JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 16, Number 3, 817–838, 2018

Leafwise symplectic structures on Lawson's foliation

YOSHIHIKO MITSUMATSU

The aim of this paper is to show that Lawson's foliation on the 5-sphere admits a smooth leafwise symplectic structure. The main part of the construction is to show that the Fermat type cubic surface admits an end-periodic symplectic structure. The results is paraphrased that the 5-sphere admits a regular Poisson structure of symplectic dimension 4.

Introduction

In this article we show the following.

Theorem A (Theorem 3.1). *Lawson's foliation on the* 5*-sphere* S^5 *admits a smooth leafwise symplectic structure.*

More generally, the Milnor fibration associated with one of the three classes of simple elliptic singularities $\widetilde{E}_6, \widetilde{E}_7$*, and* $\widetilde{E}_8$ *in complex three variables can be modified into a leafwise symplectic foliation on* S^5 *of codimension* 1*. Lawson's one is associated with* $\widetilde{E}_6$*.*

We can paraphrase this result into the following.

Corollary B. *Associated with each of the three classes* $\widetilde{E}_6, \widetilde{E}_7$*, or* $\widetilde{E}_8$ *of simple elliptic singularities in three variables, the* 5*-sphere* S^5 *admits regular Poisson structures of symplectic dimension* 4*.*

This work is motivated and inspired by the works ([SV], [MV1, MV2]) by Alberto Verjovsky and others in which they are discussing the existence of leafwise symplectic and complex structures on Lawson's foliation and on its slight modifications. The author is extremely grateful to Verjovsky for drawing his attentions to such interesting problems.

This research is partly supported by Grant-in-Aid for Scientific Research (B) 22340015.

H. B. Lawson, Jr. constructed a smooth foliation of codimension one on S^5 ([L]), which we nowadays call *Lawson's foliation*. It was achieved by a beautiful combination of the complex and differential topologies and was a breakthrough in an early stage of the history of foliations. The foliation is composed of two components. One is a tubular neighbourhood of a 3-dimensional nil-manifold and the other one is, away from the boundary, foliated by Fermat-type cubic complex surfaces. As the common boundary leaf, here appears one of Kodaira-Thurston's 4-dimensional nil-manifolds. As each Fermat cubic leaf is spiraling to this boundary leaf, its end is diffeomorphic to a cyclic covering of Kodaira-Thurston's nil-manifold. (See Section 1 for the details.)

In order to introduce a leafwise symplectic structure (for a precise definition, see Section 2), we need to find a symplectic structure on the Fermat cubic surface which (asymptotically) coincides on the end with that of the cyclic covering of the Kodaira-Thurston nil-manifold. However, as a complex surface, the Fermat cubic surface is affine and Stein, and thus its end is strictly pseudo-convex. Therefore, being quite different from periodic, the natural symplec structure on its end is 'conic' and expanding. This is the crucial point of our problem. Once we find an appropriate end-periodic symplectic structure on the Fermat cubic surface (Section 5), it is almost enough to construct a smooth leafwise symplectic structure (Section 3), because it is easy to see that a simple foliation on the tube component admits a leafwise symplectic structure (Section 2).

This paper is organized as follows. Lawson's foliation is reviewed in Section 1. In Section 2, certain symplectic structures on Kodaira-Thurston's nil-manifold and its covering are presented. According to the choice of the symplectic structures on Kodaira-Thurston's nil-manifold, a leafwise symplectic structures on the tube component is constructed. This enables us to formulate our problem in Section 3 focusing on the modification of symplectic structures on the Fermat cubic surface. Assuming this modification which is accomplished in the later sections, a construction of a leafwise symplectic structure on Lawson's foliation is given in this section. Then, after analysing natural symplectic structures on the Fermat cubic surface in Section 4, the existence of an end-periodic symplectic structure on the Fermat cubic surface is shown in Section5. In particular, Corollary 5.2 is the core of the present article, on which we will discuss further in Section 7.

In Section 6 we remark that our construction holds almost verbatimly in two other cases of the simple elliptic hypersurface singularities. In the final section, some related topics concerning the method in this paper are discussed.

1. Review of Lawson's foliation

First we review the structure of Lawson's foliation $\mathcal{L}$. For those who are familiar with the materials it is enough to check our notations.

Let us take a Fermat type homogeneous cubic polynomial $f(Z_0, Z_1, Z_2) = Z_0^3 + Z_1^3 + Z_2^3$ in three variables Z_0, Z_1, and Z_2. The complex surface $\mathbf{F}_w = \{(Z_0, Z_1, Z_2) \in \mathbb{C}^3; f(Z_0, Z_1, Z_2) = w\}$ for a complex value w is non-singular if $w \neq 0$ and $\mathbf{F}_0$ has the unique singularity at the origin. The scalar multiplication $c \cdot (Z_0, Z_1, Z_2) = (cZ_0, cZ_1, cZ_2)$ by $c \in \mathbb{C}$ maps $\mathbf{F}_w$ to $\mathbf{F}_{c^3 w}$. Hence $\mathbf{F}_0$ is preserved by such homotheties and for $w \neq 0$ $\mathbf{F}_w$ is preserved iff $c^3 = 1$.

Now we put $\tilde{F}_\theta = \bigcup_{\arg w = \theta} \mathbf{F}_w$ and $F_\theta = \tilde{F}_\theta \cap S^5$ where S^5 denotes the unit sphere in $\mathbb{C}^3$. Also, we put $N = Nil^3(-3) = \mathbf{F}_0 \cap S^5$. Let p and h denote the projection $p : \check{\mathbb{C}}^3 \to S^5$ and the Hopf fibration $h : S^5 \to \mathbb{C}P^2$. Here $\check{\mathbb{C}}^3$ denotes $\mathbb{C}^3 \setminus \{\mathrm{O}\}$. Sometimes h also denotes the composition $h \circ p : \check{\mathbb{C}}^3 \to \mathbb{C}P^2$. Let also $H(t)$ $(t \in \mathbb{R}/2\pi\mathbb{Z})$ denote the *Hopf flow* obtained by scalar multiplication by e^{it}, whose orbits are the Hopf fibres. $E_\omega = \{[z_0 : z_1 : z_2]; z_0 + z_1 + z_2 = 0\} = h(\mathbf{F}_0)$ is an elliptic curve in $\mathbb{C}P^2$ with modulus $\omega = \frac{-1+\sqrt{-3}}{2}$. The Hopf fibration restricts to $N \to E_\omega$, which is an S^1-bundle with $c_1 = -3$. Also put $\mathcal{S}_r = \cup_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} \mathbf{F}_{re^{i\theta}} = \{Z = (Z_0, Z_1, Z_2) \in \mathbb{C}^3; |f(Z)| = r\}$ for $r > 0$. $\mathcal{S}_r$ is also preserved by the Hopf flow.

The following facts are also easy to see, while they are listed as Proposition for the sake of later use.

Proposition 1.1. 1) *$f|_{S^5}$ has no critical points around N.*

2) $\arg \circ f|_{S^5 \setminus N} : S^5 \setminus N \to S^1$ *has no critical points away from N and is called the Milnor fibration. Each fibre is by F_θ $(\theta \in \mathbb{R}/2\pi\mathbb{Z})$.*

3) *The projection $p|_{\mathbf{F}_w} : \mathbf{F}_w \to F_\theta$ $(\theta = \arg w)$ is a diffeomorphism for $w \neq 0$.*

4) *The Hopf fibration $h|_{F_\theta} : F_\theta \to \mathbb{C}P^2 \setminus E_\omega$ restricted to F_θ is a three fold regular covering, and so is $\mathbf{F}_w \to \mathbb{C}P^2 \setminus E_\omega$ for $w \neq 0$.*

5) *The normal bundle to $N \hookrightarrow S^5$ is trivialized by the value of f.*

6) *The projection $p|_{\mathcal{S}_r} : \mathcal{S}_r \to S^5 \setminus K$ is a diffeomorphism for $r > 0$ and is equivariant with respect to the Hopf flow. $\mathcal{S}_r$ fibers over the circle with fibres $\mathbf{F}_{re^{i\theta}}$ $(\theta \in \mathbb{R}/2\pi\mathbb{Z})$ and the fibration structure coincides with that of the Milnor fibration through $p|_{\mathcal{S}_r}$.*

7) $H(2\pi/3)$ *gives the natural monodromy of the Milnor fibration.*

Let W_r denote the tubular neighbourhood $W_r = \{Z = (Z_0, Z_1, Z_2) \in S^5; |f(Z)| \leq r\}$ of $N \subset S^5$ for $0 < r$. Take ε $(0 < \varepsilon \ll 1)$ so small that $f|_{W_\varepsilon}$ has no critical points. (Later we will take ε again smaller for some reason.) Then let $U_\varepsilon = h(W_\varepsilon) \subset \mathbb{C}P^2$ be a tubular neighbourhood of $E_\omega \subset \mathbb{C}P^2$. W_ε is invariant under the Hopf flow. We choose further smaller constants r_0 and r_* satisfying $0 < r_0 < r_* < \varepsilon$ and take $W_* = W_{r_*}$, $W = W_{r_0}$ and $U_* = U_{r_*}$, $U = U_{r_0}$.

We decompose S^5 into W and $C = S^5 \setminus \operatorname{Int} W$, which are called the *tube* component and the *Fermat cubic* component respectively. The statement 5) in the above proposition tells that W_r is diffeomorphic to the product $N \times D_r^2$, while $E_\omega \hookrightarrow \mathbb{C}P^2$ is twisted because $[E_\omega]^2 = 9$. Here D_r^2 denotes the disk of radius r in $\mathbb{C}$.

The common boundary $\partial W = \partial C$ is diffeomorphic to $N \times S^1$, which is one of Kodaira-Thurston's 4-dimensional nil-manifolds and is well-known to be non-Kähler because $b_1 = 3$. It admits symplectic structures as well as complex structures but they are never compatible.

As the two components W and C are fibering over the circle, the following lemma (a standard process of *turbulization*) is enough in order only to obtain a smooth foliation. However, to put leafwise symplectic structures, it is helpful to describe the foliation and the turbulization in more detail.

Lemma 1.2 ([L], Lemma 1). *Let M be a compact smooth manifold with boundary ∂M and $\varphi : M \to S^1$ be a smooth submersion to the circle. Accordingly so is $\varphi|_{\partial M} : \partial M \to S^1$. Then, there exists a smooth foliation of codimension one for which the boundary ∂M is the unique compact leaf, other leaves are diffeomorphic to the interior of the fibres, and the holonomy of the compact leaf is trivial as a C^∞-jet. If we have two such submersions $\varphi_i : M_i \to S^1$ $(i = 1, 2)$ with diffeomorphic boundaries $\partial M_1 \cong \partial M_2$, then on the closed manifold $M_1 \cup_{\partial M_1 = \partial M_2} M_2$, by gluing them we obtain a smooth foliation of codimension one.*

Let us formulate the turbulization process more explicitly. Firt take small positive constants $0 < r_0 < r_1 < r_2 < r_*$ and smooth functions $g(r)$ and $h(r)$ on $\mathbb{R}_+$ satisfying the following conditions.

$$\begin{array}{llll} g \equiv 0 & (r \leq r_0), & h \equiv 1 & (r \leq r_1), \\ g = -6r & (r_1 \leq r < r_*), & h \equiv 0 & (r_2 \leq r < r_*), \\ g' < 0 & (r_0 < r < r_*), & h' < 0 & (r_1 < r < r_2). \end{array}$$

Then take a smooth non-singular vector field $X = g\frac{\partial}{\partial r} + h\frac{\partial}{\partial \theta}$ on the punctured plane $\mathbb{C} \setminus \{O\} \cong \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ where (r, θ) denotes the polar coordinates. The integral curves of X define a smooth foliation $\mathcal{F}_T$ on $\mathbb{C} \setminus \{O\}$. The constant -6 has no significance at this stage.

Now the turbulization on the side of the Fermat cubic component is described as follows. The foliation $\tilde{\mathcal{L}} = \{F_\theta\}$ on $S^5 \setminus N$ by the Milnor fibres and the pull-back foliation $f^{-1}\mathcal{F}_T$ coincide with each other on $W_* \setminus W_{r_2}$. On the Fermat cubic component $C = S^5 \setminus \operatorname{Int} W$, Lawson's foliation $\mathcal{L}|_C$ is obtained as $\mathcal{L}|_{S^5 \setminus W_{r_2}} = \tilde{\mathcal{L}}|_{S^5 \setminus W_{r_2}}$ and $\mathcal{L}|_{W_* \setminus W} = f^{-1}\mathcal{F}_T|_{W_* \setminus W}$. Let L_θ denote one of the resulting leaves which contains $F_\theta \setminus W_{r_2}$. L_θ is diffeomorphic to F_θ and only the embedding of the product end $N \times \{r \cdot e^{i\theta}\}$ is modified by the turbulization procedure. We will fix an identification of L_θ with $\mathbf{F}_{e^{i\theta}}$ in Section 3.

On the tube component W, we can describe the foliation $\mathcal{L}|_W$ by a simple turbulization as above, while it is also described by using the "Reeb component" on $S^1 \times D^2$ as follows. The tube component W is diffeomorphic to $N \times D^2$ and $N = Nil^3(-3)$ is an S^1-bundle over the elliptic curve E_ω. We take a smooth coordinate (x, y) for E_ω where $x, y \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then the projection from E_ω to $S^1 \ni x$ gives rise to a fibration of N over S^1 with fibre T^2 and the monodromy $\left(\begin{smallmatrix} 1 & -3 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore the tube component W fibers over the solid torus $S^1 \times D^2$ with the fibre T^2 and the monodoromy $\left(\begin{smallmatrix} 1 & -3 \\ 0 & 1 \end{smallmatrix}\right)$, namely, $W = \mathbb{R} \times D^2 \times T^2 / \sim$ where $(x + 2\pi, P, \left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)) \sim (x, P, \left(\begin{smallmatrix} 1 & -3 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right))$.

As $\mathcal{L}|_W$, we can take the pull-back of the standard Reeb component $\mathcal{F}_R$ on $S^1 \times D^2$ to W. Thus we obtain Lawson's foliation $\mathcal{L}$ on S^5 with a unique compact leaf which is diffeomorphic to Kodaira-Thurston's nil-manifold $K = S^1 \times N$.

Remark 1.3. The 3-dimensional nil-manifold $Nil^3(c_1)$ is often presented as the quotient $Nil^3(c_1) = \Gamma(c_1) \setminus H$ of the 3-dimensional Heisenberg group H by its lattice $\Gamma(c_1)$, which are defined as

$$H = \left\{ \begin{pmatrix} 1 & \overline{x} & \overline{z} \\ 0 & 1 & \overline{y} \\ 0 & 0 & 1 \end{pmatrix} ; \overline{x}, \overline{y}, \overline{z} \in \mathbb{R} \right\}$$
$$\supset \Gamma(c_1) = \left\{ \begin{pmatrix} 1 & \overline{x} & \overline{z} \\ 0 & 1 & \overline{y} \\ 0 & 0 & 1 \end{pmatrix} ; \overline{x}, \overline{y}, c_1\overline{z} \in \mathbb{Z} \right\}.$$

In the case $c_1 < 0$, $\overline{z}$ must be understood to have opposite sign. In this coordinate on H take $\frac{\partial}{\partial \overline{x}}$, $\frac{\partial}{\partial \overline{y}}$, and $\frac{\partial}{\partial \overline{z}}$ at the unit element, and then extend them to be X, Y, and Z as left invariant vector fields. Let $d\overline{x}$, $d\overline{y}$, and $\overline{\zeta}$ be the dual basis for the invariant 1-forms, which satisfies $d\overline{\zeta} = d\overline{x} \wedge d\overline{y}$. On our $N = Nil^3(-3)$ we have $x = 2\pi\overline{x}$, $y = 2\pi\overline{y}$, $z = 2\pi c_1\overline{z}$, and $\zeta = 2\pi c_1\overline{\zeta}$.

2. Symplectic forms on the Kodaira-Thurston nil-manifold and $\mathbf{F_0}$

In this section, we describe natural symplectic forms on Kodaira-Thurston's 4-dimensional nil-manifold and show that the tube component $\mathcal{L}|_W$ admits a smooth leafwise symplectic structure which is *tame* around the boundary.

Definition 2.1. A smooth *leafwise symplectic structure* (or *form*) on a smooth foliated manifold $(M, \mathcal{F})$ is a smooth leafwise closed 2-form β which is non-degenerate on each leaves.

More precisely, first, β is a smooth section to the smooth vector bundle $\bigwedge^2 T^*\mathcal{F}$. For smooth sections to $\bigwedge^* T^*\mathcal{F}$ naturally the exterior differential in each leaves is defined. This exterior differential is often denoted by $d_{\mathcal{F}}$. β is required to be $d_{\mathcal{F}}$-closed and is non-degenerate in each leaves, namely, $d_{\mathcal{F}}\beta = 0$ holds and $\beta^{\dim \mathcal{F}/2}$ defines a volume form on each leaves.

The existence of such β is equivalent to that of a smooth 2-form $\tilde{\beta}$ on M whose restriction to each leaf is a symplectic form of the leaf. It should be remarked that $\tilde{\beta}$ may not be closed as a 2-form on M. In this paper we do not have to distinguish β and $\tilde{\beta}$ and sometimes make abuse of these.

Definition 2.2. Let $(M, \mathcal{F})$ be a smooth foliated manifold with a boundary compact leaf ∂M and a leafwise symplectic form β. $(M, \mathcal{F}, \beta)$ is *tame* around the boundary if the triple satisfies the following condition. We also simply say that β is tame.

(1) The (one-sided) holonomy of the boundary leaf is trivial as C^∞-jet.

(2) There exists a collar neighbourhood $V \cong [0, \epsilon) \times \partial M$ of the boundary ∂M with the projection $Pr : [0, \epsilon) \times \partial M \to \partial M$ for which $\beta|_V$ coincides with the restriction to the leaves of the pull-back $Pr^*(\beta|_{\partial M})$.

Corollary 2.3. *Let $(M_i, \mathcal{F}_i, \beta_i)$ $(i = 0, 1)$ be two foliated manifolds with leafwise symplectic structures. Assume that both are tame around their boundaries and there exists a symplectomorphism*

$$\varphi : (\partial M_1, \beta_1|_{\partial M_1}) \to (\partial M_2, \beta_2|_{\partial M_2})$$

between their boundaries. Then gluing by φ *yields a smooth foliated manifold* $(M = M_1 \cup_\varphi M_2, \mathcal{F}, \beta)$ *with a smooth leafwise symplectic structure.*

We adopt this corollary for our construction. The existence of tame leafwise symplectic structures on the Fermat cubic component is discussed in Section 3, 4, and 5. In this section we show the existence of tame ones on the tube component.

For the 3-dimensional Reeb component $(S^1 \times D^2, \mathcal{F}_R)$, as the leaves are 2-dimensional, it is easy to show that there exists a tame leafwise symplectic structure. For any area form of the boundary, extend it to a collar neighbourhood so as to satisfy the tameness condition, and then further extend it to a leafwise 2-form on the whole component so that on each leaf it gives an area form. In this construction we can start with the standard area form $dx \wedge d\theta$ of the boundary under the coordinates (x, r, θ) for $S^1 \times D^2 = \{(x, r, \theta) \ ; \ x \in S^1 = \mathbb{R}/2\pi\mathbb{Z},\ 0 \le r \le r_0,\ \theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}\}$. The resulting tame leafwise symplectic form on $(S^1 \times D^2, \mathcal{F}_R)$ is denoted by β_R.

Now let us construct tame leafwise symplectic forms on the tube component, using the description of the tube component in the end of the previsous section. Let ζ denote the standard connection 1-form for the Hopf fibration $h : S^5 \to \mathbb{C}P^2$. ζ coincides with the standard contact form $\zeta = \sum_{j=1}^{3}(x_j dy_j - y_j dx_j)$ on S^5. On each fibre (with an arbitrary reference point) ζ defines an identification with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and the resulting coordinate is denoted by z in the previous section. Once ζ is restricted to $N = F \cap S^5$ it is denoted by ζ_N.

The tube component W admits a flat bundle structure $T^2 \hookrightarrow W = N \times D^2 \overset{\pi_R}{\to} S^1 \times D^2$ with the monodromy $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$. On N we have

$$d\zeta_N = -(-3)2\pi \frac{dx}{2\pi} \wedge \frac{dy}{2\pi} = \frac{3}{2\pi} dx \wedge dy$$

and hence $dy \wedge \zeta_N$ is a closed 2-form which restricts to a holonomy invariant area form on each fibre $\cong T^2$.

On $\mathcal{L}|_W$ we also have a closed 2-form ${\pi_R}^* \beta_R$ which is pulled back from the Reeb component. Therefore we obtain a tame leafwise symplectic form $\beta_{W,\lambda,\mu} = \lambda\, {\pi_R}^* \beta_R + \mu\, dy \wedge \zeta_N$ for non-zero constants λ and μ. The restriction of $\beta_{W,\lambda,\mu}$ to the boundary ∂W is presented as $\lambda\, d\theta \wedge dx + \mu\, dy \wedge \zeta_N$. Also it is easy to see that the foliation $\mathcal{L}|_W$ (in fact $\mathcal{L}$ itself) and the leafwise symplectic form $\beta_{W,\lambda,\mu}$ is invariant under the Hopf flow $H(t)$. We have established the following.

Proposition 2.4. *On the tube component Lawson's foliation $\mathcal{L}|_W$ admits a tame leafwise symplectic form $\beta_{W,\lambda,\mu} = \lambda\,{\pi_R}^*\beta_R + \mu\,dy \wedge \zeta_N$ for constants $\lambda \neq 0$ and $\mu \neq 0$, which is invariant under the Hopf flow. It restricts to $\beta_{K,\lambda,\mu} = \lambda\,d\theta \wedge dx + \mu\,dy \wedge \zeta_N$ on the boundary leaf $K = S^1 \times N$.*

In order to make a better correspondence of these computations on W with the Fermat cubic component, we introduce a new coordinate variable $\tau = 2\log\rho$ where $\rho = \sqrt{|Z_0|^2 + |Z_1|^2 + |Z_2|^2}$ on $\mathbb{C}^3$. In the following sections our Kodaira-Thurston nil-manifold $K = \partial W = S^1 \times N$ is also regarded as $K = \check{\mathbf{F}}_0/_\sim$ where in the polar coordinate $\mathbb{R}_+ \times S^5$ for $\check{\mathbb{C}}^3$ we have exactly $\check{\mathbf{F}}_0 = \mathbf{F}_0 \setminus \{\mathrm{O}\} = \mathbb{R}_+ \times N$ and the quotient is given as $P \sim Q$ for P and $Q \in \check{\mathbf{F}}_0$ iff $Q = e^{n\pi}P$ for some $n \in \mathbb{Z}$. Here, as in the $S^1 = \mathbb{R}_+/_\sim$-factor, the period is 2π in τ coordinate, τ exactly corresponds to θ and the above symplectic form is also presented as $\lambda\,d\tau \wedge dx + \mu\,dy \wedge \zeta_N$ just by replacing θ with τ.

This change of coordinates is because of the following reasons. In the quotient space $S^1 \times S^5 = \mathbb{R}_+/_\sim \times S^5$, $\mathbf{F}_1$ spirals to $K = \mathbb{R}_+/_\sim \times N$. On the Fermat cubic component C, any of the interior leaf is diffeomorphic to the Fermat cubic surface $\mathbf{F}_1$, and is spiraling to the boundary leaf K, almost exactly in the same way as above. More precise correspondence will be explained in Section 3 and in Section 4.

From K the infinite cyclic covering pulls $\beta_{K,\lambda,\mu}$ back to a periodic symplectic form $\beta_{0,\lambda,\mu} = \lambda\,d\tau \wedge dx + \mu\,dy \wedge \zeta_N = \lambda\,\frac{2}{\rho}d\rho \wedge dx + \mu\,dy \wedge \zeta_N$ on $\check{\mathbf{F}}_0$. On the other hand, $\check{\mathbf{F}}_0$ inherits a natural symplectic structure from $(\mathbb{C}^3, \beta^* = 2\sum_{j=0}^{2} dx_j \wedge dy_j)$. One of its standard Liouville forms (the primitive of symplectic form) $\lambda^* = \sum_{j=0}^{2}(x_j dy_j - y_j dx_j)$ is presented as $\rho^2\zeta_N$ in the polar coordinate. Replacing ρ^2 with $\overline{\rho}$, we see that $(\check{\mathbf{F}}_0, \beta_0 = d(\overline{\rho}\zeta_N))$ is the symplectization of the contact manifold (N, ζ_N). Replacing $\overline{\rho}$ with e^τ, we have an identification of $\check{\mathbf{F}}_0$ with $\mathbb{R} \times N \ni (\tau, P)$. We will call this the *product coordinate* on $\check{\mathbf{F}}_0$. The difference between $\beta_{0,\lambda,\mu}$ and β_0 will be the main topic.

3. Leafwise symplectic structure on the Fermat cubic component

3.1. Main theorem

Theorem 3.1. *Lawson's foliation $\mathcal{L}$ on S^5 admits a smooth leafwise symplectic form.*

This is the main result of the present article. Proposition 2.4 and Corollary 2.3 imply that this is a direct consequence of the following proposition, which we prove in this section assuming Corollary 5.2.

Proposition 3.2. *For a sufficiently small constant* $0 < \mu \ll 1$ *and a sufficiently large constant* $\lambda \gg 1$ *Lawson's foliation restricted to the Fermat cubic component admits a tame symplectic form which restricts to* $\beta_{K,\lambda,\mu} = \lambda\, d\theta \wedge dx + \mu\, dy \wedge \zeta_N$ *on the boundary leaf* K.

3.2. Coordinates

As a preparation, we start with a more precise description of tubular neighbourhoods of N in S^5 and those of $\check{\mathbf{F}}_0$ in $\check{\mathbb{C}}^3$. For the tubular neighbourhood W_ε of N in S^5, we describe an identification of W_ε with $D^2_\varepsilon \times N$. As $f : W_\varepsilon \to D^2_\varepsilon \subset \mathbb{C}$ defines the first projection, it is enought to define the projection to N.

In Section 1 we chosesmall enough. If necessary, we take ε smaller again so that the following holds. Let νN denote the normal bundle to to $N \subset S^5$. With respect to the standard metric induced from $\mathbb{C}^3$, there exists a radius $\delta > 0$ for νN such that the exponential map $\nu N \to S^5$ is diffeomorphic up to the radius δ. We choose $\varepsilon > 0$ so that W_ε is included in this diffeomorphic image. Thus the inverse of the exponential map defines the projection from W_ε to N. Simply, this projection asigns to any point in W_ε its nearest point in N. As the Hopf action is isometric and leaves N invariant, this projection is equivariant with respect to the Hopf action.

We do the same on each sphere $S^5(\rho)$ of the radius $\rho \geq 1$. Remark that once ε has been chosen small enough for $\rho = 1$, we can choose $\rho^3\varepsilon$ on $S^5(\rho)$ thanks to the homothety. But for $\rho \geq 1$, ε is enough because $\varepsilon \leq \rho^3\varepsilon$. Then the projection to $\check{\mathbf{F}}_0 \cap S^5(\rho) = \rho \cdot N$ and thus to N is defined.

Collecting these identifications with respect to each $\rho \geq 1$, it also defines an identification of the tubular neighbourhood $\check{\mathbb{C}}^3 \cap \{\rho \geq 1\} \cap \{|f| < \varepsilon\}$ of $\check{\mathbf{F}}_0 \cap \{\rho \geq 1\}$ with $(\check{\mathbf{F}}_0 \cap \{\rho \geq 1\}) \times D^2_\varepsilon$ and with $[1, \infty) \times N \times D^2_\varepsilon$. The end $\mathbf{F}_w \cap \{\rho \geq 1\}$ of the Fermat cubic surface $\mathbf{F}_w$ for $|w| \leq \varepsilon$ is exactly the graph of the constant function $\equiv w$. From this we see that these ends are diffeomorphic to $[1, \infty) \times N$, which is called the *product coordinate* of the end. When $\tau = 2 \log \rho$ replaces ρ, the identification is changed into $[0, \infty) \times N \ni (\tau, P)$ and is also called the product coordinate.

Remark 3.3. We have remarked that the identification of the end of $\mathbf{F}_w$ with that of $\check{\mathbf{F}}_0$ is equivariant with respect to the Hopf flow. Therefore the product coordinate is also equivariant under the Hopf flow $H(t)$ because the action of $H(t)$ on $\mathbb{R}_+ \times N \times D^2_\varepsilon$ is described as $(\rho, P, z) \mapsto (\rho, H(t) \cdot P, e^{i3t}z])$ where $H(t) \cdot P$ is the restriction of the Hopf flow on $N \subset S^5$ and is nothing but the fibrewise multiplication by e^{it} on each fibre of $S^1 \hookrightarrow N \to E_\omega$. Similarly, on $W_\varepsilon \setminus N\, (\cong N \times D^2_\varepsilon) \cong N \times (0, \varepsilon) \times S^1$, the Hopf action is indicated as $H(t)(P, r, \theta) = (H(t) \cdot P, r, \theta + 3t)$.

Remark 3.4. For any $0 < \epsilon \leq \varepsilon$, as $\mathbf{F}_\epsilon$ triply covers $\mathbb{C}P^2 \setminus E_\omega$ and the product coordinate is equivariant with respect to the monodromy $H(\frac{2\pi}{3}\mathbb{Z})$, the product coordinate $\mathbf{F}_\epsilon \cong [0, \infty) \times N$ induces a product coordinate of the end of $\mathbb{C}P^2 \setminus E_\omega \cong [0, \infty) \times N'$, where $N' = -\partial U \cong N/_{\mathbb{Z}/_3} \cong Nil^3(-9)$. We will make use of this in Corollary 4.2 and in Lemma 5.3.

3.3. Proof of main proposition

We prove Proposition 3.2 assuming Corollary 5.2.

First let us introduce a leafwise symplectic sturcture on the Milnor fibration $(S^5 \setminus N,\ \{F_\theta\})$. Then we will check that along the turbulization process in Section 1 the leafwise symplectic sturcture on the Milnor fibration naturally gives rise to what we are looking for.

Following Corollary 5.2, take and fix a symplectic form $\beta_{\lambda,\mu}$ on $\mathbf{F}_{\varepsilon'}$. (The constant ε' is given in Theorem 4.1 and in Proposition 4.3.) Each leaf F_θ is identified with $\mathbf{F}_{\varepsilon'}$ through the projection and the Hopf actions; $\mathbf{F}_{\varepsilon'} \xrightarrow{p} F_0 \xrightarrow{H(t)} F_\theta$ for $t = \frac{\theta + 2k\pi}{3}$ (mod $2\pi\mathbb{Z}$) ($k = 0,\, 1,\, 2$). Let $p_{\theta,k} = H(t) \circ p = p \circ H(t) : \mathbf{F}_{\varepsilon'} \to F_\theta$ denote this identification. Respecting the continuity of the identification, we can not decide which one to choose among three values of t. However as $\beta_{\lambda,\mu}$ is invariant under the action of $H(\frac{2\pi}{3}\mathbb{Z})$ on $\mathbf{F}_{\varepsilon'}$, these indentifications induce a well-defined symplectic form on each F_θ from $(\mathbf{F}_{\varepsilon'},\ \beta_{\lambda,\mu})$, which gives rise to a smooth leafwise symplectic structure $\beta_{\tilde{\mathcal{L}}}$ on the Milnor fibration.

Next we go back to the turbulizing process of obtaining the Fermat cubic component. Here we need a pointwise identification of each interior leaf L_θ of the Fermat cubic component with F_θ. On $W_{r^*} \setminus N \cong N \times (D^2_{r^*} \setminus \{0\})$, take a vector field $\tilde{X}$ which is defined as $(0, X)$, where X is the vector field defined on $D^2_{r^*}$ in Section 1 for the turbulization. Also take a vector field $\tilde{R} = (0, -6r\frac{\partial}{\partial r})$.

We identify L_0 with F_0 and with $\mathbf{F}_{\varepsilon'}$ as follows. The core part $L_0 \setminus W_{r_2}$ is exactly identical with $F_0 \setminus W_{r_2}$. For $t > -\frac{1}{6}(\log r_* - \log r_2)$, the point $\exp(t\tilde{X})(P, r_2, 0)$ of L_0 is identified with the point $\exp(t\tilde{R})(P, r_2, 0)$ of F_0.

Accordingly, these points are identified with the point (τ_2+t, P) of $\mathbf{F}_1$ in the product coordinate, where $\tau_2 = -\frac{2}{3}\log r_2$. Similarly $\exp(t\tilde{X})(P, r_2, \theta) \in L_\theta$ and $\exp(t\tilde{R})(P, r_2, \theta) \in F_\theta$ are identified. Let $lm_\theta : F_\theta \to L_\theta$ denote this identification. Remark that through these identifications a point (τ, P) in the end of $\mathbf{F}_{\varepsilon'}$ for large enough $\tau \gg 0$ is sent to a point $lm_0 \circ p(\tau, P) = (P, r(\tau), \tau + c_0)$ in $L_0 \cap W_{r_2}$ for some function $r(\tau)$ and some constant c_0. Also we have $lm_\theta \circ p_{\theta,k}(\tau, P) = (H(\frac{\theta+2k\pi}{3})P, r(\tau), \tau + c_0 + \theta)$.

The leafwise symplectic form $\beta_{\tilde{\mathcal{L}}}$ on the Milnor fibration is thus transplanted on the interior of the Fermat cubic component C of Lawson's foliation $\mathcal{L}$ to be a leafwise symplectic form β_C. What remains to prove is that $\beta_C|_{W_{r_1}\setminus\overline{W}}$ coincides with $Pr^*\beta_{K,\lambda,\mu}$ where Pr denotes the projection of the end $W_{r_1}\setminus\overline{W} \cong N \times (r_0, r_1) \times S^1$ of C to the boundary $\partial C \cong N \times \{r_0\} \times S^1 \cong N \times S^1 (= K)$. Then we obtain a tame symplectic form on C with the restriction $\beta_{K,\lambda,\mu}$ to the boundary. By Corollary 2.3 the proof will be completed.

From the above preparations, we see that the composition of the maps $Pr \circ lm_\theta \circ p_{\theta,k}$: [the end of $\mathbf{F}_{\varepsilon'}$] $= (T, \infty) \times N \to \partial C = N \times S^1$ sends the points as $(\tau, P) \mapsto (H(\frac{\theta+2k\pi}{3})P, \tau + c_0 + \theta))$ for some $T \gg 0$. As is mentioned in Corollary 5.2, $\beta_{\lambda,\mu}|_{(T,\infty)\times N}$ is invariant under the Hopf flow and the τ-translation, for any θ and $k \in \mathbb{Z}$, $(Pr \circ lm_\theta \circ p_{\theta,k})_*\beta_{\lambda,\mu}|_{(T,\infty)\times N}$ coincide with each other and in fact with $\beta_{K,\lambda,\mu}$. □

4. Natural symplectic structure on Fermat cubic surface

4.1. Statements and notations

The natural symplectic structure which $\mathbf{F}_{\varepsilon'}$ inherits from $\mathbb{C}^3$ coincides with that of $\check{\mathbf{F}}_0$ on the product ends after a small isotopy inside $\mathbf{F}_{\varepsilon'}$ and this fact is not difficult to show. However, for our purpose the following weaker result suffices and is much easier to prove.

Theorem 4.1. *For a sufficiently small $\varepsilon' > 0$, there exists a symplectic form β_1 on the Fermat cubic surface $\mathbf{F}_{\varepsilon'}$ which satisfies the following properties.*

(1) *On the end $\mathbf{F}_{\varepsilon'} \cap \{\tau \geq 2\pi\}$ in the product coordinate $[0, \infty) \times N$, $\beta_1|_{[2\pi,\infty)\times N} = d(e^\tau \zeta_N)$.*

(2) *β_1 is invariant under the Hopf action $H(t)$ for $t \in \frac{2\pi}{3}\mathbb{Z}$.*

Corollary 4.2. *For the same $\varepsilon' > 0$ as above, there exists a symplectic form β' on $\mathbb{C}P^2 \setminus E_\omega$ whose restriction to the the product end satisfies*

$$\beta'|_{[2\pi,\infty)\times N'} = d(e^\tau \zeta_{N'})$$

with respect to the product coordinate $[0,\infty) \times N'$.

See Remark 3.4 for the product coordinates for the ends of $\mathbb{C}P^2 \setminus E_\omega$ and N'. The natural contact 1-form $\zeta_{N'}$ is obtained as $\overline{\zeta}$ in Remark 1.3 and is also obtained as the quotient $(N', \zeta_{N'}) = (N, \zeta_N)/_{\mathbb{Z}/3}$. Because $\mathbb{C}P^2 \setminus E_\omega$ is regarded as the quotient of $\mathbf{F}_1$ by the Hopf action restricted to $\mathbb{Z}/3$, on its end the product coordinate $\cong (T,\infty) \times N'$ is also naturally induced from the product coordinate $(T,\infty) \times N$ for the end of $\mathbf{F}_1$ by simply regarding $N' = N/_{\mathbb{Z}/3}$.

4.2. Re-embedding of Fermat cubic surfaces

We prove Theorem 4.1. On the tubular neighbourhood $\{|f| \leq \varepsilon\} \cap \{\tau \geq 0\}$ of $\check{\mathbf{F}}_0 \cap \{\tau \geq 0\}$ in $\mathbb{C}^3 \cap \{\tau \geq 0\}$, take the product coordinate $[0,\infty) \times N \times D^2_\varepsilon$. The end of an affine surface $\{\tau \geq 1\} \cap \mathbf{F}_w$ is the graph of the constant function $c_w \equiv w : [0,\infty) \times N \to D^2_\varepsilon$ in the product coordinate.

Take a smooth function $\phi : [0,\infty) \to [0,1]$ satisfying the conditions

$$\phi(\tau) \equiv 1 \quad \text{for} \quad \tau \in [0,1] \quad \text{and} \quad \phi(\tau) \equiv 0 \quad \text{for} \quad \tau \in [\pi,\infty)$$

and consider the graphs of the functions

$$\sigma_w : [0,\infty) \times N \to D^2_\varepsilon, \quad \sigma_w(\tau, P) = \phi(\tau) w$$

for $w \in D^2_\varepsilon$. The part $\{\tau \geq \pi\}$ of the graph coincides with $\check{\mathbf{F}}_0 \cap \{\tau \geq \pi\}$ and thus symplectic as submanifold of $(\mathbb{C}^3, \beta^*)$. (See the last paragraph of Section 2 for β^*.) The family $\{\sigma_w \,;\, w \in D^2_\varepsilon\}$ apparently depends smoothly on w and the graphs converge to $\mathbf{F}_0 \cap \{\tau \geq 0\}$ when $w \to 0$. Therefore the following proposition follows easily from the compactness of $[0, 2\pi] \times N$.

Proposition 4.3. *There exists $0 < \varepsilon' \leq \varepsilon$ such that the graph of σ_w for $|w| \leq \varepsilon'$ is a symplectic submanifold of $(\mathbb{C}^3, \beta^*)$.*

For example, if we take $w = \varepsilon'$, the affine surface $\mathbf{F}_{\varepsilon'}$ has another smooth embedding into $\mathbb{C}^3$ which is modified from the original one only on the end $\{\tau \geq 0\}$ by the graph of $\sigma_{\varepsilon'}$ which is again, a symplectic submanifold and coincides with $\check{\mathbf{F}}_0$ on the end. Because the new embedding is respecting the product coordinate, it is equivariant under the Hopf action $H(t)$ for $t \in \frac{2\pi}{3}\mathbb{Z}$. The core part $\mathbf{F}_{\varepsilon'} \cap \{\tau \leq 1\}$ is unchanged, it is also invariant under $H(\frac{2\pi}{3}\mathbb{Z})$. It is clear that β^* is invariant under the Hopf flow and $\beta^*|_{\check{\mathbf{F}}_0} = d(e^\tau \zeta_N)$. Therefore the induced symplectic form β_1 from $(\mathbb{C}^3, \beta^*)$ by the new embedding is the desired one in Theorem 4.1. □

5. End-periodic symplectic form on Fermat cubic surface

Based on the preparations in the preceding sections, we prove the following results, which are the core part of this article. We use the product coordinate $(\tau, P') \in (-\frac{2}{3}\log\varepsilon, \infty) \times N' \cong U \setminus E_\omega \subset \mathbb{C}P^2 \setminus E_\omega$.

Theorem 5.1. *For a sufficiently small constant $0 < \mu \ll 1$ and sufficiently large constants $T \gg 2\pi$ and $\lambda \gg 1$, there exists a symplectic form $\beta'_{\lambda,\mu}$ on $\mathbb{C}P^2 \setminus E_\omega$ which restricts to $\lambda\, d\tau \wedge dx + \mu\, dy \wedge \zeta_{N'}$ on its end $(T, \infty) \times N'$.*

Corollary 5.2. *For the same constants as above, there exists a symplectic form $\beta_{\lambda,\mu}$ on the Fermat cubic surface $\mathbf{F}_{\varepsilon'}$ which restricts to $\lambda\, d\tau \wedge dx + \mu\, dy \wedge \zeta_{N'}$ on its end $(T, \infty) \times N$. $\beta_{\lambda,\mu}$ is invariant under the Hopf action of $H(t)$ for $t \in \frac{2\pi}{3}\mathbb{Z}$. On the end $(T, \infty) \times N$ naturally $\beta_{\lambda,\mu}$ admits more symmetries, namely, it is invariant under the translations in τ-direction and also under the Hopf flow $H(t)$ for any $t \in \mathbb{R}$.*

Being independent of the main result Theorem 3.1 of this article, this result might have an interest and an importance by itself. In the final section, we will make a brief discussion on the generalization of this result, namely, the (non-)existence of an end-periodic symplectic structure on Stein or globally convex symplectic manifolds. In the rest of this section, we prove the above theorem.

Lemma 5.3. *There exists a closed 2-form κ on $\mathbb{C}P^2 \setminus E_\omega$ which restricts to $dy \wedge \zeta_{N'}$ on the product end.*

Proof of Lemma 5.3. As $dy \wedge \zeta_{N'}$ is closed, it defines a de Rham cohomology class $[dy \wedge \zeta_{N'}] \in H^2(N') \cong \mathbb{R}^2$. Let us look at the Meyer-Vietoris exact sequence for the cohomologies of $\mathbb{C}P^2 = \overline{U} \cup (\mathbb{C}P^2 \setminus U)$. It is easy to see that the inclusion $N' \hookrightarrow \overline{U}$ induces a trivial map $0 : H^2(U) \to H^2(N')$. Therefore the fact $H^3(\mathbb{C}P^2) = 0$ and the long exact sequence tells that the inclusion to the other side induces a surjective homomorphism $H^2(\mathbb{C}P^2 \setminus U) \twoheadrightarrow H^2(N')$.

This also implies that the closed 2-form $dy \wedge \zeta_{N'}$ on the product end extends to the whole $\mathbb{C}P^2 \setminus E_\omega$ as a closed 2-form κ. □

Remark 5.4. The origin of $\mathbf{F}_0$ is an isolated singularity of *simple elliptic* type. Up to 3-fold branched covering U is orientation-reversing diffeomorphic to the minimal blowing up resolution of $\mathbf{F}_0$. On the other hand, the 3-fold covering of $\mathbb{C}P^2 \setminus E_\omega$ is biholomorphic to the Milnor fibre $\mathbf{F}_1$. The above lemma reflects the fact that the resolution and the Milnor fibre are quite different to each other. Such a phenomenon does not happen for *simple* singularities. For singularity theory, see for example [D].

Let us proceed to construct an end-periodic symplectic form on $\mathbb{C}P^2 \setminus E_\omega$. First take a positive constant μ small enough so that $\beta' + \mu\, dy \wedge \zeta_{N'}$ is still a symplectic form. On the product end $\{\tau \geq 2\pi\}$, from Corollary 4.2 we know $\beta' = d(e^\tau \zeta_{N'}) = e^\tau d\tau \wedge \zeta_{N'} + e^\tau \frac{3}{2\pi} dx \wedge dy$. This implies $\beta' \wedge dy \wedge \zeta_{N'} = 0$ and hence we have $(\beta' + \mu\, dy \wedge \zeta_{N'})^2 = \beta'^2$ on the product end. Therefore if we choose μ small enough, we can assure that even on the compact core $\mathbb{C}P^2 \setminus U$, the closed 2-form $\beta' + \mu\, dy \wedge \zeta_{N'}$ is non-degenerate. We fix such μ.

Next take constants $2\pi < T_0 < T_1 < T_2 < T_3 = T$ and non-negative smooth functions $k(\tau)$ and $l(\tau)$ of τ on $[T_0, \infty)$ satisfying the following conditions:

$$
\begin{array}{rll}
k(\tau) = e^\tau, & l(\tau) \equiv 0 \ : & T_0 \leq \tau \leq T_1, \\
k'(\tau) > 0\,, & l(\tau) > 0 \ : & T_1 \leq \tau < T_2, \\
k(\tau) > 0\,, & l(\tau) \equiv \lambda \ : & T_2 \leq \tau \leq T_3, \\
k(\tau) \equiv 0\,, & l(\tau) \equiv \lambda \ : & T_3 \leq \tau.
\end{array}
$$

This is done as follows. First choose such a smooth function k. Then take a constant $\lambda > 0$ satisfying $\max\{-\frac{3k'(\tau)k(\tau)}{4\mu\pi}\,;\, T_2 \leq \tau \leq T_3\} < \lambda$. Then it is easy to find a smooth function $l(\tau)$ which satisfies all of the above conditions.

Now we are ready to construct an end-periodic symplectic form. First modify β' on the product end. We can define $\beta'_\sharp$ as

$$
\beta'_\sharp = \begin{cases} \beta' & \text{on } \mathbb{C}P^2 \setminus U\,, \\ d(k(\tau)\zeta_{N'}) + l(\tau) d\tau \wedge dx & \text{on } [T_0, \infty) \times N' \end{cases}
$$

because the two presentations of $\beta'_\sharp$ coincide with each other on $[T_0, T_1] \times N'$. Finally we put $\beta'_{\lambda,\mu} = \beta'_\sharp + \mu\kappa$. This is the desired symplectic form on $\mathbb{C}P^2 \setminus E_\omega$ because of the following reasons. First of all, $\beta'_{\lambda,\mu}$ is closed and coincides with $\lambda\, d\tau \wedge dx + \mu\, dy \wedge \zeta_{N'}$ on $[T_3, \infty) \times N'$ and with $\beta' + \mu\, dy \wedge \zeta_{N'}$ on $\mathbb{C}P^2 \setminus U$. Therefore it is non-degenerate on $\mathbb{C}P^2 \setminus U$ as already remarked above. On the product end, as $d(k(\tau)\zeta_{N'})$ and $l(\tau)d\tau \wedge dx + \mu\, dy \wedge \zeta_{N'}$ do not interact at all under the exterior product, we have

$$(\beta'_{\lambda,\mu})^2 = \left(\frac{3k'(\tau)k(\tau)}{2\pi} + 2l(\tau)\mu\right) d\tau \wedge dx \wedge dy \wedge \zeta_{N'}.$$

Therefore $\beta'_{\lambda,\mu}$ is non-degenerate on the product end as well. □

6. $\widetilde{E}_7$ and $\widetilde{E}_8$

Among simple elliptic singularities, the following three deformation classes $\widetilde{E}_l$ $(l = 6, 7, 8)$ are known to be realized as isolated hypersurface singularities and their links are ismorphic to $Nil^3(-3)$, $Nil^3(-2)$, and to $Nil^3(-1)$ respectively. They are defined by the following polynomials except for finitely many values of λ.

$$\begin{aligned} f_{\widetilde{E}_6} &= Z_0^3 + Z_1^3 + Z_2^3 \ \ (+\lambda Z_0 Z_1 Z_2) \\ f_{\widetilde{E}_7} &= Z_0^4 + Z_1^4 + Z_2^2 \ \ (+\lambda Z_0 Z_1 Z_2) \\ f_{\widetilde{E}_8} &= Z_0^6 + Z_1^3 + Z_2^2 \ \ (+\lambda Z_0 Z_1 Z_2) \end{aligned}$$

As the smooth topology of these objects does not depend on the choice of the constant λ, in this paper we take it to be 0. Each of our constructions in this paper for the Fermat cubic, *i.e.*, the $\widetilde{E}_6$ polynomial also works in the other two cases. In this section we verify this fact, by briefly reviewing the topology of these singularities. For basic facts about hypersurface singularities, the readers may refer to Milnor's seminal text book [M] as well as Dimca's book [D].

The notations in §1 are used and interpreted in parallel or slightly modified meanings according to the context, unless otherwise specified. For $f = f_{\widetilde{E}_l}$ $(l = 7, 8)$ the origin is an isolated and in fact unique critical point of the polynomial f. Instead of scalar multiplication, we define the weighted homogeneous action of $\lambda \in \mathbb{C}^\times$ on $\mathbb{C}^3$ by $\lambda \cdot (Z_0, Z_1, Z_2) = (\lambda^{w_0} Z_0, \lambda^{w_1} Z_1, \lambda^{w_2} Z_2)$ where the weight vector $\mathbf{w} = (w_0, w_1, w_2)$ takes value $(2, 1, 1)$ [*resp.* $(3, 2, 1)$] for $\widetilde{E}_l$ $(l = 7, 8)$. By this action we have $\lambda \cdot \mathbf{F}_w = \mathbf{F}_{\lambda^4 w}$ [*resp.* $\lambda \cdot \mathbf{F}_w = \mathbf{F}_{\lambda^6 w}$] for $l = 7$ [*resp.* $l = 8$]. The weighted homogeneous action by positive real numbers $\lambda \in \mathbb{R}_+$ plays the role of the euclidean homotheties in the Fermat cubic case. The action by unit complex numbers $\lambda = e^{it}$ $(t \in \mathbb{R})$ restricts to an action on S^5 and is again denoted by $H(t)$ and called the *weighted Hopf* action or flow. The quotient space $P^2_{\mathbf{w}}$ by this weighted Hopf action is called the *weighted projective space*, which is a complex analytic orbifold.

The quotient map $h : S^5 \to P^2_{\mathbf{w}}$ is called the *weighted Hopf fibration*, which is a Siefert fibration. We take $\mathbb{C}P^2 = \{[X_0 : X_1 : X_2]\}$ as a quotient of $P^2_{\mathbf{w}}$ as follows. Define a map $\Phi : \check{\mathbb{C}}^3 \to \mathbb{C}P^2$ as $\Phi : (Z_0, Z_1, Z_2) \mapsto [X_0 : X_1 : X_2] = [Z_0^{d_0} : Z_1^{d_1} : Z_2^{d_2}]$ where $(d_0, d_1, d_2) = (1, 2, 2)$ [*resp.* $(2, 3, 6)$] for $\widetilde{E}_7$ [*resp.* $\widetilde{E}_8$]. Then Φ factors into $\Phi|_{S^5} = \Psi \circ h$ for some $\Psi : P^2_{\mathbf{w}} \to \mathbb{C}P^2$. The homogeneous equations

$$g_{\widetilde{E}_7} = X_0^2 + X_1^2 + X_2^2 = 0$$
$$g_{\widetilde{E}_8} = X_0 + X_1 + X_2 = 0$$

on $\mathbb{C}P^2$ rewrites $f_{\widetilde{E}_l} = 0$ as $g_{\widetilde{E}_l} \circ \Phi = 0$ $(l = 7, 8)$.

The first important fact to notice is that the open set of S^5 consisting of all regular orbits of the weighted Hopf flow $H(t)$ contains $N = \mathbf{F}_0 \cap S^5$. Therefore the orbit space $E_{(l)} = N/_H$ is a non-singular holomorphic curve which sits in the regular part of $P^2_{\mathbf{w}}$.

'$g_{\widetilde{E}_7} = 0$' defines a non-singular projective curve of degree 2 and '$g_{\widetilde{E}_8} = 0$' a projetive line. Both of them are biholomorphic to $\mathbb{C}P^1$. Comparing Φ and $h|_{E_{(l)}}$, we easily see that $\Psi|_{E_{(7)}} : E_{(7)} \to \{X_0^2 + X_1^2 + X_2^2 = 0\}$ is a 2-fold branched covering over the rational curve with 4 branched points $\{X_1 = 0 \text{ or } X_2 = 0\} \cap \{X_0^2 + X_1^2 + X_2^2 = 0\}$ like the Weierstrass $\wp$-function and $E_{(7)}$ is seen to be an elliptic curve. In the case of $\widetilde{E}_{(8)}$, $\Psi|_{E_{(8)}} : E_{(8)} \to \{X_0 + X_1 + X_2 = 0\}$ is a 6-fold branched covering, branching over 3 points $\{X_0 = 0\}$, $\{X_1 = 0\}$, and $\{X_2 = 0\}$ with branch indices 2, 3, and 6 respectively. From this we also see that $E_{(8)}$ is an elliptic curve.

Similarly it is easy to see that the self-intersection (the c_1 of the normal bundle) of $E_{(7)}$ [*resp.* $E_{(8)}$] in $P^2_{\mathbf{w}}$ is 8 [*resp.* 6] and that the c_1 (the euler class) of the weighted Hopf fibrations over $E_{(7)}$ [*resp.* $E_{(8)}$] is -2 [*resp.* -1].

Like in the case of $\widetilde{E}_{(6)}$, in both of the other two cases the weighted projection $\check{\mathbb{C}}^3 \to S^5 = \check{\mathbb{C}}^3/\mathbb{R}_+$ by positive real numbers restricts to a diffeomorphism from $\mathbf{F}_{re^{i\theta}}$ to the Milnor fibre L_θ. $h|_{L_\theta} : L_\theta \to P^2_{\mathbf{w}} \setminus \widetilde{E}_{(l)}$ is a branched covering, but the number of branched points is finite and around the ends it is a 4-fold [*resp.* 6-fold] regular covering for $l = 7$ [*resp.* $l = 8$].

We also remark here that the link N has a product type tubular neighbourhood W in $P^2_{\mathbf{w}}$ because f gives the trivialization. The boundary ∂W is a Kodaira-Thurston nil-manifold and $\check{\mathbf{F}}_0$ can be considered as its cyclic covering.

Now let us verify that our constructions are transplanted to the cases of $\widetilde{E}_7$ and $\widetilde{E}_8$. From the descriptions of the link N, the Milnor fibres L_θ, and of $\mathbf{F}_1$, the contents in Section 1 and 2 are recovered. The fact that the weighted Hopf flow preserves the standard sphere $S^5(\rho)$ of radius ρ and the standard symplectic form β^* implies that the product coordinates introduced in Section 3 can play the same role as in the case of $\widetilde{E}_6$ and the arguments in Section 4 are valid without modifications.

As to the results in Section 5, once a parallel result to Lemma 5.3 is verified, then the manipulations of differential forms on the product end holds without major modifications. Together with the commutative diagram below, the fact that the rational (or real) cohomology of $P^2_{\mathbf{w}}$ is isomorphic to that of $\mathbb{C}P^2$ (see *e.g.*, [D]) tells that a parallel statement to Lemma 5.3 holds.

$$\begin{array}{ccccc} (T,\infty)\times N & \cong & \text{end of } \check{\mathbf{F}}_0 \cong \text{end of } \mathbf{F}_1 & \hookrightarrow & \mathbf{F}_1 \cong L_0 \\ \downarrow & & \downarrow & & \downarrow h|_{L_0} \\ (T,\infty)\times \partial U & \cong & \text{end of } P^2_{\mathbf{w}} \setminus E_{(l)} & \hookrightarrow & P^2_{\mathbf{w}} \setminus E_{(l)} \end{array}$$

The left and the middle vertical arrows are regular coverings and the right one is a branched covering.

7. Concluding remarks

To close the article, we make some comments and raise some questions related to our construction.

7.1. End-periodic symplectic structures on Stein or globally convex symplectic manifolds

The construction of leafwise symplectic structure in this paper seems to stand on a intersection of some fortunes.

Apart from constructing leafwise symplectic foliations of codimension one, as is mentioned in the previous section, the existence of end-periodic symplectic structures on Stein or globally convex symplectic manifolds might be of an independent interest. While the possibility of such situations seems to be limited. let us discuss it.

Example 7.1 (A trivial example). The Stein manifold $\mathbb{C}$ (or the upper half plane $\mathbb{H}$) carries an end-periodic symplectic form.

This example is in many senses trivial, because, first of all the fact itself is trivial. Especially we do not have to change the symplectic form. Also, as this Stein manifold is not really convex, we should say this is a meaningless example. The convexity of symplectic structures must be discussed on manifolds of dimension ≥ 4 (see [EG]). However, this example still exhibits a clear contrast to the following case.

Example 7.2. The Stein manifold $\mathbb{C}^n$ $(n \geq 2)$ does not admit an end-periodic symplectic structure, because $S^1 \times S^{2n-1}$ does not admit a symplectic structure.

As the non-existence this example is generalized to many cases.

For the case of symplectic dimension 4, the recent result by Friedl and Vidussi, which has been known as Taubes' conjecture, provides a strong constraint.

Theorem 7.3 (Taubes' conjecture, Friedl-Vidussi [FV]). *For a closed 3-manifold M, the 4-manifold $W^4 = S^1 \times M^3$ admits a symplectic structure if and only if M fibers over the circle.*

Now in order to make the implication of this theorem clearer, let us take the following definition.

Definition 7.4. Assume that an open $2n$-manifold W has an end which is diffeomorphic to $\mathbb{R}_+ \times M^{2n-1}$ for some closed oriented manifold M. An *end-periodic symplectic structure* on W is a symplectic structure on W whose restriction to the end is invariant under the action of non-negative integers $\mathbb{N}_0$ where $m \in \mathbb{N}_0$ acts on $\mathbb{R}_+ \times M$ as $(t, x) \mapsto (t + m, \varphi^m(x))$ for some fixed monodromy diffeomorphism $\varphi : M \to M$.

It follows directly from the definition that the mapping cylinder M_φ admits a symplectic structure. If the monodromy belongs to a mapping class of finite order, $S^1 \times M$ also admits a symplectic structure.

The above theorem tells that in the case of trivial monodromy, M^3 fibres over the circle. The same construction as in the case of the Kodaira-Thurston nil-manifold in Section 2 gives rise to symplectic structures on the 4-manifold $S^1 \times M$. The virtue of the theorem of Friedl-Vidushi is of course in the converse implication.

Example 7.5 (Surfaces of higher degrees). Instead of taking the Fermat cubic surface as in the present article, if we take, *e.g.*, a Fermat type quartic surface, then the end is diffeomorphic to $\mathbb{R}_+ \times M^3$, where M^3 is an

S^1-bundle over the closed oriented surface Σ_3 of genus 3 with euler class 4. As this 3-manifold apparently does not fiber over the circle, the Fermat quartic surface does not admit an end-periodic symplectic structure with at most finite monodromy. The same applies to the Brieskorn type hyperbolic singularities $\{Z_0^p + Z_1^q + Z_2^r = 0\}$ with $1/p + 1/q + 1/r < 1$.

Example 7.6 (Cusp singularities). For $1/p + 1/q + 1/r < 1$, the polynomials equation $Z_0^p + Z_1^q + Z_2^r + Z_0 Z_1 Z_2 = 0$ defines one of so called the *cusp* singularities at the origin. Remark that a cubic term is added to the above Brieskorn type polynomial, which changes the topology of links and fibres in this case. The link is a solv-3-manifold, which are explained in the next example. Other than the simple elliptic singularities which we treated in this paper, this seems to be only the possible case where the Milnor fibration is modified into a codimension 1 leafwise symplectic foliation. In fact it turns out to be possible, but it requires different arguments so that it will be done in a forthcoming paper [Mi2]. See also [Mo] and the next subsection.

If we extend our scope from Stein manifolds to globally convex symplectic manifolds (see [EG] for details of this notion) we find one more example for the existence of end-periodic symplectic structure, which is slightly less trivial than Example 7.1.

Example 7.7 (Solvable manifold). Let $M^3 = \mathit{Solv}$ be a 3-dimensional compact solv-manifold, namely, the mapping cylinder of a hyperbolic automorphism of T^2.

Then it carries an algebraic Anosov flow of suspension type, whose strong (un)stable direction corresponds to an eigenvalue of the monodromy. Then from [Mi] we know that $\mathbb{R} \times M$ admits a globally convex symplectic structure. As it has a disconnected end, it is not Stein. On the other hand, apparently $\mathbb{R} \times M$ admits an end-periodic symplectic structure because $S^1 \times M$ admits a symplectic structure.

Some of higher genus surface bundles over the circle with pseudo-Anosov monodromy admit Anosov flows. See [Go] and [FH] for Anosov flow on hyperbolic 3-manifolds. Of course in this formulation, Thurston's virtual fibration conjecture (now it is a theorem) is involved. Then for such a 3-manifold M, by [Mi] we know that $\mathbb{R} \times M$ admits a globally convex symplectic structure while it also admits an end-periodic symplectic structure because M fibres over the circle.

These examples are again disappointing because the compact core part has no more topology than the end.

7.2. Foliations on spheres

Meersseman and Verjovsky proved in [MV2] that the tube component of Lawson's foliation does not admit a leafwise complex structure. As it is easy to show that the Fermat cubic component admits a leafwise complex structure, the situation makes a clear contrast against the case of leafwise symplectic structures.

At present, it still seems to the author that the problem on the existence of a foliation of codimension-one on S^5 with smooth leafwise complex structure is still totally open.

Also the existence of codimension-one foliation with leafwise symplectic structures on higher odd dimensional spheres is unknown as well. As mentioned in the previous subsection, on S^5 we can modify the Milnor fibrations of the cusp singularities into foliations of codimension 1 with leafwise symplectic structures ([Mi2] and [Mo]). In higher dimensions the link of isolated hypersurface singularities in the holomorphic category are simply connected ([M]), so that our way of modification of Milnor fibrations into codimension 1 foliations does not apply.

A recent result by G. Meigniez ([Me]) claims that in dimension ≥ 4, once a smooth foliation of codimension one exists on a closed manifold, it can be modified into a minimal one, namely, one with every leaf dense. Especially it follows that in higher dimensions there is no more direct analogue of Novikov's theorem, that is, for any foliation of codimension 1 on S^3 there exists a compact leaf. It is also mentioned by Meigniez that under the presence of some geometric structures, like leafwise symplectic structures, it might be of some interest to ask whether some similar statement to Novikov's theorem holds or not.

It is also an interesting and hard problem whether if there exists a 2-calibrated foliation on S^5, namely, a codimension 1 foliation and a closed 2-form ω on S^5 whose restriction to every leaf is symplectic. Thanks to Meigniez's result, we can eliminate a complact leaf from our foliation, which an apparent obstruction for the 2-calibration, while it seems very difficult to carry out Meigniez's modification respecting leafwise symplectic structures.

References

[D] Alexandru Dimca, Singularities and Topology of Hypersurfaces, Universitext, Springer-Verlag, Berlin (1992).

[EG] Yakov Eliashberg and Mikhael Gromov, *Convex symplectic manifolds*, in: Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., A.M.S. **52** Part 2, (1991), 135–162.

[FH] Patrick Foulon and Boris Hasselblatt, *Contact Anosov flows on hyperbolic* 3*-manifolds*, Geom. Topol. **17** (2013), no. 2, 1225–1252.

[FV] Stefan Friedl and Stefano Vidussi, *Twisted Alexander polynomials detect fibered* 3*-manifolds*, Ann. of Math. (2) **173** (2011), no. 3, 1587–1643.

[Go] Sue Goodman, *Dehn surgery on Anosov flows*, Geometric Dynamics (Rio de Janeiro, 1981), Lecture Notes in Math. **1007**, Springer, Berlin, (1983), 300–307.

[Gr] John W. Gray, *Some global properties of contact structures*, Ann. of Math. (2) **69** (1959), no. 3, 421–450.

[L] H. Blaine Lawson, Jr., *Codimension-one foliations of spheres*, Ann. of Math. (2) **94** (1971), no. 3, 494–503.

[MV1] Laurent Meersseman and Alberto Verjovsky, *A smooth foliation of the* 5*-sphere by complex surfaces*, Ann. of Math. (2) **156** (2002), no. 3, 915–930. See also *Corrigendum to "A smooth foliation of the* 5*-sphere by complex surfaces"*, Ann. of Math. (2) **174** (2011), no. 3, 1951–1952.

[MV2] Laurent Meersseman and Alberto Verjovsky, *On the moduli space of certain smooth codimension-one foliations of the* 5*-sphere by complex surfaces*, J. Reine Angew. Math. **632** (2009), 143–202.

[Me] Gaël Meigniez, *Regularization and minimization of* Γ_1*-structures*, J. Diff. Geom. **107** (2017), no. 1, 157–202.

[M] John W. Milnor, Singular Points of Complex Hypersurfaces, Annals of Mathematics Studies **61**, Princeton University Press, New Jersey, (1968).

[Mi] Yoshihiko Mitsumatsu, *Anosov flows and non-Stein symplectic manifolds*, Ann. Inst. Fourier, Grenoble **45** (1995), no. 5, 1407–1421.

[Mi2] Yoshihiko Mitsumatsu, *Topological flexibility of symplectic convexity and symplectic foliations*, preprint, in preparation.

[Mo] Atsuhide Mori, *A note on Mitsumatsu's construction of a leafwise symplectic foliation*, Int. Math. Res. Not. IMRN, (2018), electronic version, rnx321.

[SV] Pablo Suárez-Serrato and Alberto Verjovsky, private communications on *A smooth codimension-one foliation of the five-sphere by symplectic leaves.*

Department of Mathematics, Chuo University
1-13-27 Kasuga Bunkyo-ku, Tokyo, 112-8551, Japan
E-mail address: yoshi@math.chuo-u.ac.jp

Received July 21, 2015
Accepted January 10, 2017