JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 16, Number 3, 645–699, 2018

Global action-angle variables for non-commutative integrable systems

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In this paper we analyze the obstructions to the existence of global action-angle variables for regular non-commutative integrable systems (regular NCI systems) on Poisson manifolds. In contrast with local action-angle variables, which always exist in the neighborhood of a compact connected component of the regular fibers of the momentum map, global action-angle variables rarely exist. The fact that there are obstructions to the existence of global action-angle variables was first observed and analyzed by Duistermaat in the case of Liouville integrable systems on symplectic manifolds and later by Dazord-Delzant in the case of non-commutative integrable systems on symplectic manifolds. In our more general case where phase space is an arbitrary Poisson manifold, there are more obstructions. Our approach makes use of a few new features which we introduce: the action bundle and the action lattice bundle of the NCI system (these bundles are canonically defined) and three foliations (the action, angle and transverse foliation), whose existence is also subject to obstructions, often of a cohomological nature.

1. Introduction

The notion of a Liouville integrable system on a symplectic manifold [3, Ch. 10] has two natural generalizations, namely the notion of a Liouville integrable system on a Poisson manifold [1, Ch. 4] and the notion of a non-commutative integrable system on a symplectic manifold [4, 13, 14, 21]. These two concepts were merged in [18], where the notion of a non-commutative integrable system on a Poisson manifold was introduced.

A **non-commutative integrable system** (NCI system) on an n-dimensional Poisson manifold (M, Π) is a family $f_1, \ldots, f_s$ of smooth functions on M, such that the first $n-s$ functions are in involution (Poisson commute) with every function in the family:

$$\{f_i, f_j\} = 0, \qquad \text{for } 1 \leqslant i \leqslant n-s,\ 1 \leqslant j \leqslant s,$$

and satisfy an independence condition which will be stated below. The number $r := n - s$ is called the **rank** of the NCI system. The classical case of Liouville integrable systems on a symplectic manifold corresponds to the case where $r = s = n/2$, while the case of superintegrable systems (on a symplectic or Poisson manifold) corresponds to $r = 1$; for other NCI systems, r can be any integer satisfying $2 \leqslant 2r \leqslant n$.

One usually thinks of an NCI system on an n-dimensional phase space M as a Hamiltonian dynamical system X_h on M, associated with some function h, admitting the functions $h = f_1, f_2, \ldots, f_s$, as **first integrals**, i.e., $X_h f_i = \{f_i, h\} = 0$ for $1 \leqslant i \leqslant s$. Then the above definition of an NCI system can be understood as follows: (i) one can first reduce the dynamics of X_h to a generic common level set of all the first integrals $f_1, \ldots, f_s$, thereby reducing the dimension of the phase space by s; (ii) since the first integrals $f_1, \ldots, f_r$ are in involution with all the above first integrals, the flows of the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_r}$ define a local $\mathbb{R}^r$-action which preserves this common level set, so one can further reduce the dimension of the system by r by passing to the quotient space of the level set by the action. Altogether, one can reduce the dimension by $r + s = n$, the dimension of the phase space, which justifies the name "integrable". To be precise, the above dimension count is correct only if we assume independence of the first integrals:

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_s \neq 0, \tag{1.1}$$

as well as of the Hamiltonian vector fields generating the local $\mathbb{R}^r$-action:

$$X_{f_1} \wedge \cdots \wedge X_{f_r} \neq 0. \tag{1.2}$$

The latter condition, in general, does not follow from the former condition because the Poisson tensor may have a non-trivial kernel. We will deal in this paper solely with **regular NCI systems**, i.e., NCI systems such that conditions (1.1) and (1.2) hold at every point of M. The study of singularities of NCI systems (points where at least one of the above conditions fails) is a very important topic in the global theory of completely integrable systems. In the case of Liouville integrable systems there has been substantial progress in the past twenty years in relating singularities and global properties of the system in two degrees of freedom, including the topological theory due to Fomenko and his school (see, e.g., [5]) or the case of semi-toric integrable systems introduced by Pelayo and Vu Ngoc ([22, 23]). We defer the study of singularities of NCI systems to future works.

Examples of NCI systems include, besides Liouville integrable systems, many classical systems such as the motion in a central force field, the Kepler problem, the Euler-Poinsot top and the Gelfand-Cetlin system. Each one of these systems has singularities, but by removing some appropriate closed subset which contains them, we obtain a regular NCI system to which the theory developed here applies.

We assemble the first integrals of an NCI system in a single map $\mathbf{F} = (f_1, \ldots, f_s) : M \to \mathbb{R}^s$, called the **momentum map** of the NCI system. Notice that $\mathbf{F}$ is a submersion when the NCI system is regular. The first important, non-trivial, fact about NCI systems on Poisson manifolds is the action-angle theorem, which was proved in full generality in [18]. We state it here for regular NCI systems for which the fibers of its momentum map are compact and connected; see [18, Theorem 1.1] for a more general statement.

Theorem 1.1 (Existence of local action-angle variables). Let $(M, \Pi, \mathbf{F})$ be a regular NCI system of dimension n and rank $r = n - s$ with compact connected fibers. For any b in the image of $\mathbf{F}$, there exists an open neighborhood U of b in $\mathbb{R}^s$, an open neighborhood V of $\mathbf{F}^{-1}(b)$ in M and an open embedding $\Psi : V \to T^*\mathbb{T}^r \times \mathbb{R}^{s-r}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{F}^{-1}(U) \supset V & \xrightarrow{\ \Psi\ } & T^*\mathbb{T}^r \times \mathbb{R}^{s-r} \\ {\scriptstyle \mathbf{F}_{|V}}\big\downarrow & & \big\downarrow \\ U & \longrightarrow & \mathbb{R}^s \end{array}$$

Moreover, Ψ is a Poisson map if we consider on $T^*\mathbb{T}^r \times \mathbb{R}^{s-r}$ the product of the canonical symplectic structure on $T^*\mathbb{T}^r$ with an appropriate Poisson structure on an open subset of $\mathbb{R}^{s-r}$.

The above theorem is semi-local in the sense that it describes such NCI systems in the neighborhood of any fiber of the momentum map $\mathbf{F}$; such a fiber is diffeomorphic to an r-dimensional torus $\mathbb{T}^r$, just like in the classical Liouville theorem. In terms of the natural coordinates (θ_i, p_i, z_j) on an open subset of $\mathbb{T}^r \times \mathbb{R}^r \times \mathbb{R}^{n-2r} \simeq T^*\mathbb{T}^r \times \mathbb{R}^{s-r}$, the Poisson structure on M takes the following form:

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{1 \leqslant j < k \leqslant s-r} c_{jk}(z) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k},$$

where the second sum is absent in the case of Liouville integrable systems on regular Poisson manifolds of rank $2r$, such as symplectic manifolds. The variables θ_i, p_i and z_j, in that order, are called **angle, action** and **transverse variables** (or **coordinates**; it is understood that the θ_i are S^1-valued).

According to the theorem, the phase space of a regular NCI system (with compact connected fibers) can be covered with charts equipped with action-angle-transverse variables. Of course, these local variables are highly non-unique. Therefore, the question asking whether for a given NCI system these local variables can be glued to yield global variables is a non-trivial one. The main focus in this paper is to describe the different obstructions for this passage from local to global. As an intermediate step, we will also consider the obstructions to the existence of action, angle and transverse foliations, which are weaker than the obstructions for the existence of the corresponding variables, but are in general easier to compute. For each of these obstructions, we will prove their non-triviality in some concrete examples. Our results generalize the ones obtained by Dazord-Delzant [12], who consider the case of non-commutative integrable systems on symplectic manifolds (i.e., M is a symplectic manifold); their result, in turn, generalizes the ones obtained by Duistermaat [10] in the special case of Liouville integrable system on symplectic manifolds.

We now give an outline of the paper and describe the main results.

In Section 2 we recall the notion of a non-commutative integrable system on a Poisson manifold, which we reformulate in geometrical terms (in terms of a foliation) and we initiate the study of the Poisson geometry of such a system. The upshot is that we view regular NCI systems as Poisson maps $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$, whose fibers define a rank r foliation with compact leaves.

A key novelty which is introduced in Section 3 is the **action bundle** E, which is a vector bundle of rank r on B and whose sections generate, upon using the Poisson structure Π, the action vector fields, i.e., the commuting, integrable vector fields which are tangent to the fibers of the momentum map (Proposition 3.2 and Lemma 3.1). When the fibers of the momentum map ϕ are compact and connected, E contains a lattice bundle $L_B \to B$, the **action lattice bundle**, whose sections generate periodic vector fields of period 1; it implies that M is a torus bundle over B (see Section 3.3). A set of action variables of the NCI system is a collection of r functions on B which define a global trivialization of the action lattice bundle (making the torus bundle $\phi : M \to B$ into a principal $\mathbb{T}^r$-bundle); the obstruction to their existence lies in $H^1(B, \mathcal{C}as_B^M)$, where $\mathcal{C}as_B^M$ is the sheaf of functions on B which pull back to Casimir functions on M (Theorem 3.8). Action variables define a (transversely integral affine) foliation on B, which leads to the notion of an **action foliation**. When the action lattice bundle admits a trivialization on B, it defines a cohomology class in $H^1(B, \mathcal{C}as_B^M/\mathbb{R})$, whose nullity is equivalent to the existence of an action foliation. This class is, of course, closely related to $H^1(B, \mathcal{C}as_B^M)$, which is decisive for the existence of action variables (Proposition 3.11).

The existence of angle variables is discussed in Section 4. Interestingly, the notion of angle variables can be defined in terms of the canonically defined action lattice bundle, hence their (global) existence can be studied independently of the existence of action variables, or of a choice of such variables. We show that global angle variables exist if and only if the action lattice bundle is trivial and the momentum map $\phi : M \to B$ admits a section whose image is coisotropic (Theorem 4.7). The latter condition is in an essential way non-linear, hence does not lead to a cohomological obstruction class, as in the case of action variables. However, a set of angle variables defines a pair of foliations, an **angle foliation** (which is a foliation of M) and a **transverse foliation** (which is a foliation of B, transverse to every action foliation, see Propositions 4.8 and 4.9). The obstructions to the existence of such a pair of foliations then leads to obstructions of the existence of global angle variables, which are weaker than the existence of a section whose image is coisotropic, but easier to compute explicitly. We finish Section 4 with a theorem which gives an explicit description of every NCI system for which action-angle variables do exist, under the assumption that all leaves of the action and the transverse foliation intersect in a unique point (Theorem 4.11): in terms of angle variables θ_i and action variables p_i, the

Poisson structure on its phase space M then takes the canonical form

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \pi_{|_A},$$

where A is any leaf of the action foliation (which turns out to be a Poisson submanifold of B).

We finish the paper with a non-trivial example where global action-angle variables exist. Other examples, which are scattered throughout the text, include artificially constructed mathematical examples which illustrate the non-triviality of the obstructions, as well as examples coming from classical mechanics, which turn out to exhibit a large spectrum of phenomena which have a definite impact on the global geometry of NCI systems.

Conventions

In this paper, all manifolds are real, smooth, connected, Hausdorff and without boundary; the objects considered on them are real and smooth. When Π is a Poisson structure on a manifold M, we write $\{f, g\}$ for $\Pi(\mathrm{d}f, \mathrm{d}g)$ and we denote the Hamiltonian vector field associated to $h \in C^\infty(M)$ by X_h. The vector bundle map induced by Π is denoted by $\Pi^\sharp : T^*M \to TM$. Our sign convention is that $X_h(g) = \mathrm{d}g(X_h) = \{g, h\}$ for $g \in C^\infty(M)$ and $\Pi^\sharp(\mathrm{d}h) = -X_h$. For a foliation $\mathfrak{F}$ on a manifold M the tangent space to $\mathfrak{F}$ at m is denoted by $T_m\mathfrak{F}$, while its annihilator is denoted by $(T_m\mathfrak{F})^\circ$. It leads to subbundles $T\mathfrak{F}$ of TM and $(T\mathfrak{F})^\circ$ of T^*M. For a vector bundle E over M, the module of (smooth) sections of E is denoted by $\Gamma(E)$. We denote by $\Omega^k(M)$ (respectively by $\mathfrak{X}^k(M)$) the module $\Gamma(\wedge^k T^*M)$ of k-forms (respectively the module $\Gamma(\wedge^k TM)$ of k-vector fields) on M. For $\omega \in \Omega^k(M)$ we denote by ω_m or $\omega|_m$ its value at $m \in M$ and similarly for elements of $\mathfrak{X}^k(M)$. For a vector field X on M, we denote by L_X the Lie derivative with respect to X of elements of $\Omega^k(M)$ or of $\mathfrak{X}^k(M)$. The r-dimensional torus $(\mathbb{R}/\mathbb{Z})^r$ is denoted by $\mathbb{T}^r$.

Acknowledgment

Rui Loja Fernandes was partially supported by NSF grants DMS 13-08472 and DMS 14-05671, FCT/Portugal and the *Ciências Sem Fronteiras* program. All authors were supported by the PHC Pessoa project “Poisson geometry and applications” (2010-2011). The number is 22729TK.

The authors are grateful to the referee for his comments and suggestions that significantly improved the paper. The authors would also like to thank Raquel Caseiro who contributed with several ideas at the initial stage of this project.

2. Non-commutative integrable systems on Poisson manifolds

2.1. NCI systems

In this section, we discuss the main notion relevant to this paper, the notion of a non-commutative integrable system on a Poisson manifold. We first recall from [18] the more concrete definition, which we reformulate in geometrical terms (in terms of a foliation), to arrive at an abstract definition; it is the latter definition which will play a dominant role in the rest of the paper, because it is most adapted to the study of the global geometry of such a system.

Definition 2.1. Let (M, Π) be a Poisson manifold of dimension n. Let $\mathbf{F} = (f_1, \dots, f_s)$ be an s-tuple of functions on M, where $2s \geqslant n$ and set $r := n - s$. Suppose the following:

(1) The functions $f_1, \dots, f_r$ are in involution with the functions $f_1, \dots, f_s$:

$$\{f_i, f_j\} = 0, \quad (1 \leqslant i \leqslant r \text{ and } 1 \leqslant j \leqslant s);$$

(2) For m in a dense open subset of M:

$$\mathrm{d}_m f_1 \wedge \cdots \wedge \mathrm{d}_m f_s \neq 0 \quad \text{and} \quad X_{f_1}|_m \wedge \cdots \wedge X_{f_r}|_m \neq 0.$$

Then the triplet $(M, \Pi, \mathbf{F})$ is called a **non-commutative integrable system** (NCI system) of **rank** r and $\mathbf{F}$, viewed as a map $\mathbf{F} : M \to \mathbb{R}^s$, is called its **momentum map**.

The classical case of a **Liouville integrable system** corresponds to the particular case where r is half the (maximal) rank of Π; this implies that *all* the functions $f_1, \dots, f_s$ are pairwise in involution,

$$\{f_i, f_j\} = 0 \quad (1 \leqslant i, j \leqslant s).$$

Also, when M is symplectic, the second condition in (2) above is a consequence of the first condition in (2).

A point $m \in M$ where the two conditions in (2) hold is called a **regular point** of the NCI system, the other points are called **singular points** of the NCI system. When all points of M are regular one speaks of a **regular NCI system**. We will mainly study regular NCI systems, though we will see that singular points are present in basically all the examples; we will then be led to restricting the Poisson manifold underlying the NCI system to an appropriate open subset, on which the NCI system restricts to a regular NCI system.

We start with an example from classical mechanics (see [25, Ch. 4.48]).

Example 2.1. Consider a particle of mass m in $\mathbb{R}^3$ which is subject to a central force, derived from a potential function $V = V(r)$ which depends only on the distance r from the origin of $\mathbb{R}^3$. The Hamiltonian which describes the total energy of the particle is given by

$$H = \frac{1}{2m}\sum_{i=1}^{3} p_i^2 + V(r),$$

where $r^2 = \sum_{i=1}^{3} q_i^2$ and where (q_1, q_2, q_3) and (p_1, p_2, p_3) respectively stand for the position coordinates and for the corresponding momenta of the particle. The Poisson structure is the canonical structure on $T^*\mathbb{R}^3 \simeq \mathbb{R}^6$, to wit

$$\Pi = \sum_{i=1}^{3} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

The Hamiltonian vector field X_H whose integral curves describe the motion of the particle is given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} = -q_i \frac{V'(r)}{r}. \tag{2.1}$$

Consider the three linear momenta $\mu_{ij} := q_i p_j - q_j p_i$, where $1 \leqslant i < j \leqslant 3$. It follows at once from (2.1) that $\dot{\mu}_{ij} = 0$, so that each of these momenta is a constant of motion, and so $L := \mu_{12}^2 + \mu_{13}^2 + \mu_{23}^2$ is also a constant of motion; moreover, the latter has the virtue of being in involution with all the linear momenta μ_{ij}. Letting $\mathbf{F} := (H, L, \mu_{12}, \mu_{23})$ it follows that $(T^*\mathbb{R}^3, \Pi, \mathbf{F})$ is an NCI system of rank 2 with momentum map $\mathbf{F}$.

Next, we give a family of examples of regular NCI systems which are important for the theory which will be developed in this paper, because they to provide local models for any regular NCI system (see Proposition 2.4 below).

Example 2.2. Let $M := \mathbb{R}^{2r} \times \mathbb{R}^{s-r}$ with coordinates (q_i, p_i, z_j) be equipped with a Poisson structure Π of the form:

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \pi,$$

where π is any Poisson structure on $\mathbb{R}^{s-r}$,

$$\pi = \sum_{1 \leqslant j < k \leqslant s-r} c_{jk}(z) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k}. \tag{2.2}$$

Letting $\mathbf{F} := (p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ it is clear that $(M, \Pi, \mathbf{F})$ is a regular NCI system of rank r with momentum map $\mathbf{F}$. It is a Liouville integrable system if and only $\pi = 0$ (equivalently, all functions c_{ij} are zero).

A slight modification of this example yields a family of examples of regular NCI systems with compact fibers, which are semi-local models for regular NCI systems with compact fibers (see Theorem 2.12 below).

Example 2.3. Let $M = T^*\mathbb{T}^r \times \mathbb{R}^{s-r} \simeq \mathbb{T}^r \times \mathbb{R}^r \times \mathbb{R}^{s-r}$ with coordinates (θ_i, p_i, z_j) be equipped with a Poisson structure Π of the form:

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \pi,$$

where π is any Poisson structure on $\mathbb{R}^{s-r}$, as in (2.2). Letting $\mathbf{F} := (p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ we have as above that $(M, \Pi, \mathbf{F})$ is a regular NCI system of rank r with momentum map $\mathbf{F}$.

We finish our list of examples with a more involved example which also comes from classical mechanics, the Euler-Poinsot top. We will come back to this example in the future sections when we discuss the obstructions to the existence of action-angle variables and foliations.

Example 2.4. The configuration space of the Euler-Poinsot top is the Lie group $\mathbf{G} := \mathrm{SO}(3)$ of real orthogonal 3×3 matrices, so its phase space is the cotangent bundle $T^*\mathbf{G}$, equipped with its canonical symplectic structure. Denoting the Lie algebra of $\mathbf{G}$ by $\mathfrak{g}$, we have that $T^*\mathbf{G} \simeq \mathbf{G} \times \mathfrak{g}^*$, where the isomorphism is constructed by using left translation on $\mathbf{G}$. It is well-known that the symplectic manifold $\mathbf{G} \times \mathfrak{g}^*$ is a symplectic groupoid in the sense of [7], with target map $t : \mathbf{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$ the (coadjoint) action map $(g,\xi)\mapsto \mathrm{Ad}^*_g\,\xi$ and source map $s:\mathbf{G}\times\mathfrak{g}^*\to\mathfrak{g}^*$ the projection onto the second component, $(g,\xi)\mapsto\xi$. Like for any symplectic groupoid,

- The source map s is a Poisson map onto $\mathfrak{g}^*$, equipped with its Lie-Poisson structure;
- The target map t is an anti-Poisson map onto the same space;
- For every pair of functions F,G on $\mathfrak{g}^*$, the functions s^*F and t^*G are in involution on $\mathbf{G}\times\mathfrak{g}^*$.

It is convenient to identify $\mathfrak{g}^*$ with $\mathbb{R}^3$. First, we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by using the Killing form. Next, $\mathfrak{g}$ is the Lie algebra $\mathfrak{so}(3)$ of real skew-symmetric 3×3 matrices, which we can identify with $\mathbb{R}^3$ by assigning to $(x,y,z)\in\mathbb{R}^3$ the skew-symmetric matrix $\begin{pmatrix}0 & z & -x\\ -z & 0 & y\\ z & -y & 0\end{pmatrix}$. Under these identifications:

- The coadjoint action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)^*$ becomes the canonical action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$;
- The Lie bracket on $\mathfrak{g}$ becomes the vector product on $\mathbb{R}^3$;
- The Lie-Poisson structure on $\mathfrak{g}^*$ becomes the linear Poisson structure on $\mathbb{R}^3$, given in terms of the natural coordinates (x,y,z) on $\mathbb{R}^3$ by:

$$\{x,y\}_{\mathfrak{g}^*}=z\,,\quad \{y,z\}_{\mathfrak{g}^*}=x\,,\quad \{z,x\}_{\mathfrak{g}^*}=y. \tag{2.3}$$

A Casimir of this Poisson structure is given by $C:=x^2+y^2+z^2$.

The upshot is that $\mathrm{SO}(3)\times\mathbb{R}^3$ is a symplectic manifold, comes equipped with two maps $s,t:\mathrm{SO}(3)\times\mathbb{R}^3\to\mathbb{R}^3$ which are defined by $s(R,m)=m$ and $t(R,m)=Rm$ and which are Poisson, resp. anti-Poisson maps. Also, for every pair of functions F,G on $\mathbb{R}$, the functions s^*F and t^*G are in involution. In turn, this implies that for any function H on $\mathbb{R}^3$, the map ϕ_H, defined by

$$\begin{array}{cccc}\phi_H & : & \mathrm{SO}(3)\times\mathbb{R}^3 & \mapsto & \mathbb{R}^3\times\mathbb{R}\\ & & (R,m) & \to & \big(Rm,H(m)\big).\end{array}$$

is a Poisson map, when $\mathbb{R}^3\times\mathbb{R}$ is equipped with the Poisson structure $\pi=\{\cdot\,,\cdot\}_B$, which is the product of the linear Poisson structure (2.3) on $\mathbb{R}^3$ with the trivial Poisson structure on $\mathbb{R}$. The symplectic Poisson structure on $\mathrm{SO}(3)\times\mathbb{R}^3$ is denoted by $\Pi=\{\cdot\,,\cdot\}$.

The *Euler-Poinsot top* corresponds to the choice

$$H := \frac{1}{2}\left(\frac{x^2}{I_x} + \frac{y^2}{I_y} + \frac{z^2}{I_z}\right).$$

where $I_x\, I_y$ and I_z are positive parameters, describing the top. In what follows we assume that these parameters are different and that the coordinates are ordered such that $I_x > I_y > I_z$. Consider the functions s^*H, t^*C, t^*x, t^*y and t^*z on $\mathrm{SO}(3) \times \mathbb{R}^3$ and consider the Hamiltonian vector fields X_{s^*H} and X_{t^*C}. On the one hand, $\{s^*H, t^*C\} = 0$, so these vector fields commute; moreover, they are independent at a a generic point of $\mathrm{SO}(3) \times \mathbb{R}^3$. On the other hand, the functions t^*x, t^*y and t^*z are in involution with s^*H as well as with t^*C. It follows that[1] (s^*H, t^*C, t^*y, t^*z) defines a non-commutative integrable system of rank 2 on $\mathrm{SO}(3) \times \mathbb{R}^3$.

For our purposes we need to restrict phase space to an open subset on which the NCI system is regular. Let us denote by $||\cdot||$ the standard norm on $\mathbb{R}^3$, so for $m = (x, y, z) \in \mathbb{R}^3$ we have $||m||^2 = x^2 + y^2 + z^2$. The inequalities $I_x > I_y > I_z > 0$ imply that the image of H is the closed interval

$$\mathrm{Im}(\phi_H) = \left\{(v, h) \mid \frac{||m||^2}{2I_x} \leqslant h \leqslant \frac{||m||^2}{2I_z}\right\}.$$

Let B and B' denote the open subsets of $\mathbb{R}^3 \times \mathbb{R}$, defined by

$$B := \left\{(v, h) \mid \frac{||v||^2}{2I_x} < h < \frac{||v||^2}{2I_y}\right\},$$
$$B' := \left\{(v, h) \mid \frac{||v||^2}{2I_y} < h < \frac{||v||^2}{2I_z}\right\}.$$

We denote by $M \subset \mathrm{SO}(3) \times \mathbb{R}^3$ the inverse image $\phi_H^{-1}(B)$, consisting of all (R, m) for which $(m, H(m)) \in B$; the analysis done below can be repeated with minor changes for $M' := \phi_H^{-1}(B')$. On M the NCI system is regular; more precisely $(M, \Pi) \stackrel{\phi_H}{\to} (B, \pi)$ is a rank two NCI system with momentum map. The fibers of ϕ_H are compact but not connected: the fiber over each point of B consists of two disjoint two-dimensional tori $\mathbb{T}^2$. Since, for our analysis, we need the fibers of the momentum map (see Definition 2.9 below) to be connected, we need to do a further restriction on phase space: we define M_+ as the subset of M whose points (R, m), with $m = (x, y, z)$, satisfy $x > 0$.

[1]In this list of functions one can replace t^*y or t^*z by t^*x.

Now $(M_+,\Pi) \stackrel{\phi_H}{\to} (B,\pi)$ is a regular NCI system of rank two with compact connected fibers.

For explicitness, we give a geometrical description of these fibers as two-dimensional tori. Let $(v,h) \in B \subset \mathbb{R}^3 \times \mathbb{R}$ and let $c := ||v||$. The fiber in M_+ over (v,h) is given by

$$\phi_H^{-1}(v,h) = \left\{(R,m) \in \mathrm{SO}(3) \times \mathbb{R}^3 \mid Rm = v,\ H(m) = h\right\}.$$

Notice that when $(R,m) \in \phi_H^{-1}(v,h)$, the point m belongs to one of the two connected components of the intersection of the sphere $||m||^2 = c$ and the ellipsoid $H(m) = h$. This component, which corresponds to the component lying in the half-space $x > 0$ (see the above definition of M_+) is a smooth curve S, diffeomorphic[2] to the circle S^1. Notice also that if R_v is any rotation with center O which fixes v then (R_vR,m) belongs to the same fiber of ϕ_H. This leads to two actions of S^1 on $\phi_H^{-1}(v,h)$. The first one leaves m unchanged and is the above left multiplication of R by the unique rotation R_v over a given angle. For the action of the other component S^1 one fixes a diffeomorphism between S and S^1; the action on m, denoted $\theta \cdot m$ is then given by the standard action of S^1 on itself, while the action on R can be taken as right multiplication of R with the unique rotation which sends $\theta \cdot m$ to m. Clearly these two actions of S^1 commute and they define an action of $\mathbb{T}^2$ which is transitive and has trivial stabilizer. It allows us to identify (topologically) $\phi_H^{-1}(v,h)$ with $\mathbb{T}^2$.

2.2. Abstract NCI systems

To a regular NCI system $(M,\Pi,\mathbf{F})$ one naturally associates an r-dimensional foliation of M: by the regularity assumption, $\mathbf{F} : M \to \mathbb{R}^s$ is a submersion onto some open subset of $\mathbb{R}^s$, so that the connected components of the fibers of $\mathbf{F}$, which are r-dimensional, are the leaves of a foliation $\mathfrak{F}$ of M. In the case of Example 2.2 (resp. Example 2.3), these leaves are r-dimensional affine spaces $\mathbb{R}^r$ (resp. r-dimensional tori $\mathbb{T}^r$).

In the following proposition we rewrite the key elements of the definition of a regular NCI system in terms of the foliation which is associated to it. Before doing this, let us recall that a locally defined function which is constant on the leaves of a foliation $\mathfrak{F}$ is called a **local first integral** of $\mathfrak{F}$. These functions are characterized by the property that they are annihilated

[2]The *complex* intersection of these two quadrics is a smooth complex elliptic curve.

by any set of local vector fields which span locally the tangent bundle $T\mathfrak{F}$ to $\mathfrak{F}$. In Example 2.2 (resp. Example 2.3), the local first integrals of the foliation defined by $\mathbf{F}$ are the local functions which are independent of $q_1, \ldots, q_r$ (resp. of $\theta_1, \ldots, \theta_r$).

Proposition 2.2. Let $(M, \Pi, \mathbf{F})$ be a regular NCI system of dimension n and rank r and let $\mathfrak{F}$ denote the foliation whose leaves are the connected components of the non-empty fibers of its momentum map $\mathbf{F} : M \to \mathbb{R}^s$. Then $T\mathfrak{F}$ is spanned by Hamiltonian vector fields associated to local first integrals of $\mathfrak{F}$, i.e., for each $m \in M$ there exist local first integrals of $\mathfrak{F}$, namely $f_1, \ldots, f_r$, whose Hamiltonian vector fields span $T_{m'}\mathfrak{F}$, for m' in a neighborhood of m in M. In particular, every leaf of $\mathfrak{F}$ is contained in a symplectic leaf of Π.

Proof. Item (1) in Definition 2.1 implies that the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_r}$ are tangent to the fibers of $\mathbf{F}$ (i.e., to the leaves of $\mathfrak{F}$), while item (2) implies that they actually span the tangent spaces to these fibers at every regular point, i.e., at every point (since it is assumed that the NCI system is regular). This shows that $T\mathfrak{F}$ is spanned by the Hamiltonian vector fields associated to the local first integrals $f_1, \ldots f_r$ of $\mathfrak{F}$. As a consequence, every leaf of $\mathfrak{F}$ is contained in a symplectic leaf of Π. □

The above proposition leads to the following more abstract notion of an NCI system and of morphisms between such systems:

Definition 2.3. Let (M, Π) be a Poisson manifold. An **abstract non-commutative integrable system** (abstract NCI system) of rank r is an r-dimensional foliation $\mathfrak{F}$ of M, whose tangent bundle $T\mathfrak{F}$ is spanned by Hamiltonian vector fields associated to local first integrals of $\mathfrak{F}$.

A **morphism** between two abstract NCI systems $(M, \Pi, \mathfrak{F})$ and $(N, \Theta, \mathfrak{G})$ is a Poisson map $\phi : M \to N$ which is transverse to $\mathfrak{G}$ and such that $\phi^*(\mathfrak{G}) = \mathfrak{F}$.

Example 2.5. Let (M, ω) be a symplectic manifold and let $\Pi := \omega^{-1}$ denote the Poisson structure corresponding to ω. It follows from Proposition 2.8 below that an r-dimensional foliation $\mathfrak{F}$ of M defines an abstract NCI system if and only if the leaves are isotropic submanifolds and the symplectic orthogonal distribution $(T\mathfrak{F})^{\perp_\omega} = \Pi^\sharp(T\mathfrak{F})^\circ$ is an integrable distribution. Therefore, our notion of an abstract NCI system generalizes the notion introduced by Dazord and Delzant [12] in the case of symplectic manifolds.

Note also that in the case of a Lagrangian foliation of a symplectic manifold one has $(T\mathfrak{F})^{\perp_\omega} = T\mathfrak{F}$ and such a foliation corresponds to the abstract notion of a Liouville integrable system (on a symplectic manifold).

Example 2.6. A **Lagrangian foliation** of a Poisson manifold (M,Π) is a foliation $\mathfrak{F}$ of M for which $T\mathfrak{F} = \Pi^\sharp(T\mathfrak{F})^\circ$. In the terminology of Definition 2.6 and Example 2.13 this amounts to saying that $\mathfrak{F}$ is both isotropic and coisotropic; it implies that Π is regular, of rank twice the dimension of $\mathfrak{F}$. For a Lagrangian foliation $\mathfrak{F}$, the Hamiltonian vector fields associated to all its local first integrals are both tangent to $T\mathfrak{F}$ and span $T\mathfrak{F}$. In particular, $(M,\Pi,\mathfrak{F})$ is an abstract NCI system. It is the abstract version of a Liouville integrable system (on a regular Poisson manifold).

Example 2.7. Let (M,Π) be a Poisson manifold. Any nowhere vanishing Hamiltonian vector field X_h defines a foliation $\mathfrak{F}$, making $(M,\Pi,\mathfrak{F})$ into an abstract NCI system of rank 1. In this case, the local first integrals of $\mathfrak{F}$ are precisely the local first integrals of X_h.

We show in the following proposition how regular and abstract NCI systems are related.

Proposition 2.4. Let $(M,\Pi,\mathfrak{F})$ be an abstract NCI system of dimension n and rank r. Let m be an arbitrary point of M. There exist on a neighborhood U of m coordinates $q_1,\dots,q_r,p_1,\dots,p_r,z_1,\dots,z_{n-2r}$ such that the foliation $\mathfrak{F}$ is defined on U by the functions $p_1,\dots,p_r,z_1,\dots,z_{n-2r}$ and such that Π is given, on U, by

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{1\leqslant j<k\leqslant n-2r} c_{jk}(z)\frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k}, \tag{2.4}$$

where the functions c_{jk} are independent of $q_1,\dots,q_r,p_1,\dots,p_r$. In particular, setting $\mathbf{F} := (p_1,\dots,p_r,z_1,\dots,z_{n-2r})$ we have that $(U,\Pi_{|U},\mathbf{F})$ is a regular NCI system of rank r. Conversely, if $(M,\Pi,\mathbf{F})$ is a regular NCI system and $\mathfrak{F}$ its associated foliation, then $(M,\Pi,\mathfrak{F})$ is an abstract NCI system.

Proof. The proof is a direct application of the Carathéodory-Jacobi-Lie theorem for Poisson manifolds (see [18, Sect. 2] for a proof). This theorem says that if (M,Π) is any Poisson manifold of dimension n on which r functions $p_1,\dots,p_r$ are given, which are pairwise in involution and have independent Hamiltonian vector fields at some point $m\in M$, then these functions can be extended to a coordinate system $q_1,\dots,q_r,p_1,\dots,p_r,z_1,\dots,z_{n-2r}$ on a neighborhood U of m, such that Π takes on U the form (2.4). In order to apply this theorem in the present case, we take any point m of M and we choose as functions $p_1, \dots, p_r$ local first integrals of $\mathfrak{F}$ whose Hamiltonian vector fields span $T\mathfrak{F}$ in a neighborhood of m. These r functions are in involution so the theorem can be applied. Notice that in view of (2.4) the tangent space to $\mathfrak{F}$ is spanned by the vector fields $\partial/\partial q_1, \dots, \partial/\partial q_r$, so the local first integrals of $\mathfrak{F}$ are the local functions which are independent of $q_1, \dots, q_r$ and $\mathfrak{F}$ is locally defined by the functions $p_1, \dots, p_r, z_1, \dots, z_{n-2r}$. This proves the first part of the proposition; the proof of the converse is an immediate consequence of Proposition 2.2. □

According to the proposition, every regular NCI system is naturally associated to an abstract NCI system; by definition, a **morphism** between regular NCI systems is a morphism between their associated abstract NCI systems.

In order to give another example of an abstract NCI system, we need a result which is interesting in its own right.

Corollary 2.5. Let $(M, \Pi, \mathfrak{F})$ be an abstract NCI system of dimension n and rank r. If $\mathcal{V}$ is a local Hamiltonian vector field which is tangent to the fibers of $\mathfrak{F}$, then every Hamiltonian of $\mathcal{V}$ is a local first integral of $\mathfrak{F}$.

Proof. The proof follows at once from Proposition 2.4. We give a direct proof. Let m be an arbitrary point of M. In view of the definition of an abstract NCI system, there exist on a neighborhood U of m first integrals $f_1, \dots, f_r$ of $\mathfrak{F}$ whose Hamiltonian vector fields span $T\mathfrak{F}$ (on U). Thus, a function f on U is a local first integral of $\mathfrak{F}$ if and only if $X_{f_i}(f) = 0$, for $i = 1, \dots, r$. Suppose that h is a local function on M whose Hamiltonian vector field $\mathcal{V} := X_h$ is tangent to $\mathfrak{F}$. Then

$$X_{f_i}(h) = \{h, f_i\} = -X_h(f_i) = -\mathcal{V}(f_i) = 0,$$

so that h is a local first integral of $\mathfrak{F}$. □

Example 2.8. Let $G \times M \to M$ be a Hamiltonian action of a Lie group G (with Lie algebra $\mathfrak{g}$) on a Poisson manifold (M, Π). Recall that this means that there exists a Lie algebra homomorphism $\mu : (\mathfrak{g}, [\cdot\,,\cdot]) \to (C^\infty(M), \{\cdot\,,\cdot\})$ such that for every $x \in \mathfrak{g}$, the function $\mu(x)$ is a Hamiltonian for the fundamental vector field $\underline{x}$ associated to x. We assume that the isotropy groups of the action have constant dimension, so that the orbits are the leaves of a foliation $\mathfrak{F}$. We claim that the following conditions are equivalent:

(i) $(M, \Pi, \mathfrak{F})$ is an abstract NCI system;

(ii) For every $x \in \mathfrak{g}$, the function $\mu(x)$ is a first integral of $\mathfrak{F}$;

(iii) $\mu([\mathfrak{g}, \mathfrak{g}]) = 0$.

The implication (i) $\Rightarrow$ (ii) follows from Corollary 2.5, applied to the Hamiltonian $\mu(x)$ of $\underline{x}$. Conversely, when (ii) holds $T\mathfrak{F}$ is spanned by the Hamiltonian vector fields associated to certain first integrals of $\mathfrak{F}$, namely the functions $\mu(x)$ with $x \in \mathfrak{g}$, so $(M, \Pi, \mathfrak{F})$ is an abstract NCI system. For $x, y \in \mathfrak{g}$ we have that

$$\underline{y}(\mu(x)) = \{\mu(x), \mu(y)\} = \mu([x, y]),$$

from which the equivalence of (ii) and (iii) follows at once. Notice that (iii) is trivially satisfied when $\mathfrak{g}$ is abelian. Moreover, when the action is locally free, (iii) is equivalent to $[\mathfrak{g}, \mathfrak{g}] = 0$, i.e., to $\mathfrak{g}$ being abelian.

2.3. Poisson complete isotropic foliations

The foliation of an abstract NCI system has two main features, which we first define and illustrate with some basic examples.

Definition 2.6. Let (M, Π) be a Poisson manifold and suppose that $\mathfrak{F}$ is a foliation of M.

(1) We say that $\mathfrak{F}$ is **Poisson complete** if the Poisson bracket of two local first integrals of $\mathfrak{F}$ is a local first integral of $\mathfrak{F}$;

(2) We say that $\mathfrak{F}$ is **isotropic** if $T\mathfrak{F} \subset \Pi^\sharp(T\mathfrak{F})^\circ$.

Example 2.9. Let (M, ω) be a symplectic manifold and let $\Pi := \omega^{-1}$ denote the Poisson structure corresponding to ω. Given a foliation $\mathfrak{F}$ of M we have that $\Pi^\sharp(T\mathfrak{F})^\circ = (T\mathfrak{F})^{\perp_\omega}$. It follows that (i) $\mathfrak{F}$ is Poisson complete if and only if the distribution $(T\mathfrak{F})^{\perp_\omega}$ is integrable and that (ii) $\mathfrak{F}$ is isotropic if and only if the leaves of $\mathfrak{F}$ are isotropic in the usual sense.

Example 2.10. Suppose the Poisson manifold (M, Π) admits a 1-form α for which the vector field $\Pi^\sharp(\alpha)$ is nowhere vanishing. The corresponding 1-dimensional foliation $\mathfrak{F}$ is isotropic, since $\Pi^\sharp(\alpha)$ spans $T\mathfrak{F}$ at every point of M and $\alpha \in (T\mathfrak{F})^\circ$. We will see in Proposition 2.7 below that, if $\mathfrak{F}$ is Poisson complete, then the distribution $\Pi^\sharp(T\mathfrak{F})^\circ$ is integrable. The distribution $\Pi^\sharp(T\mathfrak{F})^\circ$ coincides with $\operatorname{Ker} \alpha$, so it is integrable if and only if $\alpha \wedge \mathrm{d}\alpha = 0$. In fact, it is easy to check that $\mathfrak{F}$ is Poisson complete if and only if $\alpha \wedge \mathrm{d}\alpha = 0$.

Example 2.11. Let $\phi : (M, \Pi) \to (M', \pi)$ be any Poisson submersion between two Poisson manifolds. Then the connected components of the fibers of ϕ define a Poisson complete foliation $\mathfrak{F}$ of M. This follows from the fact that the local first integrals of $\mathfrak{F}$ are of the form $g \circ \phi$, with $g \in C^\infty(M')$, and functions of this form are closed under the Poisson bracket since ϕ is a Poisson map.

Example 2.12. As a particular example of the previous one, consider on $\mathbb{R}^2$, with coordinates (x, y), the following Poisson structure:

$$\Pi := x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The projection on the x-axis, $(x, y) \mapsto x$ is a Poisson map, so the foliation by vertical lines is Poisson complete. On the other hand, this foliation is not isotropic since the Poisson structure vanishes on the vertical line $x = 0$, so on points of this line the inclusion $T\mathfrak{F} \subset \Pi^\sharp(T\mathfrak{F})^\circ$ does not hold.

Example 2.13. A foliation $\mathfrak{F}$ of a Poisson manifold (M, Π) is said to be **coisotropic** if $\Pi^\sharp(T\mathfrak{F})^\circ \subset T\mathfrak{F}$. A necessary and sufficient condition for a foliation $\mathfrak{F}$ of M to be coisotropic is that every pair of local first integrals of the foliation is in involution. Thus, coisotropic foliations are Poisson complete.

We give in the following proposition a characterization of Poisson complete foliations.

Proposition 2.7. Let $\mathfrak{F}$ be an r-dimensional foliation of a Poisson manifold (M, Π) of dimension n. The following statements are equivalent:

(i) $\mathfrak{F}$ is Poisson complete;

(ii) $(T\mathfrak{F})^\circ$ is a Lie subalgebroid of T^*M.

For any foliation $\mathfrak{F}$ on (M, Π) satisfying these conditions, the singular distribution $\Pi^\sharp(T\mathfrak{F})^\circ$ is integrable.

Proof. We first recall how the Poisson structure on M makes T^*M into a Lie algebroid (see [6] for background and details). For sections $\alpha, \beta \in \Omega^1(M)$ their Lie bracket is defined by

$$[\alpha, \beta] := L_{\Pi^\sharp(\alpha)}\beta - L_{\Pi^\sharp(\beta)}\alpha - \mathrm{d}(\Pi(\alpha, \beta)). \tag{2.5}$$

For local sections $f_i \mathrm{d}g_i$, where f_i is a smooth function, (2.5) amounts to

$$[f_1\, \mathrm{d}g_1, f_2\, \mathrm{d}g_2] = f_1 f_2\, \mathrm{d}\,\{g_1, g_2\} + f_1\,\{g_1, f_2\}\,\mathrm{d}g_2 - f_2\,\{g_2, f_1\}\,\mathrm{d}g_1. \tag{2.6}$$

The anchor of the Lie algebroid T^*M is the map $\Pi^\sharp : T^*M \to TM$. Let g_1 and g_2 be two local first integrals of $\mathfrak{F}$ and suppose that $(T\mathfrak{F})^\circ$ is a Lie subalgebroid of T^*M. Then (2.6) says that $\mathrm{d}\,\{g_1, g_2\}$ is a section of $(T\mathfrak{F})^\circ$, which means that $\{g_1, g_2\}$ is a local first integral of $\mathfrak{F}$. This shows that *(ii)* implies *(i)*. The converse implication also follows at once from (2.6) upon using that every section of $(T\mathfrak{F})^\circ$ is locally of the form $\sum_i f_i \mathrm{d}g_i$, where each g_i is a local first integral of $\mathfrak{F}$ and the f_i are arbitrary functions.

The final claim is a consequence of (ii) because for any Lie algebroid the image of the anchor map is an integrable (possibly singular) distribution. □

Proposition 2.8. Suppose that $\mathfrak{F}$ is an r-dimensional foliation of a Poisson manifold (M, Π).

(1) If $(M, \Pi, \mathfrak{F})$ is an abstract NCI system then $\mathfrak{F}$ is both Poisson complete and isotropic.

(2) If Π is regular and $\mathfrak{F}$ is both Poisson complete and isotropic, then $(M, \Pi, \mathfrak{F})$ is an abstract NCI system.

Proof. (1) Poisson completeness and isotropy of a foliation are local properties, hence these properties can be proven easily by using Proposition 2.4. Again, we give a direct proof. Let m be an arbitrary point of M and on a neighborhood U of m, let $f_1, \dots, f_r$ be first integrals of $\mathfrak{F}$ whose Hamiltonian vector fields span $T\mathfrak{F}$. If g and h are first integrals of $\mathfrak{F}$ on U, we have in view of the Jacobi identity for Π:

$$X_{f_i}(\{g, h\}) = \{X_{f_i}(g), h\} + \{g, X_{f_i}(h)\} = 0,$$

for $i = 1, \dots, r$. This shows that the Poisson bracket $\{g, h\}$ is a local first integral of $\mathfrak{F}$, so $\mathfrak{F}$ is Poisson complete. Also, since each f_i is a local first integral of $\mathfrak{F}$, each $\mathrm{d}f_i$ is a local section of $(T\mathfrak{F})^\circ$ and the fact that $T\mathfrak{F}$ is spanned by $X_{f_1}, \dots, X_{f_r}$ implies that $T\mathfrak{F} \subset \Pi^\sharp(T\mathfrak{F})^\circ$, so $\mathfrak{F}$ is isotropic.

(2) If $\mathfrak{F}$ is isotropic then $T\mathfrak{F} \subset \Pi^\sharp(T\mathfrak{F})^\circ$, so that $\operatorname{Ker}\Pi^\sharp \subset (T\mathfrak{F})^\circ$. Since Π is regular,

$$\frac{(T\mathfrak{F})^\circ}{\operatorname{Ker}\Pi^\sharp} \simeq \Pi^\sharp(T\mathfrak{F})^\circ$$

is a distribution, whose rank is $\operatorname{rank}(\Pi) - r$. It is generated by the Hamiltonian vector fields $\Pi^\sharp(\mathrm{d}f)$ with f a local first integral of $\mathfrak{F}$, and these functions are closed under the Poisson bracket, by Poisson completeness. According to Proposition 2.7, this implies that $\Pi^\sharp(T\mathfrak{F})^\circ$ is integrable, leading to a foliation $\mathfrak{G}$. If g is a local first integral of $\mathfrak{G}$, then X_g is tangent to $\mathfrak{F}$. Indeed, if f is a local first integral of $\mathfrak{F}$, then $\Pi^\sharp(\mathrm{d}f)$ is tangent to $\mathfrak{G}$, so that $\mathrm{d}f(\Pi^\sharp(\mathrm{d}g)) = -\mathrm{d}g(\Pi^\sharp(\mathrm{d}f)) = 0$. Note that $\mathfrak{G}$ is contained in the symplectic foliation of Π and has dimension $\operatorname{rank}(\Pi) + r$. Hence, for any point m of M, we can choose functions constant on $\mathfrak{G}$ such that at the point m:

$$\mathrm{d}_m f_1 \wedge \cdots \wedge \mathrm{d}_m f_r \neq 0 \quad \text{and} \quad X_{f_1}|_m \wedge \cdots \wedge X_{f_r}|_m \neq 0.$$

Hence, in some neighborhood of m, the functions $f_1, \ldots, f_r$ are constant on $\mathfrak{F}$ and their Hamiltonian vector fields span $T\mathfrak{F}$. This shows that $(M, \Pi, \mathfrak{F})$ is an abstract NCI system. □

Example 2.14. We show in the present example that not every Poisson complete isotropic foliation is an abstract NCI system. Consider the trivial circle bundle $M := S^1 \times \mathbb{R}^3 \to \mathbb{R}^3$ over $\mathbb{R}^3$. Denoting the coordinates on S^1 and on $\mathbb{R}^3$ by θ and x, y, z respectively, we consider on M the Poisson structure

$$\Pi := \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial z} + \pi,$$

where π is the Poisson structure on $\mathbb{R}^3$ (or on M), given by

$$\pi := \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right) \wedge \frac{\partial}{\partial z} + (x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Using the fact that $\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right)(x^2 + y^2) = 0$, one easily checks that π and Π are indeed Poisson structures. Also, by construction, the canonical projection $\phi : (M, \Pi) \to (\mathbb{R}^3, \pi)$ is a Poisson map. According to Example 2.11, the fibers of ϕ, which are circles, define a Poisson complete foliation $\mathfrak{F}$ of (M, Π). To see that $\mathfrak{F}$ is isotropic, take a point $m = (\theta_0, x_0, y_0, z_0)$ of M and consider $\alpha_m = a\,\mathrm{d}x + b\,\mathrm{d}y - \mathrm{d}z$, where $a, b \in \mathbb{R}$. By a direct computation we find that $\Pi^\sharp(\alpha_m) = \partial/\partial\theta + \pi^\sharp(\alpha_m) = \partial/\partial\theta$ when a and b are taken as

$$a = \frac{x_0}{x_0^2 + y_0^2}, \qquad b = \frac{y_0}{x_0^2 + y_0^2};$$

for $x_0 = y_0 = 0$ these formulas do not make sense, but in that case any values of a and b do the job. Since clearly $\alpha_m \in (T_m\mathfrak{F})^\circ$, this shows that $\mathfrak{F}$ is isotropic. We now show that in a neighborhood U of $m = (\theta_0, 0, 0, z_0)$ there exists no function f, constant on the leaves of $\mathfrak{F}$, whose Hamiltonian vector field X_f span $T\mathfrak{F}$ on U. The first condition means that f is independent of θ, so that

$$\mathrm{d}\theta(X_f) = \frac{\partial f}{\partial z}, \qquad \mathrm{d}x(X_f) = y\,\frac{\partial f}{\partial z} + (x^2+y^2)\frac{\partial f}{\partial y}. \tag{2.7}$$

The second condition means that $X_f = g\,\partial/\partial\theta$, for some nowhere vanishing function g on U, so that $\mathrm{d}\theta(X_f) \neq 0$ and $\mathrm{d}x(X_f) = 0$ on U. In view of (2.7) this is impossible.

2.4. Momentum map and Poisson structure on the leaf space

When the leaf space B of an abstract NCI system $(M, \Pi, \mathfrak{F})$ is a manifold (recall that our manifolds are assumed to be Hausdorff), the leaves of $\mathfrak{F}$ are the fibers of the quotient map $\phi : M \to B$, which is a fibration (with connected fibers).

Definition 2.9. We say that an abstract NCI system $(M, \Pi, \mathfrak{F})$ has a **momentum map** $\phi : M \to B$ if the leaf space B of $\mathfrak{F}$ is a (smooth, Hausdorff) manifold.

Example 2.15. Suppose that $(M, \Pi, \mathbf{F})$ is a regular NCI system of dimension n and rank r with connected fibers, i.e., the fibers of $\mathbf{F}$ are connected. Denoting by $\mathfrak{F}$ the associated foliation and by $B \subset \mathbb{R}^{n-r}$ the image of $\mathbf{F}$, the abstract NCI system $(M, \Pi, \mathfrak{F})$ has a momentum map, which is $\mathbf{F} : M \to B$.

Remark 2.16. Despite the terminology, an abstract NCI system is in general not integrable by quadratures, but an abstract NCI system with momentum map is. The proof of this fact is essentially the same as in the case of a Liouville integrable system on a Poisson manifold, see [1, Sect. 4.2].

Proposition 2.10. Let $(M, \Pi, \mathfrak{F})$ be an NCI system of dimension n and rank r with momentum map $\phi : M \to B$.

(1) B has a unique Poisson structure π for which $\phi : (M, \Pi) \to (B, \pi)$ is a Poisson map;

(2) Let f be a local function, whose Hamiltonian vector field is tangent to the leaves of $\mathfrak{F}$. The smooth function g on B, defined by $f := g \circ \phi$, is a local Casimir function of π;

(3) For every $m \in M$, $\operatorname{rank}(\pi_{\phi(m)}) = \operatorname{rank}(\Pi_m) - 2r$.

Because of this proposition, we usually simply speak of an NCI system $(M,\Pi) \stackrel{\phi}{\to} (B,\pi)$.

Proof of Proposition 2.10. Since ϕ is a submersion with connected fibers, the smooth functions on B can be identified with the first integrals of $\mathfrak{F}$ upon identifying $h \in C^\infty(B)$ with $h \circ \phi \in C^\infty(M)$. Thus, the Poisson completeness of $\mathfrak{F}$ leads to (1). It also implies that if f is a local function whose Hamiltonian vector field is tangent to $\mathfrak{F}$, so that f is a local first integral of $\mathfrak{F}$, we can write f as $g \circ \phi$ for some function g on B. For any local function h on B we have that $\{h, g\}_B \circ \phi = \{h \circ \phi, f\} = X_f(h \circ \phi) = 0$, since $h \circ \phi$ is a first integral of $\mathfrak{F}$. This shows that g is a Casimir function of $\{\cdot\,,\cdot\}_B = \pi$, which is the content of (2).

Let $m \in M$. On a neighborhood of m we can choose local functions $f_1, \ldots, f_r$ whose Hamiltonian vector fields span $T\mathfrak{F}$. In view of (2) the functions g_i, defined on a neighborhood of $\phi(m)$ by $f_i = g_i \circ \phi$, are Casimirs of π. We denote the differentials of these functions at m and at $\phi(m)$ by $\alpha_i := \mathrm{d}_m f_i$ and $\xi_i := \mathrm{d}_{\phi(m)} g_i$. Since the functions f_i are in involution with respect to Π, their (independent) differentials satisfy $\Pi_m(\alpha_i, \alpha_j) = 0$ for $1 \leqslant i, j \leqslant r$. They can be completed into a basis $\alpha_1, \ldots, \alpha_r$, $\eta_1, \ldots, \eta_r$, $\rho_1, \ldots, \rho_{n-2r}$ for $T^*_m M$, and since Π_m is skew-symmetric, this can be done such that the matrix of Π_m with respect to this basis is

$$\begin{pmatrix} 0 & I_r & 0 \\ -I_r & 0 & 0 \\ 0 & 0 & Z \end{pmatrix}$$

where $Z_{ij} = \Pi_m(\rho_i, \rho_j)$. Each one of the ρ_i belongs to $(T_m\mathfrak{F})^\circ$, since

$$\left\langle \rho_i, \Pi^\sharp_m(\alpha_j) \right\rangle = \Pi_m(\alpha_j, \rho_i) = 0$$

for all $j = 1, \ldots, r$ and since the vectors $\Pi^\sharp_m(\alpha_j)$ span $T_m\mathfrak{F}$. Therefore, there exist $\sigma_1, \ldots, \sigma_{n-2r} \in T^*_{\phi(m)}B$ such that $\rho_i = \phi^*(\sigma_i)$. In terms of the basis $\xi_1, \ldots, \xi_r, \sigma_1, \ldots, \sigma_{n-2r}$ for $T^*_{\phi(m)}B$, the matrix of $\pi_{\phi(m)}$ takes the form

$$\begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}$$

so that the rank of $\pi_{\phi(m)}$ is $2r$ less than the rank of Π_m, as asserted in (3). □

Notice that item (3) of the above proposition implies that when Π is regular (for example when M is a symplectic manifold), π is also regular.

We will show in the following proposition that very abstract NCI system admits a (foliated) atlas, consisting of NCI systems (in the sense of Definition 2.1), hence it admits locally a momentum map. First we recall (for example from [8, Ch. 1]) that an r-dimensional foliation $\mathfrak{F}$ of a manifold M of dimension n can be specified by a **regular foliated atlas** $(U_\alpha, \psi_\alpha \times \phi_\alpha)_{\alpha\in I}$: the $(U_\alpha)_{\alpha\in I}$ form an open cover[3] of M and each $\phi_\alpha : U_\alpha \to \mathbb{R}^{n-r}$ is a submersion, whose fibers define the leaves of $\mathfrak{F}$ locally; also, each $\psi_\alpha : U_\alpha \to \mathbb{R}^n$ is a submersion and the product map $\psi_\alpha \times \phi_\alpha : U_\alpha \to \mathbb{R}^r \times \mathbb{R}^{n-r} \simeq \mathbb{R}^n$ is a coordinate map. Moreover, the submersions ϕ_α are linked by diffeomorphisms $\phi_{\alpha\beta} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ such that:

$$\phi_{\alpha\beta} \circ \phi_\beta|_{U_\alpha\cap U_\beta} = \phi_\alpha|_{U_\alpha\cap U_\beta}.$$

It is clear that these diffeomorphisms are uniquely determined by the submersions ϕ_α.

Proposition 2.11. Let $\mathfrak{F}$ be an r-dimensional foliation of a Poisson manifold (M, Π) of dimension n. The following statements are equivalent:

(i) $\mathfrak{F}$ is an abstract NCI system;

(ii) $\mathfrak{F}$ admits a regular foliated atlas $(U_\alpha, \psi_\alpha \times \phi_\alpha)$ consisting of NCI systems $(U_\alpha, \Pi|_{U_\alpha}, \phi_\alpha)$ of rank r.

Proof. The implication (ii) $\Rightarrow$ (i) is straightforward because the foliation defined by a regular NCI system is an abstract NCI system and because being an abstract NCI system is a local property. Thus, let us suppose that $\mathfrak{F}$ is an abstract NCI system on (M, Π). We choose a regular foliated cover $(U_\alpha)_{\alpha\in I}$ of M subordinate to a cover $(U_\beta)_{\beta\in J}$ having the property that on each open subset $U_\beta eta$ there exist r first integrals of $\mathfrak{F}$ (restricted to U_β) whose Hamiltonian vector fields span $T\mathfrak{F}$ at every point of U_β. Let $\alpha \in I$; we show that $(U_\alpha, \Pi_{|U_\alpha}, \psi_\alpha \times \phi_\alpha)$ is a regular NCI system. Since the leaves of $\mathfrak{F}$, restricted to U_α, are the leaves of the foliation of U_α, defined by ϕ_α, we may identify local first integrals of $\mathfrak{F}$, defined on a neighborhood of a point of U_α with local first integrals of the foliation defined by the submersion ϕ_α. By construction, there exist first integrals $f_1, \dots, f_r$ on U_α whose Hamiltonian vector fields are independent in every point of U_α (they span $T\mathfrak{F}$ on U_α), in particular their differentials are independent in every point of U_α. Since ϕ_α is a submersion, there exist extra local first integrals $f_{r+1}, \dots, f_s$ of $\mathfrak{F}$, such that $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_s \neq 0$ on U_α. We have that $\{f_i, f_j\} = 0$ for $1 \leqslant i \leqslant$

[3]The cover can be chosen subordinate to any given open cover of M.

r and $1 \leqslant j \leqslant s$), so that $(U_\alpha, \Pi|_{U_\alpha}, (f_1, \ldots, f_s))$ is a regular NCI system, hence also $(U_\alpha, \Pi|_{U_\alpha}, \psi_\alpha \times \phi_\alpha)$. □

2.5. Semi-local model and local action-angle variables

The existence of local action-angle variables, proved in full generality in [18], can be translated into the following result, stating that Example 2.3 gives the semi-local model of an abstract NCI system $(M, \Pi, \mathfrak{F})$ with a momentum map, in the neighborhood of a compact fiber:

Theorem 2.12 (Semi-local model). Let $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ be an NCI system of rank $r = n - s$, where n and s are the dimensions of M and B, respectively, and assume that the fiber $\phi^{-1}(b_0)$ is compact and connected. Then there exist open neighborhoods $b_0 \in U \subset B$ and $0 \in V \subset \mathbb{R}^s$, a Poisson structure π_0 on $\mathbb{R}^s$ and an isomorphism Ψ of NCI systems:

$$\begin{array}{ccc}
(\phi^{-1}(U), \Pi) & \xrightarrow{\Psi} & (\phi_0^{-1}(V), \Pi_0) \\
\phi \big\downarrow & & \big\downarrow \phi_0 \\
(U, \pi) & \xrightarrow[\psi]{} & (V, \pi_0)
\end{array}$$

In this commutative diagram, Π_0, π_0 and ϕ_0 are the Poisson structures and the Poisson map, defined in Example 2.3.

Remark 2.13. In the literature ([13, 18]) one can find a definition of abstract NCI systems which requires the existence of a pair of foliations $\mathfrak{F} \subset \mathfrak{G}$ of (M, Π) such that $T\mathfrak{F} = \Pi^\sharp(T\mathfrak{G})^\circ$ (one says that $\mathfrak{F}$ is polar to $\mathfrak{G}$). For a regular NCI system $(M, \Pi, \mathbf{F})$ of rank r these foliations are respectively given by the connected components of the fibers of $\mathbf{F} = (f_1, \ldots, f_s)$ and of $\mathbf{G} = (f_1, \ldots, f_r)$. The proof of Theorem 2.12 given in [18] shows that the isomorphism of NCI systems which puts a given NCI system in a canonical form (providing action-angle coordinates) always respects the foliation $\mathfrak{F}$, but does not respect $\mathfrak{G}$, in general; notice also that although such a foliation $\mathfrak{G}$ always exists locally, it may not exist globally (see also Remark 3.4). For these reasons, we avoid throughout this paper the assumption of existence of a foliation $\mathfrak{G}$ to which $\mathfrak{F}$ is polar.

3. Action variables

In this section we consider NCI systems with a momentum map. Recall from Section 2.4 that this means that we have an abstract NCI system $(M, \Pi, \mathfrak{F})$, whose leaf space B is a (smooth, Hausdorff) manifold. The latter manifold inherits from (M, Π) a Poisson structure π such that the quotient map $\phi : (M, \Pi) \to (B, \pi)$ is a Poisson map. The foliation $\mathfrak{F}$ is isotropic and Poisson complete. The fibers of ϕ, which are the leaves of $\mathfrak{F}$, are connected. We will refer to the system simply as $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$.

3.1. The action bundle

Suppose that we have an NCI system $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ of rank r. We construct on B a canonical vector bundle E of rank r, which is closely related to action variables for the NCI system, as defined below. To do this, we consider two natural sheaves on B whose quotient essentially represents, pointwise, the covectors which yield the tangent space to the fibers of ϕ, upon using the Poisson structure Π.

Since the bundle map $\pi^\sharp : T^*B \to TB$ may not have constant rank, it is better to view $\pi^\sharp$ as a sheaf homomorphism $\pi^\sharp \in \mathcal{H}om(\Omega^1_B, \mathfrak{X}^1_B)$ from the sheaf Ω^1_B of differential 1-forms on B to the sheaf $\mathfrak{X}^1_B$ of vector fields on B. Precisely, $\pi^\sharp$ is a homomorphism of sheaves of C^∞_B-modules: for each open subset V of B, we have a $C^\infty_B(V)$-linear map

$$\pi^\sharp_V : \Omega^1_B(V) \to \mathfrak{X}^1_B(V),$$

which commutes with the restriction maps. The kernel of $\pi^\sharp$ is the subsheaf $\operatorname{Ker}\pi^\sharp \subset \Omega^1_B$ which to each (non-empty) open subset V of B associates the $C^\infty_B(V)$-module

$$(\operatorname{Ker}\pi^\sharp)(V) := \left\{\omega \in \Omega^1_B(V) \mid \pi^\sharp_V(\omega) = 0\right\}.$$

We also consider another subsheaf $\operatorname{Ker}(\Pi^\sharp \circ \phi^*) \subset \Omega^1_B$ which to each (non-empty) open subset V of B associates the $C^\infty_B(V)$-module

$$\operatorname{Ker}(\Pi^\sharp \circ \phi^*)(V) := \left\{\omega \in \Omega^1_B(V) \mid \Pi^\sharp_{\phi^{-1}(V)}(\phi^*\omega) = 0\right\}.$$

Since ϕ is a surjective Poisson submersion, $\operatorname{Ker}(\Pi^\sharp \circ \phi^*)(V) \subset (\operatorname{Ker}\pi^\sharp)(V)$, for every open subset V of B. As a consequence, $\operatorname{Ker}(\Pi^\sharp \circ \phi^*)$ is a subsheaf of $\operatorname{Ker}\pi^\sharp$, and we can form the quotient sheaf $\mathcal{E}_B$, which is also a sheaf of

C^∞_B-modules on B. These sheaves fit together in the following exact sequence of sheaves on B:

$$0 \longrightarrow \operatorname{Ker}(\Pi^\sharp \circ \phi^*) \longrightarrow \operatorname{Ker}\pi^\sharp \longrightarrow \mathcal{E}_B \longrightarrow 0.$$

Recall from the general theory of sheaves that, for every open subset V of B, an element of $\mathcal{E}_B(V)$ is a collection $(V_i, s_i)_{i\in I}$, where $(V_i)_{i\in I}$ is an open cover of V and $s_i \in (\operatorname{Ker}\pi^\sharp)(V_i)$ for every $i \in I$; these sections are demanded to satisfy $s_i|_{V_i\cap V_j} - s_j|_{V_i\cap V_j} \in \operatorname{Ker}(\Pi^\sharp \circ \phi^*)(V_i \cap V_j)$ whenever $V_i \cap V_j \neq \emptyset$. For $\omega \in (\operatorname{Ker}\pi^\sharp)(V)$ we denote its image in $\mathcal{E}_B(V)$ by $[\omega]$.

Remark 3.1. When M is symplectic, one has $\mathcal{E}_B = \operatorname{Ker}\pi^\#$, since $\Pi^\#$ and ϕ^* are injective. Moreover, π is then of constant rank (see item (3) of Proposition 2.10), so that $\mathcal{E}_B$ is the space of sections of a vector bundle on B; since $\operatorname{Ker}\pi^\# = (\operatorname{Im}\pi^\#)^\circ$, this vector bundle is the annihilator of the symplectic foliation of (B,π). When M is not symplectic, $\mathcal{E}_B \neq \operatorname{Ker}\pi^\#$ and the rank of $\pi^\#$ may not be constant, so the use of sheaf language is unavoidable.

The main reason why the non-symplectic case is still tractable is the following somewhat surprising result, stating that the sheaf $\mathcal{E}_B$ is always the sheaf of sections of a vector bundle $E \to B$.

Proposition 3.2. Let $(M,\Pi) \stackrel{\phi}{\to} (B,\pi)$ be an NCI system of rank r. The quotient sheaf $\mathcal{E}_B := \operatorname{Ker}\pi^\sharp / \operatorname{Ker}(\Pi^\sharp \circ \phi^*)$ is the sheaf of sections of a vector bundle E on B of rank r. We call $\mathcal{E}_B$ the **action sheaf** and $E \to B$ the **action bundle** of the NCI system.

Proof. We need to show that $\mathcal{E}_B$ is a locally free sheaf of C^∞_B-modules of rank r. Let $b \in B$ and denote, as before, by $\mathfrak{F}$ the foliation of M defined by the fibers of ϕ. According to the definition of an NCI system and Proposition 2.10 (2) there exist, on a neighborhood V of b, Casimir functions $g_1, \dots, g_r$ of π such that $T\mathfrak{F}$ is spanned at each point of $\phi^{-1}(V)$ by $X_{f_1}, \dots, X_{f_r}$, where $f_i := g_i \circ \phi$, for $i = 1, \dots, r$. Let $s \in \mathcal{E}_B(V)$. By definition, s is given by a collection $(V_i, s_i)_{i\in I}$, where $(V_i)_{i\in I}$ is an open cover of V and $s_i \in (\operatorname{Ker}\pi^\sharp)(V_i) \subset \Omega^1_B(V_i)$ for every $i \in I$; also $s_i|_{V_i\cap V_j} - s_j|_{V_i\cap V_j} \in \operatorname{Ker}(\Pi^\sharp \circ \phi^*)(V_i \cap V_j)$ whenever $V_i \cap V_j \neq \emptyset$. Since the vector fields $\Pi^\sharp(\phi^* s_i)$ are tangent to the fibers of ϕ, there exist unique smooth functions λ_{il} on $\phi^{-1}(V)$, such that

$$\Pi^\sharp(\phi^* s_i) = \sum_{l=1}^{r} \lambda_{il} \Pi^\sharp(\mathrm{d} f_l). \tag{3.1}$$

We show that these functions are ϕ-basic (i.e., constant on the fibers of ϕ). To do this, we show that $X_{f_k}(\lambda_{il}) = 0$ for $i \in I$ and $k, l = 1, \dots, r$. Since X_{f_k} is tangent to $\mathfrak{F}$,

$$\left[X_{f_k}, \Pi^\sharp(\phi^* s_i)\right] = \Pi^\sharp(\mathcal{L}_{X_{f_k}} \phi^* s_i) = 0,$$

so that

$$\sum_{l=1}^{r} X_{f_k}(\lambda_{il}) \Pi^\sharp(\mathrm{d} f_l) = 0.$$

This shows our claim because the vector fields $\Pi^\sharp(\mathrm{d}f_1), \dots, \Pi^\sharp(\mathrm{d}f_r)$ are linearly independent at every point of $\phi^{-1}(V)$. Since the fibers of ϕ are connected, it follows that there exist (unique) smooth functions σ_{il} on V such that $\lambda_{il} = \sigma_{il} \circ \phi$. Substituted in (3.1), we find that

$$\Pi^\sharp \phi^* \left(s_i - \sum_{l=1}^{r} \sigma_{il} \mathrm{d} g_l \right) = 0,$$

so that $s_i - \sum_{l=1}^{r} \sigma_{il} \mathrm{d} g_l \in \mathrm{Ker}(\Pi^\sharp \circ \phi^*)(V_i)$. For i, j such that $V_i \cap V_j \neq \emptyset$ we have that $s_i|_{V_i \cap V_j} - s_j|_{V_i \cap V_j} \in \mathrm{Ker}(\Pi^\sharp \circ \phi^*)(V_i \cap V_j)$, so that $\sigma_{il} = \sigma_{jl}$ on $V_i \cap V_j$ for all l. Thus, the functions $(\sigma_{il})_{i \in I}$ glue together to a global function $\sigma_l \in C^\infty_B(V)$ and we can write $s = \sum_{l=1}^{r} \sigma_l [\mathrm{d} g_l]$ for some unique smooth functions σ_l on V, as required. □

For $b \in B$, the fiber E_b of the vector bundle $E \to B$ corresponding to $\mathcal{E}_B$ can be recovered from $\mathcal{E}_B$ as

$$E_b = \frac{\mathcal{E}_B(V)}{C^\infty_b(V) \mathcal{E}_B(V)}, \tag{3.2}$$

where $C^\infty_b(V)$ stands for the ideal of $C^\infty_B(V)$ containing all smooth functions on V which vanish at b and V is any open subset of B containing b and such that $\mathcal{E}_B(V)$ is a free $C^\infty_B(V)$-module. Let m be any point in the fiber of ϕ over b. We show that the following sequence of vector spaces is exact:

$$0 \longrightarrow C^\infty_b(V) \mathcal{E}_B(V) \longrightarrow \mathcal{E}_B(V) \xrightarrow{\rho_b} \frac{\mathrm{Ker}(\pi^\sharp_b)}{\mathrm{Ker}(\Pi^\sharp_m \circ \phi^*)} \longrightarrow 0. \tag{3.3}$$

To do this, we first show that if $m, m' \in \phi^{-1}(b)$ then

$$\mathrm{Ker}(\Pi^\sharp_m \circ \phi^*) = \mathrm{Ker}(\Pi^\sharp_{m'} \circ \phi^*),$$

so that the latter space is independent of the choice of m in $\phi^{-1}(b)$. Since the fibers of ϕ are connected it is enough to prove the equality for m' in a neighborhood of m. There exist, in a neighborhood of $\phi(m)$ in B, Casimir functions $g_1, \ldots, g_r$ such that the Hamiltonian vector fields of $f_1 := g_1 \circ \phi, \ldots, f_r := g_r \circ \phi$ span $T\mathfrak{F}$ in a neighborhood of m in M. The (local) flows of these vector fields commute, since $\left[X_{f_i}, X_{f_j}\right] = -X_{\{g_i,g_j\}\circ\phi} = 0$. These flows therefore define a (local) action of $\mathbb{R}^r$, by Poisson diffeomorphisms, which is transitive in a neighborhood of m. In particular, we obtain a local Poisson diffeomorphism Ψ such that $\phi \circ \Psi = \phi$ and $\Psi(m) = m'$. It follows that:

$$\Pi^\sharp_{m'} \circ \phi^* = \mathrm{d}_m\Psi \circ \Pi^\sharp_m \circ (\mathrm{d}_m\Psi)^* \circ \phi^* = \mathrm{d}_m\Psi \circ \Pi^\sharp_m \circ \phi^*.$$

This implies our claim since $\mathrm{d}_m\Psi$ is an isomorphism. We can now prove that (3.3) is a short exact sequence. Since the injectivity of the first arrow and the surjectivity of the last arrow are clear, we only prove the exactness at $\mathcal{E}_B(V)$. Let s be an element of $\mathcal{E}_B(V)$. As we have seen in the proof of Proposition 3.2, s can be written as $s = \sum_{l=1}^r \sigma_l[\mathrm{d}g_l]$ for some unique smooth functions σ_l on V. Exactness then follows from the fact that $\rho_b(s) = \sum_{l=1}^r \sigma_l(b)[\mathrm{d}_b g_l]$ where, by a slight abuse of notation, $[\mathrm{d}_b g_l]$ stands for the class of $\mathrm{d}_b g_l$ in $\mathrm{Ker}(\pi^\sharp_b)/\mathrm{Ker}(\Pi^\sharp_m \circ \phi^*)$.

The exactness of (3.3), combined with (3.2), provides a natural identification of E_b with $\frac{\mathrm{Ker}(\pi^\sharp_b)}{\mathrm{Ker}(\Pi^\sharp_m \circ \phi^*)}$. As we show next, the Poisson structure Π also induces a natural identification of E_b with $T_m\mathfrak{F}$, which is the tangent space to $\phi^{-1}(b)$ at m, where m is an arbitrary point in $\phi^{-1}(b)$. Indeed, every $\alpha \in E_b$ defines a smooth vector field X_α on the fiber $\phi^{-1}(b)$ over b, by

$$X_\alpha(m) := \Pi^\sharp_m(\phi^*\alpha)$$

for all $m \in \phi^{-1}(b)$. We call X_α the **action vector field** associated to α.

Lemma 3.1. *Let $(M,\Pi) \stackrel{\phi}{\to} (B,\pi)$ be an NCI system of rank r. Let $m \in M$ and denote $b := \phi(m) \in B$.*

1) *For every $\alpha, \alpha' \in E_b$, the action vector fields X_α and $X_{\alpha'}$ commute;*
2) *For every basis $\alpha_1, \ldots, \alpha_r$ of E_b the vector fields $X_{\alpha_1}, \ldots, X_{\alpha_r}$ form a basis of $T_m\mathfrak{F}$. In particular, X_α is nowhere vanishing when $\alpha \neq 0$.*

Proof. On a neighborhood V of b in B there exist Casimirs $g_1, \ldots, g_r$ such that their associated vector fields $\Pi^\sharp(\mathrm{d}(g_i \circ \phi))$ span the tangent space to $\mathfrak{F}$ on $\phi^{-1}(V)$. It follows that $[\mathrm{d}_b g_1], \ldots, [\mathrm{d}_b g_r]$ are independent, hence form a basis for E_b and (2) follows. The vector fields $X_{g_1\circ\phi},\ldots,X_{g_r\circ\phi}$ are tangent to the fibers of ϕ over V and they commute, as we have seen above. In particular, the vector fields X_α commute. □

In view of item (2) above, the map $E_{\phi(m)} \to T_m\mathfrak{F}$ which sends $\alpha \in E_{\phi(m)}$ to X_α is an isomorphism and we may think of $E_{\phi(m)}$ as being the tangent space to the fiber of ϕ at m.

The notation X_α which we introduced above for elements α of E_b will also be used for (local) sections of $E \to B$: for a section $e \in \mathcal{E}_B(V)$, the **action vector field** X_e is a vector field which is defined on $\phi^{-1}(V)$ and it is tangent to the fibers of ϕ: for $b \in V$, the restriction of X_e to $\phi^{-1}(b)$ is $X_{e(b)}$. For arbitrary sections $e, e' \in \mathcal{E}_B(V)$ the vector fields X_e and $X_{e'}$ commute, in view of Lemma 3.1 (1).

3.2. Holonomic sections of the action bundle

Suppose that we have an NCI system $(M,\Pi) \stackrel{\phi}{\to} (B,\pi)$ of rank r. We denote its action sheaf by $\mathcal{E}_B$.

Definition 3.3. For $V \subset B$, we call an element $e \in \mathcal{E}_B(V)$ **locally holonomic** if for every point $b \in V$, there exists a Casimir function g of π, defined on a neighborhood $W \subset V$ of b in B, such that $e_{|W} = [\mathrm{d}g]$. For an open subset V of B, an element e of $\mathcal{E}_B(V)$ is called a **globally holonomic** section if $e = [\mathrm{d}g]$ for some Casimir function g on V.

Notice that when two such neighborhoods W_1 and W_2 intersect, the Casimir functions g_1 and g_2 which define e satisfy $[\mathrm{d}(g_1 - g_2)] = 0$ on $W_1 \cap W_2$, so that $\phi^*(g_1 - g_2)$ is a Casimir function of Π (on $\phi^{-1}(W_1 \cap W_2)$). Therefore, we introduce three more sheaves on B, by letting for every open subset V of B:

$$\begin{aligned}
\mathcal{C}as_B(V) &:= \{F \in C^\infty(V) \mid F \text{ is a Casimir function of } \pi|_V\},\\
\mathcal{C}as_B^M(V) &:= \left\{F \in C^\infty(V) \mid F\circ\phi \text{ is a Casimir function of } \Pi|_{\phi^{-1}(V)}\right\},\\
\mathcal{E}_B^0(V) &:= \{e \in \mathcal{E}_B(V) \mid e \text{ is locally holonomic}\}.
\end{aligned}$$

$\mathcal{C}as_B$ is the sheaf of Casimir functions on B, while $\mathcal{C}as_B^M$ is the sheaf of **Casimir-basic functions**, that is local functions on B whose pullback to M are Casimir functions.

Example 3.2. When M is symplectic, so (B,π) is regular, $\mathcal{C}as_B$ is the sheaf of functions on B locally constant along the symplectic foliation of

(B,π), while $\mathcal{C}as_B^M$ is simply the sheaf of constant functions on B. In this case, as we pointed out before, $\mathcal{E}$ is the sheaf of sections of $\ker\pi$, i.e., the sheaf of 1-forms on B, annihilating the symplectic foliation of (B,π), while $\mathcal{E}_B^0$ is the sheaf of closed 1-forms on B, annihilating the symplectic foliation of (B,π).

Notice that, contrary to the sheaves which were introduced in the previous subsection, the sheaves $\mathcal{C}as_B$, $\mathcal{C}as_B^M$ and $\mathcal{E}_V^0$ are simply sheaves of $\mathbb{R}$-vector spaces and not of C_B^∞-modules. Since ϕ is a surjective Poisson morphism, $\mathcal{C}as_B^M$ is included in $\mathcal{C}as_B$, and the above argument shows that $\mathcal{E}_B^0$ is the quotient sheaf $\mathcal{C}as_B/\mathcal{C}as_B^M$, i.e., the following sequence of sheaves of vector spaces on B is exact:

$$0 \longrightarrow \mathcal{C}as_B^M \longrightarrow \mathcal{C}as_B \xrightarrow{[\mathrm{d}\cdot]} \mathcal{E}_B^0 \longrightarrow 0. \tag{3.4}$$

We will be particularly interested in globally holonomic sections which are defined on all of B. In order to characterize these sections, we consider the long cohomology sequence associated to (3.4), which is given in part by

$$\cdots \longrightarrow H^0(B,\mathcal{C}as_B) \longrightarrow H^0(B,\mathcal{E}_B^0) \xrightarrow{\mathrm{Obs}} H^1(B,\mathcal{C}as_B^M) \longrightarrow \cdots$$

The connecting homomorphism defines a map, which we denote by Obs and which we call the **holonomy obstruction** (of the NCI system). The locally holonomic elements of $\mathcal{E}_B(B)$ are precisely the elements of $H^0(B,\mathcal{E}_B^0)$, while the globally holonomic elements of $\mathcal{E}_B(B)$ are the elements in the image of $H^0(B,\mathcal{C}as_B) \to H^0(B,\mathcal{E}_B^0)$. When M is symplectic, $\mathcal{C}as_B^M$ is a constant sheaf, so Obs takes values in the first De Rham cohomology group of M.

Exactness of the above long exact sequence leads to the following proposition.

Proposition 3.4. Let e be a global section of $\mathcal{E}_B$ which is locally holonomic. Then e is globally holonomic if and only if $\mathrm{Obs}(e)=0$.

For future use, we give the following exact sequence of sheaves, which derives from the exact sequence (3.4):

$$0 \longrightarrow \mathcal{C}as_B^M/\mathbb{R} \longrightarrow \mathcal{C}as_B/\mathbb{R} \xrightarrow{[\mathrm{d}\cdot]} \mathcal{E}_B^0 \longrightarrow 0; \tag{3.5}$$

here, and in all further sheaf contexts, $\mathbb{R}$ stands for the sheaf of locally constant functions on the manifold under consideration, in this case B.

Example 3.3. Let M be a symplectic manifold, so (B, π) is regular, and recall that in this case $\mathcal{C}as_B^M = \mathbb{R}_B$ (cf. Example 3.2). Then the proposition is rather tautological: it says that if e is a closed 1-form annihilating the symplectic foliation of (B, π) (i.e. e is a locally holonomic global section of $\mathcal{E}_B$), then $e = \mathrm{d}g$ for some Casimir g (i.e. e is globally holonomic) if and only if the class $[e] \in H^1(B, \mathbb{R})$ vanishes.

3.3. Action lattice bundle and integral affine structure

We say that an NCI system $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ has **compact fibers** when all the fibers of ϕ are compact. In this case, all the vector fields $X_{g\circ\phi}$, with g a local Casimir function of π, are complete. In particular, the action vector fields X_α, with $\alpha \in E_b$ are complete and we can consider their time 1 flow. In view of Lemma 3.1 the action vector fields associated to two elements of E_b (with $b \in B$) commute, hence the time 1 flow defines an action of E_b on $\phi^{-1}(b)$. By the same lemma, the action is locally free, hence transitive (recall that by definition the fibers of ϕ are connected). It follows that there is for each $b \in E_b$ a canonically defined lattice $L_b \subset E_b$, namely the subset of all points $\alpha \in E_b$ such that the time 1 flow of X_α is the identity map.

$$L_b = \{\alpha \in E_b : \phi^1_{X_\alpha}(m) = m, \forall m \in \phi^{-1}(b)\}.$$

We call $L_b \subset E_b$ the **action lattice** at b. As b runs through B, these lattices L_b fit nicely together in a group bundle L over B, with fiber isomorphic to $\mathbb{Z}^r$; for the proof of this fact, we refer to [18, Sect. 3.4]. We call L the **action lattice bundle** of the NCI system.

We will find it convenient to view the local sections of $L \to B$ as a sheaf on B, which we denote by $\mathcal{L}_B$ and which we call the **action lattice sheaf**. Thus, for any open subset V of B we denote by $\mathcal{L}_B(V)$ the space of sections of $L \to B$ over V. It is clear that $\mathcal{L}_B$ is a sheaf of $\mathbb{Z}$-modules on B: locally, $\mathcal{L}_B$ is isomorphic to the constant sheaf $\mathbb{Z}^r$ on B. An isomorphism between the restrictions of $\mathcal{L}_B$ and $\mathbb{Z}^r$ to $V \subset B$ is called a **trivialization** of $\mathcal{L}_B$ on V. Such as isomorphism is defined by r sections of $\mathcal{L}_B$ over V.

We can now define the notion of action variables in terms of the above terminology.

Definition 3.5. Let V be an open subset of B. We say that an r-tuple $(p_1, \ldots, p_r)$ of functions on V are a set of **local action variables** (on V) if $[\mathrm{d}p_1], \ldots, [\mathrm{d}p_r]$ define a trivialization of $\mathcal{L}_B$ on V. Local action variables on $V = B$ are called **global action variables**.

In view of Proposition 2.10 (2), local action variables are local Casimir functions of (B, π). Since, as we pointed out above, we can identify functions on B with functions on M which are constant on the fibers of ϕ, we will also call the functions $\phi^* p_i$ a set of (local or global) action variables.

Remark 3.6. By construction, if $(p_1, \ldots, p_r)$ are a set of action variables on V, then the Hamiltonian vector fields of $\phi^* p_1, \ldots, \phi^* p_r$ are periodic of period one and they commute; in particular $\phi^* p_1, \ldots, \phi^* p_r$ are the components of a momentum map of a $\mathbb{T}^r$-action on V. These properties justify the terminology *action variables.*

Theorem 2.12 leads immediately to the following:

Proposition 3.7. Let $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ be an NCI system with compact fibers.

(1) Local action variables exist on a neighborhood V of every point $b \in B$;

(2) $\mathcal{L}_B$ is a subsheaf of $\mathcal{E}^0_B$, where both sheaves are viewed as sheaves of $\mathbb{Z}$-modules. Said differently, if V is an open subset of B and $\ell \in \mathcal{L}_B(V)$, then ℓ is locally holonomic.

The following theorem gives a cohomological condition for the existence of global action variables.

Theorem 3.8. Let $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ be an NCI system of rank r with compact fibers. The following properties are equivalent:

(i) There exists a set of global action variables;

(ii) The action lattice sheaf $\mathcal{L}_B$ admits a (global) trivialization and every global section ℓ of $\mathcal{L}_B$ satisfies $\mathrm{Obs}(\ell) = 0$.

Proof. Let $(p_1, \ldots, p_r)$ be a set of global action variables for the NCI system. By definition, $[\mathrm{d}p_1], \ldots, [\mathrm{d}p_r]$ define a trivialization of $\mathcal{L}_B$ on B, hence every global section ℓ of $\mathcal{L}_B$ is of the form

$$\ell = \sum_{i=1}^{r} n_i [\mathrm{d}p_i]$$

for some integers $n_1, \ldots, n_r$. This implies that $\ell = [\mathrm{d}(\sum_{i=1}^r n_i p_i)]$ is in the image of $\mathcal{C}as_B(B) \to \mathcal{E}^0_B(B)$, so that $\mathrm{Obs}(\ell) = 0$. This proves that (i) implies (ii).

Conversely, suppose that $\ell_1, \dots, \ell_r$ define a trivialization of $\mathcal{L}_B$ on B and that $\mathrm{Obs}(\ell_i) = 0$ for $i = 1, \dots, r$. According to Proposition 3.4, there exist Casimir functions $p_1, \dots, p_r$ such that $\ell_i = [\mathrm{d}p_i]$ for $i = 1, \dots, r$. By definition, $(p_1, \dots, p_r)$ is a set of global action variables for the NCI system. □

Remark 3.9. Notice that saying that $\mathcal{L}_B$ admits a (global) trivialization is equivalent to saying that $M \to B$ is a principal $\mathbb{T}^r$-bundle; it is also equivalent to saying that the class defined by $\mathcal{L}_B$ in $H^1(M, GL_r(\mathbb{Z}))$ is trivial.

3.4. Action foliations

Let $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ be an NCI system of rank r with compact fibers. A foliation $\mathfrak{A}$ of B is said to be an **action foliation** of the NCI system when $\mathfrak{A}$ is defined in the neighborhood of every point by local action variables. It means that on a neighborhood V of any point $b \in B$ we can find functions $p_1, \dots, p_r$ such that

(1) $[\mathrm{d}p_1], \dots, [\mathrm{d}p_r]$ define a trivialization of $\mathcal{L}_B$ on V;

(2) The foliation $\mathfrak{A}$, restricted to V, is defined by $p_1, \dots, p_r$.

Although the foliation defined by action variables is always an action foliation, an action foliation may not be defined by global action variables, as we will see later.

Remark 3.4. The foliation $\mathfrak{F}$ of the NCI system is polar to the pullback $\phi^{-1}(\mathfrak{A})$ of any action foliation $\mathfrak{A}$ (see Remark 2.13). Conversely, if $\mathfrak{F}$ is polar to some foliation $\mathfrak{G}$, then $\mathfrak{G}$ is the pullback of a foliation $\phi(\mathfrak{G})$, but this foliation, in general, will fail to be an action foliation. One can show that this is the case if and only if $\mathfrak{G}$ is locally given around its leaves by the kernel of basic closed 1-forms $\alpha_1, \dots, \alpha_r \in \Omega^1(M)$ with the property that the the vector fields $\Pi^\sharp(\alpha_1), \dots, \Pi^\sharp(\alpha_1)$ have all their orbits periodic with period 1. Hence, the existence of an action foliation requires the existence of a polar foliation of a very special nature.

We denote by $\mathcal{C}as_{\mathfrak{A}}$ the sheaf of local first integrals of $\mathfrak{A}$; the notation is motivated by the first item in the following proposition:

Proposition 3.10. Let $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ be an NCI system of rank r with compact fibers. Suppose that it has an action foliation $\mathfrak{A}$. Then the following properties are satisfied:

(1) $\mathcal{C}as_{\mathfrak{A}}$ is a subsheaf of $\mathcal{C}as_B$; said differently, $\mathfrak{A}$ contains the symplectic foliation of (B, π);

(2) $\mathfrak{A}$ is a transversely integral affine foliation.

Proof. Item (1) follows from the fact that $\mathfrak{A}$ is locally defined by action variables, which are local Casimir functions of π. In order to prove (2), consider a cover of B by open sets on which $\mathfrak{A}$ is defined by local action variables. Let V and V' be two intersecting subsets of the cover and let $(p_1, \ldots, p_r)$ (resp. $(p'_1, \ldots, p'_r)$) be a set of action variables on V (resp. on V') which define $\mathfrak{A}$. Then we can write on a connected neighborhood W of any $b \in V \cap V'$ the functions $p'_1, \ldots, p'_r$ in terms of $p_1, \ldots, p_r$. Taking the differential, we get

$$\mathrm{d}p'_i = \sum_{k=1}^{r} \frac{\partial p'_i}{\partial p_k} \mathrm{d}p_k, \qquad (i = 1, \ldots, r).$$

Since both $([\mathrm{d}p_1], \ldots, [\mathrm{d}p_r])$ and $([\mathrm{d}p'_1], \ldots, [\mathrm{d}p'_r])$ define a trivialization of $\mathcal{L}_B$ on W, the above relations imply that the functions $a_{ij} := \frac{\partial p'_i}{\partial p_j}$ are constant and take values in $\mathbb{Z}$, for all $i, j = 1, \ldots, r$. Since W is connected, it follows that each one of the functions $p'_1, \ldots, p'_r$ is, up to real a constant, a linear combination with integral coefficients of the functions $p_1, \ldots, p_r$; this is precisely the property which defines transversely integral affine foliations. □

Example 3.5. When M is symplectic there exists a unique action foliation, namely the symplectic foliation of (B, π). Indeed, Proposition 2.10 (2) implies that the Poisson structure π on B is regular, with symplectic leaves of dimension $\dim B - r$. Hence, *every* set of local action variables defines the symplectic foliation. The previous theorem then shows that the symplectic foliation has a transversely integral affine structure, a result that plays a fundamental role in the work of Dazord and Delzant [12]. A very special instance of this is the well known fact that the base of a Lagrangian fibration (i.e. a Liouville integrable system) is an integral affine manifold, since in this case $\pi \equiv 0$ and the symplectic leaves are just the points of B.

Example 3.6. We now get back to our main example, the Euler-Poinsot top (see Example 2.4) and address the question of the existence of action variables and foliations on M_+. First, since M_+ is a symplectic manifold, the symplectic foliation on B is regular and is the only action foliation (see Remark 3.5), in particular there exists an action foliation. Moreover, since B is simply-connected there are no obstructions to extend the action variables which define locally the action foliation into global action variables. Thus, global action variables exist also.

We will next analyse the obstruction to the existence of an action foliation when $\mathcal{L}_B$ admits a trivialization over B. Associated to the following short exact sequence of sheaves on B:

$$0 \to \mathbb{R} \to \mathcal{C}as_B^M \to \mathcal{C}as_B^M/\mathbb{R} \to 0,$$

there is the long exact sequence

$$\cdots \to H^1(B,\mathbb{R}) \to H^1(B,\mathcal{C}as_B^M) \to H^1(B,\mathcal{C}as_B^M/\mathbb{R}) \to \cdots \tag{3.6}$$

We say that a class in $H^1(B,\mathcal{C}as_B^M)$ is **representable by constants** if it lies in the image of $H^1(B,\mathbb{R}) \to H^1(B,\mathcal{C}as_B^M)$, or, equivalently, in the kernel of $H^1(B,\mathcal{C}as_B^M) \to H^1(B,\mathcal{C}as_B^M/\mathbb{R})$. A class is representable by constants if and only if it can be represented by a cocycle valued in locally constant functions, hence the name.

Proposition 3.11. Let $(M,\Pi) \xrightarrow{\phi} (B,\pi)$ be an NCI system of rank r with compact fibers. Suppose that its action lattice sheaf $\mathcal{L}_B$ admits a trivialization on B, defined by sections $\ell_1,\dots,\ell_r$ of $\mathcal{L}_B$ over B. Then the following conditions are equivalent:

(i) There exists a global action foliation for the NCI system;

(ii) For $i = 1,\dots,r$, the class $\mathrm{Obs}(\ell_i) \in H^1(B,\mathcal{C}as_B^M)$ is representable by constants.

Proof. The connecting morphism Obs of the exact sequence (3.4) and the connecting morphism δ of the exact sequence (3.5) are related through the following commutative diagram:

$$\begin{array}{ccccc} & & H^0(B,\mathcal{E}_B^0) & & \\ & & \big\downarrow{\scriptstyle \mathrm{Obs}} & \searrow^{\delta} & \\ H^1(B,\mathbb{R}) & \longrightarrow & H^1(B,\mathcal{C}as_B^M) & \longrightarrow & H^1(B,\mathcal{C}as_B^M/\mathbb{R}) \end{array}$$

Since the horizontal line of this diagram is exact, $\mathrm{Obs}(\ell_i)$ is representable by constants if and only if $\delta(\ell_i) = 0$.

Suppose that there exists a global action foliation $\mathfrak{A}$. Then there exist in the neighborhood of every point of B Casimir functions $p_1,\dots,p_r$, such that $[\mathrm{d}p_i] = \ell_i$ for $i = 1, \dots, r$. As in the proof of Proposition 3.10 the functions p_i and p'_i differ on overlapping open subsets only by locally constant functions, hence the cocycle which is defined by ℓ_i is trivial in $H^1(B, \mathcal{C}as_B^M)$, i.e., $\delta(\ell_i) = 0$, so that $\mathrm{Obs}(\ell_i)$ is representable by constants. This shows that (i) implies (ii).

Suppose now that each $\mathrm{Obs}(\ell_i)$ is representable by constants. Then there exists a cover of B by open subsets $(U_j)_{j\in J}$ and Casimir functions $p_{1j}, \dots, p_{rj}$ on each U_j, such that for every $i = 1, \dots, r$,

(1) $[\mathrm{d}p_{ij}] = \ell_i$ on U_j, for all $j \in J$;

(2) On non-empty overlaps $U_j \cap U_k$, which are supposed connected, $p_{ij} - p_{ik}$ is constant.

The first condition implies that for fixed $j \in J$ the functions $p_{1j}, \dots, p_{rj}$ define an action foliation on U_j, while the second condition implies that the action foliations on U_j and U_k concide on $U_j \cap U_k$, hence define a global action foliation on B. This shows that (ii) implies (i). □

Example 3.7. We now give two examples of NCI systems which have compact fibers and trivial action lattice sheaf, yet fail to have global action variables; the two examples differ in the existence of an action foliation. We also show that the existence of global action variables, defining an action foliation may depend on the choice of action foliation.

Let $M := S^1 \times B$ where B is a manifold equipped with a nowhere vanishing vector field $\mathcal{V}$. The foliation of B, defined by $\mathcal{V}$, is denoted by $\mathfrak{T}$. Consider the Poisson structure on M defined by

$$\Pi := \frac{\partial}{\partial\theta} \wedge \mathcal{V},$$

where θ is the parameter on S^1, viewed as a function on M. Let $\phi : S^1 \times B \to B$ denote the projection on the second component. The tangent space to the fibers of ϕ is spanned by $\partial/\partial\theta$, which is a locally Hamiltonian vector field: for any local function p on B we have that $X_{\phi^* p} = \phi^*(\mathcal{V}(p))\, \partial/\partial\theta$. Thus, $(M, \Pi) \stackrel{\phi}{\to} (B, 0)$ is an NCI system of rank 1. The fibers of its momentum map are circles. For every point $b \in B$, only one of the two generators of the action lattice L_b at the point b corresponds to the vector field $\frac{\partial}{\partial\theta}$. The action lattice, therefore, admits a global section e, in particular the action lattice sheaf $\mathcal{L}_B$ is trivial.

Claim. When B is compact, the NCI system $(M,\Pi) \stackrel{\phi}{\to} (B,0)$ above does not admit global action variables. When B is moreover simply-connected, it even does not admit an action foliation.

Indeed, when B is compact, every function on B has points where its differential vanishes. Such a function can never be an action variable, which shows the first statement. Assume now that there exists a global action foliation $\mathfrak{A}$ on B. Since $\mathfrak{A}$ is a transverse integral affine foliation of codimension 1 on a simply-connected manifold, it must be given by the kernel of a *closed* 1-form. But since B is simply-connected, $H^1(B,\mathbb{R})=0$, so this form is exact and its kernel cannot define a regular foliation. This shows our claim.

The second part of this argument can be reformulated in terms of the obstruction theory of Section 3.4 as follows: according to Proposition 3.11, an action foliation exists iff $\mathrm{Obs}([e])$ is representable by constants. When B is simply-connected, $H^1(B,\mathbb{R})=0$, so $\mathrm{Obs}([e])$ is representable by constants if and only if $\mathrm{Obs}([e])=0$, which is according to Theorem 3.7 equivalent to the existence of a global action variable. But we know from the first part that such a global variable does not exist.

Let us apply the proposition to $B=S^3$, equipped with the fundamental vector field $\mathcal{V}$ of the Hopf fibration $S^3\to S^2$, i.e., the fundamental vector field of the natural S^1-action on S^3. Since S^3 is both compact and simply-connected, the above claim shows that this NCI system does not admit an action foliation.

We next apply the above claim to $B=S^1$, with its natural translation invariant vector field $\partial/\partial\psi$, so that $\omega=\mathrm{d}\psi$. Since S^1 is compact, the claim shows that this system does not admit an action variable. However, since ω is closed (but not exact!), it defines an action foliation.

To finish this example, we consider $B:=S^1\times\mathbb{R}$ (a cylinder) equipped with an S^1-valued coordinate ψ and an $\mathbb{R}$-valued coordinate p, corresponding to the first and second projections. Any foliation $\mathfrak{A}$ of B, transverse to $\mathcal{V}:=\partial/\partial p$ is an action foliation since $\mathfrak{A}$ can locally be defined by a function $\tilde{p}$ such that $\frac{\partial\tilde{p}}{\partial p}=1$, i.e., a local action variable. Thus the two foliations, defined by the vector fields

$$\frac{\partial}{\partial\psi} \qquad \text{and} \qquad \frac{\partial}{\partial\psi}+p\frac{\partial}{\partial p}$$

are action foliations. The first foliation is defined by the function p, which is an action variable. However, the second foliation has as leaves the circle $C_0:=\{p=0\}$ and a family of curves which are transverse to $\partial/\partial p$ and spiral towards C_0. It is not a foliation defined by a function, so there is no global action variable defining it.

Remark 3.8. Proposition 3.11 can be generalized to the case where the lattice sheaf is not trivial as follows: let $(M,\Pi) \stackrel{\phi}{\to} (B,\pi)$ be an NCI system of rank r with compact fibers and let $\mathcal{L}_B$ denote its lattice sheaf. The following statements are then equivalent:

(i) There exists an action foliation for $(M,\Pi) \stackrel{\phi}{\to} (B,\pi)$;

(ii) The cohomology class $\mathrm{Obs}(\mathcal{L}_B) \in H^1(B, \mathcal{H}om_{\mathbb{Z}}(\mathcal{L}_B, \mathcal{C}as_B^M/\mathbb{R}))$ vanishes.

Let us define the class and the cohomology space that appear in (ii). For any sheaf of abelian groups $\mathcal{F}$ over B, we denote by $\mathcal{H}om_{\mathbb{Z}}(\mathcal{L}_B, \mathcal{F})$ the sheaf whose sections over an open subset $U \subset B$ is the set of all group morphisms from $\mathcal{L}_B(U)$ to $\mathcal{F}(U)$; thus, $\mathcal{H}om_{\mathbb{Z}}(\mathcal{L}_B, \mathcal{F})$ is itself a sheaf of abelian groups. Applying to the exact sequence (3.5) the exact functor $\mathcal{H}om_{\mathbb{Z}}(\mathcal{L}_B, \cdot)$ yields an exact sequence:

$$(3.7) \qquad \begin{aligned} 0 &\to \mathcal{H}om_{\mathbb{Z}}(\mathcal{L}_B, \mathcal{C}as_B^M/\mathbb{R}) \\ &\to \mathcal{H}om_{\mathbb{Z}}(\mathcal{L}_B, \mathcal{C}as_B/\mathbb{R}) \stackrel{[\mathrm{d}\cdot]}{\to} \mathcal{H}om_{\mathbb{Z}}(\mathcal{L}_B, \mathcal{E}_B^0) \to 0. \end{aligned}$$

Now, the canonical inclusion $\mathcal{L}_B \hookrightarrow \mathcal{E}_B^0$ can be seen as an element in $H^0(B, \mathcal{H}om_{\mathbb{Z}}(\mathcal{L}_B, \mathcal{E}_B^0))$, to which the connecting morphism of (3.7) can be applied, giving a class in $H^1(B, \mathcal{H}om_{\mathbb{Z}}(\mathcal{L}_B, \mathcal{C}as_B^M/\mathbb{R}))$, which we denote by $\mathrm{Obs}(\mathcal{L}_B)$.

The proof of the equivalence between (i) and (ii) follows essentially the same lines as the proof of proposition 3.11, upon noticing that $\delta(\imath_{\mathcal{L}}) = 0$ is tantamount to the existence of a sheaf homomorphism $\jmath_{\mathcal{L}}$ from $\mathcal{L}_B$ to $\mathcal{C}as_B/\mathbb{R}$ which makes the following diagram commutative:

$$(3.8) \qquad \begin{array}{ccc} \mathcal{L}_B & \stackrel{\jmath_{\mathcal{L}}}{\dashrightarrow} & \mathcal{C}as_B/\mathbb{R} \\ & {\scriptstyle \imath_{\mathcal{L}}} \searrow \quad \swarrow {\scriptstyle [\mathrm{d}\cdot]} & \\ & \mathcal{E}_B^0 & \end{array}$$

while the existence of $\jmath_{\mathcal{L}}$ can be checked to be equivalent to the existence of an action foliation.

4. Angle variables and transverse structure

In this section, we suppose that we have an NCI system $(M,\Pi) \stackrel{\phi}{\to} (B,\pi)$ with compact fibers. Recall from Section 2.4 that this means in particular that both M and B are manifolds and that the fibers of ϕ are compact and connected. As before, we denote its action lattice sheaf by $\mathcal{L}_B$ and we denote the action vector field associated to a local section e of $\mathcal{L}_B$ by X_e.

4.1. Angle variables

We first define the notion of angle variables.

Definition 4.1. Let $(e_1,\dots,e_r)$ be a trivialization of $\mathcal{L}_B(V)$ where V is some open subset of B. An r-tuple of $\mathbb{R}/\mathbb{Z}$-valued functions $(\theta_1,\dots,\theta_r)$ defined on $\phi^{-1}(V)$ is called a set of **local angle variables** on $\phi^{-1}(V)$, adapted to $(e_1,\dots,e_r)$, if

$$\{\theta_i,\theta_j\}=0, \qquad X_{e_i}(\theta_j)=\delta_{i,j}, \tag{4.1}$$

for all $1\leqslant i,j\leqslant r$.

Notice that, given a set of local angle variables, the trivialization of $\mathcal{L}_B(V)$ with respect to which it is adapted is uniquely determined by it, so we may speak of local angle variables without specifying a (local) trivialization of $\mathcal{L}_B$. As a consequence, given an r-tuple of $\mathbb{R}/\mathbb{Z}$-valued functions $(\theta_1,\dots,\theta_r)$ on M, which are local angle variables in the neighborhood of every point of B, there exists a global trivialization $(e_1,\dots,e_r)$ of $\mathcal{L}_B(B)$ such that $(\theta_1,\dots,\theta_r)$ are angle variables on M, adapted to it. We then call $(\theta_1,\dots,\theta_r)$ **global angle variables**.

The following proposition is a corollary of the local action-angle theorem (Theorem 2.12):

Proposition 4.2. Every point $b\in B$ is contained in an open neighborhood V such that there exists a trivialization $e=(e_1,\dots,e_r)$ of $\mathcal{L}_B(V)$ and a set of local angle variables on $\phi^{-1}(V)$ adapted to e.

In order to show how two different sets of local angle variables are related, we first construct r vector fields Y_{θ_i} on $V\subset B$ which represent the

Hamiltonian vector fields, associated to a set of local angle variables[4] of the lattice sheaf.

Proposition 4.3. Let V be an open subset of B and suppose that $(\theta_1,\dots,\theta_r)$ is a set of local angle variables on $\phi^{-1}(V)$, adapted to some trivialization $(e_1,\dots,e_r)$ of $\mathcal{L}_B(V)$. The Hamiltonian vector fields $X_{\theta_1},\dots,X_{\theta_r}$ are ϕ-related to commuting Poisson vector fields $Y_{\theta_1},\dots,Y_{\theta_r}$ on V.

Proof. As we have seen in Section 3.3, the sections e_i of $\mathcal{L}_B(V)$ are locally of the form $[\mathrm{d}p_i]$, where each p_i is a local Casimir on B. Thus, $X_{e_i}=X_{\phi^*p_i}$, so that the vector fields X_{e_i} are locally Hamiltonian vector fields, hence globally Poisson vector fields, on $\phi^{-1}(V)$. It implies that for every function H on $\phi^{-1}(V)$

$$[X_{e_i},X_H]=X_{X_{e_i}(H)}.$$

In view of (4.1), this shows that $[X_{e_i},X_{\theta_j}]=0$ for $i,j=1,\dots,r$. In turn, this implies that for F a function on V, the function $X_{\theta_j}(\phi^*F)$ is a ϕ-basic function on $\phi^{-1}(V)$; indeed, for any $i=1,\dots,r$,

$$X_{e_i}\left(X_{\theta_j}(\phi^*F)\right)=X_{\theta_j}\left(X_{e_i}(\phi^*F)\right)=0.$$

As a consequence, there exists a unique function G_j such that $\phi^*G_j=X_{\theta_j}(\phi^*F)$. The map $F\mapsto G_j$ is clearly a derivation, hence defines a vector field on V which we denote by Y_{θ_j}. By construction, the vector fields X_{θ_j} and Y_{θ_j} are ϕ-related, $\phi^*\circ Y_{\theta_j}=X_{\theta_j}\circ\phi^*$. The fact that each Y_{θ_i} is a Poisson vector field follows from the fact that ϕ is a Poisson submersion from M to B, and that X_{θ_i}, which is a Hamiltonian, hence Poisson vector field, is ϕ-related to Y_{θ_i}. They commute in view of the commutativity of the vector fields X_{θ_i} to which they are ϕ-related, with ϕ being a submersion. □

We show in the following lemma how two different sets of local angle variables of an NCI system $(M,\Pi)\stackrel{\phi}{\to}(B,\pi)$ are related.

Lemma 4.1. *Let V be an open subset of B and suppose that $\theta=(\theta_1,\dots,\theta_r)$ and $\theta'=(\theta'_1,\dots,\theta'_r)$ are two sets of local angle variables adapted to the same trivialization $(e_1,\dots,e_r)$ of $\mathcal{L}_B(V)$. Let $Y_{\theta_1},\dots,Y_{\theta_r}$ be the vector fields on V defined in Proposition 4.3 using the set of angle variables $\theta_1,\dots,\theta_r$. Then there exist functions $F_1,\dots,F_r$ on V such that:*

[4]We will see in Section 4.3 that these vector fields define an integrable distribution of rank r which depends only on the foliation, defined by the angle variables and which is transverse to every local action foliation.

(1) $\theta'_i = \theta_i + \phi^* F_i$;

(2) $\{F_i, F_j\} = Y_{\theta_j}(F_i) - Y_{\theta_i}(F_j)$.

Proof. In view of (4.1), $X_{e_j}(\theta_i - \theta'_i) = 0$ for all $i, j = 1, \ldots, r$, which yields the existence of unique functions $F_1, \ldots, F_r$ on V, satisfying (1). Since $\left\{\theta'_i, \theta'_j\right\} = \{\theta_i, \theta_j\} = 0$ for $i, j = 1, \ldots, r$, (1) implies:

$$0 = \left\{\theta'_i, \theta'_j\right\} - \{\theta_i, \theta_j\} = \{\theta_i, \phi^* F_j\} + \{\phi^* F_i, \theta_j\} + \{\phi^* F_i, \phi^* F_j\}.$$

Now, by definition of the vector fields Y_{θ_i} and since ϕ is a Poisson map, this amounts to:

$$\phi^*(Y_{\theta_i}(F_j)) - \phi^*(Y_{\theta_j}(F_i)) - \phi^* \{F_i, F_j\} = 0.$$

This gives the second relation. Conversely, given a set of angle variables $\theta_1, \ldots, \theta_r$ and functions $F_1, \ldots, F_r$ on V, satisfying (2), the above computation shows that the functions θ'_i, defined by (1), are a set of angle variables adapted to the same trivialization $(e_1, \ldots, e_r)$ of $\mathcal{L}_B(V)$. □

Remark 4.2. For a given trivialization $e = (e_1, \ldots, e_r)$ of $\mathcal{L}_B(V)$, each one of the action variables p_i satisfying $e_i = [\mathrm{d}p_i]$ is uniquely determined up to an element of $\mathcal{C}as_B^M(V)$. Therefore, if a set of action variables adapted to e exists, the space of all sets of action variables adapted to e is an affine space of rank r over the ring $\mathcal{C}as_B^M(V)$. There is no similar property for angle variables adapted to $(e_1, \ldots, e_r)$: it is not an affine space, since the transformation which relates two of them (formulas (1) and (2) above) is non-linear.

4.2. Angle foliations

For a given set of local angle variables $\theta = (\theta_1, \ldots, \theta_r)$ on $\phi^{-1}(V)$, the level sets of the map $\theta : \phi^{-1}(V) \to (\mathbb{R}/\mathbb{Z})^r$ define a foliation $\mathfrak{G}_\theta$ of $\phi^{-1}(V)$, transverse to the fibers of ϕ, and having the following two properties:

(1) $\mathfrak{G}_\theta$ is invariant under the flow of the action vector field associated to any element of $\mathcal{L}_B(V)$;

(2) $\mathfrak{G}_\theta$ is coisotropic, i.e., every leaf of $\mathfrak{G}_\theta$ is a coisotropic submanifold of (M, Π).

For the proof of (1), one needs to check that the Lie derivative with respect to the action vector fields X_{e_i} of every local first integral of $\mathfrak{G}_\theta$ is a local first integral of $\mathfrak{G}_\theta$; this is clear because the leaves of $\mathfrak{G}_\theta$ are defined by $\theta_j =$ constant and $\mathcal{L}_{X_{e_i}}(\theta_j) = X_{e_i}(\theta_j)$ is constant for all i and j, in view of (4.1). The proof of (2) follows from the fact that the functions θ_j, which define $\mathfrak{G}_\theta$, are in involution, again according to (4.1).

Making abstraction of these properties leads to the following definition.

Definition 4.4. Let V be an open subset of B. A foliation $\mathfrak{G}$ of $\phi^{-1}(V)$ is called an **angle foliation** if it has the following properties:

(1) $\mathfrak{G}$ is transverse to the fibers of ϕ;

(2) $\mathfrak{G}$ is invariant under the flow of the action vector field associated to any element of $\mathcal{L}_B(V)$;

(3) $\mathfrak{G}$ is coisotropic.

As we will see in Example 4.3 below (the Euler-Poinsot top), even when M is symplectic the existence of an angle foliation is not guaranteed, despite the fact that in the symplectic case an action foliation always exists (see Remark 3.5).

According to Proposition 4.2, angle variables exist semi-locally, i.e., on an open neighborhood of any fiber of ϕ, hence action foliations exist semi-locally. We show in the following proposition that every angle foliation is defined semi-locally by angle variables.

Proposition 4.5. Let V be an open subset of B. We suppose that we are given on V a trivialization $(e_1, \ldots, e_r)$ of $\mathcal{L}_B(V)$ and on $\phi^{-1}(V)$ an angle foliation $\mathfrak{G}$. Let $b \in V$. There exists a neighborhood V_0 of b, contained in V, and there exist local angle variables $\theta = (\theta_1, \ldots, \theta_r)$ on $\phi^{-1}(V_0)$, adapted to $(e_1, \ldots, e_r)$, such that $\mathfrak{G}_\theta = \mathfrak{G}$ on $\phi^{-1}(V_0)$.

Proof. It follows from (2) in Definition 4.4 that the flow of the commuting action vector fields X_{e_i} defines a diffeomorphism between $\phi^{-1}(V_0)$ and $\mathbb{T}^r \times V_0$ where V_0 is an open subset of V which contains b. By construction, this diffeomorphism has the following two properties: first, the fundamental vector fields of the natural action of $\mathbb{T}^r$ on $\mathbb{T}^r \times V_0$ coincide with the action vector fields X_{e_i}. Second, the leaves of $\mathfrak{G}$ correspond to the fibers of the projection map $\theta : \phi^{-1}(V_0) \simeq \mathbb{T}^r \times V_0 \to \mathbb{T}^r$; in particular, the foliations $\mathfrak{G}_\theta$ and $\mathfrak{G}$ coincide over points of V_0. Writing $\theta = (\theta_1, \ldots, \theta_r)$ yields local angle coordinates on $\phi^{-1}(V_0)$ adapted to $(e_1, \ldots, e_r)$. Indeed, by construction, $X_{e_i}(\theta_j) = \delta_{i,j}$ for $i, j = 1, \ldots, r$ and the functions θ_i are in involution because $\mathfrak{G}$ is coisotropic. $\square$

The set of angle variables defining a given angle foliation is unique up to adding locally constant functions and taking integer-valued linear transformations. This is shown in the following proposition.

Proposition 4.6. Let $\mathfrak{G}$ be an angle foliation on $\phi^{-1}(V)$, where V is an open subset of B. Let $e = (e_1, \ldots, e_r)$ and $e' = (e'_1, \ldots, e'_r)$ of $\mathcal{L}_B(V)$ be two local trivializations of V and denote by C the invertible integer-valued matrix such that $e' = eC$. Let θ and θ' be two sets of angle variables defining $\mathfrak{G}$ and adapted to e and e' respectively. There exists a vector of locally constant $\mathbb{R}/\mathbb{Z}$-valued functions $c = (c_1, \ldots, c_r)$ on $\phi^{-1}(V)$, such that

$$\theta' = \theta(C^t)^{-1} + c. \tag{4.2}$$

Proof. Suppose first that $e = e'$. Since both θ and θ' define the same foliation $\mathfrak{G}$, we have, in a neighborhood of any point of $\phi^{-1}(V)$, $\theta'_i = K_i(\theta_1, \ldots, \theta_r)$ for some function K_i. Applying X_{e_j} to both sides of the previous equation amounts to:

$$\delta_{i,j} = \sum_{k=1}^{r} \frac{\partial K_i}{\partial x_k} X_{e_j}(\theta_k) = \frac{\partial K_i}{\partial x_j}.$$

This implies that $\theta'_i - \theta_i$ is a locally constant function, which proves (4.2) in case $C = I_r$. In general (i.e., without assuming that $e = e'$) the angle variables θ' and $\theta(C^t)^{-1}$ are both adapted to e', so that they differ by locally constant functions. □

The next theorem gives a necessary and sufficient condition for the existence of angle variables. We use angle foliations in its proof, in order to clarify the argument.

Theorem 4.7. Let $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ be an NCI system with compact fibers. The following statements are equivalent:

(i) There exist global angle variables;

(ii) The action lattice sheaf $\mathcal{L}_B$ admits a global trivialization and there exists a section of $\phi : M \to B$ whose image is a coisotropic submanifold of (M, Π).

Proof. As pointed out after Definition 4.1, if there exists a set of global angle variables $(\theta_1, \ldots, \theta_r)$, then the action lattice sheaf $\mathcal{L}_B$ admits a global trivialization. The zero locus $\theta_1 = \cdots = \theta_r = 0$ is a submanifold B_0 which is transverse to the fibers of $\phi : M \to B$. Since the restriction of ϕ is a bijection from B_0 to B, it is the image of some section σ of $\phi : M \to B$. Since the foliation $\mathfrak{G}_\theta$ which is associated to θ is coisotropic, B_0 is coisotropic. This proves $(i) \implies (ii)$.

Let us prove that (ii) implies (i). A choice of global trivialization $(e_1, \ldots, e_r)$ of $\mathcal{L}_B$ turns $M \to B$ into a principal $\mathbb{T}^r$-bundle; we denote by $(s, m) \to s \cdot m$ the action of $s \in \mathbb{T}^r$ on $m \in M$. Let $\sigma : B \to M$ be a section of $\phi : M \to B$ whose image $B_0 := \sigma(B)$ is coisotropic. Consider the unique $\mathbb{T}^r$-invariant foliation $\mathfrak{G}$ on M admitting B_0 as a leaf, i.e., consider the foliation admitting the submanifolds $s \cdot B_0$ with $s \in \mathbb{T}^r$ as leaves. By construction, $\mathfrak{G}$ is transverse to all fibers of ϕ. Also, $\mathfrak{G}$ is $\mathbb{T}^r$-invariant, so that it is invariant under all the action fields associated to elements of $\mathcal{L}_B(B)$. Since for all $s \in \mathbb{T}^r$, the map $m \to s \cdot m$ is a Poisson diffeomorphism of M, the fact that B_0 is a coisotropic submanifold implies that all the leaves of $\mathfrak{G}$ are coisotropic submanifolds, so that $\mathfrak{G}$ is an angle foliation.

According to Proposition 4.5, there exists for any $b \in B$ a neighborhood U_b of $\phi^{-1}(b)$ and a unique set $(\theta_1, \ldots, \theta_r)$ of angle variables on U_b, adapted to $(e_1, \ldots, e_r)$, constant on the leaves of $\mathfrak{G}$ and vanishing on B_0. The open subsets $(U_b)_{b \in B}$ form an open cover of M. Since the angle variables defined on U_b and U'_b coincide on $U_b \cap U'_b$, they lead to global angle variables. □

The difference between the existence of angle foliations and angle variables can also be stated in the following geometrical terms. Suppose that $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ is an NCI system with compact fibers and suppose that its lattice sheaf $\mathcal{L}_B$ admits a global trivialization, so that $M \to B$ is a principal $\mathbb{T}^r$-bundle. Suppose also that we have an angle foliation $\mathfrak{G}$. Any distribution, tangent to $\mathfrak{G}$ is an Ehresmann connection, which is invariant under the torus action, hence it defines a principal $\mathbb{T}^r$-connection. By construction, this distribution is integrable, which is tantamount to saying that the connection is flat. Saying that there exist angle variables, adapted to $\mathfrak{G}$ is equivalent to saying that the bundle $M \to B$ is trivial, hence is of the form $\phi : \mathbb{T}^r \times B \to B$, where ϕ is the projection on the second component and the $\mathbb{T}^r$ action is the standard one.

Example 4.3. As an application of the theorem, we show that the Euler-Poinsot top (Examples 2.4 and 3.6) does not admit an angle foliation, hence does not admit global angle variables. To do this, we show that the submersion $\phi_H : M_+ \to B$ does not admit a section whose image is coisotropic. Notice first that B is, topologically, the product of a 2-sphere by $\mathbb{R}$. In particular, it is simply-connected, i.e. $\pi_1(B) = 0$, but it is not 2-connected, i.e.

$\pi_2(B)$ is not trivial. On the contrary,

$$M_+ = SO(3) \times \left\{ (x,y,z) \neq (x,0,0) \mid x > 0 \text{ and } x^2 < \frac{I_y - I_z}{I_x - I_y} \frac{I_x}{I_z} z^2 \right\}$$

from which we see that M_+ is homeomorphic to $SO(3) \times \mathbb{R}_{>0} \times (\mathbb{R}^2 \setminus \{0\})$, so that M_+ is 2-connected but not simply-connected. The argument is now purely topological. Assume that an angle foliation exists, and denote by $\mathcal{F}$ one of its leaves. By construction, $\mathcal{F}$ is a connected submanifold and the restriction of ϕ_H to $\mathcal{F}$ is a local diffeomorphism onto B. Since B is simply-connected, the restriction of ϕ_H to $\mathcal{F}$ has to be a global diffeomorphism. Inverting the restriction of ϕ_H to $\mathcal{F}$ yields a global section of ϕ_H. But this is in turn impossible because $\pi_2(B)$ is not trivial while $\pi_2(M_+)$ is trivial, which prohibits the existence of such a section. Hence the Euler-Poinsot top admits neither a set of angle variables nor an angle foliation. The fact that angle variables for the Euler-Poinsot top do not exist was already shown by F. Fasso (see [13]).

Example 4.4. We now give an example of an NCI system which admits an angle foliation, but not angle variables; a slight modification gives another example of an NCI system which admits no angle foliation. Consider an NCI system $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ of rank $r = 1$ with compact fibers. We assume that its action lattice sheaf admits a trivialization. Recall from Remark 3.9 that this implies that $\phi : M \to B$ is a principal S^1-bundle. Notice that in the rank 1 case the image of every section of $\phi : M \to B$ is coisotropic, because this image is of codimension 1. It follows that the principal S^1-bundle $\phi : M \to B$ has the following properties:

(1) It admits a trivialization if and only if there exists a global angle variable;

(2) It admits a flat connection if and only if there exists a global angle foliation.

Indeed, Theorem 4.7 yields in the present case that a global angle variable exists if and only if a global section of ϕ exists, which is itself equivalent to the triviality of the principal S^1-bundle. This shows (1). Also, the connection form of a principal S^1-bundle is simply a nowhere vanishing one-form $\beta \in \Omega^1(M, \mathbb{R})$, and such a connection is flat if and only if β is closed, which in turn implies that the distribution $\operatorname{Ker} \beta$ is integrable, hence defines a foliation transverse to the fibers of ϕ. It is an angle foliation, because it is of codimension 1 (hence coisotropic) and because the connection form β is

S^1-invariant. Conversely, the leaves of any angle foliation of the NCI system define an integrable distribution which is transverse to the fibers of ϕ and is S^1-invariant, i.e. a flat connection. This shows (2).

Let $\phi_0 : M_0 \to B_0$ be a principal S^1-bundle and denote the fundamental vector field of the S^1-action on M_0 by $\mathcal{W}$. We associate to it an NCI system $(M, \Pi) \stackrel{\phi}{\to} (B, 0)$ of rank 1 by setting $M := M_0 \times \mathbb{R}$, $B := B_0 \times \mathbb{R}$ and $\phi := \phi_0 \times \mathrm{Id}_{\mathbb{R}}$. The Poisson structure on M is given by $\Pi := \frac{\partial}{\partial p} \wedge \mathcal{W}$, where p is the parameter on $\mathbb{R}$. Clearly, the NCI system has compact fibers and its action lattice sheaf admits a trivialization; indeed, $\phi : M \to B$ is a principal S^1-bundle. This bundle admits a flat connection (respectively, is trivial) if and only if $\phi_0 : M_0 \to B_0$ admits a flat connection (respectively, is trivial). Therefore, in order to construct an NCI system with compact fibers which admits no angle foliation and an NCI system with compact fibers that admits an angle foliation but no angle variables, it suffices to find:

(A) A principal S^1-bundle which does not admit a flat connection;

(B) A non-trivial principal S^1-bundle which admits a flat connection.

The Hopf fibration $S^3 \to S^2$ is an example of (A). In order to give an example of (B) we consider on $S^2 \times S^1$ the equivalence relation R defined by $(x, y) \sim (-x, -y)$. The quotient map $S^2 \to \mathbb{RP}^2$ leads to a map $\phi_0 : (S^2 \times S^1)/R \to \mathbb{RP}^2$ which makes it into a non-trivial principal S^1-bundle. The standard vector field $\partial/\partial\theta$ on S^1 is invariant under $y \mapsto -y$, hence leads to a non-vanishing vector field on $(S^2 \times S^1)/R$ which is both S^1-invariant and transverse to the fibers of ϕ_0. It defines a distribution on $(S^2 \times S^1)/R$ which is a flat connection.

As in the case of action variables, an NCI system may have two different angle foliations, where one can be defined by angle variables while the other one can't. In view of the above analysis, an example of this for $r = 1$ can be constructed from a trivial S^1-bundle $M = S^1 \times B$ with two flat connections, one which is associated to a trivialization but not the other one. We can take $B := S^1$ and choose for the second connection a translation invariant distribution on the torus M whose leaves spiral at least twice around the torus.

Example 4.5. We have seen in Theorem 4.7 that global angle variables can only exist when the momentum map has a section whose image is coisotropic. We now show that a section of the momentum map, whose image is coisotropic, may fail to exist even when the momentum map has a section. Our example admits both an action foliation and a trivialization of its action lattice sheaf.

We consider the NCI system $(M,\Pi) \stackrel{\phi}{\to} (B,0)$ where $M := \mathbb{T}^2 \times B$, where ϕ is the projection on the second component and $B := \mathbb{T}^2$. Also, Π is given by

$$\Pi := \frac{\partial}{\partial\theta_1} \wedge \frac{\partial}{\partial\psi_1} + \frac{\partial}{\partial\theta_2} \wedge \frac{\partial}{\partial\psi_2} + \alpha\frac{\partial}{\partial\theta_1} \wedge \frac{\partial}{\partial\theta_2}.$$

where $\alpha \in \mathbb{R}^*$, the standard (S^1-valued) coordinates on B are denoted by (ψ_1,ψ_2) and those on the first factor of M by (θ_1,θ_2). Throughout the example we identify S^1 with $\mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^2$ with $S^1 \times S^1$. The action lattice sheaf $\mathcal{L}_B$ admits $(e_1,e_2) := ([\mathrm{d}\psi_1],[\mathrm{d}\psi_2])$ as trivialization and we have $X_{e_i}(\theta_j) = \delta_{i,j}$. However, (θ_1,θ_2) is not a set of angle variables because $\{\theta_1,\theta_2\} = \alpha$. If (θ_1',θ_2') is a set of angle variables adapted to the trivialization (e_1,e_2), then $\theta_i' = \theta_i + \phi^* F_i$, for some S^1-valued functions F_1, F_2 on B; also, if we want that the submanifold which is defined by $\theta_1' = \theta_2' = 0$ is coisotropic, we must have $\{\theta_1',\theta_2'\} = 0$, to wit

$$\alpha - \frac{\partial F_1}{\partial\psi_2} + \frac{\partial F_2}{\partial\psi_1} = 0. \tag{4.3}$$

Let F be any smooth map from $S^1 = \mathbb{R}/\mathbb{Z}$ to itself. Since any two smooth liftings $\tilde F : \mathbb{R} \to \mathbb{R}$ differ by an integer, the integral $\int_{S^1} F\mathrm{d}\psi$ is well-defined up to an integer and $\int_{S^1} \frac{\partial F}{\partial\psi}\mathrm{d}\psi \in \mathbb{Z}$. Therefore,

$$\int_{S^1} \frac{\partial F_1}{\partial\psi_2}\mathrm{d}\psi_2 \in \mathbb{Z} \quad \text{and} \quad \int_{S^1} \frac{\partial F_2}{\partial\psi_1}\mathrm{d}\psi_1 \in \mathbb{Z},$$

so that

$$\iint_{S^1\times S^1} \left(\frac{\partial F_1}{\partial\psi_2} - \frac{\partial F_2}{\partial\psi_1}\right)\mathrm{d}\psi_1\mathrm{d}\psi_2 \in \mathbb{Z}.$$

However, $\iint_{S^1\times S^1} \alpha\mathrm{d}\psi_1\mathrm{d}\psi_2 = \alpha$, so there is no solution to Equation (4.3) unless $\alpha \in \mathbb{Z}$. This shows that a set of angle variables adapted to the trivialization (e_1,e_2) does not exist, hence no set of angle variables exists (see Proposition 4.6). In, turn, this implies that the image of a section of the momentum map of this NCI system cannot be coisotropic.

4.3. The transverse foliation

We have seen in Section 4.1 that we can associate to a set of local angle variables $\theta = (\theta_1,\dots,\theta_r)$ on $\phi^{-1}(V)$ vector fields $Y_{\theta_1},\dots,Y_{\theta_r}$ on $V \subset B$. We now show that they define a distribution of rank r on V which depends only on the angle foliation, defined by the angle variables. For a given set of local angle variables, let us denote by D_θ the (a priori singular) distribution on V, defined by the vector fields $Y_{\theta_1}, \ldots, Y_{\theta_r}$, where $Y_{\theta_i} := \phi_* X_{\theta_i}$ and by L_θ the (a priori singular) lattice subbundle of D_θ, generated by these vector fields.

Proposition 4.8. Let V be an open subset of B and suppose that $\mathfrak{G}$ is an angle foliation on $\phi^{-1}(V)$, where V is an open subset of B. Suppose that $\mathfrak{G}$ is defined by local angle variables $\theta = (\theta_1, \ldots, \theta_r)$.

(1) D_θ is an integrable distribution of rank r on V;

(2) D_θ and L_θ are independent of the choice of θ, defining $\mathfrak{G}$.

Therefore, $\mathfrak{G}$ defines an r-dimensional foliation $\mathfrak{T}_{\mathfrak{G}}$ of V and a lattice bundle $L_{\mathfrak{G}}$ on V, which we call the **transverse foliation**, respectively the **transverse lattice bundle** of the NCI system.

Proof. Using the angle foliation $\mathfrak{G}$ we can define an r-dimensional subspace D'_m of T_mM at very point $m \in \phi^{-1}(V)$ by setting $D'_m := \Pi^\sharp_m((T_m\mathfrak{G})^0)$. It leads to a distribution D' on $\phi^{-1}(V)$, which is spanned by the r independent commuting vector fields X_{θ_i} at m, where $\theta = (\theta_1, \ldots, \theta_r)$ is any set of local angle variables defining $\mathfrak{G}$ around m. Thus, its projection under ϕ, whose fibers are transverse to $\mathfrak{G}$, is a distribution which is spanned by the r vector fields Y_{θ_i} on B, hence it is the distribution D_θ. It follows that D_θ is an integrable distribution of rank r on V and that D_θ is independent of the choice of θ, defining $\mathfrak{G}$. The integral manifolds of D_θ are the leaves of an r-dimensional foliation of V, denoted by $\mathfrak{T}_{\mathfrak{G}}$. In view of (4.2), two different choices θ and θ' are related by $\theta' = \theta(C^t)^{-1} + c$, where C is an integer-valued matrix and c is a constant vector. It follows that L_θ and $L_{\theta'}$ define the same lattice bundle in $D_\theta = D_{\theta'}$. □

We show in the following proposition how an action and an angle foliation, if they exist, are related.

Proposition 4.9. Let V be an open subset of B. Suppose that we have on V an action foliation $\mathfrak{A}$ and on $\phi^{-1}(V)$ an angle foliation $\mathfrak{G}$.

(1) $\mathfrak{T}_{\mathfrak{G}}$ is transverse to $\mathfrak{A}$;

(2) The tangent space to $\mathfrak{T}_{\mathfrak{G}}$ is spanned by local Poisson vector fields which preserve $\mathfrak{A}$.

Proof. In a neighborhood V_0 of any point of V, there exist action-angle variables $p_1, \ldots, p_r, \theta_1, \ldots, \theta_r$ such that $([\mathrm{d}p_1], \ldots, [\mathrm{d}p_r])$ is a trivialization of $\mathcal{L}_B(V_0)$. Hence:

$$\phi^*(Y_{\theta_i}(p_j)) = X_{\theta_i}(\phi^* p_j) = -X_{\phi^* p_j}(\theta_i) = -X_{e_j}(\theta_i) = -\delta_{i,j}, \tag{4.4}$$

which implies both items (1) and (2). □

Consider a foliation $\mathfrak{G}$ of $\phi^{-1}(V)$ transverse to the fibers of the surjective submersion $\phi : M \to B$, where V is an open subset of B. For any leaf G of $\mathfrak{G}$, ϕ is a local diffeomorphism from G to B, so that a multivector field on B induces a multivector field on the leaf G. Making this construction for all the leaves of $\mathfrak{G}$ simultaneously, yields a graded Lie algebra morphism $\phi^*_{\mathfrak{G}}$ from the space of multivector fields on V to the space of multivector fields on $\phi^{-1}(V)$ tangent to the foliation $\mathfrak{G}$, where both spaces are equipped with the Schouten bracket.

We apply this to the case of an angle foliation $\mathfrak{G}$ on $\phi^{-1}(V)$ with V an open subset of B, to construct two Poisson structures on $\phi^{-1}(V)$, to wit $\phi^*_{\mathfrak{G}}(\pi)$ (with π the Poisson structure on B) and

$$\Pi_{\mathfrak{G}} := \sum_{i=1}^{r} X_{e_i} \wedge \phi^*_{\mathfrak{G}}(Y_{\theta_i}). \tag{4.5}$$

In this formula, the θ_i stand for any set of local action variables, defined in a neighborhood W of some point of V and $e = (e_1, \ldots, e_r)$ stands for the corresponding trivialization of $\mathcal{L}_B(W)$ and $Y_{\theta_1}, \ldots, Y_{\theta_r}$ are the vector fields on W, defined in Proposition 4.3; the right hand side of (4.5) does not depend on the choice of θ_i because the θ_i, and hence the vector fields Y_{θ_i}, are dual to the trivialization e. It follows that the right hand side of (4.5) is a well-defined bivector field on $\phi^{-1}(V)$.

Proposition 4.10. Let V be an open subset of B and suppose that $\mathfrak{G}$ is an angle foliation on $\phi^{-1}(V)$.

(1) The bivector field $\Pi_{\mathfrak{G}}$ is a regular Poisson structure on $\phi^{-1}(V)$ of rank $2r$.

(2) The Poisson structures Π, $\Pi_{\mathfrak{G}}$ and $\phi^*_{\mathfrak{G}}(\pi)$ are related by:

$$\Pi = \Pi_{\mathfrak{G}} + \phi^*_{\mathfrak{G}}(\pi).$$

Proof. Let us first rewrite the local expression of $\Pi_{\mathfrak{G}}$ given in formula (4.5) in a more convenient way. Choose a trivialization $e = (e_1, \ldots, e_r)$ of $\mathcal{L}_B(V)$, a set of local angle variables $(\theta_1, \dots, \theta_r)$ adapted to e defining $\mathfrak{G}$, and a set of local action variables $p = (p_1, \dots, p_r)$ satisfying $e_i = [\mathrm{d}p_i]$. For $i = 1, \dots, r$, the identity $X_{\phi^* p_i} = X_{e_i}$ holds. Also, Y_{θ_i} is ϕ-related to X_{θ_i}, which is tangent to $\mathfrak{G}$, so that $X_{\theta_i} = \phi^*_{\mathfrak{G}}(Y_{\theta_i})$. It follows that (4.5) can be written as

$$\Pi_{\mathfrak{G}} = \sum_{i=1}^{r} X_{\phi^* p_i} \wedge X_{\theta_i}. \tag{4.6}$$

Since the $2r$ vector fields $X_{\phi^* p_1}, \dots, X_{\phi^* p_r}, X_{\theta_1}, \dots, X_{\theta_r}$ are pairwise commuting, $\Pi_{\mathfrak{G}}$ is a Poisson structure. Also, (4.6) implies that $\Pi_{\mathfrak{G}}(\mathrm{d}p_j, \mathrm{d}\theta_i) = \delta_{i,j}$ while $\Pi_{\mathfrak{G}}(\mathrm{d}\theta_i, \mathrm{d}\theta_j) = \Pi_{\mathfrak{G}}(\mathrm{d}p_i, \mathrm{d}p_j) = 0$, which proves that $\Pi_{\mathfrak{G}}$ is a regular bivector field of rank $2r$. This proves (1).

The bivector field $P := \Pi - \Pi_{\mathfrak{G}}$ is tangent to $\mathfrak{G}$, i.e. $P_m \in \wedge^2 T_m \mathfrak{G}$ for every $m \in \phi^{-1}(V)$. Indeed, we have in view of (4.6) that $\Pi^\sharp_{\mathfrak{G}}(\mathrm{d}\theta_j) = -X_{\theta_j} = \Pi^\sharp(\mathrm{d}\theta_j)$. Also, $\wedge^2 T_m \phi(\Pi_{\mathfrak{G}})_m = 0$ so that $\wedge^2 T_m \phi(\Pi_m) = \pi_{\phi(m)}$. This shows that on $\phi^{-1}(V)$ both bivector fields P and $\phi^*_{\mathfrak{G}}(\pi)$ are tangent to $\mathfrak{G}$ and project to π, so they are equal and (2) follows. □

4.4. The transverse Poisson manifold

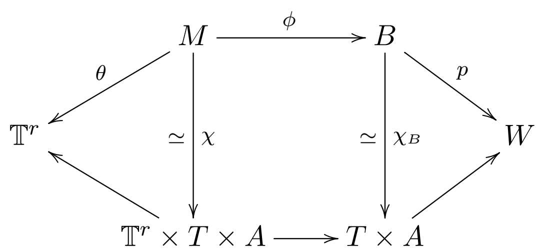
In this paragraph we give necessary and sufficient conditions for (M, Π) to be Poisson diffeomorphic with the product $\mathbb{T}^r \times T \times A$, where A is a leaf of $\mathfrak{A}$, equipped with the Poisson structure inherited from (B, π) (as a Poisson submanifold) and T is a leaf of $\mathfrak{T}_{\mathfrak{G}}$, the Poisson structure on $\mathbb{T}^r \times W$ being the canonical Poisson structure defined by a set of global action-angle variables, which we assume to exist.

In order to do this, we first recall a basic result from foliation theory. Suppose that $\mathfrak{A}$ and $\mathfrak{T}$ are two foliations of a manifold B which intersect transversally (as the notations suggest, we will use the result when $\mathfrak{A}$ and $\mathfrak{T}$ are the action and transverse foliations on B, defined by the action-angle variables). We say that $\mathfrak{A}$ and $\mathfrak{T}$ have the **unique intersection property** if any leaf of $\mathfrak{A}$ has exactly one point in common with any leaf of $\mathfrak{T}$. Fix a point $b \in B$ and denote by A and T the leaves of $\mathfrak{A}$ resp. of $\mathfrak{T}$, passing through b. There is a neighborhood V_b of b in B and a unique diffeomorphism Φ_b from V_b to $A_b \times T_b$ with A_b and T_b a neighborhood of b in A resp. in T, under which the foliations $\mathfrak{A}$ and $\mathfrak{T}$ become the fibers of the projections onto the first and second components respectively. Since this diffeomorphism on V_b is unique, it leads to a global diffeomorphism between B and $A \times T$ if and only if the foliations $\mathfrak{A}$ and $\mathfrak{T}$ of B have the unique intersection property.

Theorem 4.11. Let $(M, \Pi) \stackrel{\phi}{\to} (B, \pi)$ be a NCI system with compact fibers, equipped with a set of angle variables $\theta := (\theta_1, \ldots, \theta_r)$ and a set of action variables $p := (p_1, \ldots, p_r)$. We set $W := p(B)$, which is a connected open subset of $\mathbb{R}^r$. Choose a point $b \in B$ and let A and T denote the leaves through b of the action foliation $\mathfrak{A}$, associated to p and of the transverse foliation $\mathfrak{T}_{\mathfrak{G}}$, associated to θ. Then the following are equivalent:

(i) The map p restricts to a bijection from T to W, and the foliations $\mathfrak{A}$ and $\mathfrak{T}_{\mathfrak{G}}$ have the unique intersection property.

(ii) There exist diffeomorphisms χ and χ_B making the following diagram commutative:



Moreover, when these conditions are satisfied,

$$\chi_*(\Pi) = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \pi_{|_A} \quad \text{and} \quad (\chi_B)_*\pi = \pi_{|_A}. \tag{4.7}$$

Proof. Recall that the action and transverse foliations, when they exist, are transverse. We assume here to be given global action-angle variables, hence both foliations exist and we can apply the above remarks on transversally intersecting foliations to prove the equivalence of (i) with the existence of χ_B in (ii), making the rightmost triangle in the above diagram commutative. In view of the existence of action-angle variables, M is a trivial $\mathbb{T}^r$-bundle over B, allowing us to complete the diagram. This shows the equivalence of (i) and (ii).

Locally, χ_B is a a Poisson diffeomorphism between an open neighborhood in B and open neighborhoods in the leaves A and T, when $A \times T$ is equipped with the product of π restricted to the Poisson submanifold A and the trivial Poisson structure on T. This follows from the fact that the foliation $\mathfrak{T}_{\mathfrak{G}}$ is spanned by Poisson vector fields which preserve the foliation $\mathfrak{A}$ (see Propositions 4.8 and 4.9). Since χ_B is a (global) diffeomorphism, it is a Poisson diffeomorphism, leading to the second formula in (4.7). The first formula in (4.7) follows from Proposition 4.10. ∎

It deserves to be noticed that the angle and the action foliations satisfy the unique intersection property when $W := p(B)$ is a connected and simply-connected open subset of $\mathbb{R}^r$ and $p : B \to W$ is a proper map. Also, under these assumptions, all leaves of the angle foliation are diffeomorphic to W. This follows from the simple observation that the distribution generated by the Hamiltonian vector fields of $\theta_1, \ldots, \theta_r$ is then a flat Ehresmann connection (i.e., an integrable distribution transverse to the fibers of p such that for any $b \in B$, any path on W starting at $p(b)$ can be lifted to a horizontal path starting from b) . Since W is simply connected, this makes the fibration $p : B \to W$ trivial, and isomorphic to $A \times W$, with A some fiber of p, isomorphism into which the leaves of the action and angle foliation turn to be the fibers of the projections on the second and first components respectively. In particular, they satisfy the unique intersection property and all the leaves of the action foliation are isomorphic to W.

5. The Gelfand-Cetlin system

We finish the paper with a non-trivial example which is non-symplectic, yet global action-angle variables exist. The results in this section are due to A. Giacobbe and we refer to his original paper [15] for details and proofs. We present them in our framework.

The phase space of the Gelfand-Cetlin system is the real vector space of $n \times n$ hermitian matrices $\mathfrak{H}_n$. It has a linear Poisson structure, since it can be viewed as the dual of the Lie algebra of unitary matrices $\mathfrak{u}_n$. Explicitly, the Poisson structure Π is given for smooth functions F, G on $\mathfrak{H}_n$ at $X \in \mathfrak{H}_n$ by

$$\{F, G\}(X) := \langle [\nabla F(X), \nabla G(X)] \mid X \rangle ,$$

where the inner product is defined for $X, Y \in \mathfrak{H}_n$ by $\langle X \mid Y \rangle := i \operatorname{Trace} XY$ and $\nabla F(X)$ is the differential of F at X, viewed as an element of $\mathfrak{H}_n$ (using the inner product). The rank of this Poisson structure is $n(n-1)$, to be compared with $\dim \mathfrak{H}_n = n^2$. When one removes from $X \in \mathfrak{H}_n$ the last $n-i$ rows and columns one obtains an element of $\mathfrak{H}_{n-i}$, which is denoted by $X^{(i)}$. For $i = 1, \ldots, n$ the i eigenvalues of $X^{(i)}$ are denoted by $\mu_p^i(X)$; they are ordered such that $\mu_1^i(X) \leqslant \mu_2^i(X) < \cdots \leqslant \mu_{n-i}^i(X)$. They satisfy

$$\mu_p^{i+1}(X) \leqslant \mu_p^i(X) \leqslant \mu_{p+1}^{i+1}(X). \tag{5.1}$$

Let M be the open subset of $\mathfrak{H}_n$ where each $X^{(i)}$ has simple spectrum and where the eigenvalues of $X^{(i)}$ are different from the eigenvalues of

$X^{(i+1)}$. On M the maps $X \mapsto \mu_p^i(X)$ define $N := n(n+1)/2$ smooth functions, which are independent, leading to a submersion $\phi : M \to B$, where B is the sector in $\mathbb{R}^N$, defined by replacing in (5.1) the inequalities by strict inequalities. Moreover, these functions are in involution and the NCI system $(M, \Pi) \xrightarrow{\phi} (B, 0)$ is regular. The fibers of ϕ are compact and connected, i.e., they are diffeomorphic to tori of dimension $r := n(n-1)/2 = \operatorname{rank} \Pi/2$.

The n functions $\mu_1^n, \mu_2^n, \ldots, \mu_n^n$ are Casimirs of Π, while the other $N/2$ functions μ_p^i $(i < n)$ have independent periodic flows of period 1. Thus, they provide a set of action variables. The construction of the global angle variables is slightly more involved. For given i such that $0 < i < n$ we explain how to compute the angle variables φ_p^i which are conjugate to μ_p^i, for $p = 1, \ldots, p$. The main operation involved in computing $\varphi_p^i(X)$ for $X \in \mathfrak{H}_n$ is to conjugate X by a unitary block matrix of the form $\Lambda := \begin{pmatrix} P & 0 \\ 0 & I_{n-i} \end{pmatrix}$ such that $\Lambda X \bar{\Lambda}^t$ is of the form $X' := \begin{pmatrix} \Delta & * \\ * & * \end{pmatrix}$, where Δ is diagonal, i.e., $\Delta = \operatorname{diag}(\mu_1^i, \ldots, \mu_i^i)$. Of course, such a matrix P is not unique, but all entries of its last row are non-zero and a unique P can be selected by demanding that all these entries are strictly positive real numbers and that the columns have norm 1. With this choice of P, the angle variable $\varphi_p^i(X)$ is the argument of the complex number $X'_{p,i+1}$. Combined, the set of (μ_p^i, φ_p^i), where i ranges from 1 to $n-1$ and p from 1 to i, provide a set of global action-angle variables for the Gelfand-Cetlin system.

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Received June 28, 2015
Accepted September 15, 2016