

Two constructions of virtually contact structures

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Motivated by recent developments in proving the Weinstein conjecture we introduce the notion of covering contact connected sum for virtually contact manifolds and construct virtually contact structures on boundaries of subcritical handle bodies.

1. Introduction

Virtually contact structures naturally appear in classical mechanics in the study of magnetic flows on compact Riemannian manifolds (Q, h) of negative sectional curvature. The appearance of the magnetic 2-form σ on Q is reflected in the use of the twisted symplectic form on T^*Q obtained by adding the pull back of σ along the cotangent bundle projection to $d\mathbf{p} \wedge d\mathbf{q}$. As it turns out, energy surfaces $M \subset T^*Q$ of twisted cotangent bundles need not to be of contact type in general.

It was pointed out by Cieliebak–Frauenfelder–Parternain [8] that in many interesting cases a certain covering $\pi: M' \rightarrow M$ of the energy surface $M \subset T^*Q$ admits a contact form α whose Reeb flow projects to the Hamiltonian flow on the energy surface $M \subset T^*Q$ up to parametrization. Moreover, the contact form α admits uniform upper and lower bounds with respect to a lifted metric. In this situation, the manifold M together with the odd-dimensional symplectic form ω obtained by restriction of the twisted symplectic form to TM is called a virtually contact manifold. In particular, questions about periodic orbits on virtually contact manifolds (M, ω) can be answered on the covering space M' with help of the contact form α .

If the covering space M' of a virtually contact manifold (M, ω) is compact, and hence the covering π is finite, the energy surface M will be of contact type. The existence question about periodic orbits in this case is subject to the Weinstein conjecture, see [26], and the virtually contact manifold

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(M, ω) is called to be trivial. If the covering π is infinite with a non-amenable covering group, one is intended to study periodic orbits on a non-compact contact manifold (M', α) . This is because the covered energy surface M is not necessarily of contact type.

In general, open contact manifolds do admit aperiodic Reeb flows as the standard contact form $dz + \mathbf{y}d\mathbf{x}$ on Euclidean spaces shows. In order to achieve existence of periodic Reeb orbits additional conditions are required, cf. [1, 6–8, 23, 24]. It was asked by G. P. Paternain whether virtually contact manifolds have to admit periodic orbits. The question was answered positively in many instances by Cieliebak–Frauenfelder–Paternain [8] and, more recently, by Bae–Wiegand–Zehmisch [2]. The content of the following theorem is to give a large class of examples to which the existence theory developed in [2] applies.

Theorem 1.1. *For all $n \geq 2$ there exist non-trivial closed virtually contact manifolds M of dimension $2n - 1$ which topologically are a connected sum such that the corresponding belt sphere represents a non-trivial homotopy class in $\pi_{2n-2}M$. The involved covering space M' is obtained by covering contact connected sum.*

The virtually contact structures studied by Cieliebak–Frauenfelder–Paternain [8] are diffeomorphic to unit cotangent bundles of negatively curved manifolds. The examples we are going to construct in Section 2.6 are obtained by covering connected sum, which is an extension of the contact connected sum operation to the class of virtually contact manifolds. In Section 2.7 we will show that unit cotangent bundles of aspherical manifolds are prime. This implies that the covering connected sum produces virtually contact structure that differ from those studied in [8].

Motivated by Hofer’s [19] verification of the Weinstein conjecture for closed overtwisted contact 3-manifolds Bae [1] constructed virtually contact manifolds in dimension 3 using a covering version of the Lutz twist. The topology of the base manifold of the covering thereby stays unchanged. The total space of the resulting covering is an overtwisted contact manifold and the virtually contact structure will be non-trivial. In Proposition 2.6.2 we present a tool to produce more examples of that nature. Let us remind that non-trivially here and in Theorem 1.1 means that the symplectic form on the odd-dimensional manifold is not the differential of a contact form.

The verification of the Weinstein conjecture by Hofer [19] for closed reducible 3-manifolds suggests the question about the existence of non-trivial virtually contact 3-manifolds with non-vanishing π_2 . This question

is answered by Theorem 1.1. In fact, the results in [11–15] motivated the definition of the covering contact connected sum. Extending the work of Geiges–Zehmisch [13] the existence of periodic orbits for virtually contact structures addressed by Theorem 1.1 that in addition admit a C^3 -bounded contact form on the total space of the covering is shown in [2].

In Section 3 we will give a second construction of virtually contact structures that will be obtained via energy surfaces of classical Hamiltonian functions in twisted cotangent bundles. The corresponding energy will be below the Mañé critical value of the involved magnetic system so that the energy surfaces intersect the zero section of the cotangent bundle. The topology of the energy surface is determined by the potential function on the configuration space according to Morse theoretical considerations.

Theorem 1.2. *For any $n \geq 2$ and given $b \in \mathbb{N}$ there exists a closed virtually contact manifold M of dimension $2n - 1$ such that $\pi_n M$ and the image in $H_n M$ under the Hurewicz homomorphism, resp., contain a subgroup isomorphic to \mathbb{Z}^b . The virtually contact manifold M appears as the energy surface of a classical Hamiltonian function in a twisted cotangent bundle T^*Q . The rank b of the subgroup \mathbb{Z}^b is the first Betti number of the configuration space Q . If $n \geq 3$ the virtually contact structure on M is non-trivial.*

Based on the work of Ghiggini–Niederkrüger–Wendl [15] existence of periodic solutions in the context of Theorem 1.2 can be shown provided that the magnetic form has a C^3 -bounded primitive on the universal cover of Q , see [2, Theorem 1.1 and 1.2]. Furthermore, by the classification obtained by Barth–Geiges–Zehmisch in [4, Theorem 1.2.(a)] the contact structure on M obtained by homotoping the magnetic term of the twisted cotangent bundle T^*Q to zero is different from the standard contact structure on the unit cotangent bundle ST^*P of any Riemannian manifold P .

2. A construction via surgery

This section is devoted to a proof of Theorem 1.1.

2.1. Definitions

The following terminology was introduced in [1, 8]. Let M be a $(2n - 1)$ -dimensional manifold for $n \geq 2$. A closed 2-form ω on M is called **symplectic** if $\ker \omega$ is a 1-dimensional distribution. The pair (M, ω) is an **odd-dimensional symplectic manifold**. It is called **virtually contact** if the following two conditions are satisfied:

Primitive: There exist a covering $\pi: M' \rightarrow M$ and a contact form α on M' such that $\pi^*\omega = d\alpha$, so that α is a primitive of the lift of ω and α defines a contact structure $\xi = \ker \alpha$ on the covering space M' .

Bounded geometry: There exist a metric g of bounded geometry on M and a constant $c > 0$ subject to the following geometric bounds:

$$(gb_1) \quad \sup_{M'} |\alpha|_{(\pi^*g)^b} < \infty$$

with respect to the dual of the pull back metric π^*g ; and for all $v \in \ker d\alpha$

$$(gb_2) \quad |\alpha(v)| > c|v|_{\pi^*g}.$$

If the manifold M is closed any metric g will be of **bounded geometry**, i.e. the injectivity radius $\text{inj}_g > 0$ of (M, g) is positive and the absolute value of the sectional curvature $|\text{sec}_g|$ is bounded.

The tuple

$$(\pi: M' \rightarrow M, \alpha, \omega, g)$$

is called **virtually contact structure** and (M, ω) a **virtually contact manifold**. A virtually contact manifold is **non-trivial** if ω is not the differential of a contact form on M . In particular, the covering π of a non-trivial virtually contact structure is infinite and M has a non-amenable fundamental group. A virtually contact structure is called **somewhere contact** if there exist an open subset U of M and a contact form α_U on U such that $\pi^*\alpha_U = \alpha$ on $\pi^{-1}(U)$.

2.2. Covering connected sum

For $i = 1, 2$ we consider two somewhere contact virtually contact structures

$$(\pi_i: M'_i \rightarrow M_i, \alpha_i, \omega_i, g_i).$$

Denote by $U_i, i = 1, 2$, an open subset of M_i on which a contact form α_{U_i} exists according to the definition of being somewhere contact. Given a bijection b between the fibers of the coverings π_1 and π_2 over the respective base points of M_1 and M_2 we define a covering connected sum as follows:

Let $D_i^{2n-1}, i = 1, 2$, be a closed embedded disc contained in U_i such that a neighbourhood of the disc is equipped with Darboux coordinates for the contact form α_{U_i} . We perform contact index-1 surgery as described in [10] identifying ∂D_i^{2n-1} with the boundary $\{i\} \times S^{2n-2}$ of the upper boundary of $[1, 2] \times S^{2n-2}$ the 1-handle $[1, 2] \times D^{2n-1}$. The resulting contact form on

the connected sum $U_1 \# U_2$ is denoted by $\alpha_{U_1} \# \alpha_{U_2}$. Notice, that $\alpha_{U_1} \# \alpha_{U_2}$ coincides with α_{U_i} on $U_i \setminus D_i^{2n-1}$. Let ω be the odd-dimensional symplectic form on $M_1 \# M_2$ that coincides with ω_i on $M_i \setminus U_i$ for $i = 1, 2$ and with $d(\alpha_{U_1} \# \alpha_{U_2})$ on $U_1 \# U_2$. Similarly, a metric g of bounded geometry can be defined via extension of g_1 and g_2 over the handle part.

In order to define a connected sum of the coverings π_i we may assume that the base point x_i of M_i lies on the boundary of D_i^{2n-1} . Moreover, we choose the subset U_i , $i = 1, 2$, so small such that $\pi_i^{-1}(U_i)$ decomposes into a disjoint union of open sets U_i^y , $y \in \pi_i^{-1}(x_i)$, and that the restriction of π_i to U_i^y is an embedding into M_i for all $y \in \pi_i^{-1}(x_i)$. Then, topologically, we define a family of connected sums $U_1^y \# U_2^{b(y)}$ according to the bijection b between the fibers over the base points.

The restrictions of the contact forms $\alpha_i|_{U_i^y}$ correspond to the local contact form α_{U_i} diffeomorphically via π_i , $i = 1, 2$. A contact form on $U_1^y \# U_2^{b(y)}$ can be defined equivariantly via contact connected sum as follows: Let $M'_1 \#_b M'_2$ be the manifold obtained by gluing $M'_i \setminus \pi_i^{-1}(U_i)$, $i = 1, 2$, with $U_1^y \# U_2^{b(y)}$, $y \in \pi_i^{-1}(x_1)$, along their boundaries in the obvious way. We obtain a covering

$$\pi: M'_1 \#_b M'_2 \longrightarrow M_1 \# M_2$$

that restricts to π_i on $M'_i \setminus \pi_i^{-1}(U_i)$, $i = 1, 2$, and defines the trivial covering over the handle parts being the identity restricted to each of the sheets. Then $M'_1 \#_b M'_2$ carries a contact form α whose restriction to the union of the $U_1^y \# U_2^{b(y)}$, $y \in \pi_1^{-1}(x_1)$, coincides with $\pi^*(\alpha_{U_1} \# \alpha_{U_2})$ and that restricts to α_i on $M'_i \setminus \pi_i^{-1}(U_i)$, $i = 1, 2$. Because each of the involved handles is compact the covering $\pi: M' \rightarrow M$ of $M = M_1 \# M_2$ by $M' = M'_1 \#_b M'_2$ defines a virtually contact structure given by $(\pi: M' \rightarrow M, \alpha, \omega, g)$.

Remark 2.2.1. Observe, that the model contact handle used for the contact connected sum carries obvious periodic characteristics of

$$\ker(d(\alpha_{U_1} \# \alpha_{U_2}))$$

inside the **belt sphere** $\{3/2\} \times S^{2n-2}$. The situation changes after a perturbation of $\alpha_{U_1} \# \alpha_{U_2}$ obtained by a multiplication with a positive function that is constantly equal to 1 in the complement of the handle. This operation changes the virtually contact structure on the connected sum $M = M_1 \# M_2$ but not the contact structure $\xi = \ker \alpha$ on the covering M' . Still, there exists a contact embedding of the model contact handle into (M', ξ) .

Lemma 2.2.2. *For $i = 1, 2$ let $(\pi_i: M'_i \rightarrow M_i, \alpha_i, \omega_i, g_i)$ be a somewhere contact virtually contact structure. If ω_1 is non-exact, then the odd-dimensional symplectic form ω on $M_1 \# M_2$ corresponding to the virtually contact structure*

$$(\pi: M' \rightarrow M, \alpha, \omega, g)$$

obtained by covering contact connected sum is non-exact.

Proof. We argue by contradiction and continue the use of notation from above. Suppose that the symplectic form ω on the $(2n - 1)$ -dimensional connected sum $M = M_1 \# M_2$ has a primitive. Then the restriction ω_1 of ω to $M_1 \setminus D_1^{2n-1}$ does. An interpolation argument for primitives in terms of Mayer–Vietoris sequence in de Rham cohomology using $H^1_{\text{dR}}(S^{2n-2}) = 0$ shows that the odd-dimensional symplectic form ω_1 on M_1 has a primitive.

A more elementary argument goes as follows: Denote the primitive of the restriction of ω_1 to $M_1 \setminus D_1^{2n-1}$ by λ . Observe that $\lambda|_U$ is a closed 1-form and, hence, exact in a neighbourhood D' of the disc D_1^{2n-1} . Cutting a primitive function of $\lambda|_{D'}$ down to zero in radial direction we can assume that λ vanishes near $\partial D_1^{2n-1} \subset U \subset M$. In other words, a perturbation of λ extends over D_1^{2n-1} by zero resulting in a primitive of ω_1 . This is a contradiction. \square

2.3. Magnetic flows

Virtually contact structures appear naturally on energy surfaces of classical Hamiltonians on twisted cotangent bundles. We briefly recall the construction following [5, 8].

Let (Q, h) be a closed n -dimensional Riemannian manifold and let σ be a closed 2-form on Q , which is called the **magnetic form**. The **Liouville form** on the cotangent bundle $\tau: T^*Q \rightarrow Q$ is the 1-form λ on the total space T^*Q that is given by $\lambda_u = u \circ T\tau$ for all covectors $u \in T^*Q$. The **twisted symplectic form** by definition is

$$\omega_\sigma = d\lambda + \tau^*\sigma.$$

For a smooth function V on Q , the so-called **potential**, and the dual metric h^\flat of h we consider the Hamiltonian of classical mechanics

$$H(u) = \frac{1}{2}|u|_{h^\flat}^2 + V(\tau(u)).$$

For energies $k > \max_Q V$ we consider the energy surfaces $\{H = k\}$, which are regular and in fact diffeomorphic to the unit cotangent bundle ST^*Q via a diffeomorphism induced by a fibrewise radial isotopy.

It is of particular interest whether the Lorentz force induced by the magnetic 2-form σ comes from a potential 1-form. Up to lifting σ to a certain cover this will be the case at least for so-called weakly exact 2-forms: Denoting by $\mu: \tilde{Q} \rightarrow Q$ the universal covering of Q we call the 2-form σ on Q **weakly exact** if there exists a 1-form θ on \tilde{Q} such that $\mu^*\sigma = d\theta$. In the following we will assume that the magnetic form σ is weakly exact. Therefore, it is natural to lift the Hamiltonian system to the universal cover.

The covering map μ induces a natural map $T^*\mu: T^*\tilde{Q} \rightarrow T^*Q$ that is given by

$$\tilde{u} \longmapsto \tilde{u} \circ (T\mu_{\tilde{\tau}(\tilde{u})})^{-1},$$

where $\tilde{\tau}: T^*\tilde{Q} \rightarrow \tilde{Q}$ denotes the cotangent bundle of \tilde{Q} and $\mu_{\tilde{\tau}(\tilde{u})}$ is the germ of local diffeomorphism at $\tilde{\tau}(\tilde{u})$ that coincides with μ near $\tilde{\tau}(\tilde{u})$. Naturallity can be expressed by saying that $\mu \circ \tilde{\tau} = \tau \circ T^*\mu$ so that

$$(T^*\mu)^*\lambda = \tilde{\lambda},$$

where $\tilde{\lambda}$ denotes the Liouville form on $T^*\tilde{Q}$. Moreover, $T^*\mu$ itself is a covering, which because of the homotopy equivalence $T^*\tilde{Q} \simeq \tilde{Q}$ can be used to represent the universal covering of T^*Q . The lifted Hamiltonian $\tilde{H} = H \circ T^*\mu$ is a Hamiltonian of classical mechanics

$$\tilde{H}(\tilde{u}) = \frac{1}{2}|\tilde{u}|_{(\tilde{h})}^2 + \tilde{V}(\tilde{\tau}(\tilde{u})),$$

$\tilde{u} \in T^*\tilde{Q}$, with respect to the lifted metric $\tilde{h} = \mu^*h$ and the lifted potential energy function $\tilde{V} = V \circ \mu$. The preimage of $\{H = k\}$ under $T^*\mu$ is equal to $\{\tilde{H} = k\}$. In fact, an application of the implicit function theorem yields that the restriction

$$\pi = T^*\mu|_{\{\tilde{H}=k\}}$$

defines a covering projection

$$M' = \{\tilde{H} = k\} \longrightarrow \{H = k\} = M.$$

Because there exists a 1-form θ on \tilde{Q} such that $\mu^*\sigma = d\theta$ we find that

$$(T^*\mu)^*\tau^*\sigma = d(\tilde{\tau}^*\theta),$$

so that

$$(T^*\mu)^*\omega_\sigma = d\tilde{\lambda} + \tilde{\tau}^*d\theta =: \tilde{\omega}_{d\theta}$$

has primitive $\tilde{\lambda} + \tilde{\tau}^*\theta$. The restriction to TM' is denoted by

$$\alpha = (\tilde{\lambda} + \tilde{\tau}^*\theta)|_{TM'}.$$

Setting $\omega = \omega_\sigma|_{TM}$ we obtain a map

$$\pi: (M', d\alpha) \longrightarrow (M, \omega)$$

of odd-dimensional symplectic manifolds. The question that we will address in the following is under which conditions the 1-form α will be a contact form on M' .

Remark 2.3.1. The topology of the covering π can be determined as follows. By the choice $k > \max_Q V$ the covering space M' is diffeomorphic to $ST^*\tilde{Q}$ so that M' carries the structure of a S^{n-1} -bundle over \tilde{Q} . The long exact sequence of the induced Serre fibration shows that the inclusion $S^{n-1} \rightarrow M'$ of the typical fibre yields a surjection of fundamental groups. Therefore, if Q is not a surface, i.e. $n > 2$, then M' is simply connected and π the universal covering. If Q is a surface, then in view of uniformization π is a covering of $M = ST^*Q$ with covering space M' equal to $\mathbb{R}^2 \times S^1$ for $Q \neq S^2$; otherwise, if $Q = S^2$, then π is the trivial one-sheeted covering of $\mathbb{R}P^3$.

2.4. Bounded primitive

We assume that the primitive θ of $\mu^*\sigma$, viewed as a section $\tilde{Q} \rightarrow T^*\tilde{Q}$ of $\tilde{\tau}$, is **bounded** with respect to the lifted metric \tilde{h} , i.e.

$$\sup_{\tilde{Q}} |\theta|_{(\tilde{h})^\flat} < \infty.$$

This will be the case for negatively curved Riemannian manifolds (Q, h) as it was pointed out by Gromov [17], see Example 2.4.2 below. By compactness of Q the lifted potential \tilde{V} is bounded on \tilde{Q} so that the function $\tilde{H} \circ \theta: \tilde{Q} \rightarrow \mathbb{R}$ is bounded from above, i.e.

$$\sup_{\tilde{Q}} \tilde{H}(\theta) < \infty.$$

The following proposition is contained in [8, Lemma 5.1].

Proposition 2.4.1. *We assume the situation described in Section 2.3. Let g be a metric on M . If $\mu^*\sigma$ has a bounded primitive θ , then for all $k > \sup_{\tilde{Q}} \tilde{H}(\theta)$ the tuple $(\pi: M' \rightarrow M, \alpha, \omega, g)$ is a virtually contact structure. The odd-dimensional symplectic form ω of the virtually contact structure is non-exact provided $\dim Q \geq 3$ and the magnetic form σ on Q is not exact. On closed hyperbolic surfaces Q there exist magnetic forms σ on Q for which the construction yields non-trivial virtually contact structures.*

Proof. Choose k such that $k > \sup_{\tilde{Q}} \tilde{H}(\theta)$. As in [8, Lemma 5.1] we find a $\varepsilon > 0$ such that

$$|\theta|_{(\tilde{h})^\flat} + \varepsilon \leq \sqrt{2(k - V)}$$

uniformly on \tilde{Q} . Notice, that

$$(\tilde{\lambda} + \tilde{\tau}^*\theta)(X_{\tilde{H}})(\tilde{u}) = |\tilde{u}|_{(\tilde{h})^\flat}^2 + (\tilde{h})^\flat(\tilde{u}, \theta) \geq |\tilde{u}|_{(\tilde{h})^\flat} \left(|\tilde{u}|_{(\tilde{h})^\flat} - |\theta|_{(\tilde{h})^\flat} \right),$$

where $X_{\tilde{H}}$ is the Hamiltonian vector field of the Hamiltonian system $(\tilde{\omega}_{d\theta}, \tilde{H})$. Because M' is the regular level set $\{\tilde{H} = k\}$ we get $\alpha(X_{\tilde{H}}) \geq \varepsilon^2$ on M' . In particular, α is a contact form on M' , see [20, Chapter 4.3]. Because $(\tilde{\omega}_{d\theta}, \tilde{H})$ is the lift of (ω_σ, H) via $T^*\mu$ we obtain $T(T^*\mu)(X_{\tilde{H}}) = X_H$. Hence, the restriction of $X_{\tilde{H}}$ to M' is bounded for any choice of metric on M , which by construction is a closed manifold. This implies (gb₂).

In order to verify (gb₁) we choose the metric on the total space $T^*\tilde{Q}$ induced by the splitting into horizontal and vertical distribution with respect to the Levi-Civita connection of \tilde{h} . This induces a metric on M' and turns $T\tilde{\tau}$ into an orthogonal projection operator, whose operator norm is bounded by 1. Hence, $\tilde{\tau}^*\theta = \theta_{\tilde{\tau}} \circ T\tilde{\tau}$ and $\tilde{\lambda}_{\tilde{u}} = \tilde{u} \circ T\tilde{\tau}$ are uniformly bounded because θ and $\frac{1}{2}|\tilde{u}|_{(\tilde{h})^\flat}^2 = k - \tilde{V}(\tilde{\tau}(\tilde{u}))$ are. This shows that the contact form α is bounded.

Therefore, $(\pi: M' \rightarrow M, \alpha, \omega, g)$ is a virtually contact structure. It remains to show that the virtually contact structure has a non-exact odd-dimensional symplectic form provided that $n \geq 3$ and σ is not exact. Observe that as in Remark 2.3.1 one verifies that M is an S^{n-1} -bundle over Q . The Gysin sequence yields an injection $(\tau|_M)^*$ from the second de Rham cohomology of Q into the one of M . Hence, $\tau^*\sigma|_{TM}$ is non-exact too so that the restriction ω of the twisted symplectic form ω_σ to TM is non-exact. This shows non-exactness of the symplectic form of the resulting virtually contact structures for $n \geq 3$.

We discuss non-triviality of the virtually contact structure for $n = 2$. Only closed orientable surfaces Q admit non-exact 2-forms. By the Gysin

sequence the 2-form $\tau^*\sigma|_{TM}$ is non-exact only for the 2-torus. The argumentation from [17, Example 0.1.A] shows that any primitive of $\mu^*\sigma$ on the cover \mathbb{R}^2 is unbounded and, therefore, can not result into a virtually contact structure. This excludes the case that Q is a torus. By Remark 2.3.1 we also can ignore the case Q being S^2 . For the remaining hyperbolic surfaces it was shown in [9, Theorem B.1] that there are induced virtually contact structures $(\pi: M' \rightarrow M, \alpha, \omega, g)$ that are non-trivial, cf. [8, p. 1833, (ii)] and [20, Chapter 4.3]. We remark that examples of contact type are constructed in [16]. \square

Example 2.4.2. Let (Q, h) be a closed Riemannian manifold of negative sectional curvature and let σ be a closed 2-form on Q . Then the lift $\mu^*\sigma$ along the universal covering $\mu: \tilde{Q} \rightarrow Q$ has a bounded primitive θ on (\tilde{Q}, \tilde{h}) , see [17, 0.2.A.] or [3, Proposition 8.4]. We remark that by the theorem of Hadamard–Cartan \tilde{Q} is diffeomorphic to \mathbb{R}^n so that $M' = \mathbb{R}^n \times S^{n-1}$ and Q is an aspherical manifold.

By Preissmann’s theorem the product $Q_1 \times Q_2$ of two negatively curved manifolds does not admit a metric of negative sectional curvature. But still such a product $Q_1 \times Q_2$ is aspherical and any closed 2-form of the form $\sigma_1 \oplus \sigma_2$ has a bounded primitive on the universal cover of $Q_1 \times Q_2$.

For more examples the reader is referred to [21].

2.5. Somewhere contact

We will use Proposition 2.4.1 for a construction of somewhere contact virtually contact structures. The main observation for that is that if the magnetic term σ vanishes, then the restriction of λ to TM defines a contact form on $M = \{H = k\}$ for all $k > \max_Q V$. Indeed, for $\varepsilon > 0$ and $u \in M$ satisfying $\frac{1}{2}\varepsilon^2 \leq k - V(\tau(u))$ we get

$$\lambda(X_H)(u) = |u|_h^2 \geq \varepsilon^2$$

so that [20, Chapter 4.3] applies. The same holds true for the Hamiltonian system that is obtained via a lift along μ , or if Q is replaced by a relatively compact open subset U of Q .

We consider a closed 2-form σ on Q such that $\{\sigma = 0\}$ contains a non-empty relatively compact open subset U . If the lift of σ along μ has a bounded primitive θ that vanishes on $\mu^{-1}(U)$, then the resulting virtually contact structure that is described in Proposition 2.4.1 will be somewhere

contact. Indeed, the restriction of the contact form

$$\alpha = (\tilde{\lambda} + \tilde{\tau}^*\theta)|_{TM'}$$

to $M' \cap (T^*\mu)^{-1}(T^*U)$ equals the one of $\tilde{\lambda}|_{TM'}$, which is mapped to the contact form

$$\lambda|_{T(M \cap T^*U)}$$

via $\pi = T^*\mu|_{M'}$.

Lemma 2.5.1. *Let σ be a closed 2-form on Q and V be a non-empty relatively compact open subset of Q such that $\sigma|_V = 0$. Let θ be a bounded primitive of $\mu^*\sigma$. Then there exist an open subset $U \subset \bar{U} \subset V$ of Q and a bounded primitive $\hat{\theta}$ of $\mu^*\sigma$ that coincides with θ on the complement of $\mu^{-1}(V)$ and vanishes on $\mu^{-1}(U)$ such that the virtually contact structure*

$$\left(\pi: M' \rightarrow M, \alpha = (\tilde{\lambda} + \tilde{\tau}^*\hat{\theta})|_{TM'}, \omega, g\right)$$

obtained in Proposition 2.4.1 is somewhere contact for all $k > \sup_{\tilde{Q}} \tilde{H}(\hat{\theta})$. The odd-dimensional symplectic form ω of the virtually contact structure is non-exact provided $\dim Q \geq 3$ and the magnetic form σ on Q is not exact.

Proof. In view of the preceding remarks it is enough to show that $\mu^*\sigma$ has a bounded primitive that vanishes on $\mu^{-1}(U)$ for an open subset U of Q . In order to do so we will assume that σ vanishes on an embedded closed disc $D^n \cong V \subset Q$. The open set U is taken to be the Euclidean ball $B_{1/2}(0)$ inside D^n . Additionally, we choose V so small that $\mu^{-1}(V)$ decomposes into a disjoint union of subsets V^p of the universal cover of Q where the union is taken over all $p \in \mu^{-1}(q)$, $q \equiv 0$, so that

$$\mu^p := \mu|_{V^p}: V^p \longrightarrow V$$

is a diffeomorphism for all p . In a similar way the preimage of U is decomposed into sets denoted by U^p . Taking the metric $\tilde{h} = \mu^*h$ on \tilde{Q} the maps μ^p are in fact isometries.

Consider the given bounded primitive θ of $\mu^*\sigma$ and denote the restriction of θ to V^p by $\theta^p := \theta|_{V^p}$. Notice, that $d\theta^p = 0$ for all p . By the Poincaré–Lemma there exists a function $f^p: V^p \rightarrow \mathbb{R}$ such that $df^p = \theta^p$. Choose a cut-off function χ on Q that vanishes on $B_{1/2}(0) \cong U$ and is identically 1 in

a neighbourhood of $Q \setminus \text{Int } V$. Set $\chi^p = \chi \circ \mu^p$ and

$$\hat{\theta}^p = d(\chi^p f^p)$$

and observe that $\hat{\theta}^p|_{U^p} = 0$. This defines a 1-form $\hat{\theta}$ on \tilde{Q} that is equal to θ in the complement of the V^p 's and coincides with $\hat{\theta}^p$ on each V^p . By construction $\hat{\theta}$ is a primitive of $\mu^* \sigma$ that vanishes on $\mu^{-1}(U)$.

It remains to show boundedness of $\hat{\theta}$ on (\tilde{Q}, \tilde{h}) . For this it will suffice to obtain a bound for

$$\hat{\theta}^p = f^p d\chi^p + \chi^p \theta^p$$

independently of p . Of course χ^p is bounded by 1. By chain rule we have

$$d\chi^p = d\chi \circ T\mu^p.$$

Because μ^p is an isometry we obtain a uniform bound on $|d\chi^p|_{(\tilde{h})^p}$. Moreover, $|\theta^p|_{(\tilde{h})^p}$ can be estimated by the supremum of $|\theta|_{(\tilde{h})^p}$, which is bounded by assumption. Therefore, in order to obtain a uniform bound on $|\hat{\theta}^p|_{(\tilde{h})^p}$ we need a uniform bound on $|f^p|$.

For this recall the Poincaré–Lemma. Identify $D^n \cong V$ with V^p isometrically via μ^p . In order to simplify the following computation in local coordinates we suppress the superscript p from the notation. The 1-form θ , which got identified with θ^p , is closed. Write $\theta_{\mathbf{x}} = \theta_j(\mathbf{x})dx^j$ using summation convention for $\mathbf{x} = (x^1, \dots, x^n)$ in D^n . For $t \in [0, 1]$ we get $\theta_{t\mathbf{x}}(\mathbf{x}) = \theta_j(t\mathbf{x})x^j$ so that a primitive of θ is given by

$$f(\mathbf{x}) = \int_0^1 \theta_{t\mathbf{x}}(\mathbf{x}) dt.$$

Hence, by the mean value theorem there exists $t_0 \in [0, 1]$ such that

$$|f(\mathbf{x})| \leq |\theta_{t_0\mathbf{x}}(\mathbf{x})| \leq \|\theta\|_{h_{t_0\mathbf{x}}} |\mathbf{x}|_{h_{t_0\mathbf{x}}}.$$

Observe that the operator norm $\|\theta\|_h$ equals $|\theta|_{(\tilde{h})^p}$ pointwise and is, therefore, uniformly bounded. Moreover, by compactness of D^n the restriction of the metric h to D^n is uniformly equivalent to the Euclidean metric so that $|\mathbf{x}|_{h_{t_0\mathbf{x}}}$ admits a uniform bound. Therefore, the same holds true for $|f(\mathbf{x})|$. Consequently, the perturbed primitive $\hat{\theta}$ of $\mu^* \sigma$ is bounded.

In order to finish the proof of the lemma we have to verify non-exactness of the odd-dimensional symplectic form of the resulting virtually contact structure if $\dim Q \geq 3$ and σ is non-exact. But this follows exactly as for Proposition 2.4.1. □

2.6. Proof of Theorem 1.1

In view of Example 2.4.2 we choose a closed Riemannian manifold (Q, h) that is not simply connected. Moreover, choose a closed non-exact 2-form σ on Q whose lift to the universal cover has a bounded primitive. By a use of a cut-off function χ as in the proof of Lemma 2.5.1 we can cut-off a local primitive θ_V of $\sigma|_V$ for an embedded closed disc V . Setting σ equal to $d(\chi\theta_V)$ on V this results into a new magnetic 2-form that vanishes somewhere. Notice, that the cohomology class of σ is unchanged and the lift of σ still has a bounded primitive. In this situation Lemma 2.5.1 yields a somewhere contact virtually contact structure $(\pi: M' \rightarrow M, \alpha, \omega, g)$ with ω being non-exact if $\dim Q \geq 3$ and with M being not simply connected, cf. Remark 2.3.1. With these preliminaries Theorem 1.1 will be a consequence of the following proposition if $n \geq 3$.

Proposition 2.6.1. *Let $(\pi: M' \rightarrow M, \alpha, \omega, g)$ be a somewhere contact virtually contact structure with non-exact ω and denote by $(T, \ker \alpha_T)$ a contact manifold. Assume that M and T are of dimension $2n - 1$. Then the connected sums $M \# M$ and $M \# T$ admit somewhere contact virtually contact structures whose odd-dimensional symplectic forms are non-exact. Moreover, if M and T are not simply connected, then the belt spheres of the connected sums $M \# M$ and $M \# T$ represent non-trivial elements in π_{2n-2} .*

Proof. Denote by $x \in U$ the base point of M where U is an open subset of M according to the definition of being somewhere contact, see Section 2.1. Performing a covering connected sum of $(\pi: M' \rightarrow M, \alpha, \omega, g)$ with itself for any bijection b of the base point fibre $\pi^{-1}(x)$ yields a virtually contact structure on $M \# M$, see Section 2.2. In order to obtain a virtually contact structure on $M \# T$ consider the covering obtained by the disjoint union of $(T \times \{y_j\}, \alpha_T)$, $y \in \pi^{-1}(x)$ and perform covering connected sum. Non-exactness of the odd-dimensional symplectic form of the constructed virtually contact structures follows with Lemma 2.2.2. Further, in both cases the resulting covering contact manifold admits a contact embedding of the upper boundary of a standard symplectic 1-handle as it is discussed in Remark 2.2.1. In particular, the virtually contact structures on the surgered manifolds are somewhere contact. Moreover, if M and T both are not simply connected, then the belt sphere represents a non-trivial homotopy class in π_{2n-2} by the proof of [18, Proposition 3.10]. □

This finishes the proof of Theorem 1.1 if $n \geq 3$. The reason why the above argumentation does not work for $n = 2$ is that the odd-dimensional

symplectic structure obtained from a twisted cotangent bundle of a surface Q is necessarily exact if Q is not a 2-torus, cf. the discussion on the end of the proof of Proposition 2.4.1. In order to construct non-trivial virtually contact structures in dimension 3 that are a non-trivial connected sum we make the following observations:

Proposition 2.6.2. *Let $(M, \ker \alpha_M)$ be a closed connected contact manifold. Assume that M carries a metric of negative sectional curvature and a non-exact closed 2-form η . Then there exists a somewhere contact virtually contact structure $(\pi: M' \rightarrow M, \alpha, \omega, g)$ on M such that ω is cohomologous to a positive multiple of η .*

Proof. By using a suitable local cut-off of η we assume that there exists an open subset $V \subset M$ such that $\eta|_V = 0$. This does not change the cohomology class of η . As explained in the proof of Lemma 2.5.1 we can further assume that $\theta|_{\pi^{-1}(U)} = 0$ for an open subset $U \subset \bar{U} \subset V$ of M . With [17, 0.2.A.] $\pi^*\eta$ has a bounded primitive θ on the universal cover denoting by π the corresponding covering map. For $\varepsilon > 0$ sufficiently small the lift of the 2-form $\omega = d\alpha_M + \varepsilon\eta$ along π has a bounded primitive $\alpha = \pi^*\alpha_M + \varepsilon\theta$ in the sense of (gb_1) that is a contact form. By shrinking $\varepsilon > 0$ if necessary the contact form α satisfies (gb_2) as an argumentation by contradiction shows. \square

Observe that M is aspherical in contrast to the examples given in Proposition 2.6.1 and that by the theorem of Hadamard–Cartan the compact manifold M can not be simply connected. Examples in dimension 3 can be obtained as follows:

Example 2.6.3. Let M be the mapping torus of a closed orientable surface of higher genus with monodromy diffeomorphism being pseudo-Anosov. By a theorem of Thurston [25] M is hyperbolic. Moreover, the Betti numbers $b_1 = b_2$ of M are non-zero so that a non-exact closed 2-form η can be found. By Martinet’s theorem [10, Theorem 4.1.1] M has a contact form α_M .

A covering contact connected sum of the somewhere contact virtually contact manifold M obtained with Example 2.6.3 and Proposition 2.6.2 as described in Proposition 2.6.1 results in a non-trivial virtually contact manifold, such that the related belt sphere represents a non-trivial class in π_{2n-2} . This finishes the proof of Theorem 1.1. Q.E.D.

2.7. Being prime

Recall that a closed connected manifold M is called **prime** if whenever written as a connected sum $M = M_1 \# M_2$ one of the summands M_1 and M_2 is a homotopy sphere. The connected sum with a homotopy sphere is called to be **trivial**. We remark that the virtually contact manifolds constructed in Section 2.6 are obtained by a non-trivial connected sum and are, therefore, not prime. This follows from the corresponding belt sphere not to be contractible inside the surgered manifold.

The aim of the following proposition is to show that the examples of virtually contact structures given in Section 2.6 differ from the one obtained on unit cotangent bundles $M \cong ST^*Q$ of n -dimensional Riemannian manifolds of negative sectional curvature studied in Section 2.4. Recall, that by Hadamard–Cartan’s theorem the universal cover of a Riemannian manifold of non-positive sectional curvature is diffeomorphic to \mathbb{R}^n .

Proposition 2.7.1. *The total space ST^*Q of the unit cotangent bundle of a closed connected aspherical n -dimensional manifold Q with respect to any metric on Q is prime.*

Proof. As Q is aspherical by Whitehead’s theorem the universal cover \tilde{Q} of Q contracts to its base point, see [18, Theorem 4.5]. Therefore, the cotangent bundle of \tilde{Q} is trivial and $ST^*\tilde{Q}$, which is diffeomorphic to $\tilde{Q} \times S^{n-1}$, is homotopy equivalent to S^{n-1} .

If $n = 2$, then the universal cover of ST^*Q is \mathbb{R}^3 , see Remark 2.3.1. By Alexander’s theorem \mathbb{R}^3 is **irreducible**, i.e. any embedded 2-sphere bounds a ball, see [18, Theorem 1.1]. With [18, Proposition 1.6] the closed 3-manifold ST^*Q itself is irreducible and, therefore, prime.

If $n \geq 3$, then the universal cover of ST^*Q is diffeomorphic to $\tilde{Q} \times S^{n-1}$. Consider an embedded $(2n - 2)$ -sphere S_b in ST^*Q thinking of it as the belt sphere of a connected sum decomposition of ST^*Q . Let \tilde{S}_b be a lift of S_b to the universal cover of ST^*Q . Because the homology of the universal cover of ST^*Q vanishes in degree $2n - 2$ any lift of S_b is the boundary of a bounded domain whose closure we denote by Ω_0 . We choose \tilde{S}_b so that Ω_0 does not contain any other of the lifts of S_b . The closure of the unbounded component of the complement of \tilde{S}_b is denoted by Ω_1 . Therefore, we obtain

$$\tilde{Q} \times S^{n-1} \cong \widetilde{ST^*Q} = \Omega_0 \cup_{\tilde{S}_b} \Omega_1.$$

By Seifert–van Kampen’s theorem Ω_0 must be simply connected. Moreover, the boundary operator of the Mayer–Vietoris sequence with respect to the

above decomposition vanishes in all positive degrees. Indeed, we can take the image of $\{q\} \times S^{n-1}$, for $q \in \tilde{Q} \simeq \{*\}$, as a generator of the homology in degree $n - 1$ so that its intersection with Ω_0 , and hence with \tilde{S}_b , is empty. Therefore, the Mayer–Vietoris sequence reduces to the following short exact sequences

$$0 \rightarrow H_k \tilde{S}_b \rightarrow H_k \Omega_0 \oplus H_k \Omega_1 \rightarrow H_k(\tilde{Q} \times S^{n-1}) \rightarrow 0$$

for all positive k . This implies that Ω_0 has the homology of a ball. To see this for $k = n - 1$ notice that the generator of the homology in degree $n - 1$ of the universal cover of ST^*Q is chosen to be contained in Ω_1 . The vanishing in degree $2n - 2$ follows with $\tilde{S}_b \cong S^{2n-2}$ being the boundary of Ω_0 . Therefore, Ω_0 is a simply connected $(2n - 1)$ -dimensional homology ball with boundary S^{2n-2} . With [22, p. 108, Proposition A and p. 110, Proposition C] it follows that Ω_0 is diffeomorphic to a $(2n - 1)$ -dimensional disc. With the arguments used in the proofs of [18, Proposition 1.6 and Proposition 3.10] this yields that S_b bounds a $(2n - 1)$ -dimensional disc in ST^*Q meaning that the assumed connected sum decomposition is trivial. After all, we see that ST^*Q has to be prime. □

3. Morse potentials

This section is devoted to a proof of Theorem 1.2.

3.1. Morsification

We consider the Hamiltonian function

$$H(u) = \frac{1}{2}|u|_{h^b}^2 + V(\tau(u))$$

of classical mechanics on T^*Q , where $\tau: T^*Q \rightarrow Q$ is the cotangent bundle and (Q, h) is a closed oriented connected Riemannian manifold. The linearization of H at a point $u \in T^*Q$ can be written as

$$T_u H = h^b(u, K_u(\cdot)) + T_{\tau(u)} V \circ T_u \tau,$$

where $K_u: T_u(T^*Q) \rightarrow T_{\tau(u)}^*Q$ is the connection operator of h^b . In particular, u is a critical point of H if and only if u is contained in the zero section Q of T^*Q and is a critical point of the potential $V: Q \rightarrow \mathbb{R}$.

This is of particular interest if V is a Morse function what we will assume in the following. Then H will be a Morse function too. This is because to

the potential V a positive definite quadratic form with respect to the fibre direction is added. In particular, the Morse indices of a critical point are the same for both functions V and H .

3.2. Topology of the energy surface

We choose a Morse function V on Q that has a unique local maximum. We assume that the maximum of V is equal to 1 and that all critical points of index less or equal than $n - 1$ have critical value smaller than -1 . For the regular value 0 we consider the energy surface $M = \{H = 0\}$.

The sublevel set $W = \{H \leq 0\}$ is a CW-complex of dimension less or equal than $n - 1$. In particular, $H_k W = 0$ for all $k \geq n$ and $H_{n-1} W$ is torsion-free. Hence, the boundary operator of the long exact sequence of the pair (W, M) induces an isomorphism $H_{n+1}(W, M) \rightarrow H_n M$. Moreover, by the universal coefficient theorem and Poincaré duality $H_{n-1} W$ injects into $H_{n+1}(W, M)$ naturally. In fact, the Poincaré duality isomorphism

$$H^{n-1} W \rightarrow H_{n+1}(W, M)$$

can be given in terms of the Morse functions meaning that the classes in $H_{n+1}(W, M)$ can be represented by cocore discs $\{*\} \times D^{n+1}$, see [22, Remark on p. 35/36 and Theorem 7.5]. Therefore, the corresponding belt spheres $\{*\} \times S^n$ generate a free subgroup of $H_n M$ that is isomorphic to $H_{n-1} W$ as an application of the boundary operator shows.

The **negative set** $N = \{V \leq 0\} \subset Q$ is a deformation retract of W . Hence, $H_{n-1} W$ and $H_{n-1} N$ are isomorphic. By the assumptions on the Morse function V we have $N \simeq Q \setminus \{*\}$ so that $H_{n-1} N = H_{n-1} Q$. Therefore, $H_{n-1} Q$ injects into $H_n M$ whose image is freely generated by belt spheres. Denoting by $b_k Q$ the Betti numbers of Q and using $b_1 Q = b_{n-1} Q$ the Hurewicz homomorphism yields

$$\pi_n M \geq \mathbb{Z}^{b_1 Q}.$$

This verifies the claim on the n -th homotopy group in Theorem 1.2.

Example 3.2.1. If Q is a closed Riemann surface of genus g , then M is equal to the connected sum $S^3 \# (2g)(S^1 \times S^2)$.

3.3. Virtually contact type

Let σ be a 2-form on Q that vanishes on $\{V > -1\}$ and consider the twisted symplectic form $\omega_\sigma = d\lambda + \tau^*\sigma$ on T^*Q . Let θ be a bounded primitive of $\mu^*\sigma$ denoting by $\mu: \tilde{Q} \rightarrow Q$ the universal covering. By the proof of Lemma 2.5.1 we can assume that θ vanishes on $\mu^{-1}(\{V > -1\})$.

By multiplying σ with a small positive constant we achieve that

$$\frac{1}{2}|\theta|_{(\tilde{h})^b}^2 < \frac{1}{2}.$$

This implies that $\tilde{H}(\theta)$ is negative on $\mu^{-1}(\{V \leq -1\})$. Therefore, as in the proof of Proposition 2.4.1,

$$(\tilde{\lambda} + \tilde{\tau}^*\theta)|_{TM'}$$

is a contact form on the intersection of M' with $(T^*\mu)^{-1}(T^*\{V \leq -1\})$ satisfying (gb₁) and (gb₂).

Over the remaining part $U := \{V > -1\}$ we perturb the Liouville form as follows: Choose a function F on T^*Q whose support is contained in T^*U such that $(\lambda + dF)(X_H) > 0$ on $M \cap T^*U$, see [8, Lemma 5.2] or [24, p. 137]. Therefore, $(\lambda + dF)|_{TM}$ defines a contact form on $M \cap T^*U$. Consequently,

$$\alpha = (\tilde{\lambda} + \tilde{\tau}^*\theta + d\tilde{F})|_{TM'}$$

is a contact form on M' , where $\tilde{F} = F \circ T^*\mu$. Observe, that \bar{U} is a compact set and that the magnetic term σ and the chosen primitive θ of the lift $\mu^*\sigma$ vanish over $\mu^{-1}(U)$. Hence, all involved differential forms are lifts of differential forms that are defined on a compact set. In other words, (gb₁) and (gb₂) are satisfied along $M' \cap T^*(\mu^{-1}(U))$ so that α defines a virtually contact structure.

Remark 3.3.1. The Mañé critical value of the described magnetic system equals 1 as the maximum of V is always a lower bound.

3.4. Exactness

The resulting odd-dimensional symplectic form on M is equal to

$$\omega = (d\lambda + \tau^*\sigma)|_{TM}.$$

This form is exact precisely if $\tau^*\sigma|_{TM}$ is exact, which is the case provided that σ restricts to an exact form on \bar{N} . Invoking de Rham's theorem and $N \simeq Q \setminus \{*\}$ we see that ω will be exact in dimension $2n - 1 = 3$. If $n \geq 3$ the exactness of $\tau^*\sigma|_{TM}$ is equivalent to the one of σ on Q . This follows with the Gysin sequence for the unit cotangent bundle of $\{V \leq -1\}$, for which the map induced by τ is injective in degree 2, and an extension argument for primitive 1-forms over U , which is diffeomorphic to D^n .

In other words, for $n = 2$ the odd-dimensional symplectic form ω is always exact; for $n \geq 3$ the odd-dimensional symplectic form ω can be chosen to be non-exact precisely if the Betti number b_2Q does not vanish.

3.5. Proof of Theorem 1.2

According to the construction given in Sections 3.2, 3.3, and 3.4 and Example 2.4.2 it suffices to find oriented closed manifolds Q with non-trivial Betti numbers b_1Q and b_2Q that allow a Riemannian metric and a closed non-exact 2-form σ such that the lift of σ has a bounded primitive.

In dimension $n = 2$ we can take any closed oriented hyperbolic surface and any 2-form as magnetic term. With Example 2.6.3 the case $n = 3$ can be treated similarly. In view of Künneth's formula taking products in the sense of Example 2.4.2 yields higher dimensional examples. Because for any $b \in \mathbb{N}$ we find a manifold Q with the above listed properties satisfying $b_1Q \geq b$ the claim of Theorem 3.5 follows. Q.E.D.

Remark 3.5.1. For $b \geq 2$ the manifold M constructed in Section 3.5 is not diffeomorphic to a unit cotangent bundle of a closed aspherical manifold Q as such a S^{n-1} -bundle over Q has vanishing π_2 if $n = 2$, π_3 equal to \mathbb{Z}_2 if $n = 3$, and π_n equal to \mathbb{Z} if $n \geq 4$.

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