifted loop of Hamiltonian diffeomorphisms has infinite order on the fundamental group of the group of Hamiltonian diffeomorphisms of the blown up manifold.

1. Introduction

The rational homotopy type of the group of Hamiltonian diffeomorphisms of the symplectic one-point blow up $(\widetilde{M}, \widetilde{\omega}_\rho)$ of weight ρ is known for only a special class of symplectic manifolds. In [1], M. Abreu and D. McDuff computed the rational homotopy type of the group of symplectic diffeomorphisms of the symplectic one-point blow up of $(\mathbb{C}P^2, \omega_{FS})$. In [4], F. Lalonde and M. Pinsonnault computed the rational homotopy type of the above group for the one-point blow up of $(S^2 \times S^2, \omega \oplus \mu\omega)$ for $1 \leq \mu \leq 2$; and in [9] M. Pinsonnault worked out the case of the one-point blow up of rational ruled symplectic 4-manifolds; see also [2]. The case of multiple points blown up simultaneously has also been considered. J.D. Evans [3] considered the group of symplectic diffeomorphisms that act trivially on homology for the case of $(\mathbb{C}P^2, \omega_{FS})$ blown up at 3, 4 and 5 generic points. The reason that all the above examples are in dimension 4, has to do with the special behavior of holomorphic curves in 4 dimensional symplectic manifolds. Apart from these cases, only partial information is known about the homotopy type of $\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho)$. For example in [5], D. McDuff showed that if the Hurewicz morphism $\pi_2(M) \rightarrow H_2(M; \mathbb{Q})$ is non trivial then there exists a non trivial morphism $\pi_2(M) \rightarrow \pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$.

In this paper we will focus on determining that $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$ is non trivial for some particular class of symplectic manifolds (M, ω) . Moreover, the way that we show that $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$ is non trivial is by considering a particular class of loops of Hamiltonian diffeomorphisms in $\text{Ham}(M, \omega)$,

lift it to a loop in $\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho)$ and then use Weinstein's morphism to show that is not null homotopic. What is surprising is that in some cases a non constant null homotopic loop in $\text{Ham}(M, \omega)$ lifts to a loop that is not null homotopic in $\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho)$.

To be more precise about our statements, fix a base point $x_0 \in M$ and a symplectic embedding $\iota : (B_\rho, \omega_0) \rightarrow (M, \omega)$ of the closed ball of radius ρ in $(\mathbb{R}^{2n}, \omega_0)$ such that $\iota(0) = x_0$. Relative to the embedding ι we have the symplectic one-point blow up $(\widetilde{M}, \widetilde{\omega}_\rho)$ at x_0 of weight ρ and the blow up map $\pi : \widetilde{M} \rightarrow M$. In Section 2 we review the construction of the symplectic one-point blow up. Denote by \mathcal{H}_ρ^U the subgroup of Hamiltonian diffeomorphisms ψ of (M, ω) such that

- a) $\psi(x_0) = x_0$, and
- b) ψ acts in a $U(n)$ -way in a neighborhood of ιB_ρ .

(When we say that ψ behaves in a $U(n)$ -way, we mean with respect to the coordinates induced by the symplectic embedding.) Let $\mathcal{H}_{\rho,0}^U$ be the connected component of \mathcal{H}_ρ^U that contains the identity map and $\Phi_\rho : \mathcal{H}_\rho^U \rightarrow \text{Ham}(M, \omega)$ the inclusion morphism. It is well known that a diffeomorphism ψ that fixes the base point x_0 and behaves in a $U(n)$ -way near ιB_ρ induces a unique diffeomorphism $\widetilde{\psi}$ of the one-point blow up \widetilde{M} such that $\pi \circ \widetilde{\psi} = \pi \circ \psi$. In this case we say that $\widetilde{\psi}$ *lifts* ψ . Now consider the symplectic structure in the process of lifting diffeomorphisms; in Section 3 we show that $\widetilde{\psi}$ is symplectic if ψ is symplectic; moreover if $\psi \in \mathcal{H}_{\rho,0}^U$ then $\widetilde{\psi}$ is a Hamiltonian diffeomorphism of $(\widetilde{M}, \widetilde{\omega}_\rho)$. This gives rise to a group morphism $\Psi_\rho : \mathcal{H}_{\rho,0}^U \rightarrow \text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho)$ that consist of lifting a Hamiltonian ψ of (M, ω) to a Hamiltonian $\widetilde{\psi}$ of $(\widetilde{M}, \widetilde{\omega}_\rho)$. Notice that elements in the image of Φ_ρ are the ones that lift to Hamiltonian diffeomorphisms of $(\widetilde{M}, \widetilde{\omega}_\rho)$ via the morphism Ψ_ρ . The map Ψ_ρ is known to be a homotopy equivalence in some cases [4]; for example in the case $(S^2 \times S^2, \omega \oplus \mu\omega)$ for $\mu \geq 1$ and $0 < \rho < 1$. Indeed, this is part of the argument that F. Lalonde and M. Pinsonnault use in the computation of the rational homotopy type of the group of Hamiltonian diffeomorphisms of the one-point blow up of $(S^2 \times S^2, \omega \oplus \mu\omega)$.

Now consider the induced maps $\Phi_{\rho,*}$ and $\Psi_{\rho,*}$ on fundamental groups. Thus we are interested in the image of $\Phi_{\rho,*} : \pi_1(\mathcal{H}_{\rho,0}^U) \rightarrow \pi_1(\text{Ham}(M, \omega))$. Contrary to the case when the lift of a single Hamiltonian diffeomorphism is unique, the lift of $\psi = [\{\psi_t\}] \in \text{Im}(\Phi_{\rho,*})$ to $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$ is not unique. The way to single out one element in $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$ when ψ is in the image of $\Phi_{\rho,*}$ is by fixing a representative $\{\psi_t\}$ of ψ in $\mathcal{H}_{\rho,0}^U$. In other words the map

Φ_ρ is injective, but the map $\Phi_{\rho,*}$ is not necessarily injective. Once we fixed a representative, by Proposition 3.7 we obtain a loop $\{\tilde{\psi}_t\}$ in $\text{Ham}(\widetilde{M}, \tilde{\omega}_\rho)$ and define the lifted element as $\tilde{\psi} := [\{\tilde{\psi}_t\}]$.

$$\begin{array}{ccc}
 \pi_1(\mathcal{H}_{\rho,0}^U) & \xrightarrow{\Phi_{\rho,*}} & \pi_1(\Phi_\rho(\mathcal{H}_{\rho,0}^U)) \subset \pi_1(\text{Ham}(M, \omega)) \\
 \Psi_{\rho,*} \downarrow & & \swarrow \\
 \pi_1(\text{Ham}(\widetilde{M}, \tilde{\omega}_\rho)) & &
 \end{array}$$

The argument we use to show that a Hamiltonian loop is not null homotopic is by using Weinstein’s morphism [11],

$$\mathcal{A} : \pi_1(\text{Ham}(M, \omega)) \rightarrow \mathbb{R}/\mathcal{P}(M, \omega).$$

Here $\mathcal{P}(M, \omega)$ is the period group of (M, ω) . In Section 4 we review Weinstein’s morphism.

Throughout, we have a fixed symplectic embedding $\iota : (B_\rho, \omega_0) \rightarrow (M, \omega)$ such that $\iota(0) = x_0$. This embedding is used to define the symplectic one-point blow up of (M, ω) at x_0 as a coordinate chart about x_0 and to define the lift of Hamiltonian diffeomorphisms.

Theorem 1.1. *Let (M, ω) be a closed symplectic manifold and ψ an element in $\pi_1(\text{Ham}(M, \omega))$ such that $\Phi_{\rho,*}([\{\psi_t\}]) = \psi$, where the loop $\{\psi_t\}$ is given by the normalized Hamiltonian H_t . Then for $\tilde{\psi} = \Psi_{\rho,*}([\{\psi_t\}])$ in $\pi_1(\text{Ham}(\widetilde{M}, \tilde{\omega}_\rho))$ we have*

$$(1) \quad \mathcal{A}(\tilde{\psi}) = \left[\mathcal{A}(\psi) + \frac{1}{\text{Vol}(\widetilde{M}, \tilde{\omega}_\rho^n)} \int_0^1 \int_{\iota B_\rho} H_t \omega^n dt \right]$$

in $\mathbb{R}/\mathcal{P}(\widetilde{M}, \tilde{\omega}_\rho)$.

There are two things to notice about expression (1) of $\mathcal{A}(\tilde{\psi})$. The second term on the right hand side depends on local information of ψ about x_0 ; and it also reflects the choice of the representative of ψ in order to lift it to $\tilde{\psi}$, namely the Hamiltonian function H_t .

In the special case when the normalized Hamiltonian function H_t of the loop ψ takes the form

$$(2) \quad H_t(z_1, \dots, z_n) := -\pi \sum_{j=1}^n m_j |z_j|^2 + c_t$$

on ιB_ρ where $m_1, \dots, m_n \in \mathbb{Z}$ and $c_t \in \mathbb{R}$, Eq. (1) can be rewritten as

$$(3) \quad \mathcal{A}(\tilde{\psi}) = \left[\mathcal{A}(\psi) - \frac{m_1 + \dots + m_n}{(n+1)!} \frac{\pi^{n+1} \rho^{2n+2}}{\text{Vol}(M, \omega_\rho^n) - \pi^n \rho^{2n}} + \frac{C \pi^n \rho^{2n}}{\text{Vol}(M, \omega_\rho^n) - \pi^n \rho^{2n}} \right]$$

in $\mathbb{R}/\mathcal{P}(\widetilde{M}, \widetilde{\omega}_\rho)$, where $C = \int_0^1 c_t dt$.

A loop ψ in $\text{Ham}(M, \omega)$ based at the identity map is called ι -circle loop if on ιB_ρ the corresponding normalized Hamiltonian takes the form of Eq. (2). Denote by $n(\psi)$ the order of $\mathcal{A}([\psi])$ in $\mathbb{R}/\mathcal{P}(M, \omega)$

Theorem 1.2. *Let (M, ω) be a closed symplectic manifold such that ω is rational. If ψ_1, \dots, ψ_k are ι -circle loops in $\text{Ham}(M, \omega)$ and $\{n(\psi_1), \dots, n(\psi_k)\}$ are pairwise relative prime, then*

$$\text{rank } \pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_{\rho_0})) \geq k$$

for some small $\rho_0 < \rho$. Furthermore, the classes $[\tilde{\psi}_1], \dots, [\tilde{\psi}_k]$ of the lifted loops generate an abelian subgroup of rank k of $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_{\rho_0}))$.

The idea behind the proof of Theorem 1.2, is that from expression (3) of $\mathcal{A}(\tilde{\psi})$ we obtain a polynomial in $\pi \rho^2$ with rational coefficients. Hence the hypothesis that the symplectic form must be rational. Then the fact that $\mathcal{A}(\tilde{\psi})$ has infinite order in $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$, is equivalent to the fact that $\pi \rho^2$ is not a root of this polynomial. Hence the value ρ_0 in Theorem 1.2 is subject to the condition that $\pi \rho_0^2$ must be a transcendental number.

The most common examples of Hamiltonian loops are Hamiltonian S^1 -actions. Recall that the fixed point set of a Hamiltonian circle action on a closed symplectic manifold is non empty. Hence if ψ is a Hamiltonian circle action on (M, ω) , then by blowing up a fixed point the above result guarantees that $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$ has positive rank for some values of ρ as long as $\mathcal{A}([\psi])$ has finite order. This is true for $(\mathbb{C}P^n, \omega)$, where the symplectic form is normalized to be rational. Hence by Theorem 1.2, the rank of

$\pi_1(\text{Ham}(\widetilde{\mathbb{C}P^n}, \widetilde{\omega}_\rho))$ is greater than or equal to one. The results of D. McDuff in [5] already imply that the rank of $\pi_1(\text{Ham}(\widetilde{\mathbb{C}P^n}, \widetilde{\omega}_\rho))$ is greater than or equal to one; we provide an alternative solution and show that such element of infinite order is induced from an element in $\pi_1(\text{Ham}(\mathbb{C}P^n, \omega))$ of finite order.

Corollary 1.3. *Let (M, ω) be a closed symplectic manifold such that ω is rational. If ψ is a non trivial Hamiltonian circle action on (M, ω) such that $\mathcal{A}([\psi])$ has finite order, then for some small ρ the rank of $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$ is positive.*

Remark. The conclusions of Theorem 1.2 and Corollary 1.3 are also valid for the group $\pi_1(\mathcal{H}_{\rho,0}^U)$; since the elements guaranteed by these results are induced by the map $\Phi_{\rho*} : \pi_1(\mathcal{H}_{\rho,0}^U) \rightarrow \pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$.

We conjectured that for any closed symplectic manifold (M, ω) , the rank of $\pi_1(\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho))$ must be positive for ρ small. More intriguing is to know if for every positive integer k there exists a closed symplectic 4-manifold such that the rank of $\pi_1(\text{Ham}(M, \omega))$ is precisely k .

The methods mentioned above, lifting Hamiltonian diffeomorphisms and the relation between $\mathcal{A}(\psi)$ and $\mathcal{A}(\widetilde{\psi})$, also work in the case when k points are blown up simultaneously. As well, in the non compact case a relation analogous to Eq. (1) can be obtained for Calabi’s morphism, namely

$$\text{Cal}(\widetilde{\psi}) = \text{Cal}(\psi) - \frac{1}{n!} \int_0^1 \int_{\iota B_\rho} H_t \omega^n dt.$$

Here ψ is a loop in $\text{Ham}^c(M, \omega)$, H_t its Hamiltonian function with compact support and $\widetilde{\psi}$ a lift of the loop. As in Theorem 1.1, the lift $\widetilde{\psi}$ is induced by the representative of ψ given by the Hamiltonian H_t .

Finally, we make some comments on our notation. In order to simplify notation we use ψ to denote either a loop $\{\psi_t\}_{0 \leq t \leq 1}$ of diffeomorphisms based at the identity map, or an element $[\{\psi_t\}_{0 \leq t \leq 1}]$ in the fundamental group or a single diffeomorphism. From context it will be clear which of these three objects ψ stands for.

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2. The symplectic blow up

In this section we review the symplectic one-point blow up of a manifold, with the intention of setting up notation that will be used throughout the paper.

To that end, first consider the blow up of \mathbb{C}^n at the origin $\Phi : \widetilde{\mathbb{C}^n} \rightarrow \mathbb{C}^n$, where

$$\widetilde{\mathbb{C}^n} := \{(z, \ell) : z \in \mathbb{C}^n, \ell \in \mathbb{C}P^{n-1} \text{ and } z \in \ell\}$$

and $\Phi(z, \ell) = z$. Recall that $\widetilde{\mathbb{C}^n}$ can also be identified with the tautological line bundle $pr : \widetilde{\mathbb{C}^n} \rightarrow \mathbb{C}P^{n-1}$, where $pr(z, \ell) = \ell$. For the closed ball $B_r \subset \mathbb{C}^n$ of radius r centred at the origin, set $L_r := \Phi^{-1}(B_r)$. Let (M, ω) be a symplectic manifold, ω_0 is the standard symplectic form on \mathbb{C}^n and $\iota : (B_r, \omega_0) \rightarrow (M, \omega)$ a symplectic embedding such that $\iota(0) = x_0$. Then as a smooth manifold, the blow up of M at x_0 is defined as

$$\widetilde{M} := (M \setminus \{x_0\}) \cup L_r / \sim,$$

where $x = \iota(z) \in \iota B_r \subset M \setminus \{x_0\}$ is identified with $\Phi^{-1}(z)$ for $z \neq 0$. The projection map $\pi : \widetilde{M} \rightarrow M$, is such that $\pi^{-1}(x_0) = E$ is the exceptional divisor and it induces a diffeomorphism $\widetilde{M} \setminus E \rightarrow M \setminus \{x_0\}$.

As for the symplectic form on the blow up manifold, first we note that $\widetilde{\mathbb{C}^n}$ carries a family of Kähler forms

$$(4) \quad \omega(\rho) := \Phi^*(\omega_0) + \rho^2 pr^*(\omega_{FS})$$

where $\rho > 0$ and the Fubini-Study form on $(\mathbb{C}P^{n-1}, \omega_{FS})$ is normalized so that the area of any line is π . In order to define a symplectic form on the blow up manifold \widetilde{M} , let $\rho < r$; then the symplectic form $\omega(\rho)$ on L_r is perturbed so that near the boundary of L_r agrees with the canonical symplectic form ω_0 . Let $\beta_\rho : [0, r] \rightarrow [\rho, r]$ be defined as

$$\beta_\rho(s) := \begin{cases} \sqrt{\rho^2 + s^2} & \text{for } 0 \leq s \leq \delta \\ s & \text{for } r - \delta \leq s \leq r \end{cases}$$

and on interval $[\delta, r - \delta]$ is defined in any smooth way as long as $0 < \beta'_\rho(s) \leq 1$ for $0 < s \leq r - \delta$. Define the diffeomorphism $F_\rho : L_r \setminus E \rightarrow B_r \setminus B_\rho$ as

$$F_\rho(z) := \beta_\rho(|z|) \frac{z}{|z|}$$

and set $\tilde{\omega}(\rho) := F_\rho^*(\omega_0)$. So defined $\tilde{\omega}(\rho)$ is a symplectic form such that it equals ω_0 on $L_r \setminus L_{r-\delta}$ and $\omega(\rho)$ on L_δ . We call $(L_r, \tilde{\omega}(\rho))$ the local model of the symplectic blow up. Now we can define a symplectic form on the blow up manifold. The symplectic form of weight $\rho < r$ on \tilde{M} is defined as

$$\tilde{\omega}_\rho := \begin{cases} \omega & \text{on } \pi^{-1}(M \setminus \iota B_{\sqrt{\rho^2 + \delta^2}}) \\ \tilde{\omega}(\rho) & \text{on } L_r. \end{cases}$$

For further details on the symplectic blow up see [7] and [8]. The above observations are summarized in the next proposition.

Proposition 2.4. *Let $\iota : (B_r, \omega_0) \rightarrow (M, \omega)$ be a symplectic embedding such that $\iota(0) = x_0$, and $(\tilde{M}, \tilde{\omega}_\rho)$ the symplectic blow up of weight $\rho < r$. Then*

- 1) $\pi : \tilde{M} \setminus E \rightarrow M \setminus \{x_0\}$ is a diffeomorphism,
- 2) $\pi^*(\omega) = \tilde{\omega}_\rho$ on $\pi^{-1}(M \setminus \iota B_r)$, and
- 3) the area of any line in E is $\rho^2 \pi$.

3. Symplectic and Hamiltonian diffeomorphisms on the blow up

In order to lift a symplectic diffeomorphism ψ on (M, ω) to a symplectic diffeomorphism $\tilde{\psi}$ on $(\tilde{M}, \tilde{\omega}_\rho)$, that is in order for the relation $\pi \circ \tilde{\psi} = \psi \circ \pi$ to hold, we must focus on the behavior of ψ on the embedded ball $\iota B_r \subset M$. A necessary condition to lift ψ is that it must map the boundary of ιB_ρ to it self. This is so because the relation $\pi \circ \tilde{\psi} = \psi \circ \pi$ implies that $\tilde{\psi}$ maps the divisor to itself. Hence, ψ maps ιB_ρ to itself.

As expected the problem of lifting a symplectic diffeomorphism on M to a diffeomorphism on the blow up is of local nature. For that matter we consider ψ as a symplectic diffeomorphism of (B_r, ω_0) such that $\psi(0) = 0$. Further assume that

$$\psi : (B_r, \omega_0) \rightarrow (B_r, \omega_0)$$

is given by unitary linear map $\psi = A \in U(n)$. In this case we define $\tilde{\psi} : L_r \rightarrow L_r$ by

$$(5) \quad \tilde{\psi}(z, \ell) = (A(z), A(\ell)).$$

Recall the classification theorem of several complex variables of Cartan [10]; a holomorphic map on \mathbb{C}^n that maps the ball to it self and fixes the origin must be given by a unitary matrix.

Lemma 3.5. *The map $\tilde{\psi}$ defined in (5) preserves the symplectic form $\tilde{\omega}_\rho$.*

Proof. From the definitions of F_ρ and $\tilde{\psi}$ we have that $F_\rho \circ \tilde{\psi} = \psi \circ F_\rho$ on $L_r \setminus E$. Since $\tilde{\omega}_\rho = F_\rho^*(\omega_0)$, then

$$(\tilde{\psi})^*(\tilde{\omega}_\rho) = (\tilde{\psi})^* \circ F_\rho^*(\omega_0) = F_\rho^* \circ \psi^*(\omega_0) = \tilde{\omega}_\rho$$

on $L_r \setminus E$. Finally since $A \in U(n)$, then $\tilde{\psi}$ preserves the Kähler form $\omega(\rho)$. In particular the symplectic form $\tilde{\omega}_\rho$ on E . □

We say that a symplectic diffeomorphism ψ of (M, ω) is *liftable* to $(\tilde{M}, \tilde{\omega}_\rho)$ if

- $\psi(\iota B_r) = \iota B_r$, and
- $\iota^{-1} \circ \psi \circ \iota : B_r \rightarrow B_r$ is given by a unitary matrix

where $\rho < r$. This is exactly the description of \mathcal{H}_ρ^U given in Section 1 for the case when ψ is Hamiltonian. Thus if ψ admits a lift, by Lemma 3.5 we have that $\tilde{\psi}$ is a symplectic diffeomorphism of $(\tilde{M}, \tilde{\omega}_\rho)$. It is important to note that the above definition depends on the symplectic embedding $\iota : (B_r, \omega_0) \rightarrow (M, \omega)$. Thus from now on we fix a symplectic embedding and the lifted diffeomorphisms will be with respect to it.

Now we take into account the problem determining that the lift of a Hamiltonian diffeomorphism is Hamiltonian. Again, we focus on the local picture. Thus let $\psi : B_r \rightarrow B_r$ be liftable and Hamiltonian and assume that there is a Hamiltonian path $\{\psi_t\}$, with $\psi_0 = 1$, $\psi_1 = \psi$ and ψ_t liftable for each t . Let $H_t : (B_r, \omega) \rightarrow \mathbb{R}$ and X_t be the Hamiltonian function and time-dependent vector field induced by the path $\{\psi_t\}$. Since the path $\{\psi_t\}$ is liftable, it is actually a path in $U(n)$; hence X_t is tangent to the sphere centered at the origin. As for the Hamiltonian function we have the following.

Lemma 3.6. *Let $\psi_t : (B_r, \omega_0) \rightarrow (B_r, \omega_0)$ as above; that is, a path of unitary matrices starting at the identity matrix. Then $H_t(z) = H_t(\lambda z)$ for $z \in B_r$ and $\lambda \in S^1$.*

Proof. Denote by X_t the time-dependent vector field of the path $\{\psi_t\}$. For $\lambda \in S^1$, let $\phi_\lambda : B_r \rightarrow B_r$ be matrix multiplication by λI . Since ϕ_λ is in the center of $U(n)$,

$$X_t \circ \phi_\lambda = \left. \frac{d}{ds} \right|_{s=t} \psi_s \circ \psi_t^{-1} \circ \phi_\lambda = \left. \frac{d}{ds} \right|_{s=t} \phi_\lambda \circ \psi_s \circ \psi_t^{-1} = (\phi_\lambda)_* X_t.$$

Therefore

$$d(H_t \circ \phi_\lambda) = \omega_0(X_t, (\phi_\lambda)_*(\cdot)) = (\phi_\lambda)^* \omega_0(X_t, \cdot) = dH_t.$$

Note that both functions H_t and $H_t \circ \phi_\lambda$ agree at the origin, thus $H_t(z) = H_t(\lambda z)$. □

Since ψ_t is liftable, we have a symplectic path $\{\tilde{\psi}_t\}$ on $(L_r, \tilde{\omega}_\rho)$ that starts at the identity and ends at $\tilde{\psi}_1 = \tilde{\psi}$. Moreover if \tilde{X}_t is the vector field induced by $\{\tilde{\psi}_t\}$, then we have that $\pi_*(\tilde{X}_t) = X_t$ since ψ_t is the lift of ψ_t . Now define the function $\tilde{H}_t : (L_r, \tilde{\omega}_\rho) \rightarrow \mathbb{R}$ as

$$(6) \quad \tilde{H}_t(z, \ell) := \begin{cases} H_t \circ F_\rho(z) & \text{if } (z, \ell) \in L_r \setminus E \\ H_t\left(\frac{\rho}{|w|}w\right) & \text{if } z = 0 \text{ and } [w] = \ell. \end{cases}$$

It follows by Lemma 3.6 that \tilde{H}_t is well-defined and smooth. That is, is independent of the representative of ℓ when evaluated at points in the exceptional divisor.

Proposition 3.7. *The Hamiltonian function $\tilde{H}_t : (L_r, \tilde{\omega}_\rho) \rightarrow \mathbb{R}$ defined above induces the path of Hamiltonian diffeomorphisms $\{\tilde{\psi}_t\}$ that is the lift of the path $\{\psi_t\}$. Moreover \tilde{X}_t is such that $\pi_*(\tilde{X}_t) = X_t$.*

Proof. We already showed that the vector fields \tilde{X}_t and X_t are related by the blow up map. It only remains to show that $\iota(\tilde{X}_t)\tilde{\omega}_\rho = d\tilde{H}_t$. First note that

$$(F_\rho)_{*,x}(X) = \beta_\rho(x)X + d\beta_\rho(X)x.$$

Since β_ρ is radial, the kernel of $d\beta_\rho$ agrees with the tangent space to the sphere centred at the origin. Now ψ_t is defined by a unitary matrix, thus

outside the origin and E , the vector fields X_t and \tilde{X}_t lie in the tangent space of the sphere. Thus $F_{\rho,*}(\tilde{X}_t) = \beta_\rho X_t$ and

$$\begin{aligned} \tilde{\omega}_\rho(\tilde{X}_t, \cdot) &= F_\rho^*(\omega_0)(\tilde{X}_t, \cdot) \\ &= \omega_0(F_{\rho,*}\tilde{X}_t, F_{\rho,*}(\cdot)) \\ &= \omega_0(\beta_\rho \cdot X_t, F_{\rho,*}(\cdot)) \\ &= \beta_\rho \omega_0(X_t, \beta_\rho^{-1} \cdot F_{\rho,*}(\cdot)) \\ &= \beta_\rho(dH_t) \circ \beta_\rho^{-1} \cdot F_{\rho,*} \\ &= d(H_t \circ F_\rho). \end{aligned}$$

on $L_r \setminus E$. □

Hence if $\{\psi_t\}$ is a Hamiltonian path on (M, ω) with Hamiltonian function H_t and each ψ_t is liftable, that is a path in $\mathcal{H}_{\rho,0}^U$, then the lift $\{\tilde{\psi}_t\}$ is a Hamiltonian path with Hamiltonian function $\tilde{H}_t : (\tilde{M}, \tilde{\omega}_\rho) \rightarrow \mathbb{R}$ given by

$$(7) \quad \tilde{H}_t(x) := \begin{cases} H_t \circ \pi(x) & \text{if } \pi(x) \notin \iota B_r \\ H_t \circ \iota \circ F_\rho \circ \iota^{-1} \circ \pi(x) & \text{if } \pi(x) \in \iota B_r \setminus \{x_0\} \\ H_t\left(\frac{\rho}{|w|}w\right) & \text{if } x = [x_0, \ell] \in E \text{ and } [w] = \ell. \end{cases}$$

Thus Proposition 3.7 can be stated in global terms.

Proposition 3.8. *Let $\{\psi_t\}$ be a path of Hamiltonian diffeomorphisms in $\mathcal{H}_{\rho,0}^U$ with Hamiltonian function H_t . Then the lifted path $\{\tilde{\psi}_t\}$ is a Hamiltonian path on $(\tilde{M}, \tilde{\omega}_\rho)$ with Hamiltonian function \tilde{H}_t given by (7).*

Remark. The Hamiltonian diffeomorphism on $(\tilde{M}, \tilde{\omega}_\rho)$ induced by the map $H_t \circ \pi$, is *not* the one that lifts the Hamiltonian diffeomorphism of the base manifold. Most importantly to our interest, if H_t generates a loop of Hamiltonian diffeomorphisms in (M, ω) , then $H_t \circ \pi$ induces a path and *not* a loop of Hamiltonian diffeomorphisms on $(\tilde{M}, \tilde{\omega}_\rho)$; the time-one Hamiltonian diffeomorphism of $H_t \circ \pi$ is *not* the identity map. Notice also that $H_t \circ \pi$ is independent of ρ , whereas \tilde{H}_t depends on F_ρ .

There are two typical examples of symplectic diffeomorphisms that are liftable. The first class of examples is when the support of ψ is disjoint from ιB_r . In this case the matrix representation of ψ on ιB_r is the identity matrix. Another example is a circle action with x_0 a fixed point of the action. In this case there is a Darboux chart about x_0 so that the action can be described by a loop of unitary matrices.

Example. In this example we see how the definition of the Hamiltonian function \tilde{H}_t given in (7) coincides with the natural Hamiltonian function on $\widetilde{\mathbb{C}^n}$, in the case of a linear circle action on \mathbb{C}^n . To that end, consider a linear circle action on (\mathbb{C}^n, ω_0) with Hamiltonian function $H : \mathbb{C}^n \rightarrow \mathbb{R}$ given by

$$H(z_1, \dots, z_n) := -\pi \sum_{j=1}^n m_j |z_j|^2,$$

where $m_1, \dots, m_n \in \mathbb{Z}$ and $\omega_0(X, \cdot) = dH$. Since the action is linear, it induces a Hamiltonian circle action on $(\mathbb{C}P^{n-1}, \omega_{FS})$ with Hamiltonian function

$$H'([z_1 : \dots : z_n]) := -\pi \sum_{j=1}^n m_j \frac{|z_j|^2}{|z|^2}$$

and $\omega_{FS}(X', \cdot) = dH'$.

Thus we have a circle action on $\mathbb{C}^n \times \mathbb{C}P^{n-1}$ given by the diagonal action. Furthermore $\widetilde{\mathbb{C}^n}$ is invariant under the action. Recall the symplectic form $\omega(\rho) = \Phi^*(\omega_0) + \rho^2 pr^*(\omega_{FS})$ on $\widetilde{\mathbb{C}^n}$. Then the circle action on $(\widetilde{\mathbb{C}^n}, \omega(\rho))$ is Hamiltonian, with Hamiltonian function $H + \rho^2 H'$ restricted to $\widetilde{\mathbb{C}^n}$.

Now we compute the Hamiltonian function \tilde{H} on a small neighborhood U of the exceptional divisor, following the definition given by Eq. (7). Recall that the symplectic form $\tilde{\omega}_\rho$ on $\widetilde{\mathbb{C}^n}$ equals $\omega(\rho)$ on U . Then for $(z, [z_1 : \dots : z_n]) \in U \setminus E$ we have

$$\begin{aligned} \tilde{H}(z, [z_1 : \dots : z_n]) &= H \circ F_\rho(z) \\ &= H\left(\sqrt{\rho^2 + |z|^2} \frac{z}{|z|}\right) = -\pi \frac{\rho^2 + |z|^2}{|z|^2} \sum_{j=1}^n m_j |z_j|^2 \\ &= -\pi \sum_{j=1}^n m_j |z_j|^2 - \rho^2 \pi \sum_{j=1}^n m_j \frac{|z_j|^2}{|z|^2}. \end{aligned}$$

Now for $(0, [w_1 : \dots : w_n])$ in the exceptional divisor

$$\begin{aligned} \tilde{H}(0, [w_1 : \dots : w_n]) &= H\left(\frac{\rho}{|w|} w\right) \\ &= -\pi \sum_{j=1}^n m_j \rho^2 \frac{|w_j|^2}{|w|^2}. \end{aligned}$$

That is $\tilde{H} = H + \rho^2 H'$ in a small neighborhood of the exceptional divisor.

The process of blowing up a point has an alternative description than the one presented in Section 2. Heuristically, the blow up of a point of weight ρ can be described as removing the interior of the embedded ball B_ρ and collapsing its boundary to $\mathbb{C}P^{n-1}$ via the Hopf fibration. The next result is a consequence of this fact.

Lemma 3.9. *Let $H : (M, \omega) \rightarrow \mathbb{R}$ be a smooth function with compact support and $\tilde{H} : (\tilde{M}, \tilde{\omega}_\rho) \rightarrow \mathbb{R}$ defined as in (7). Then,*

$$\int_{\tilde{M}} \tilde{H} \tilde{\omega}_\rho^n = \int_M H \omega^n - \int_{\iota B_\rho} H \omega^n.$$

Proof. By Proposition 2.4 the blow up map induces a symplectic diffeomorphism between $(\tilde{M} \setminus \pi^{-1}(\iota B_r), \tilde{\omega}_\rho)$ and $(M \setminus \iota B_r, \omega)$. Since $\tilde{H} = H \circ \pi$ on $\tilde{M} \setminus \pi^{-1}(\iota B_r)$ we get

$$\int_{\tilde{M}} \tilde{H} \tilde{\omega}_\rho^n = \int_{M \setminus \iota B_r} H \omega^n + \int_{\pi^{-1}(\iota B_r)} \tilde{H} \tilde{\omega}_\rho^n.$$

By the definition of \tilde{H} on $\pi^{-1}(\iota B_r)$, the fact that $F_\rho^*(\omega_0) = \tilde{\omega}_\rho$ on $\iota B_r \setminus \iota B_\rho$ and removing the exceptional divisor from the domain of the second integral, the claim follows:

$$\begin{aligned} \int_{\tilde{M}} \tilde{H} \tilde{\omega}_\rho^n &= \int_{M \setminus \iota B_r} H \omega^n + \int_{\pi^{-1}(\iota B_r) \setminus E} H \circ F_\rho \tilde{\omega}_\rho^n \\ &= \int_{M \setminus \iota B_r} H \omega^n + \int_{\iota B_r \setminus \iota B_\rho} H \omega^n \\ &= \int_M H \omega^n - \int_{\iota B_\rho} H \omega^n. \end{aligned}$$

□

Remark. Remember that we fix a symplectic embedding $\iota : (B_r, \omega_0) \rightarrow (M, \omega)$ and respect to this embedding we have lifted symplectic and Hamiltonian diffeomorphisms. Clearly a different embedding might yield a different set of diffeomorphisms that are liftable. Recall from [6], that if the embeddings are isotopic via symplectic embeddings then the symplectic blow ups are symplectomorphic. Since we are interested in the topology of the group $\text{Ham}(\tilde{M}, \tilde{\omega}_\rho)$, for our purpose it suffices to fix a symplectic embedding and to require the unitary condition on diffeomorphisms in a neighborhood of ιB_ρ and not on all ιB_r .

4. Weinstein’s morphism on $(\widetilde{M}, \widetilde{\omega}_\rho)$

Now we consider the case of loops in $\text{Ham}(\widetilde{M}, \widetilde{\omega}_\rho)$ when (M, ω) is a closed manifold and $\pi\rho^2 < c_G(M, \omega)$, where $c_G(M, \omega)$ stands for the Gromov’s width of (M, ω) . As in Section 3, let $x_0 \in M$ be a based point and $\iota : (B_r, \omega_0) \rightarrow (M, \omega)$ be a fixed symplectic embedding such that $\iota(0) = x_0$ and $\pi\rho^2 < \pi r^2 < c_G(M, \omega)$.

Recall that the period group $\mathcal{P}(M, \omega)$ of (M, ω) is defined as the image of the pairing $[\omega] \cdot H_2(M; \mathbb{Z}) \rightarrow \mathbb{R}$. Weinstein’s morphism [11],

$$\mathcal{A} : \pi_1(\text{Ham}(M)) \rightarrow \mathbb{R}/\mathcal{P}(M, \omega)$$

is defined via the action functional as

$$\mathcal{A}(\psi) = - \int_D u^*(\omega) + \int_0^1 H_t(\psi_t(x_0)) dt.$$

Here D is the unit closed disk and $u : D \rightarrow M$ is a smooth function such that $u(\partial D)$ is the loop $\{\psi_t(x_0)\}$ and H_t is the 1-periodic Hamiltonian function induced by the Hamiltonian loop $\{\psi_t\}$ subject to the normalized condition

$$\int_M H_t \omega^n = 0$$

for every $t \in [0, 1]$.

Remember that the dimension of (M, ω) is greater than two. Then for the one-point blow up $(\widetilde{M}, \widetilde{\omega}_\rho)$, we have that $H_2(\widetilde{M}; \mathbb{Z}) \simeq H_2(M; \mathbb{Z}) + \mathbb{Z}\langle L \rangle$ where $L \subset E$ if the class of a line in the exceptional divisor of $(\widetilde{M}, \widetilde{\omega}_\rho)$. Note also that any class in $H_2(M; \mathbb{Z})$ can be represented by a cycle away from the embedded ball ιB_r . Hence $\langle [\omega], c \rangle = \langle [\widetilde{\omega}_\rho], \pi^{-1}(c) \rangle$ for any $c \in H_2(M; \mathbb{Z})$. By definition of the symplectic form $\widetilde{\omega}_\rho$ on the blow up, $L \subset (\widetilde{M}, \widetilde{\omega}_\rho)$ has symplectic area $\rho^2\pi$ and

$$\mathcal{P}(\widetilde{M}, \widetilde{\omega}_\rho) = \mathcal{P}(M, \omega) + \mathbb{Z}\langle \pi\rho^2 \rangle \subset \mathbb{R}.$$

Now we give the proofs of the results mentioned at the Introduction.

Proof of Theorem. 1.1. Let $\{\psi_t\}$ be a loop in $\text{Ham}(M, \omega)$ that is liftable with respect to the symplectic embedding $\iota : (B_r, \omega_0) \rightarrow (M, \omega)$. That is $\psi_t \in \mathcal{H}_{\rho,0}^U$ for every t . Fix $p_0 \in M$ outside the embedded ball ιB_r , since ψ is liftable the loop $\gamma := \{\psi_t(p_0)\}$ in M lies outside the embedded ball. Hence $\widetilde{\gamma} := \{\psi_t(\pi^{-1}(p_0))\}$ is a loop in \widetilde{M} that covers γ .

Now let $u : D \rightarrow M$ be a smooth map such that $u(\partial D) = \gamma$. Since M has dimension greater than two, by the excision theorem for homotopy groups we can assume that $u(D)$ is disjoint from ιB_r . Hence there is a smooth map $\tilde{u} : D \rightarrow \widetilde{M}$, such that $\tilde{u}(\partial D) = \tilde{\gamma}$ and $\pi \circ \tilde{u} = u$. Since $\pi^* \omega = \tilde{\omega}_\rho$ on $\widetilde{M} \setminus \pi^{-1}(\iota B_r)$, we get

$$\int_D u^*(\omega) = \int_D \tilde{u}^*(\tilde{\omega}_\rho).$$

Let $H_t : (M, \omega) \rightarrow \mathbb{R}$ be the normalized Hamiltonian function induced by the loop ψ . Then by Lemma 3.9 the normalized Hamiltonian of the lifted loop $\tilde{\psi}$ is $\tilde{H}_t + c_\rho(M, \omega, H_t)$ where \tilde{H}_t is given by Eq. (7) and

$$c_\rho(M, \omega, H_t) := -\frac{1}{\text{Vol}(\widetilde{M}, \tilde{\omega}_\rho^n)} \int_{\widetilde{M}} \tilde{H}_t \tilde{\omega}_\rho^n = \frac{1}{\text{Vol}(\widetilde{M}, \tilde{\omega}_\rho^n)} \int_{\iota B_\rho} H_t \omega^n.$$

Hence,

$$\begin{aligned} \mathcal{A}(\tilde{\psi}) &= -\int_D \tilde{u}^*(\tilde{\omega}_\rho) + \int_0^1 (\tilde{H}_t + c_\rho(M, \omega, H_t))(\tilde{\psi}_t(\pi^{-1}(p_0))) dt \\ &= -\int_D u^*(\omega) + \int_0^1 H_t(\psi_t(p_0)) dt + \int_0^1 c_\rho(M, \omega, H_t) dt \\ &= \left[\mathcal{A}(\psi) + \frac{1}{\text{Vol}(\widetilde{M}, \tilde{\omega}_\rho^n)} \int_0^1 \int_{\iota B_\rho} H_t \omega^n dt \right]. \end{aligned}$$

□

In the case when the normalized Hamiltonian function takes the form

$$(8) \quad H_t(z_1, \dots, z_n) := -\pi \sum_{j=1}^n m_j |z_j|^2 + c_t$$

on ιB_r , we have

$$\int_0^1 \int_{\iota B_\rho} H_t \omega^n dt = -(m_1 + \dots + m_n) \frac{\pi^{n+1} \rho^{2n+2}}{(n+1)!} + \text{Vol}(B_\rho, \omega_0^n) \int_0^1 c_t.$$

Since the volume of (B_ρ, ω_0^n) is $\pi^n \rho^{2n}$, then in this case $\mathcal{A}(\tilde{\psi})$ takes the form

$$(9) \quad \mathcal{A}(\tilde{\psi}) = \left[\mathcal{A}(\psi) - \frac{m_1 + \dots + m_n}{(n+1)!} \frac{\pi^{n+1} \rho^{2n+2}}{\text{Vol}(M, \omega_\rho^n) - \pi^n \rho^{2n}} + \frac{C \pi^n \rho^{2n}}{\text{Vol}(M, \omega_\rho^n) - \pi^n \rho^{2n}} \right]$$

where $C = \int_0^1 c_t$.

As mentioned at the Introduction, the proof of Theorem 1.2 relies on some polynomials with rational coefficients. In part, we take care of this by assuming that the symplectic form must be rational. However some work needs to be done in order to guarantee that the constant C that appears in Eq. (9) is in fact a rational number.

Recall that an ι -circle loop ψ in $\text{Ham}(M, \omega)$, is a loop of Hamiltonian diffeomorphisms based at the identity map such that its normalized Hamiltonian function on ιB_ρ takes the form as in Eq. (8).

Lemma 4.10. *Let ψ be an ι -circle loop in $\text{Ham}(M, \omega)$. Then*

$$\mathcal{A}(\psi) = [C].$$

Proof. Let H_t be the normalized Hamiltonian function of the loop. Since the loop ψ is ι -circle, then H_t is given by Eq. (8) on ιB_ρ . In particular x_0 is fixed by each Hamiltonian in the loop. Therefore,

$$\mathcal{A}(\psi) = \left[\int_0^1 H_t(x_0) dt \right] = \left[\int_0^1 c_t dt \right] = [C]. \quad \square$$

If ψ is an ι -circle loop in $\text{Ham}(M, \omega)$, denote by $K(\psi, x_0)$ the sum of its weights $m_1 + \dots + m_n$ at x_0 .

Proof of Theorem 1.2. Since the symplectic form ω is rational and (M, ω) is closed, the period group $\mathcal{P}(M, \omega)$ is discrete and $V := \text{Vol}(M, \omega^n)$ is a rational number. Moreover $\mathcal{P}(M, \omega) = \mathbb{Z}\langle a \rangle$ for some $a \in \mathbb{Q} \setminus \{0\}$.

Let $\rho_0 > 0$ be such that $\pi\rho_0^2$ is transcendental and less than the Gromov width of (M, ω) . Then consider the blow up of (M, ω) at x_0 of weight ρ_0 . Since ψ_j is a loop in \mathcal{H}_0^U , it follows by Proposition 3.8 that it can be lifted to a Hamiltonian loop $\tilde{\psi}_j$ on $(\tilde{M}, \tilde{\omega}_{\rho_0})$. By hypothesis, there are positive integers $n_j := n(\psi_j)$ such that $\mathcal{A}([\psi_j]) = [a/n_j]$ in $\mathbb{R}/\mathbb{Z}\langle a \rangle$ for $1 \leq j \leq k$. Then by Lemma 4.10, we have that $\mathcal{A}([\psi_j]) = [C_j] = [a/n_j]$. Hence by Eq. (9) we get

$$(10) \quad \mathcal{A}([\tilde{\psi}_j]) = \left[\frac{a}{n_j} \left(1 + \frac{(\pi\rho_0^2)^n}{V - (\pi\rho_0^2)^n} \right) - \frac{K(\gamma_j, x_0)}{(n+1)!} \frac{(\pi\rho_0^2)^{n+1}}{V - (\pi\rho_0^2)^n} \right]$$

in $\mathbb{R}/\mathbb{Z}\langle a, \pi\rho_0^2 \rangle$.

For $k \in \mathbb{Z}$ non zero the expression $k\mathcal{A}([\tilde{\psi}_j]) = 0$ is equivalent to the fact that $k\mathcal{A}([\tilde{\psi}_j])$ lies in $\mathbb{Z}\langle a, \pi\rho_0^2 \rangle$; that after multiplying it by $V - (\pi\rho_0^2)^n$ gives a polynomial of degree $n + 1$ in $\pi\rho_0^2$ with rational coefficients equal to zero. Since $\pi\rho_0^2$ is assumed to be a transcendental number, $k\mathcal{A}([\tilde{\psi}_j]) \neq 0$ and each

$[\tilde{\psi}_j]$ has infinite order in $\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_{\rho_0}))$. Note that ρ_0 is independent of j .

The rest of the proof is devoted to show that $[\tilde{\psi}_1], \dots, [\tilde{\psi}_k]$ generate a subgroup of $\pi_1(\text{Ham}(\tilde{M}, \tilde{\omega}_{\rho_0}))$ isomorphic to \mathbb{Z}^k . Assume that $\mathcal{A}([\tilde{\psi}_r])$ is in the group generated by $\{\mathcal{A}([\tilde{\psi}_j]) | j \neq r\}$. Then there exist $\alpha_j \in \mathbb{Z}$ such that

$$(11) \quad \mathcal{A}([\tilde{\psi}_r]) = \alpha_1 \mathcal{A}([\tilde{\psi}_1]) + \dots + \alpha_k \mathcal{A}([\tilde{\psi}_k])$$

in $\mathbb{R}/\mathbb{Z}\langle a, \pi\rho_0^2 \rangle$, where there is no r -term on the right hand side. Substituting Eq. (10) in each term of Eq. (11), we get that

$$(12) \quad a \left(\frac{1}{n_r} - \frac{\alpha_1}{n_1} - \dots - \frac{\alpha_k}{n_k} \right) \left(1 + \frac{(\pi\rho_0^2)^n}{V - (\pi\rho_0^2)^n} \right) + \frac{-K(\gamma_j, x_0) + \alpha_1 K(\gamma_1, x_0) + \dots + \alpha_k K(\gamma_k, x_0)}{(n+1)!} \frac{(\pi\rho_0^2)^{n+1}}{V - (\pi\rho_0^2)^n}$$

belongs to $\mathbb{Z}\langle a, \pi\rho_0^2 \rangle$. In order to handle the above expression, set

$$\alpha_0 := \left(\frac{1}{n_r} - \frac{\alpha_1}{n_1} - \dots - \frac{\alpha_k}{n_k} \right) \in \mathbb{Q}$$

and

$$K_0 := \frac{-K(\gamma_j, x_0) + \alpha_1 K(\gamma_1, x_0) + \dots + \alpha_k K(\gamma_k, x_0)}{(n+1)!} \in \mathbb{Q}.$$

Then expression (12) takes the form

$$(13) \quad a\alpha_0 \left(1 + \frac{(\pi\rho_0^2)^n}{V - (\pi\rho_0^2)^n} \right) + K_0 \frac{(\pi\rho_0^2)^{n+1}}{V - (\pi\rho_0^2)^n} = aA + \pi\rho_0^2 B.$$

for some $A, B \in \mathbb{Z}$. After multiplying Eq. (13) by $V - (\pi\rho_0^2)^n$, we obtain a polynomial of degree $n + 1$ in $\pi\rho_0^2$ with rational coefficients that is equal to zero;

$$(B + K_0)(\pi\rho_0^2)^{n+1} + aA(\pi\rho_0^2)^n - BV(\pi\rho_0^2) + (a\alpha_0 V - aAV) = 0.$$

Finally, since $\pi\rho_0^2$ is a transcendental number and a is the generator of the period group of (M, ω) , A and B must be zero. Thus $K_0 = 0$ and $\alpha_0 = 0$. Since $\{n_1, \dots, n_k\}$ are pairwise relatively prime, from Lemma 4.11 we get that $\alpha_0 \not\equiv 0$. This means that Eq. (11) holds only when the α_j 's are zero, that is $[\tilde{\psi}_1], \dots, [\tilde{\psi}_k]$ generate a subgroup of rank k . □

In the last part of the proof of Theorem 1.2 we used the following fact about integers. Only when quoting this lemma, we used the hypothesis that the n_j 's are relative prime by pairs.

Lemma 4.11. *Let n_1, \dots, n_k be integers, such that*

- $n_j \geq 2$
- $(n_1, n_j) = 1$ for all $j > 1$.

Then for any $\alpha_2, \dots, \alpha_k \in \mathbb{Z}$,

$$\frac{1}{n_1} - \frac{\alpha_2}{n_2} - \dots - \frac{\alpha_{k-1}}{n_{k-1}} - \frac{\alpha_k}{n_k}$$

is not equal to zero.

Proof. Assume that

$$\frac{1}{n_1} - \frac{\alpha_2}{n_2} - \dots - \frac{\alpha_{k-1}}{n_{k-1}} - \frac{\alpha_k}{n_k} = 0,$$

thus

$$n_2 \cdots n_k = \sum_{j=2}^k n_1 \cdots \alpha_j \cdots n_k.$$

That is $n_2 \cdots n_k$ is a multiple of n_1 , which is not possible. □

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