Seidel's morphism of toric 4–manifolds

SÍLVIA ANJOS AND RÉMI LECLERCQ

Following McDuff and Tolman's work on toric manifolds [32], we focus on 4–dimensional NEF toric manifolds and we show that even though Seidel's elements consist of infinitely many contributions, they can be expressed by closed formulas. From these formulas, we then deduce the expression of the quantum homology ring of these manifolds as well as their Landau–Ginzburg superpotential. We also give explicit formulas for the Seidel elements in some non-NEF cases. These results are closely related to recent work by Fukaya, Oh, Ohta, and Ono [15], González and Iritani [19], and Chan, Lau, Leung, and Tseng [8]. The main difference is that in the 4–dimensional case the methods we use are more elementary: they do not rely on open Gromov–Witten invariants nor mirror maps. We only use the definition of Seidel's elements and specific closed Gromov–Witten invariants which we compute via localization. This makes the resulting formulas directly readable from the moment polytope.

1. Introduction

Let (M, ω) be a closed connected symplectic manifold and let as usual $\mathrm{Ham}(M, \omega)$ denote its Hamiltonian diffeomorphism group. Under a suitable condition of semipositivity, Seidel defined in [36] a morphism, $\mathcal{S}$, from $\pi_1(\mathrm{Ham}(M, \omega))$ to – after a mild generalization due to Lalonde, McDuff, and Polterovich [28] – $QH_*(M, \omega)^\times$, the group of invertible elements of the quantum homology of (M, ω). This morphism has been extensively used in order to get information on the topology of Hamiltonian diffeomorphism groups as well as the quantum homology of symplectic manifolds. It has also been extended in various directions, see the end of the introduction for some of these extensions related to the present work.

A quantum class lying in the image of $\mathcal{S}$ is called a *Seidel element*. In [32], McDuff and Tolman were able to specify the structure of the lower order terms of Seidel's elements associated to Hamiltonian circle actions whose maximal fixed point component, $F_{\max}$, is semifree. Recall that this condition means that the action is semifree on a neighborhood of $F_{\max}$ which means, in our case, that the stabilizer of each point is trivial or the whole circle. When the codimension of $F_{\max}$ is 2, their result immediately ensures that if there exists an almost complex structure J on M so that (M, J) is Fano, i.e so that there are no J–pseudo-holomorphic spheres in M with non-positive first Chern number, all the lower order terms vanish. In the presence of J–pseudo-holomorphic spheres with vanishing first Chern number, there is a priori no reason why arbitrarily large multiple coverings of such objects should not contribute to the Seidel elements. As a matter of fact, McDuff and Tolman exhibited an example of such a phenomenon when (M, J) is a NEF pair, which by definition do not admit J–pseudo-holomorphic spheres with negative first Chern number.

In this paper, we show that even though there are indeed infinitely many contributions to the Seidel elements associated to the Hamiltonian circle actions of a NEF 4–dimensional toric manifold, these quantum classes can still be expressed by explicit closed formulas. Moreover, these formulas only depend on the relative position of representatives of elements of $\pi_2(M)$ with vanishing first Chern number as facets of the moment polytope. In particular, they are directly readable from the polytope.

More precisely, we consider (see Section 2 for precise definitions):

- a 4–dimensional closed symplectic manifold (M, ω), endowed with a toric structure and admitting a NEF almost complex structure,

- its corresponding Delzant polytope P, which is assumed to have $n \geq 4$ facets,

- a Hamiltonian action generated by a circle subgroup Λ, with moment map Φ_Λ.

We assume additionally, that the fixed point component of Λ on which Φ_Λ is maximal is a 2–sphere, $F_{\max} \subset M$, whose momentum image is a facet of P, D. We denote by $A \in H_2(M; \mathbb{Z})$ the homology class of $F_{\max}$ and by $\Phi_{\max} = \Phi_\Lambda(F_{\max})$.

In this case, McDuff–Tolman's result ensures that the Seidel element associated to Λ is

$$S(\Lambda) = A \otimes q t^{\Phi_{\max}} + \sum_{B \in H_2^S(M;\mathbb{Z})^{>0}} a_B \otimes q^{1-c_1(B)} t^{\Phi_{\max} - \omega(B)}$$

where $H_2^S(M; \mathbb{Z})^{>0}$ consists of the spherical classes of positive symplectic area, that is, $\omega(B) > 0$ and $a_B \in H_*(M; \mathbb{Z})$ denotes the contribution of B. As mentioned above, when there exists a Fano almost complex structure, all the lower order terms vanish and we end up with $S(\Lambda) = A \otimes q t^{\Phi_{\max}}$.

In the non-Fano case, one has to be careful about the number and relative position of facets, in the vicinity of D, corresponding to spheres in M with vanishing first Chern number. We denote the number of such facets by $\#\{c_1 = 0\}$. Theorem 4.4 lists all the contributions made to the Seidel element associated to Λ in the 6 cases when $\#\{c_1 = 0\} \leq 2$. We denote the facets and the corresponding homology classes in M in a cyclic way, that is, D, which we denote by D_n below, has neighbooring facets D_{n-1} on one side and $D_{n+1} = D_1$ on the other, and they respectively induce classes A_n, A_{n-1}, and $A_{n+1} = A_1$ in $H_2(M; \mathbb{Z})$.

Figure 1 shows the relevant parts of the different polytopes we need to consider. Dotted lines represent facets with non-zero first Chern number and we indicate near each facet with non-trivial contribution the homology class of the corresponding sphere in M. For example, in Case *(3c)*, only three homology classes contribute: A_{n-1}, A_n, and A_1; A_{n-1} and A_1 have vanishing first Chern number while $c_1(A_n) \neq 0$.

Theorem 4.4. *With the notation and under the assumptions above, the following homology classes have non trivial contributions to $S(\Lambda)$:*

1) A_n *contributes by* $a_{A_n} = A_n$.

2) *When* $c_1(A_n) = 0$,
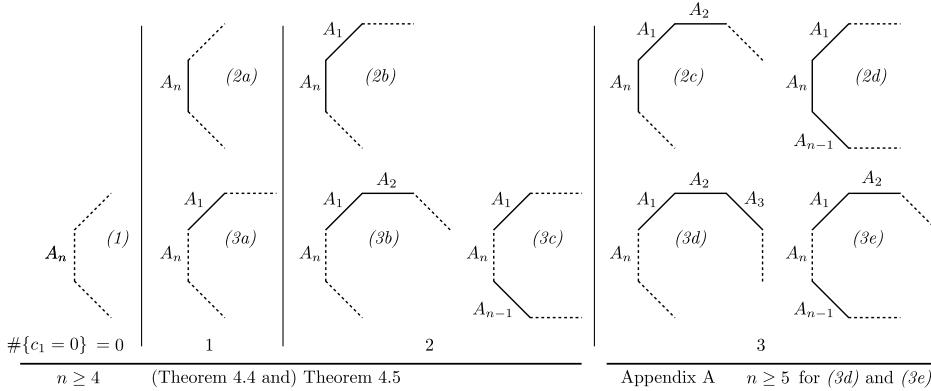   *(2a) then* kA_n *(with* $k > 0$*) contributes by* $a_{kA_n} = A_n$,

Figure 1: The cases appearing in Theorems 4.4 and 4.5, and Appendix A.

(2b) and when $c_1(A_1) = 0$, then $kA_n + lA_1$ (with $k \geq 0$ and $l > 0$) contributes and its contribution is $a_{kA_n+lA_1} = \begin{cases} A_n & \text{if } k \geq l \\ -A_1 & \text{otherwise.} \end{cases}$

3) *When $c_1(A_n) \neq 0$,*
  (3a) when $c_1(A_1) = 0$, then kA_1 (with $k > 0$) contributes by $a_{kA_1} = -A_1$,
  (3b) when $c_1(A_1) = 0$ and $c_1(A_2) = 0$, then $kA_1 + lA_2$ (with $k > 0$ and $l > 0$) also contributes, and its contribution is $a_{kA_1+lA_2} = \begin{cases} -A_1 & \text{if } k \geq l \\ A_2 & \text{otherwise,} \end{cases}$
  (3c) when $c_1(A_{n-1}) = 0$ and $c_1(A_1) = 0$, then kA_{n-1} and lA_1 (with $k > 0$ and $l > 0$) also contribute, and their contributions are given by $a_{kA_{n-1}} = -A_{n-1}$ and $a_{lA_1} = -A_1$.

Moreover, in each case, if the facets immediately next to the ones mentioned correspond to spheres with non-zero first Chern number, then these are the only non-trivial contributions.

Now, under the same assumptions, Theorem 4.5 gives the explicit expression of the Seidel element associated to Λ when $\#\{c_1 = 0\} \leq 2$. Notice that we give (without proofs) the expression of the Seidel elements for $\#\{c_1 = 0\} = 3$ in Appendix A.

Theorem 4.5. *Under the assumptions above, and in the cases described by Figure 1, the Seidel element associated to Λ is*

(1) $S(\Lambda) = A_n \otimes qt^{\Phi_{\max}}$

(2a) $S(\Lambda) = A_n \otimes q \, \dfrac{t^{\Phi_{\max}}}{1 - t^{-\omega(A_n)}}$

(2b) $S(\Lambda) = \left(A_n \otimes q \, \dfrac{t^{\Phi_{\max}}}{1 - t^{-\omega(A_n)}} - A_1 \otimes q \, \dfrac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}} \right) \dfrac{1}{1 - t^{-\omega(A_n)-\omega(A_1)}}$

(3a) $S(\Lambda) = A_n \otimes qt^{\Phi_{\max}} - A_1 \otimes q \, \dfrac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}}$

(3b) $S(\Lambda) = A_n \otimes qt^{\Phi_{\max}} - A_1 \otimes q \, \dfrac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}}$
$\qquad\qquad - \left(A_1 \otimes q \, \dfrac{t^{\Phi_{\max}}}{1 - t^{-\omega(A_1)}} - A_2 \otimes q \, \dfrac{t^{\Phi_{\max}-\omega(A_2)}}{1 - t^{-\omega(A_2)}} \right) \dfrac{t^{-\omega(A_1)-\omega(A_2)}}{1 - t^{-\omega(A_1)-\omega(A_2)}}$

(3c) $S(\Lambda) = A_n \otimes qt^{\Phi_{\max}} - A_{n-1} \otimes q \, \dfrac{t^{\Phi_{\max}-\omega(A_{n-1})}}{1 - t^{-\omega(A_{n-1})}} - A_1 \otimes q \, \dfrac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}}.$

Interest of our approach

This work is closely related to recent work by Fukaya, Oh, Ohta and Ono [14], González and Iritani [19], and Chan, Lau, Leung and Tseng [8]. Roughly speaking, for toric NEF symplectic manifolds, on one side Fukaya, Oh, Ohta, and Ono showed that quantum homology is isomorphic to the Jacobian of the open Gromov–Witten invariants generating function, $\mathrm{Jac}(W^{\mathrm{open}})$. On the other side, González and Iritani expressed the Seidel elements in terms of Batyrev's elements via mirror maps. Finally, Chan, Lau, Leung, and Tseng proved that W^{open} coincides with the Hori–Vafa superpotential. Then by using this open mirror symmetry and the aforementioned results, they showed that the Seidel elements correspond to simple explicit monomials in $\mathrm{Jac}(W^{\mathrm{open}})$. In the 4–dimensional case, these results are clearly related to ours – see for example the discussion on the Landau–Ginzburg superpotential in Example 1.3 below –, however our approach is somehow more elementary and stays on the symplectic side of the mirror.

We now sketch our approach. The Seidel element of a symplectic manifold (M, ω) associated to a loop of Hamiltonian diffeomorphisms ϕ based at identity is defined by counting pseudo-holomorphic sections of (M_ϕ, Ω) which is a symplectic fibration over S^2 with fibre M and whose monodromy along the equator is given by ϕ (this construction is called the clutching construction, see Section 2.2 for more details). To compute Seidel's elements when (M, ω) is a toric 4–dimensional symplectic manifold and $\phi = \Lambda$ is one of the distinguished circle actions, we proceed as follows.

1) Following González and Iritani [19], and Chan, Lau, Leung, and Tseng [8], we notice that (M_Λ, Ω) is a toric 6–dimensional symplectic manifold, see Proposition 2.1[1]. This allows us to reduce the computation of the Seidel elements to the computation of some 1–point Gromov–Witten invariants, see Section 4.2.

2) Then we compute the latter by induction using localization formulas from Spielberg's [37, 38] or Liu's [29] for the base cases and the splitting axiom satisfied by Gromov–Witten invariants for the inductive steps, see Section 4.4.

3) Step (2) completely ends the computation up to some particular 0–point Gromov–Witten invariants which we preliminarily compute using a localization argument, see Section 4.3.

Application in terms of Seidel's morphism and quantum homology

As mentioned above, Seidel's morphism has been extensively studied for its applications. However not many things are known concerning $\mathcal{S}$ itself, for example its injectivity. It is obvious that Seidel's morphism is trivial for symplectically aspherical manifolds since these particular manifolds do not admit non-constant pseudo-holomorphic spheres at all. In [36], Seidel showed that for all $m \geq 1$ Seidel's morphism detects an element of order $m+1$ in $\pi_1(\mathrm{Ham}(\mathbb{CP}^m, \omega_{\mathrm{st}}))$, with ω_{st} the Fubini–Study symplectic form. In the case of $\mathbb{CP}^2$ for example, this makes the Seidel morphism injective. Determining non-trivial elements of the kernel of $\mathcal{S}$ in cases when $\mathcal{S}$ is not "obviously" trivial would be interesting, for example to test the Seidel-type second order invariants introduced by Barraud and Cornea via their spectral sequence machinery [4]. In order to find such classes, one should first compute all the Seidel elements in specific cases; here are families of examples for which the present work allows such computations.

Example 1.1 (Hirzebruch surfaces). It is well-known that Hirzebruch surfaces $\mathbb{F}_{2k}$ are symplectomorphic to $S^2 \times S^2$ endowed with the split symplectic form ω_μ with area $\mu \geq 1$ for the first S^2–factor, and with area 1 for the second factor. Recall that $\mathbb{F}_0$ is Fano, $\mathbb{F}_2$ is NEF, and that for all $k \geq 2$, $\mathbb{F}_{2k}$ admits spheres with negative first Chern number. As we shall see in Section 5.3, the computations we present in this paper allow us not only to

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[1]Actually, this first step does not require M to be 4–dimensional.

compute directly the Seidel elements associated to the circle actions of $\mathbb{F}_2$, but also to compute the Seidel elements associated to the circle actions of $\mathbb{F}_{2k}$ for all $k \geq 2$, that is, in the non-NEF cases. We present explicitly the case of $\mathbb{F}_4$.

Similar computations can be made for $\mathbb{F}_{2k+1}$ which can be identified with the 1–point blow-up of $\mathbb{CP}^2$ endowed with its different symplectic forms.

Example 1.2 (2– and 3–point blow-ups of $\mathbb{CP}^2$). In the same spirit, consider the symplectic manifold obtained from $\mathbb{CP}^2$ by performing 2 or 3 blow-ups. It carries a family of symplectic forms ω_ν, where $\nu > 0$ determines the cohomology class of ω_ν. It is well-known that it is symplectomorphic to M_{μ,c_1} or M_{μ,c_1,c_2}, respectively the 1– or 2–point blow-up of $S^2 \times S^2$ endowed, as above, with the symplectic form ω_μ. Here, c_1 and c_2 are the capacities of the blow-ups.

In previous works, Pinsonnault [34], and Anjos and Pinsonnault [3] computed the homotopy algebra of the Hamiltonian diffeomorphism groups of M_{μ,c_1} and M_{μ,c_1,c_2}. In particular they showed that all the generators of its fundamental group do not depend on the symplectic form nor the size of the blow-ups provided that $\mu > 1$. In both cases, all the generators but one can be obtained as Hamiltonian circle actions associated to a Fano polytope while the last one is associated to a NEF polytope. When $\mu = 1$, the fundamental group of the Hamiltonian diffeomorphism group is generated only by the former. So the computations we present here again allow us to compute all the Seidel elements of the 2– and 3–point blow-ups of $\mathbb{CP}^2$, regardless of the symplectic form and sizes of the blow-ups.

In relation with these two examples, let us point out that we were able to exhibit a nontrivial element in the kernel of Seidel's morphism on some specific 3–point blow-ups of $\mathbb{CP}^2$. We also proved that Seidel's morphism is injective on all Hirzebruch surfaces. See [2] for more details.

Then we turn to quantum homology. Following [32], we deduce from the expression of the Seidel elements described in Theorem 4.5 a presentation of the quantum homology of 4–dimensional NEF toric manifolds. Batyrev [5] and Givental [17, 18] showed that the quantum homology of Fano toric manifolds is isomorphic to a polynomial ring quotiented by relations given as the derivatives of the well-known Landau–Ginzburg superpotential. For NEF toric manifolds see also the works by Chan and Lau [7], Fukaya, Oh, Ohta, and Ono [13, 14], Iritani [25], Usher [39], and references therein. As an application of our computations we are able to give explicit expressions for

the potential in the NEF case which can be read directly from the moment polytope, and obviously can be related with Chan and Lau's results.

Example 1.3 (4– and 5–point blow-ups of $\mathbb{CP}^2$). To illustrate what is explained above, we explicitly compute the Seidel elements of the 4– and 5– point blow-ups of $\mathbb{CP}^2$. Note that these manifolds are NEF and do not admit any Fano almost complex structure. Then we deduce their quantum homology and we give the explicit expression of the related Landau–Ginzburg superpotential, see Section 5.2. Of course, this expression agrees with Chan and Lau's result [7] and in Remark 5.5 we indicate how.

Extensions and applications

We now discuss some extensions of Seidel's morphism for which there is hope to get explicit information in the setting of and with similar techniques as the ones used in the present work.

Homotopy of Ham in higher degrees. As mentioned above, since [34] and [3] the homotopy algebra of the Hamiltonian diffeomorphism groups of the 2– and 3–point blow-ups of $\mathbb{CP}^2$ is completely understood. It would be interesting in this case to compute explicitly some invariants of the higher-degree homotopy groups generalizing Seidel's construction: the Floer-theoretic invariants for families defined by Hutchings in [23] and the quantum characteristic classes introduced by Savelyev in [35]. Briefly recall that the former are morphisms $\pi_*(\mathrm{Ham}(M,\omega)) \to \mathrm{End}_{*-1}(QH_*(M,\omega))$ obtained as higher continuation maps in Floer homology. The latter are defined via parametric Gromov–Witten invariants and lead to ring morphisms

$$H_*(\Omega\mathrm{Ham}(M,\omega),\mathbb{Q}) \to QH_{2n+*}(M,\omega).$$

Both constructions restrict to the Seidel representation, respectively, in degree 1 and 0.

Bulk extension. In this paper, what is called quantum homology should more precisely be refered to as the *small* quantum homology ring. There is also a notion of *big* quantum homology ring, obtained by considering not only the usual quantum product but also a family of deformations via even-degree cohomology classes of M, see e.g Usher [39] and Fukaya, Oh, Ohta, and Ono [15] for a precise definition. For $b \in H^{\mathrm{ev}}(M)$, one ends up with $QH_*^b(M,\omega)$ isomorphic to $QH_*(M,\omega)$ as a vector space but with a

twisted product. In [15], the authors extended Seidel's morphism to morphisms $\pi_1(\mathrm{Ham}(M,\omega)) \to QH_*^b(M,\omega)^\times$ and generalized in the toric case part of the results of McDuff and Tolman [32]. It would be interesting to see which information on the big quantum homology can be extracted from the present work.

Lagrangian setting. The Seidel morphism has been extended to the Lagrangian setting in works by Hu and Lalonde [21], and Hu, Lalonde, and Leclercq [22]. Following McDuff and Tolman [32], Hyvrier [24] computed the leading term of the Lagrangian Seidel elements associated to circle actions preserving some given monotone Lagrangian. He showed that when the latter is the real Lagrangian of a Fano toric manifold, all lower order terms vanish. It could be interesting to study the Lagrangian case in NEF toric manifolds, however the preliminary question of the structure of the lower order terms has to be tackled with different techniques than the ones used in [24] since they require the use of almost complex structures which generically lacks regularity. Let us also mention that Hyvrier's work as well as such a possible extension provide examples where one can apprehend the categorical refinement of the Lagrangian Seidel morphism due to Charette and Cornea [9].

Organization of the paper

The paper is organized as follows. In Section 2 we review the necessary background material, that is toric geometry, quantum homology, and Gromov–Witten invariants. Section 3 is devoted to the case of toric 4–dimensional NEF manifolds where we specify these notions. In Section 4, we precisely state the main theorems enumerating all the contributions to the Seidel morphism and the expression of the Seidel elements (Section 4.1) and we prove them (Section 4.2 to Section 4.4). Finally, we describe explicit examples and applications mentioned in the introduction in Section 5. In Appendix A we gather additional computations of Seidel's elements in more cases, completing Theorem 4.5.

Acknowledgments

2. Toric manifolds and quantum homology

2.1. Toric geometry: the symplectic viewpoint

Recall that a closed symplectic $2m$–dimensional manifold (M, ω) is said to be toric if it is equipped with an effective Hamiltonian action of a m–torus T and with a choice of a corresponding moment map $\Phi : M \to \mathfrak{t}^*$, where $\mathfrak{t}^*$ is the dual of the Lie algebra $\mathfrak{t}$ of the torus T. There is a natural integral lattice $\mathfrak{t}_{\mathbb{Z}}$ in $\mathfrak{t}$ whose elements H exponentiate to circles Λ_H in T, and hence also a dual lattice $\mathfrak{t}_{\mathbb{Z}}^*$ in $\mathfrak{t}^*$. The image $\Phi(M)$ is well-known to be a convex polytope P, called a *Delzant polytope*. It is *simple* (m facets meet at each vertex), *rational* (the conormal vectors $\eta_i \in \mathfrak{t}$ to each facet may be chosen to be primitive and integral), and *smooth* (at each vertex $v \in P$ the conormals to the m facets meeting at v form a basis of the lattice $\mathfrak{t}_{\mathbb{Z}}$). We describe them as follows:

$$P = P(\kappa) := \{x \in \mathfrak{t}^* | \langle \eta_i, x \rangle \leq \kappa_i, \, i = 1, \ldots, n\},$$

where P has n facets $D_1, \ldots, D_n$ with *outward*[2] *primitive integral conormals* $\eta_i \in \mathfrak{t}_{\mathbb{Z}}$ and *support constants* $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n$.

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[2]It seemed more relevant to follow the same convention as in [32] even though the polytope is often defined by $P' = \{x \in \mathfrak{t}^* | \langle \eta_i', x \rangle \geq -\kappa_i, \, i = 1, \ldots, N\}$ for inward normals η_i'.

Delzant proved in [11] that there is a one–to–one correspondence between toric manifolds and Delzant polytopes given by the map that sends the toric manifold (M, ω, T, Φ) to the polytope $\Phi(M)$. (See [27] and the references therein for more details on this background material.)

2.2. The clutching construction

Let (M, ω) be a closed symplectic manifold and $\Lambda = \{\Lambda_\theta\}$ be a loop in $\mathrm{Ham}(M, \omega)$ based at identity. Denote by M_Λ the total space of the fibration over $\mathbb{CP}^1$ with fiber M which consists of two trivial fibrations over 2–discs, glued along their boundary via Λ. Namely, we consider $\mathbb{CP}^1$ as the union of the two 2–discs

$$D_1 = \{[1 : z] \in \mathbb{CP}^1 \,|\, |z| \leq 1\} \quad \text{and} \quad D_2 = \{[z : 1] \in \mathbb{CP}^1 \,|\, |z| \leq 1\}$$

glued along their boundary

$$\partial D_1 = \{[1 : \mathrm{e}^{-2i\pi\theta}], \theta \in [0, 1[\} = \{[\mathrm{e}^{2i\pi\theta} : 1], \theta \in [0, 1[\} = \partial D_2\,.$$

The total space is

$$M_\Lambda = \left(M \times D_1 \bigsqcup M \times D_2\right) \big/ {\sim_\Lambda} \quad \text{with } (x, [1 : \mathrm{e}^{-2i\pi\theta}]) \sim (\Lambda_\theta(x), [\mathrm{e}^{2i\pi\theta} : 1]).$$

This construction only depends on the homotopy class of Λ. Moreover, Ω, the family (parameterized by S^2) of symplectic forms of the fibers, can be "horizontally completed" to give a symplectic form on M_Λ, $\omega_{\Lambda,\kappa} = \Omega + \kappa \cdot \pi^*(\omega_0)$ where ω_0 is the standard symplectic form on S^2 (with area 1), π is the projection to the base of the fibration and κ a big enough constant to make $\omega_{\Lambda,\kappa}$ non-degenerate. (Once chosen, κ will be omitted from the notation.)

So we end up with the following Hamiltonian fibration:

$$(M, \omega) \lhook\joinrel\longrightarrow (M_\Lambda, \omega_\Lambda) \xrightarrow{\;\pi\;} (S^2, \omega_0).$$

In [32], McDuff and Tolman observed that, when Λ is a circle action (with associated moment map Φ_Λ), the clutching construction can be simplified since, then, M_Λ can be seen as the quotient of $M \times S^3$ by the diagonal action of S^1, $e^{2\pi i\theta} \cdot (x, (z_1, z_2)) = (\Lambda_\theta(x), (e^{2\pi i\theta} z_1, e^{2\pi i\theta} z_2))$. The symplectic form also has an alternative description in $M \times_{S^1} S^3$. Let $\alpha \in \Omega^1(S^3)$ be the standard contact form on S^3 such that $d\alpha = \chi^*(\omega_0)$ where $\chi : S^3 \to S^2$ is the Hopf map and ω_0 is the standard area form on S^2 with total area 1. For

all $c \in \mathbb{R}$, $\omega + c d\alpha - d(\Phi_\Lambda \alpha)$ is a closed 2–form on $M \times S^3$ which descends through the projection, $p \colon M \times S^3 \to M \times_{S^1} S^3$, to a closed 2–form on M_Λ:

$$\omega_c = p_*(\omega + c d\alpha - d(\Phi_\Lambda \alpha)) \tag{1}$$

which extends Ω. Now, if $c > \max \Phi_\Lambda$, ω_c is non-degenerate and coincides with $\omega_{\Lambda, \kappa}$ for some big enough κ.

In the case of a toric symplectic manifold fiber, the same arguments lead to the fact that $(M_\Lambda, \omega_\Lambda)$ itself is toric. This fact has been already noticed and used in other instances, e.g. by Gonzáles–Iritani [19, Section 3.2] and Chan–Lau–Leung–Tseng [8, Section 4] in more general settings than what we will need in this paper, so that we only give here the specific statement which we will need, and refer the reader to the aforementioned works for details.

Proposition 2.1. *Let $(M^{2m}, \omega, T, \Phi)$ be a toric symplectic manifold with associated Delzant polytope P. Denote by $(M_\Lambda, \omega_\Lambda)$ the total space resulting from the clutching construction associated to Λ, Hamiltonian circle subgroup of T. Λ admits a representative in T given as the exponential of θb where $\theta \in [0,1]$ and $b \in \mathfrak{t}_\mathbb{Z}$, the lattice of circle subgroups of T.*

Then there exist a $(m+1)$–dimensional torus $T_\Lambda \subset \mathrm{Ham}(M_\Lambda, \omega_\Lambda)$, and a moment map $\Phi_\Lambda \colon M_\Lambda \to \mathfrak{t}_\Lambda^* \simeq \mathfrak{t}^* \times \mathbb{R}$ such that $(M_\Lambda, \omega_\Lambda, T_\Lambda, \Phi_\Lambda)$ is a toric symplectic manifold, whose associated Delzant polytope P_Λ and integral lattice $\mathfrak{t}_\mathbb{Z}^\Lambda$ are given by

$$P_\Lambda = \{(x, x_0) \in (\mathfrak{t} \times \mathbb{R})^* \mid x \in P,\ c' + \langle x, b \rangle \le x_0 \le 0\}$$

and $\mathfrak{t}_\mathbb{Z}^\Lambda = \mathfrak{t}_\mathbb{Z} \times \mathbb{Z} \subset \mathfrak{t} \times \mathbb{R}$ where $c' > \max\{\langle x, b \rangle, x \in P\}$ coincides with the constant c appearing in (1) above.

Moreover, the outward normals of P_Λ, η_Λ, are given in terms of the ones of P, η, as follows: $\eta_\Lambda = \{(\eta_i, 0), \eta_i \in \eta\} \cup \{(0,1), (b,-1)\}$.

The polytopes P and P_Λ are illustrated by Figure 2. The upper and lower facets of P_Λ correspond to two copies of P, the former horizontal, the latter orthogonal to $(b, -1) \in \mathfrak{t}^* \times \mathbb{R}$.

2.3. Toric geometry: the algebraic viewpoint

We now briefly review toric varieties since we will use this viewpoint extensively. Good basic references are Cox–Katz [10] and Batyrev [5]. There is
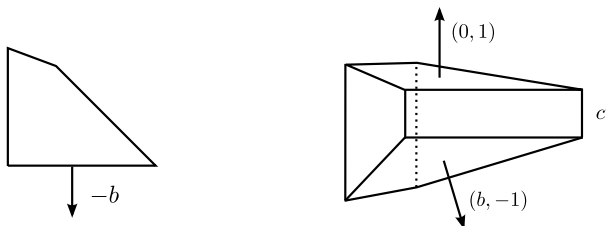
Figure 2: The polytopes associated to the fiber M (left) and to the total space M_Λ (right).

also a good summary of the definition and some properties of smooth toric varieties in Spielberg [37]. In what follows we mainly use his notation.

Let $m > 0$ be a positive integer, $\mathfrak{t}_\mathbb{Z} = \mathbb{Z}^m$ be the m–dimensional integral lattice and $\mathfrak{t}_\mathbb{Z}^* = \mathrm{Hom}(\mathfrak{t}_\mathbb{Z}, \mathbb{Z})$ be its dual space. Moreover, let $\mathfrak{t} = \mathfrak{t}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$ and $\mathfrak{t}^* = \mathfrak{t}_\mathbb{Z}^* \otimes_\mathbb{Z} \mathbb{R}$ be the $\mathbb{R}$–scalar extensions of $\mathfrak{t}_\mathbb{Z}$ and $\mathfrak{t}_\mathbb{Z}^*$, respectively.

A convex subset $\sigma \subset \mathfrak{t}$ is called a *regular k–dimensional cone* $(1 \le k \le m)$ if there exists a $\mathbb{Z}$–basis of $\mathfrak{t}_\mathbb{Z}$, $\{\eta_1, \dots, \eta_m\}$, such that the cone σ is generated by $\eta_1, \dots, \eta_k$. The vectors $\eta_1, \dots, \eta_k \in \mathfrak{t}_\mathbb{Z}$ are the *integral generators of σ*. If σ' is a (proper) face of σ, we will write $\sigma' \prec \sigma$. A finite system $\Sigma = \{\sigma_1, \dots \sigma_s\}$ of regular cones in $\mathfrak{t}$ is called a regular *m–dimensional fan* of cones if any face σ' of a cone $\sigma \in \Sigma$ is in the fan and any intersection of two cones $\sigma_1, \sigma_2 \in \Sigma$ is again in the fan. A fan Σ is called a *complete fan* if $\mathfrak{t} = \cup_i \sigma_i$. The *$k$–skeleton* $\Sigma^{(k)}$ of the fan Σ is the set of all k–dimensional cones in Σ. A subset of the 1–skeleton of a fan is called a *primitive collection* of Σ if it is not the set of generators of a cone in Σ, while any of its proper subset is. We will denote the set of primitive collections of Σ by $\mathcal{P}$.

Suppose the 1–skeleton of Σ is given by $\eta_1, \dots, \eta_d$. Let $z_1, \dots, z_d$ be a set of coordinates in $\mathbb{C}^d$ and let $\iota \colon \mathbb{C}^d \to \mathfrak{t}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ be a linear map such that $\iota(z_i) = \eta_i$. For each primitive collection $P = \{\eta_{i_1}, \dots, \eta_{i_p}\}$ we define a $(d-p)$–dimensional affine subspace in $\mathbb{C}^d$ by

$$A(P) := \{(z_1, \dots, z_d) \in \mathbb{C}^d \mid z_{i_1} = \dots = z_{i_p} = 0\}.$$

Moreover, we define the set $U(\Sigma)$ to be the open algebraic subset of $\mathbb{C}^d$ given by

$$U(\Sigma) = \mathbb{C}^d \setminus \bigcup_{p \in \mathcal{P}} A(P).$$

The map $\iota\colon \mathbb{C}^d \to \mathfrak{t}_\mathbb{C}$ induces a map between tori $(\mathbb{C}^*)^d \to (\mathbb{C}^*)^m$ that we will also call ι. Its kernel, $D(\Sigma) := \ker(\iota\colon (\mathbb{C}^*)^d \to (\mathbb{C}^*)^m)$, is a $(d-m)$–dimensional subtorus. Then the quotient $X_\Sigma := U(\Sigma)/D(\Sigma)$ is called the *toric manifold associated to* Σ. Note that there is a torus of dimension m acting on X_Σ. Moreover, Delzant [11] showed that if X_Σ is a projective simplicial toric variety then it can be constructed as a symplectic quotient and therefore it is endowed with a symplectic form ω (it is also endowed with an action of a m–dimensional torus). From the moment polytope of this symplectic toric manifold it is possible to recover the fan Σ. However, as explained in [6, Part B], changing the cohomology class of the symplectic form corresponds to changing the lengths of the edges of the polytope. The size of the faces of a polytope cannot be recovered from the fan which only encodes the combinatorics of the faces. Hence, the fan does not give the cohomology class of the symplectic form.

Standard results about toric manifolds explain how to obtain the cohomology ring of the toric variety X_Σ. Assume the moment map $\Phi : X_\Sigma \to \mathfrak{t}^*$ is chosen so that each of its components is mean-normalised. Let $P_\Sigma \subset \mathfrak{t}^*$ be the image of the moment map. Let $D_1, \ldots, D_n$ be the facets of P (the codimension–1 faces), and let $\eta_1, \ldots, \eta_n \in \mathfrak{t}_\mathbb{Z}$ denote the outward primitive integral normal vectors. Let C be the set of subsets $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $1 \leq k \leq m$ and $D_{i_1} \cap \cdots \cap D_{i_k} \neq \emptyset$. Consider the two following ideals in $\mathbb{Q}[Z_1, \ldots, Z_n]$:

$$\mathrm{Lin}(\Sigma) = \left\langle \sum (x, \eta_i) Z_i \mid x \in \mathfrak{t}_\mathbb{Z}^* \right\rangle, \quad \mathrm{SR}(\Sigma) = \langle Z_{i_1} \cdots Z_{i_k} \mid \{i_1, \ldots, i_k\} \notin C \rangle.$$

The ideal $\mathrm{Lin}(\Sigma)$ is generated by linear relations and the ideal $\mathrm{SR}(\Sigma)$ is called the *Stanley–Reisner ideal.* A subset $I \subseteq \{1, \ldots, n\}$ is called *primitive* if I is not in C but every proper subset is. Clearly,

$$\mathrm{SR}(\Sigma) = \langle Z_{i_1} \cdots Z_{i_k} \mid \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} \text{ is primitive} \rangle.$$

The map which sends Z_i to the Poincaré dual of $\Phi^{-1}(D_i)$ (which we shall also denote by $Z_i \in H^2(X_\Sigma; \mathbb{Q})$) induces an isomorphism

(2) $$H^*(X_\Sigma; \mathbb{Q}) \cong \mathbb{R}[Z_1, \ldots, Z_n]/\langle \mathrm{Lin}(\Sigma) + \mathrm{SR}(\Sigma) \rangle.$$

Moreover, there is a natural isomorphism between $H_2(X_\Sigma; \mathbb{Z})$ and the set of tuples $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $\sum a_i \eta_i = 0$, under which the pairing between such an element of $H_2(X_\Sigma; \mathbb{Z})$ and Z_i is a_i. The linear functional η_i is constant on D_i and let $\eta_i(D_i)$ denote its value. Under the isomorphism

of (2) we have

$$[\omega] = \sum_i \eta_i(D_i) Z_i \quad \text{and} \quad c_1(X_\Sigma) = \sum_i Z_i. \tag{3}$$

Dually, let $R(\Sigma) \subset \mathbb{Z}^n$ be the subgroup of $\mathbb{Z}^n$ defined by

$$R(\Sigma) := \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n \mid \gamma_1 \eta_1 + \cdots + \gamma_n \eta_n = 0\} \cong \mathbb{Z}^{n-m}. \tag{4}$$

Then the group $R(\Sigma)$ is canonically isomorphic to $H_2(X_\Sigma; \mathbb{Z})$.

2.4. Small quantum homology and Gromov–Witten invariants

Except for our application in terms of the Landau–Ginzburg potential in Section 5, we will work with the (small) quantum homology ring with coefficients in the ring $\Pi := \Pi^{\text{univ}}[q, q^{-1}]$. The variable q is of degree 2 and Π^{univ} is a generalised Laurent series ring in a variable of degree 0:

$$\Pi^{\text{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \mid r_\kappa \in \mathbb{Q}, \ \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}. \tag{5}$$

The quantum homology $QH_*(M; \Pi) = H_*(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \Pi$ is $\mathbb{Z}$–graded so that $\deg(a \otimes q^d t^\kappa) = \deg(a) + 2d$ with $a \in H_*(M)$. The *quantum intersection product* $a * b \in QH_{i+j-\dim M}(M; \Pi)$, of classes $a \in H_i(M)$ and $b \in H_j(M)$ has the form

$$a * b = \sum_{B \in H_2^S(M; \mathbb{Z})} (a * b)_B \otimes q^{-c_1(B)} t^{-\omega(B)},$$

where $H_2^S(M; \mathbb{Z})$ is the image of $\pi_2(M)$ under the Hurewicz map. The homology class $(a * b)_B \in H_{i+j-\dim M+2c_1(B)}(M)$ is defined by the requirement that

$$(a * b)_B \cdot_M c = \mathrm{GW}_{B,3}^M(a, b, c) \quad \text{for all } c \in H_*(M).$$

In this formula $\mathrm{GW}_{B,3}^M(a, b, c) \in \mathbb{Q}$ denotes the Gromov–Witten invariant that counts the number of spheres in M in class B that meet cycles representing the classes $a, b, c \in H_*(M)$. The product $*$ is extended to $QH_*(M)$ by linearity over Π, and is associative. It also respects the $\mathbb{Z}$–grading and gives $QH_*(M)$ the structure of a graded commutative ring, with unit $[M]$.

Gromov–Witten invariants can also be interpreted as homomorphisms

$$\mathrm{GW}_{A,k}^M\colon H^*(M;\mathbb{Q})^{\otimes k} \otimes H_*(\overline{\mathcal{M}}_{0,k};\mathbb{Q}) \longrightarrow \mathbb{Q}$$

$$\mathrm{GW}_{A,k}^M(a_1,\ldots,a_k;\beta) = \int_{\overline{\mathcal{M}}_{0,k}(A;J)} \mathrm{ev}_1^*a_1 \cup \ldots \mathrm{ev}_k^*a_k \cup \pi^*\mathrm{PD}(\beta),$$

where $\overline{\mathcal{M}}_{0,k}(A;J)$ is the compactified moduli space of J–holomorphic spheres with k marked points in M representing the homology class A. Let us recall that in general $\mathrm{GW}_{A,k}^M(a_1,\ldots,a_k)$ is the homomorphism

$$\mathrm{GW}_{A,k}^M\colon H^*(M;\mathbb{Q})^{\otimes k} \to \mathbb{Q}, \quad (a_1,\ldots,a_k) \mapsto \mathrm{GW}_{A,k}^M(a_1,\ldots,a_k;[\overline{\mathcal{M}}_{0,k}])$$

so that when $k = 3$, $\mathrm{GW}_{A,3}^M(a_1,a_2,a_3) = \mathrm{GW}_{A,3}^M(a_1,a_2,a_3;[\mathrm{pt}])$.

For easy reference, we gather here the properties of Gromov–Witten invariants which will be used explicitly at several places in the computations of Section 4: The first two are extracted from [31, Proposition 7.5.6] and the third is the particular case of [31, Theorem 7.5.10] for the invariants $\mathrm{GW}_{A,4}^M(a_1,\ldots,a_4;[\mathrm{pt}]) = \mathrm{GW}_{A,4}^{M,\{1,2,3,4\}}(a_1,\ldots,a_4)$ when $k = 4$ (see [31, Remark 7.5.1.(vi)]).

Proposition 2.2. *Let (M,ω) be a semipositive compact symplectic manifold, $A \in H_2(M;\mathbb{Z})$, $k \geq 1$, and $a_1,\ldots,a_k \in H^*(M;\mathbb{Q})$. Then the following properties hold.*

*(**Divisor**) If $(A,k) \neq (0,3)$ and $\deg(a_k) = 2$ then*

$$\mathrm{GW}_{A,k}^M(a_1,\ldots,a_k) = \mathrm{GW}_{A,k-1}^M(a_1,\ldots,a_{k-1}) \cdot \int_A a_k\,.$$

*(**Zero**) If $k \neq 3$ then $\mathrm{GW}_{0,k}^M = 0$. If $k = 3$ then*

$$\mathrm{GW}_{0,3}^M(a_1,a_2,a_3) = \int_M a_1 \cup a_2 \cup a_3\,.$$

*(**Splitting**) If $k = 4$ then $\mathrm{GW}_{A,4}^M(a_1,\ldots,a_4;[\mathrm{pt}])$ is equal to*

$$\sum_{A=A_0+A_1} \sum_{\nu,\mu} \mathrm{GW}_{A_0,3}^M(a_1,a_2,e_\nu)\, g^{\nu\mu}\, \mathrm{GW}_{A_1,3}^M(e_\mu,a_3,a_4)$$

where $(e_\nu)_\nu$ is a basis of $H^*(M;\mathbb{Q})$, $g_{\nu\mu}$ are the coefficients of the cup-product matrix: $g_{\nu\mu} = \int_M e_\nu \cup e_\mu$, and $g^{\nu\mu}$ the coefficients of its inverse.

2.5. Gromov–Witten invariants of toric manifolds

In this section we present Spielberg's formula from [38, Theorem 1] for the computation of Gromov–Witten invariants of toric manifolds, which we will use in Section 3.2. Note that Liu proved a more general result in [29], however since we only need to compute genus–0 Gromov–Witten invariants we will use Spielberg's formulation and notation.

Definition 2.3. [37, Definition 6.4] Let Σ be a complete regular fan in $\mathbb{Z}^m$ and let P_Σ be its dual polytope. A graph Γ is a finite 1–dimensional CW–complex with the following decorations:

1. A map $\sigma : \mathrm{Vert}(\Gamma) \to \Sigma^{(m)}$ mapping each vertex $\mathfrak{b}$ of the graph to a vertex $\sigma(\mathfrak{b})$ of P_Σ;

2. A map $d : \mathrm{Edge}(\Gamma) \to \mathbb{Z}_{>0}$ representing multiplicities of maps;

3. A map $S : \mathrm{Vert}(\Gamma) \to \mathfrak{B}(\{1,\ldots,p\})$ associating to each vertex a set of marked points.

These decorations are subject to the following compatibility conditions:

(a) If an edge $e \in \mathrm{Edge}(\Gamma)$ connects two vertices $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathrm{Vert}(\Gamma)$ labeled $\sigma(\mathfrak{b}_1)$ and $\sigma(\mathfrak{b}_2)$, then the two cones must be different and have a common $(m-1)$–dimensional face: $\sigma(\mathfrak{b}_1) \cap \sigma(\mathfrak{b}_2) \in \Sigma^{(m-1)}$;

(b) The graph represents a stable map with homology class A;

(c) The CW–complex Γ contains no loops;

(d) For any two vertices $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathrm{Vert}(\Gamma)$, the sets of associated marked points are disjoint: $S(\mathfrak{b}_1) \cap S(\mathfrak{b}_2) = \emptyset$;

(e) Every marked point is associated with some vertex.

The following notation will be useful to understand the statement of the theorem. We define the following subset of Vert:

$$\mathrm{Vert}_{t,s} := \left\{ \mathfrak{b} \in \mathrm{Vert} \,\middle|\, \mathrm{val}(\mathfrak{b}) = t,\ \deg(\mathfrak{b}) = t + s \right\},$$

where $\mathrm{val} : \mathrm{Vert}(\Gamma) \to \mathbb{Z}_{>0}$ is the function assigning to each vertex the number of outgoing edges and $\deg : \mathrm{Vert}(\Gamma) \to \mathbb{Z}_{>0}$ assigns to each vertex the number of its special points:

$$\deg(\mathfrak{b}) = \#S(\mathfrak{b}) + \#\{e \in \mathrm{Edge}(\Gamma) \,|\, \mathfrak{b} \in \partial e\}$$

where $\partial e = \{\mathfrak{b}_1(e), \mathfrak{b}_2(e)\}$ associates to an edge e the two vertices $\mathfrak{b}_1(e), \mathfrak{b}_2(e)$ it connects.

We also need the following result:

Lemma 2.4. *[37, Lemma 6.7] Let $\sigma_1, \sigma_2 \in \Sigma^{(m)}$ be two m–cones in Σ that have a common $(m-1)$–face $\tau \in \Sigma^{(m-1)}$. Let $\eta_{i_1}, \ldots, \eta_{i_{m-1}}$ be the generators of the common face τ, such that*

$$\sigma_1 = \langle \eta_{i_1}, \ldots, \eta_{i_{m-1}}, \eta_{\tau(1)} \rangle \quad and \quad \sigma_2 = \langle \eta_{i_1}, \ldots, \eta_{i_{m-1}}, \eta_{\tau(2)} \rangle.$$

Let $\omega_1, \ldots, \omega_n$ be the weights of a diagonal action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ with respect to the standard basis. The induced $\mathbb{C}^*$–action on the invariant 2–sphere V_τ has weight $\omega_{\sigma_2}^{\sigma_1}$ at the point V_{σ_1} given by

$$\omega_{\sigma_2}^{\sigma_1} := \sum_{\ell=1}^{n} \langle \eta_\ell, u_m \rangle \omega_\ell,$$

where $\{u_1, \ldots, u_m\}$ is a basis of $\mathfrak{t}_{\mathbb{Z}}^*$ dual to $\left\{ \eta_{i_1}, \ldots, \eta_{i_{m-1}}, \eta_{\tau(1)} \right\}$.

Corollary 2.5. *[37, Corollary 6.8] Let $e \in \mathrm{Edge}(\Gamma)$ and $\mathfrak{b}_1, \mathfrak{b}_2 \in \partial e$ be the vertices at its two ends. Let $\sigma_i = \sigma(\mathfrak{b}_i)$ be the m–cones of the vertices $\mathfrak{b}_i$ and $\tau = \sigma_1 \cap \sigma_2$ its common $(m-1)$–face, that are generated as in the Lemma above. For a stable map $(C; x_1, \ldots, x_p; f)$ fixed by the torus action, let C_e be the irreducible component of C corresponding to the edge e. Let $F := (\mathfrak{b}_1, e) \in \mathrm{Vert}(\Gamma) \times \mathrm{Edge}(\Gamma)$ be such that $\mathfrak{b}_1 \in \partial e$. At the point*

$$p_F := f^{-1}(V_{\sigma(\mathfrak{b}_1)}) \cap C_e,$$

the pull back to C_e of the torus action on V_τ has the weight ω_F at p_F:

$$\omega_F := \frac{1}{d_e} \sum_{\ell=1}^{n} \langle \eta_\ell, u_m \rangle \omega_\ell,$$

where d_e is the multiplicity of the component C_e and the vectors u_i are as in the lemma above.

We will introduce some more notation, grouping together certain weights on a graph Γ. We will write $\sigma_1 \diamond \sigma_2$ for the property of σ_1 and σ_2 having a

common $(m-1)$–dimensional proper face:

$$\sigma_1 \diamond \sigma_2 \Longleftrightarrow \sigma_1, \sigma_2 \in \Sigma^{(m)} \text{ and } \sigma_1 \cap \sigma_2 \in \Sigma^{(m-1)}.$$

The *total weight of a m–dimensional cone* σ is defined to be

$$\omega_{\text{total}}^{\sigma} := \prod_{\alpha \,:\, \alpha \diamond \sigma} \omega_{\alpha}^{\sigma}.$$

Finally, let $\alpha \in \Sigma^{(m)}$ be a m–cone in the fan Σ that has a common $(m-1)$–face τ with $\sigma_1 : \alpha \diamond \sigma_1$. Then α and σ_1 have $(m-1)$ generators in common; let $\eta_{i_\alpha} \in \Sigma^{(1)}$ be the generator of σ_1 that is not a generator of α. We then set $\lambda_e^\alpha := \gamma_{i_\alpha}$, where $(\gamma_1, \ldots, \gamma_n)$ represents the homology class of τ (see (4)).

Since we are interested only in 1–point Gromov–Witten invariants we will give a simplified version of Spielberg's formula.

Theorem 2.6. *[38, Theorem 1] The 1–point genus–0 Gromov–Witten invariants for a toric variety X_Σ are given by*

$$GW_{A,1}^{X_\Sigma}(Z_\ell) = \sum_{\Gamma} \frac{1}{|A_\Gamma|} T_\Gamma \cdot S_\Gamma$$

where A_Γ is the automorphism group of the graph Γ,

$$T_\Gamma = \prod_{t=1}^{\infty} \prod_{\mathfrak{b} \in \text{Vert}_{t,*}(\Gamma)} (\omega_{\text{total}}^{\sigma(\mathfrak{b})})^{t-1} \cdot \left(\prod_{i=1}^{t} \frac{1}{\omega_{F_i(\mathfrak{b})}} \right) \cdot \left(\frac{1}{\omega_{F_1(\mathfrak{b})}} + \cdots + \frac{1}{\omega_{F_t(\mathfrak{b})}} \right)^{t-3}$$

$$\cdot \prod_{\substack{e \in \text{Edge} \\ \partial e = \{\mathfrak{b}_1, \mathfrak{b}_2\}}} \left(\frac{(-1)^m m^{2m}}{(m!)^2 (\omega_{\sigma_2}^{\sigma_1})^{2m}} \prod_{\substack{\alpha \,:\, \alpha \neq \sigma_2 \\ \text{and } \alpha \diamond \sigma_1}} \frac{\prod\limits_{i=\lambda_e^\alpha+1}^{-1} \left(\omega_\alpha^{\sigma_1} - \frac{i}{m} \cdot \omega_{\sigma_2}^{\sigma_1} \right)}{\prod\limits_{i=0}^{\lambda_e^\alpha} \left(\omega_\alpha^{\sigma_1} - \frac{i}{m} \cdot \omega_{\sigma_2}^{\sigma_1} \right)} \right)$$

$$S_\Gamma = \left[\prod_{t,s} \prod_{\mathfrak{b} \in \text{Vert}_{t,s}(\Gamma)} \left(\frac{1}{\omega_{F_1(\mathfrak{b})}} + \cdots + \frac{1}{\omega_{F_t(\mathfrak{b})}} \right)^s \right] \cdot \prod_{k=1}^{n} (\omega_k^{\sigma(1)})^{l_k}$$

and where

- *we use the convention $0^0 = 1$;*
- $Z^l = Z_1^{l_1} \cdots Z_n^{l_n};$

- $\sigma(1)$ *is the fixed point the marked point is mapped to;*

- *we define* $\omega_k^{\sigma(1)} := \begin{cases} 0 & \text{if } \eta_k \notin \Sigma_{\sigma(1)}^{(1)}, \\ \omega_\alpha^{\sigma(1)} & \text{if } \alpha \diamond \sigma(1) \text{ and } \eta_k \in \Sigma_{\sigma(1)}^{(1)} \backslash \Sigma_\alpha^{(1)}. \end{cases}$

3. Toric 4–dimensional NEF manifolds

Now we restrict ourselves to the case of toric 4–dimensional NEF manifolds. We explain the construction of M_Λ and its properties including its cohomology ring. This will play a very important role in the next section.

3.1. Toric and homological data

We consider a 4–dimensional toric manifold (M, ω, T, Φ) and its moment 2–dimensional Delzant polytope P. Assume it has n facets that we denote by D_i, $i = 1, \ldots, n$. Let $v_1, \ldots, v_n$ denote the outward primitive integral normal vectors and let Λ_i denote the circle action corresponding to v_i, that is, Λ_i is the circle action whose moment map is given by $\Phi_{\Lambda_i} := \langle v_i, \Phi(\cdot) \rangle$.

We pick a ω–tame almost complex structure J and denote by $c_1(M)$ the first Chern class of (TM, J). We assume that (M, J) is NEF, that is $\langle c_1(M), B \rangle \geq 0$ for every class $B \in H_2(M, \mathbb{Z})$ with a J–pseudo-holomorphic representative.

Moreover, we consider the particular case when there are at most 2 (consecutive) facets corresponding to spheres with vanishing first Chern number and assume their normal vectors are v_n and v_1 (recall that we denote v_{n+1} by v_1 as for the $D_i's$). Since the polytope P is Delzant we can assume that the facets D_{n-1} and D_n are perpendicular. Moreover, as explained in [16, Section 2.5], the vectors v_i satisfy the relations

$$(6) \qquad\qquad v_{i-1} + v_{i+1} = d_i v_i,$$

where $-d_i = D_i \cdot D_i$ denotes the self-intersection of the facet D_i. Since the first Chern number vanishes on the facets D_n and D_1 it follows that $D_n \cdot D_n = D_1 \cdot D_1 = -2$. Therefore we can assume that the vectors v_i satisfy the following relations:

$$(7) \qquad v_{n-1} = -e_2, \quad v_n = -e_1, \quad v_1 = e_2 - 2e_1 \quad \text{and} \quad v_2 = 2e_2 - 3e_1,$$

where the vectors e_1, e_2 form the canonical basis of $\mathbb{Z}^2$.

Next, using the clutching construction described in Section 2.2, we construct the manifold M_{Λ_n} associated to the loop Λ_n which we will denote

simply by M_Λ in order to simplify the notation. As we noticed in Proposition 2.1, M_Λ is a toric manifold with moment map Φ_Λ. The moment image is a 3–dimensional polytope P_Λ with $n+2$ facets which we denote by $D_1^\Lambda, \ldots, D_n^\Lambda, D_b^\Lambda, D_t^\Lambda$ with corresponding outward primitive integral normal vectors $\eta_1, \ldots, \eta_n, \eta_b, \eta_t$. $D_1^\Lambda, \ldots, D_n^\Lambda$ are the vertical facets of P_Λ "coming from" the facets of P, while D_b^Λ and D_t^Λ are respectively the bottom and top facets. Note that the vectors $\eta_1, \ldots, \eta_n$ are induced by the normal vectors $v_1, \ldots, v_n$. More precisely, $\eta_i = (v_i, 0)$ with $i = 1, \ldots, n$. It follows from (7) together with the clutching construction that the vectors η_i satisfy the following relations

$$
\begin{array}{lll}
\eta_{n-1} = -e_2 & \eta_3 = \alpha_3 e_1 + \beta_3 e_2 & \eta_b = -e_1 - e_3 \\
\eta_n = -e_1 & \cdots & \eta_t = e_3 \\
\eta_1 = e_2 - 2e_1 & \eta_j = \alpha_j e_1 + \beta_j e_2 & \\
\eta_2 = 2e_2 - 3e_1 & \cdots &
\end{array}
$$

where now the vectors e_1, e_2, e_3 form the canonical basis of $\mathbb{Z}^3$. Clearly, it follows from the definition of η_i, with $i = 1, \ldots, n$, together with (6) that

$$
(8) \qquad\qquad\qquad \eta_{i-1} + \eta_{i+1} = d_i \eta_i.
$$

Example 3.1. Consider the second Hirzebruch surface, with a polytope with normal (outward) vectors $(0, -1), (-1, 0), (-2, 1), (1, 0)$ where the facet normal to $(-1, 0)$ corresponds to a curve of zero Chern number (in this example there is only one facet where the first Chern number vanishes). In this case the vectors η_i are the following: $\eta_1 = (-2, 1, 0), \eta_2 = (1, 0, 0), \eta_3 = (0, -1, 0), \eta_4 = (-1, 0, 0), \eta_b = (-1, 0, -1), \eta_t = (0, 0, 1)$.

The vertical facets of P_Λ and the corresponding outward normals are represented in Figure 3. Note that the polytope is closed, but in Figure 3 we only draw the facets in which we are interested.

The manifold M_Λ is 6–dimensional, hence its fan Σ lives in the lattice $\mathbb{Z}^3$. Then the 1–dimensional cones of the fan Σ are generated by the vectors η_i defined above. The set of primitive collections of the fan Σ is given by the following set:

$$
\mathcal{P} = \big\{ \{\eta_1, \eta_3\}, \ldots, \{\eta_1, \eta_{n-1}\}, \{\eta_2, \eta_4\}, \ldots, \{\eta_2, \eta_n\}, \{\eta_3, \eta_5\},
$$
$$
\ldots, \{\eta_3, \eta_n\}, \{\eta_{n-2}, \eta_n\}, \{\eta_b, \eta_t\} \big\}.
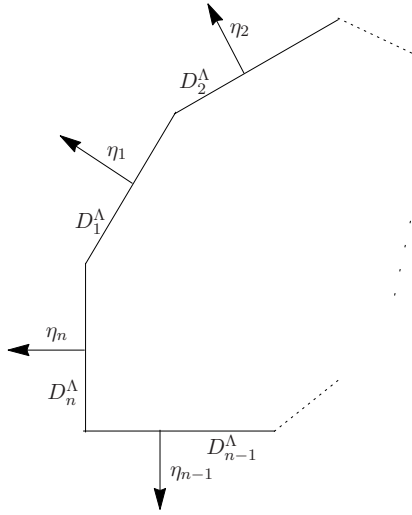$$

Figure 3: Some vertical facets of the polytope P_Λ and their outward normals.

From (2) it follows that the cohomology ring of M_Λ is given by the following isomorphism:

$$H^*(M_\Lambda; \mathbb{Q}) \cong \mathbb{Q}[Z_1, \dots, Z_n, Z_b, Z_t]/\langle \mathrm{Lin}(\Sigma) + \mathrm{SR}(\Sigma)\rangle$$

where $\mathrm{SR}(\Sigma)$ is the Stanley–Reisner ideal of Σ and $\mathrm{Lin}(\Sigma)$ is the ideal generated by the linear relations. The former is generated by the set of primitive collections:

$$(9) \quad Z_1 Z_3, \dots, Z_1 Z_{n-1}, Z_2 Z_4, \dots, Z_2 Z_n, Z_3 Z_5, \dots, Z_3 Z_n, \dots,$$
$$Z_{n-3} Z_{n-1}, Z_{n-3} Z_n, Z_{n-2} Z_n \ \text{ and } \ Z_b Z_t,$$

while the ideal $\mathrm{Lin}(\Sigma)$ is generated by the following three elements:

$$(10) \qquad Z_n + 2Z_1 + 3Z_2 - \alpha_3 Z_3 - \cdots - \alpha_{n-2} Z_{n-2} + Z_b,$$
$$(11) \qquad Z_{n-1} - Z_1 - 2Z_2 - \beta_3 Z_3 - \cdots - \beta_{n-2} Z_{n-2}, \text{ and}$$
$$(12) \qquad Z_t - Z_b.$$

In view of the relations (10)–(12), Z_{n-1}, Z_n and Z_t are linear combinations of the others, so that the set $\{Z_1, \dots, Z_{n-2}, Z_b\}$ is a basis of the degree 2 part of the cohomology ring. The degree 2 homology $H_2(M_\Lambda; \mathbb{Z})$ can be identified

with the group $R(\Sigma) \subset \mathbb{Z}^{n+2}$ given by

$$R(\Sigma) := \{(\gamma_1, \ldots, \gamma_n, \gamma_b, \gamma_t) \in \mathbb{Z}^{n+2} \mid \gamma_1 \eta_1 + \cdots + \gamma_n \eta_n + \gamma_b \eta_b + \gamma_t \eta_t = 0\},$$

where we identify η_b, η_t with η_{n+1}, η_{n+2} respectively. If follows from the definition of the vectors η_i that a basis for the degree 2 homology, $H_2(M_\Lambda; \mathbb{Z})$, can be given by the set $\{\lambda_1, \ldots, \lambda_{n-2}, \lambda_b\}$ which is dual to the basis of the degree 2 cohomology, that is, $Z_i(\lambda_j) = 1$ if $i = j$ and 0 otherwise. More precisely, the generators are given by

$$\lambda_1 = (1, 0, \ldots, 0, 1, -2, 0, 0), \qquad \lambda_2 = (0, 1, 0, \ldots, 0, 2, -3, 0, 0),$$
$$\lambda_j = (0, \ldots 0, 1, 0, \ldots, 0, \beta_j, \alpha_j, 0, 0), \quad j = 3, \ldots, n-2, \text{ and}$$
$$\lambda_b = (0, \ldots, 0, -1, 1, 1),$$

where the entry 1 in λ_j is located at the j–th entry.

From the description of the set of primitive collections, it is easy to get the set of maximal cones in Σ. Next we list some 3–dimensional cones (the ones that are going to be relevant for our computations):

$$\sigma_1 = \langle \eta_{n-2}, \eta_{n-1}, \eta_b \rangle \qquad \sigma_5 = \langle \eta_2, \eta_3, \eta_b \rangle \qquad \sigma_9 = \langle \eta_1, \eta_2, \eta_t \rangle$$
$$\sigma_2 = \langle \eta_{n-1}, \eta_n, \eta_b \rangle \qquad \sigma_6 = \langle \eta_{n-2}, \eta_{n-1}, \eta_t \rangle \qquad \sigma_{10} = \langle \eta_2, \eta_3, \eta_t \rangle$$
$$\sigma_3 = \langle \eta_1, \eta_n, \eta_b \rangle \qquad \sigma_7 = \langle \eta_{n-1}, \eta_n, \eta_t \rangle$$
$$\sigma_4 = \langle \eta_1, \eta_2, \eta_b \rangle \qquad \sigma_8 = \langle \eta_1, \eta_n, \eta_t \rangle$$

Consider now, for example, the invariant 2–sphere $V_{\sigma_2 \cap \sigma_3}$, connecting the fixed points corresponding to σ_2 and σ_3. Since $\sigma_2 = \langle \eta_{n-1}, \eta_n, \eta_b \rangle$ and $\sigma_3 = \langle \eta_1, \eta_n, \eta_b \rangle$, the homology class of $V_{\sigma_2 \cap \sigma_3}$ is Poincaré dual to $Z_n Z_b$. Hence the primitive relations yield

$$\langle Z_1, V_{\sigma_2 \cap \sigma_3} \rangle = Z_1 Z_n Z_b = Z_1 Z_2 Z_b, \quad \langle Z_b, V_{\sigma_2 \cap \sigma_3} \rangle = 0,$$
$$\langle Z_2, V_{\sigma_2 \cap \sigma_3} \rangle = 0, \quad \ldots \quad \langle Z_{n-2}, V_{\sigma_2 \cap \sigma_3} \rangle = 0.$$

Since $\{Z_1, \ldots, Z_{n-2}, Z_b\}$ is dual to $\{\lambda_1, \ldots, \lambda_{n-2}, \lambda_b\}$, it follows that $V_{\sigma_2 \cap \sigma_3} = \lambda_1$. For another example, consider the homology class of $V_{\sigma_4 \cap \sigma_5}$ which is Poincaré dual to $Z_2 Z_b$. Since $\eta_1 + \eta_3 = d_2 \eta_2$ (see (8)) it follows that $2\alpha_3 + 3\beta_3 = 1$ and $\alpha_3 + 2\beta_3 = d_2$. Using (10) and (11) one obtains

$$\langle Z_1, V_{\sigma_4 \cap \sigma_5} \rangle = Z_1 Z_2 Z_b, \langle Z_2, V_{\sigma_4 \cap \sigma_5} \rangle = Z_2^2 Z_b = -d_2 Z_1 Z_2 Z_b, \langle Z_b, V_{\sigma_4 \cap \sigma_5} \rangle = 0,$$
$$\langle Z_3, V_{\sigma_4 \cap \sigma_5} \rangle = Z_2 Z_3 Z_b = Z_1 Z_2 Z_b, \quad \langle Z_4, V_{\sigma_2 \cap \sigma_3} \rangle = 0, \quad \ldots \quad \langle Z_{n-2}, V_{\sigma_4 \cap \sigma_5} \rangle = 0.$$

Therefore $V_{\sigma_4 \cap \sigma_5} = \lambda_1 - d_2 \lambda_2 + \lambda_3$. Calculations of the homology classes of the other invariant spheres are similar. Moreover, it is not hard to check

that the ones not identified in the diagram of Figure 4, all include contributions of generators λ_i distinct from λ_1, λ_2, and λ_b.
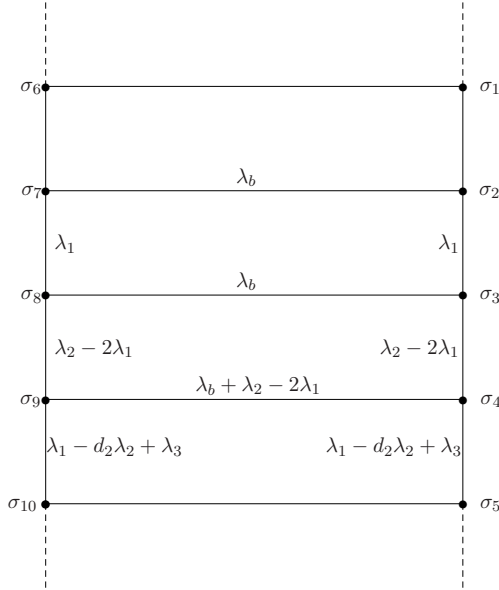

Figure 4: Diagram representing some invariant 2–spheres of the toric manifold M_Λ and their homology classes.

Let $A_i \in H_*(M; \mathbb{Z})$ with $i = 1, \ldots, n$ denote the homology class of the pre–image under the moment map Φ of the facet D_i. Since M_Λ is the total space of a fibration with fiber M, these homology classes can be identified with some invariant 2–spheres in M_Λ, $V_{\sigma_j \cap \sigma_k}$. More precisely, we have $A_n = \lambda_1$, $A_1 = \lambda_2 - 2\lambda_1$. Let[3] $A_{\max} = \lambda_b = V_{\sigma_3 \cap \sigma_8} = V_{\sigma_2 \cap \sigma_7}$. Since

$$c_1(M_\Lambda) = Z_1 + \cdots + Z_n + Z_b + Z_t,$$

where $c_1(M_\Lambda)$ is the first Chern class of the tangent bundle of M_Λ, it follows easily that $\langle c_1(M_\Lambda), \lambda_1 \rangle = \langle c_1(M_\Lambda), \lambda_2 \rangle = 0$ and $\langle c_1(M_\Lambda), \lambda_b \rangle = 1$. Therefore we have $\langle c_1(M_\Lambda), A_n \rangle = \langle c_1(M_\Lambda), A_1 \rangle = 0$ and $\langle c_1(M_\Lambda), A_{\max} \rangle = 1$.

As we shall see in Section 4.1, in order to compute certain Gromov–Witten invariants we will need to know some more information about the ring structure of the cohomology of M_Λ, namely certain relations satisfied by

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[3]The notation $A_{\max}$ is due to the fact that this is the homology class of a section of M_Λ through points on the maximal fixed point component of the action (prior to the clutching construction).

the coefficients of the cup-product matrix $G = (g_{\nu\mu})_{\nu\mu}$, with $g_{\nu\mu} = \int_{M_\Lambda} e_\nu \cup e_\mu$ (for some basis $(e_\nu)_\nu$ of the cohomology ring), and its inverse, $G^{-1} = (g^{\nu\mu})_{\nu\mu}$.

By noticing that the cohomology of M_Λ is non-zero only in even degrees, that the degree 0 and degree 6 groups are 1–dimensional (respectively generated by $\mathbb{1}$ and the fundamental class of M_Λ, $[M_\Lambda]$), and that $g_{\nu\mu} \neq 0$ only if the degrees of e_ν and e_μ sum up to 6, it is easy to see that, *as soon as $(e_\nu)_\nu$ is ordered so that the degree increases*, G decomposes as

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & & & & 0 \\
 & \mathbf{0} & \mathbf{B} & & \\
 & \mathbf{B}^T & \mathbf{0} & & \\
0 & & & & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$

with $\mathbf{B}$ the matrix composed of the $(g_{\nu\mu})_{|e_\nu|=2,|e_\mu|=4}$.

Now, let us specify the basis. Recall that the set $\{Z_1, \ldots, Z_{n-2}, Z_b\}$ is a basis of the degree 2 part of the cohomology. Notice that by (9) and (12) we have $Z_b^2 = 0$. Then the degree 4 part of the cohomology consists of all products $Z_i Z_j$ and $Z_i Z_b$ with $1 \leq i \leq j \leq n-2$. In view of the relations coming from $SR(\Sigma)$, $Z_1 Z_j = 0$ for $3 \leq j \leq Z_{n-2}$. Then, multiplying (11) by Z_1 immediately leads to the relations $Z_1^2 + 2Z_1 Z_2 = 0$. Hence, for $i = 1$, only $Z_1 Z_2$ and $Z_1 Z_b$ need to be considered. Recall that we have $2\alpha_3 + 3\beta_3 = 1$ and $\alpha_3 + 2\beta_3 = d_2$ as seen above. Then multiplying (10) and (11) by Z_2 gives $Z_2 Z_3 = Z_1 Z_2 + 2Z_2 Z_b$ and $Z_2^2 = -d_2 Z_1 Z_2 + (1 - 2d_2)Z_2 Z_b$. Thus for $i = 2$ we only have to consider $Z_2 Z_b$. Hence, we can explicitly write some part of $\mathbf{B}$:

(13)

	$Z_1 Z_2$	$Z_1 Z_b$	$Z_2 Z_b$	$\ldots \ldots$
Z_1	-2	-2	1	$0 — 0$
Z_2	1	1	$-d_2$	
Z_3	0	0	1	
Z_4	0	0	0	
$\mid$	0	0	0	
Z_{n-2}	0	0	0	
Z_b	1	0	0	

Indeed, the vanishing terms come from the relations given by the ideal $SR(\Sigma)$, while the non-zero terms can be computed using the definition. For example, since $Z_1 Z_b$ is Poincaré dual to $V_{\sigma_3 \cap \sigma_4} = \lambda_2 - 2\lambda_1$ (see Figure 4), it

follows that $Z_1 Z_2 Z_b$ is given by

$$\int_{M_\Lambda} Z_1 Z_2 Z_b = Z_2(\lambda_2 - 2\lambda_1) = 1.$$

Using this computation together with the relations given by the ideals $\mathrm{SR}(\Sigma)$ and $\mathrm{Lin}(\Sigma)$ we can obtain the other non-vanishing terms.

In order to simplify the notation, we will denote $g_{\nu\mu}$ and $g^{\nu\mu}$ by using the indices of the corresponding elements e_ν and e_μ. For example, for $e_\nu = Z_1$ and $e_\mu = Z_2 Z_b$, $g_{\nu\mu}$ will be denoted $g_{1,2b}$ and $g^{\nu\mu}$ will be denoted $g^{1,2b}$. Of course G and G^{-1} are symmetric so that $g_{\nu\mu} = g_{\mu\nu}$ and $g^{\nu\mu} = g^{\mu\nu}$. Moreover, note that by commutativity of the cup-product, permuting the indices does not change the value $g_{1,2b} = g_{b,12} = g_{2,1b}$. However, this fails for the coefficients of G^{-1}.

Since $G^{-1}G = \mathbb{1}$, we get relations between the coefficients of G and G^{-1} by multiplying particular lines of G^{-1} with columns of G. For example,

$$\sum_\nu g^{1b,\nu} g_{\nu,1b} = 1 \Longleftrightarrow -2g^{1b,1} + g^{1b,2} = 1$$

$$\sum_\nu g^{1b,\nu} g_{\nu,12} = 0 \Longleftrightarrow -2g^{1b,1} + g^{1b,2} + g^{1b,b} = 0$$

$$\sum_\nu g^{1b,\nu} g_{\nu,2b} = 0 \Longleftrightarrow g^{1b,1} - d_2 g^{1b,2} + g^{1b,3} = 0$$

which lead to the fact that $g^{1b,b} = -1$. By using the lines of G^{-1} corresponding to $Z_1 Z_2$, $Z_2 Z_b$, and again the columns of G corresponding to $Z_1 Z_2$, $Z_1 Z_b$, and $Z_2 Z_b$, we get some more relations between the coefficients of the matrix G^{-1}. We gather in the next lemma the result of these computations.

Lemma 3.2 (Some coefficients of G^{-1}).

$$\begin{cases} g^{1b,b} = -1 \\ g^{1b,2} - 2g^{1b,1} = 1 \\ g^{1b,1} - d_2 g^{1b,2} + g^{1b,3} = 0 \end{cases} \qquad \begin{cases} g^{12,b} = 1 \\ g^{12,2} = 2g^{12,1} \\ g^{12,1} - d_2 g^{12,2} + g^{12,3} = 0 \end{cases}$$

$$\begin{cases} g^{2b,b} = 0 \\ g^{2b,2} = 2g^{2b,1} \\ g^{2b,1} - d_2 g^{2b,2} + g^{2b,3} = 1 \end{cases}$$

3.2. Gromov–Witten invariants

We now compute some Gromov–Witten invariants of M_Λ using Spielberg's machinery from [38]. In particular we will use a simplified version of its main theorem which we give in Section 2.5.

We need to know the weights of the torus action at the different charts. By general theory each 3–dimensional cone gives a chart of the toric manifold near a fixed point. For our calculations it will be convenient to know the following weights, which we compute using Lemma 2.4.

$\sigma_2 = \langle \eta_{n-1}, \eta_n, \eta_b \rangle$	$\sigma_7 = \langle \eta_{n-1}, \eta_n, \eta_t \rangle$
$\omega_{\sigma_3}^{\sigma_2} = a_1$	$\omega_{\sigma_8}^{\sigma_7} = a_1$
$\omega_{\sigma_1}^{\sigma_2} = a_2 + \omega_t$	$\omega_{\sigma_6}^{\sigma_7} = a_2 + \omega_b$
$\omega_{\sigma_7}^{\sigma_2} = \omega_b - \omega_t$	$\omega_{\sigma_2}^{\sigma_7} = \omega_t - \omega_b$
$\sigma_3 = \langle \eta_1, \eta_n, \eta_b \rangle$	$\sigma_8 = \langle \eta_1, \eta_n, \eta_t \rangle$
$\omega_{\sigma_2}^{\sigma_3} = -a_1$	$\omega_{\sigma_7}^{\sigma_8} = -a_1$
$\omega_{\sigma_4}^{\sigma_3} = 2a_1 + a_2 + \omega_t$	$\omega_{\sigma_9}^{\sigma_8} = 2a_1 + a_2 + \omega_b$
$\omega_{\sigma_8}^{\sigma_3} = \omega_b - \omega_t$	$\omega_{\sigma_3}^{\sigma_8} = \omega_t - \omega_b$
$\sigma_4 = \langle \eta_1, \eta_2, \eta_b \rangle$	$\sigma_9 = \langle \eta_1, \eta_2, \eta_t \rangle$
$\omega_{\sigma_3}^{\sigma_4} = -2a_1 - a_2 - \omega_t$	$\omega_{\sigma_8}^{\sigma_9} = -2a_1 - a_2 - \omega_b$
$\omega_{\sigma_5}^{\sigma_4} = 3a_1 + 2a_2 + 2\omega_t$	$\omega_{\sigma_{10}}^{\sigma_9} = 3a_1 + 2a_2 + 2\omega_b$
$\omega_{\sigma_9}^{\sigma_4} = \omega_b - \omega_t$	$\omega_{\sigma_4}^{\sigma_9} = \omega_t - \omega_b$

where the $a_1, a_2 \in \mathbb{Z}$ are linear functions on the weights $\omega_1, \ldots, \omega_n$. Now we are ready to begin calculating Gromov–Witten invariants of this manifold. In the next lemma we will compute some invariants which will be needed later in the proof of Theorem 4.6.

Lemma 3.3 (Gromov–Witten invariants).

$$\mathrm{GW}_{A_{\max}+A_n,1}^{M_\Lambda}(Z_i Z_j) = \mathrm{GW}_{A_{\max}+A_n+A_1,1}^{M_\Lambda}(Z_i Z_j) = \begin{cases} 1 & \text{if } i=1, j=b \\ 0 & \text{if } i=1, j=2 \\ 0 & \text{if } i=2, j=b \end{cases}$$

$$\text{and} \quad \mathrm{GW}_{A_{\max}+A_1,1}^{M_\Lambda}(Z_i Z_j) = \begin{cases} 2 & \text{if } i=1, j=b \\ 2 & \text{if } i=1, j=2 \\ -1 & \text{if } i=2, j=b \end{cases}$$

Proof. We first compute the invariant $\mathrm{GW}_{A_{\max}+A_n+A_1,1}^{M_\Lambda}(Z_1 Z_b)$. We use the formula from Section 2.5. Since the marked point has to lie in the cone σ_3 or σ_4, we need to consider the graphs which contain one of these cones and

which represent the class $A_{\max} + A_n + A_1$. It follows that we should consider the following graphs:
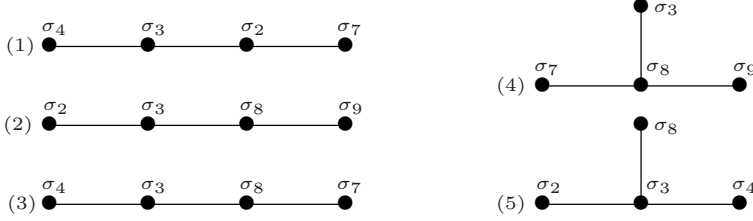
Therefore Theorem 2.6 gives the following computation

$$\mathrm{GW}^{M_\Lambda}_{A_{\max}+A_n+A_1,1}(Z_1Z_b) = (1) + \cdots + (5) = -\frac{(a_1 + a_2 + \omega_t)(a_1 + a_2 + \omega_b)}{(\omega_b - \omega_t)^2}$$

$$-\frac{(a_1 + a_2 + \omega_t)(a_2 + \omega_t)}{a_1(a_1 + \omega_b - \omega_t)} - \frac{(a_1 + 2(\omega_b - \omega_t))(a_1 + a_2 + \omega_t)(a_1 + a_2 + \omega_b)}{(\omega_b - \omega_t)^2(a_1 + \omega_b - \omega_t)}$$

$$+\frac{(a_1 + a_2 + \omega_b)^2}{(\omega_b - \omega_t)^2} + \frac{(a_1 + a_2 + \omega_t)^2(a_1 + \omega_b - \omega_t)}{(\omega_b - \omega_t)^2} = 1.$$

We can compute the invariant $\mathrm{GW}^{M_\Lambda}_{A_{\max}+A_n+A_1,1}(Z_1Z_2)$ in a similar way. In this case the marked point lies in the cone σ_4 or σ_9 so we need to consider the same graphs as in the computation above plus the following graph:

The formula now gives for $\mathrm{GW}^{M_\Lambda}_{A_{\max}+A_n+A_1,1}(Z_1Z_2) = (1) + \cdots + (6)$:

$$\mathrm{GW}^{M_\Lambda}_{A_{\max}+A_n+A_1,1}(Z_nZ_b) = \frac{(3a_1 + 2a_2 + 2\omega_t)(a_2 + \omega_t)(a_1 + a_2 + \omega_t)}{a_1(\omega_b - \omega_t)(a_1 + \omega_b - \omega_t)}$$

$$+\frac{(3a_1 + 2a_2 + 2\omega_b)(a_1 + a_2 + \omega_t)(a_1 + a_2 + \omega_b)}{(\omega_b - \omega_t)^2(a_1 + \omega_t - \omega_b)}$$

$$+\frac{(3a_1 + 2a_2 + 2\omega_t)(a_1 + a_2 + \omega_t)(a_1 + a_2 + \omega_b)}{(\omega_b - \omega_t)^2(a_1 + \omega_b - \omega_t)}$$

$$-\frac{(3a_1 + 2a_2 + 2\omega_b)(a_1 + a_2 + \omega_b)^2}{a_1(\omega_b - \omega_t)^2} - \frac{(3a_1 + 2a_2 + 2\omega_t)(a_1 + a_2 + \omega_t)^2}{a_1(\omega_b - \omega_t)^2}$$

$$+\frac{(3a_1 + 2a_2 + 2\omega_b)(a_1 + a_2 + \omega_b)(a_2 + \omega_b)}{a_1(\omega_t - \omega_b)(a_1 + \omega_t - \omega_b)} = 0.$$

The remaining invariants can be computed using the same formula, therefore we leave their computation for the interested reader. $\qquad\square$

4. Seidel morphism in the NEF case

In this section we explain how to compute the Seidel element associated to a Hamiltonian circle action fixing a facet of a toric 4–dimensional NEF symplectic manifold.

4.1. The Seidel morphism

Recall from Section 2.2 that, starting from any closed symplectic manifold (M, ω) and a loop of Hamiltonian diffeomorphisms $\Lambda \subset \mathrm{Ham}(M, \omega)$, one can construct a Hamiltonian fibration $\pi \colon (M_\Lambda, \omega_\Lambda) \to (S^2, \omega_0)$ with fiber (M, ω), where $\omega_\Lambda = \Omega + \kappa \cdot \pi^*(\omega_0)$ for some big enough κ. Then, following [36], one can define Seidel's morphism, under some appropriate semi-positivity assumption on (M, ω), by counting pseudo-holomorphic section classes in $H_2^S(M_\Lambda; \mathbb{Z})$, with respect to some arbitrary choice of such a section. This choice was made canonical in [28].

In view of our goal, we now focus on the following specific case:

(i) The manifold M admits an almost complex structure J so that (M, J) is NEF (that is, there are no J–pseudo-holomorphic spheres with $\langle c_1(M), B \rangle < 0$).

(ii) The symplectic manifold (M, ω) is a toric 4–dimensional manifold, whose associated Delzant polytope has $n \geq 4$ facets.

(iii) Λ is a circle action, with moment map Φ_Λ, whose maximal fixed point component corresponds to a divisor, denoted by $F_{\max}$.

Notation 4.1. *Since the first Chern class of M (and of M only) is extensively used in what follows, we will denote $c_1(M)$ by c_1 and $\langle c_1(M), B \rangle$ by $c_1(B)$.*

We now extract from [32] the results which will be used in this section. Notice that in our specific setting, $F_{\max}$ is semifree and has dimension 2. We denote by $\Phi_{\max} = \Phi_\Lambda(F_{\max})$ the maximal value of the moment map. Concerning the choice of the section mentioned above, recall that in the toric case it is convenient to choose $\sigma_{\max} = \{x\} \times D_1 \cup_\Lambda \{x\} \times D_2$ (see the description of the clutching construction, Section 2.2) for any fixed point of the S^1–action x lying in $F_{\max}$. If we let $A_{\max} = [\sigma_{\max}] \in H_2^S(M; \mathbb{Z})$ then all the contributions to the Seidel morphism come from the section classes $A_{\max} + B$ with $B \in H_2^S(M; \mathbb{Z})$ and are determined by counting Gromov–Witten invariants in the classes $A_{\max} + B$, see e.g [32, Definition 2.4]. Lastly,

by [32, Lemma 2.2] the sum of the weights which appear in the formula giving the Seidel morphism, as part of the exponent of the q variable, is $m_{\max} = -1$.

Theorem 4.2 (Theorem 1.10 and Lemma 3.10 of [32]). *Under the assumptions* (i)–(iii) *above, the Seidel element associated to the circle action Λ is*

$$S(\Lambda) = [F_{\max}] \otimes qt^{\Phi_{\max}} + \sum_{B \in H_2^S(M;\mathbb{Z})^{>0}} a_B \otimes q^{1-c_1(B)} t^{\Phi_{\max}-\omega(B)}$$

where $H_2^S(M;\mathbb{Z})^{>0}$ consists of the spherical classes of positive symplectic area, that is, $\omega(B) > 0$ and $a_B \in H_*(M;\mathbb{Z})$ is the contribution of the section class $A_{\max} + B$ defined by requiring that $a_B \cdot_M c = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(c)$ for all homology classes $c \in H_*(M;\mathbb{Z})$. Moreover,

(i) *If $a_B \neq 0$ either $c_1(B) = 0$ and $a_B \in H_2(M;\mathbb{Z})$ or $c_1(B) = 1$ and $a_B \in H_4(M;\mathbb{Z})$.*

(ii) *If $a_B \neq 0$ then B intersects $F_{\max}$.*

(iii) *If $c_1(B') \geq 1$ for all J–holomorphic spheres B' which intersect $F_{\max}$, then all the lower order terms vanish.*

(iv) *If $c_1(B') \geq 1$ for all J–holomorphic spheres B' which intersect $F_{\max}$ but are not included in $F_{\max}$, then $a_B \neq 0 \Rightarrow c_1(B) = 0$.*

Remark 4.3. Item *(i)* above reads: If $a_B \neq 0$ then $c_1(B) = 0$ and $|a_B| = 2$. Indeed, when M is 4–dimensional, $|a_B| = 4$ means that a_B has to be a multiple of the fundamental class $[M]$, however this case can easily be ruled out. (See for example the end of the proof of [32, Theorem 1.10].)

Item *(ii)* is [32, Lemma 3.10] and shows that, even though the formula above might contain infinitely many terms, computing the Seidel morphism is somehow "local" (that is, one does not need to know the whole polytope).

Recall the notation we introduced in Section 3: We consider the case when the polytope P, associated to M, admits $n \geq 4$ facets, $D_1, \ldots, D_n$. These facets correspond to divisors whose homology classes we respectively denote by $A_1, \ldots, A_n$. We put $A_n = [F_{\max}]$ and we see the indices mod n. For any n–tuple $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we denote by $A_{\bar{a}} = \sum_i a_i A_i$ the homology class of the union of (possibly multiply covered) spheres in M whose projection to P is given by $D_{\bar{a}} = \cup_i D_i$.

Thus Theorem 4.2, combined with Remark 4.3, implies that the Seidel element is given by

$$S(\Lambda) = A_n \otimes qt^{\Phi_{\max}} + \sum_{\bar{a}} a_{A_{\bar{a}}} \otimes qt^{\Phi_{\max} - \omega(A_{\bar{a}})}$$

where $a_{A_{\bar{a}}} \neq 0$ if and only if

1) $D_{\bar{a}}$ is connected and intersects D_n,

2) $c_1(A_{\bar{a}}) = 0$ (i.e, by NEF condition, for all i so that $a_i \neq 0$, $c_1(A_i) = 0$).

In Theorem 4.4 below, we compute each contribution $a_{A_{\bar{a}}}$ in the case of polytopes where any $D_{\bar{a}}$ satisfying (1) and (2) contains at most two facets corresponding to spheres with vanishing first Chern number. Notice that in case the facets corresponding to divisors with vanishing first Chern number are not D_n and/or D_1 (that is, Cases *(3b)* and *(3c)*), the content of Section 3.1 has to be slightly adapted.

Theorem 4.4. *Let (M, ω) be a closed NEF toric 4–dimensional symplectic manifold. Assume that its associated Delzant polytope has $n \geq 4$ facets. Let Λ be a circle action, whose maximal fixed point component is a divisor $F_{\max}$ and denote $A_n = [F_{\max}]$ its homology class. The following homology classes have non trivial contributions to $S(\Lambda)$, the Seidel element associated to Λ:*

1) *A_n contributes by $a_{A_n} = A_n$.*

2) *If $c_1(A_n) = 0$,*
   (2a) then kA_n (with $k > 0$) contributes by $a_{kA_n} = A_n$,
   (2b) and if $c_1(A_1) = 0$, then $kA_n + lA_1$ (with $k \geq 0$ and $l > 0$) con-tributes and its contribution is $a_{kA_n + lA_1} = \begin{cases} A_n & \text{if } k \geq l, \\ -A_1 & \text{otherwise.} \end{cases}$

3) *If $c_1(A_n) \neq 0$,*
   (3a) if $c_1(A_1) = 0$, then kA_1 (with $k > 0$) contributes by $a_{kA_1} = -A_1$,
   (3b) if $c_1(A_1) = 0$ and $c_1(A_2) = 0$, then $kA_1 + lA_2$ (with $k > 0$ and $l > 0$) also contributes, and its contribution is

$$a_{kA_1 + lA_2} = \begin{cases} -A_1 & \text{if } k \geq l, \\ A_2 & \text{otherwise.} \end{cases}$$

   (3c) if $c_1(A_{n-1}) = 0$ and $c_1(A_1) = 0$, then kA_{n-1} and lA_1 (with $k > 0$ and $l > 0$) also contribute, with respective contributions $a_{kA_{n-1}} = -A_{n-1}$ and $a_{lA_1} = -A_1$.

Moreover, in each case, if the facets immediately next to the ones mentioned above correspond to spheres with non-zero first Chern number, then these are the only non-trivial contributions.

As a corollary, we compute the Seidel element associated to Λ in these different cases. (See also Figure 1 in the introduction.) Recall that we also compute in Appendix A the Seidel element associated to Λ when there exist three divisors in the vicinity of A_n with vanishing first Chern number.

Theorem 4.5. *Under the assumptions and with the notation of Theorem 4.4 above, the Seidel element associated to Λ is as follows.*

1) *If $c_1(A_n)$, $c_1(A_{n-1})$ and $c_1(A_1)$ are all non-zero, then $S(\Lambda) = A_n \otimes qt^{\Phi_{\max}}$.*

2) *If $c_1(A_n) = 0$,*
   (2a) but $c_1(A_{n-1})$ and $c_1(A_1)$ are non-zero, then

$$S(\Lambda) = A_n \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega(A_n)}},$$

   (2b) and $c_1(A_1) = 0$ but $c_1(A_{n-1})$ and $c_1(A_2)$ non-zero, then

$$S(\Lambda) = \left[A_n \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega(A_n)}} - A_1 \otimes q\, \frac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}} \right] \cdot \frac{1}{1 - t^{-\omega(A_n)-\omega(A_1)}}.$$

3) *If $c_1(A_n) \neq 0$,*
   (3a) if $c_1(A_1) = 0$ and $c_1(A_{n-1}), c_1(A_2)$ non-zero, then

$$S(\Lambda) = A_n \otimes qt^{\Phi_{\max}} - A_1 \otimes q\, \frac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}},$$

   (3b) if $c_1(A_1) = c_1(A_2) = 0$ but $c_1(A_{n-1})$ and $c_1(A_3)$ non-zero, then

$$S(\Lambda) = A_n \otimes qt^{\Phi_{\max}} - A_1 \otimes q\, \frac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}}$$
$$- \left(A_1 \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega(A_1)}} - A_2 \otimes q\, \frac{t^{\Phi_{\max}-\omega(A_2)}}{1 - t^{-\omega(A_2)}} \right) \cdot \frac{t^{-\omega(A_1)-\omega(A_2)}}{1 - t^{-\omega(A_1)-\omega(A_2)}},$$

   (3c) if $c_1(A_{n-1}) = c_1(A_1) = 0$, $c_1(A_{n-2})$ and $c_1(A_2)$ non-zero, then

$$S(\Lambda) = A_n \otimes qt^{\Phi_{\max}} - A_{n-1} \otimes q\, \frac{t^{\Phi_{\max}-\omega(A_{n-1})}}{1 - t^{-\omega(A_{n-1})}} - A_1 \otimes q\, \frac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}}.$$

We start by deducing Theorem 4.5 from Theorem 4.4. The proof of the latter is postponed to the next subsection since it is much more involving.

Proof of Theorem 4.5. It is a staigthforward consequence of Theorems 4.2 and 4.4.

(1): By Theorem 4.4, only A_n contributes and its contribution is of the form $S(\Lambda) = A_n \otimes qt^{\Phi_{\max}}$.

(2a): Here A_n and its iterations induce the only non-trivial contributions. The contribution of kA_n being $A_n \otimes q^{-c_1(kA_n)}t^{\Phi_{\max}-\omega(kA_n)}$, we get the result by summing over k (starting at $k = 0$):

$$S(\Lambda) = A_n \otimes q\, t^{\Phi_{\max}} \left(\sum_{k=0}^{\infty} (t^{-\omega(A_n)})^k \right) = A_n \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega(A_n)}}$$

(3a): This case is similar to (2a) except that we sum the contributions of all the kA_1's starting at $k = 1$ (thus, the new $-\omega(A_1)$ as power of t).

(3c): This case is similar to (3a) (but for both A_{n-1} and A_1).

Now we turn to (3b). The first two terms coincide with the sum of the contributions induced by A_n and kA_1. However, we also have to count the contributions of $kA_1 + lA_2$. As before, we can see that

$$- A_1 \otimes q\, \frac{t^{\Phi_{\max}-\omega(A_1)-\omega(A_2)}}{(1 - t^{-\omega(A_1)})(1 - t^{-\omega(A_1)-\omega(A_2)})}$$

$$= \sum_{k=1,l=0}^{\infty} a_{k(A_1+A_2)+lA_1} \otimes qt^{\Phi_{\max}-(k+l)\omega(A_1)-k\omega(A_2)}$$

which sums the contributions of $k(A_1 + A_2) + lA_1$ (with $k \geq 1$ and $l \geq 0$), that is, the contributions of all terms of the form $kA_1 + lA_2$ with $k \geq l \geq 1$. In the same way,

$$A_2 \otimes q\, \frac{t^{\Phi_{\max}-2\omega(A_2)-\omega(A_1)}}{(1 - t^{-\omega(A_2)})(1 - t^{-\omega(A_1)-\omega(A_2)})}$$

$$= \sum_{k,l=1}^{\infty} a_{k(A_1+A_2)+lA_2} \otimes qt^{\Phi_{\max}-k\omega(A_1)-(k+l)\omega(A_2)}$$

which sums the contributions of all terms of the form $kA_1 + lA_2$ with $k < l$. Thus the formula given for the case (3b) is indeed the sum of all non-trivial contributions.

Finally, let us look at (2b). First decompose

$$\frac{1}{1 - t^{-\omega(A_n)-\omega(A_1)}} = 1 + \frac{t^{-\omega(A_n)-\omega(A_1)}}{1 - t^{-\omega(A_n)-\omega(A_1)}}$$

and by replacing, we check that

$$
\begin{aligned}
S(\Lambda) &= \left[A_n \otimes q \, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega(A_n)}} - A_1 \otimes q \, \frac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}} \right] \cdot \frac{1}{1 - t^{-\omega(A_n)-\omega(A_1)}} \\
&= A_n \otimes q \, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega(A_n)}} + A_n \otimes q \, \frac{t^{\Phi_{\max}-\omega(A_n)-\omega(A_1)}}{(1 - t^{-\omega(A_n)})(1 - t^{-\omega(A_n)-\omega(A_1)})} \\
&\quad - A_1 \otimes q \, \frac{t^{\Phi_{\max}-\omega(A_1)}}{1 - t^{-\omega(A_1)}} - A_1 \otimes q \, \frac{t^{\Phi_{\max}-2\omega(A_1)-\omega(A_n)}}{(1 - t^{-\omega(A_1)})(1 - t^{-\omega(A_n)-\omega(A_1)})}.
\end{aligned}
$$

Now the first term counts the contributions of all terms of the form kA_n (as in (2a) above), the second term counts the contributions of kA_1 (or $A_n + kA_1$, see above) and then the last two count (as for (3b) but with A_n playing the role of A_1 and A_1 playing the role of A_2) all the contributions of the terms of the form $kA_n + lA_1$ (with k and l both non-zero).   □

4.2. Proof of Theorem 4.4

The proof is more or less a case-by-case proof and we focus on Case *(2b)*, since all the difficulties which one might encounter are already present and since the methods used to compute the Gromov–Witten invariants are the same. Notice that Case *(2b)* is one of the spectific cases described in Section 3.

We need to determine the class a_B of Theorem 4.2 where $B = kA_n + lA_1 \in H_2(M; \mathbb{Z})$. Recall that this class is determined by the requirement that

$$a_B \cdot c = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(c), \quad \text{for all } c \in H_*(M; \mathbb{Z}).$$

In the notation for the Gromov–Witten invariant we can either use the homology class c or its Poincaré dual. We define $B_{k,l} := A_{\max} + kA_n + lA_1$. Now we claim that in order to prove the theorem in Case *(2b)* it is sufficient to compute the following Gromov–Witten invariants.

Theorem 4.6. *For any $k, l \in \mathbb{N}$ we have*

$$\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1 Z_2) = \begin{cases} 0 & \text{if } k \geq l \\ 2 & \text{if } k < l \end{cases} \qquad \mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1 Z_b) = \begin{cases} 1 & \text{if } k \geq l \\ 2 & \text{if } k < l \end{cases}$$

$$and \quad \mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_2 Z_b) = \begin{cases} 0 & \text{if } k \geq l \\ -1 & \text{if } k < l \end{cases}$$

where $Z_1, Z_2, Z_b \in H^2(M_\Lambda; \mathbb{Q})$ are defined in Section 3.1.

Since the proof of this theorem is quite long and technical, we postpone it to Sections 4.3 and 4.4, and we first finish the proof of Theorem 4.4 by proving the claim.

The class a_B is a linear combination of the homology classes of the pre-images, under the moment map Φ of the facets of the polytope $P = \Phi(M)$, that is,

$$(14) \qquad\qquad a_B = \sum_{i=1}^{n} a_i A_i,$$

where $a_i \in \mathbb{Z}$. Since the dimension of the $\mathbb{Z}$–module $H_2^S(M; \mathbb{Z})$ is $n - 2$, we can assume that two of the coefficients a_i vanish. The following lemma shows that we can choose the coefficients $a_2 = a_3 = 0$.

Lemma 4.7. *All the classes A_i are linear combinations of the basis elements $\{\lambda_1, \ldots, \lambda_{n-2}\}$, defined in Section 3.*

Proof. It is known from the diagram of Figure 4 that $A_n = \lambda_1$ and $A_1 = \lambda_2 - 2\lambda_1$ which gives $\lambda_1 = A_n$ and $\lambda_2 = 2A_n + A_1$. Recall that $\eta_i = \alpha_i e_1 + \beta_i e_2$ where $i = 1, \ldots, n$. Let $\gamma_{i,j} := \alpha_j \beta_i - \alpha_i \beta_j$. It is not hard to check that Relation (8) implies that $\gamma_{i,i+1} = 1$. Moreover $\gamma_{i,j} \neq 0$ if $j \neq i + 1$ because the polytope is convex. We can write all the $A_i's$ as linear combinations of the basis elements λ_i, using the same argument as we use in Section 3 for A_n and A_1, which yields:

$$\begin{aligned} A_{n-1} &= \lambda_{n-2}, & A_4 &= \lambda_3 + \gamma_{5,3}\,\lambda_4 + \lambda_5, \\ A_{n-2} &= \lambda_{n-3} + \gamma_{n-1,n-3}\,\lambda_{n-2}, & \cdots \quad A_3 &= \lambda_2 + \gamma_{4,2}\,\lambda_3 + \lambda_4, \\ A_{n-3} &= \lambda_{n-4} + \gamma_{n-2,n-4}\,\lambda_{n-3} + \lambda_{n-2}, & A_2 &= \lambda_1 - d_2\,\lambda_2 + \lambda_3. \end{aligned}$$

Since $\lambda_{n-2} = A_{n-1}$ it follows from the second equation that $\lambda_{n-3} = A_{n-2} - \gamma_{n-1,n-3}A_{n-1}$. Substituting this in the third equation we can find an expression of λ_{n-4} as a linear combination of A_{n-2} and A_{n-1}. Going around the polytope we easily see that we can, recursively, determine an expression of each λ_i as a linear combination of the $A_i's$ with $i \neq 2, 3$. In particular, we obtain expressions for λ_3 and λ_4 which implies, by the last two equations, that A_2 and A_3 are linear combinations of the remaining $A_i's$.   □

Therefore, from now on, we assume $a_2 = a_3 = 0$ in the linear combination (14). Recall that

$$a_B \cdot c = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(\mathrm{PD}(c))$$

for $c \in H_2(M;\mathbb{Z})$. If c does not contain A_{n-1}, A_n, A_1, A_2 then it is clear that the Gromov–Witten invariant $\mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(\mathrm{PD}(c))$ vanishes when $B = kA_n + lA_1$. Therefore

$$0 = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(\mathrm{PD}(A_3)) = a_B \cdot A_3 = a_4$$

because $a_2 = a_3 = 0$. Then, using that $a_4 = 0$, we get

$$0 = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(\mathrm{PD}(A_4)) = a_B \cdot A_4 = a_5$$

and by repeating the process around the polytope we get for all k, $3 \leq k \leq n-2$,

$$0 = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(\mathrm{PD}(A_k)) = a_B \cdot A_k = a_{k+1}$$

so that all the coefficients vanish except a_n, a_1. That is, we obtain $a_B = a_n A_n + a_1 A_1$ for some $a_n, a_1 \in \mathbb{Z}$ when $B = kA_n + lA_1$. Since $\mathrm{PD}(A_2) = Z_2 Z_b$ and $\mathrm{PD}(A_1) = Z_1 Z_b$ it follows from Theorem 4.6 that if $k \geq l$ then

$$0 = \mathrm{GW}^{M_\Lambda}_{\sigma_{B_{k,l}},1}(Z_2 Z_b) = a_B \cdot A_2 = (a_n A_n + a_1 A_1) \cdot A_2 = a_1,$$
$$1 = \mathrm{GW}^{M_\Lambda}_{\sigma_{B_{k,l}},1}(Z_1 Z_b) = a_B \cdot A_1 = (a_n A_n + a_1 A_1) \cdot A_1 = a_n - 2a_1.$$

We conclude that $a_n = 1$, $a_1 = 0$ and $a_B = A_n$ in this case. If $k < l$ then we obtain

$$-1 = \mathrm{GW}^{M_\Lambda}_{\sigma_{B_{k,l}},1}(Z_2 Z_b) = a_1 \quad \text{and} \quad 2 = \mathrm{GW}^{M_\Lambda}_{\sigma_{B_{k,l}},1}(Z_1 Z_b) = a_n - 2a_1.$$

Therefore, in this case, $a_n = 0$, $a_1 = -1$ and $a_B = -A_1$. This concludes the proof of Theorem 4.4, Case *(2b)*.

4.3. An intermediate result

Before giving the proof of Theorem 4.6, we first need an intermediate result about some particular 0–point Gromov–Witten invariants. Recall that, by the divisor axiom, the 0–point invariant $\mathrm{GW}_0^{M_\Lambda}(A)$, for $A \neq 0 \in H_2(M_\Lambda; \mathbb{Z})$, is given by

$$\mathrm{GW}_0^{M_\Lambda}(A) = \frac{1}{h(A)^3} \mathrm{GW}_{A,3}^{M_\Lambda}(h, h, h)$$

where $h \in H^2(M_\Lambda; \mathbb{Q})$ is such that $h(A) = \int_A h \neq 0$. From now on we will suppress the indication of the number of marked points when that number is clear from the context and the expression for the Gromov–Witten invariant.

Proposition 4.8. *Let k and l be non-negative integers. Then*

$$\mathrm{GW}^{M_\Lambda}(kA_n + lA_1) = \begin{cases} -\dfrac{1}{k^3} & \text{if } l = 0, \\ -\dfrac{1}{l^3} & \text{if } k = 0, \\ -\dfrac{1}{k^3} & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In Steps 1 and 2 below, we prove the result in the first two cases. Then, in Step 3., we prove the result in the remaining cases by adapting Steps 1 and 2. A good reference for what follows is [29].

Step 1. Let $k > 0$. We begin with some preliminaries about moduli spaces of stable curves. Let $\overline{\mathcal{M}}_{0,n}(\mathbb{CP}^1, k)$ denote the moduli space of genus 0, n–pointed, degree k stable maps to $\mathbb{CP}^1$. Let

$$p : \overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1, k) \to \overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1, k)$$

be the universal curve, and let $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1, k) \to \mathbb{CP}^1$ be the evaluation map at the marked point. $\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1, k)$ is a smooth Deligne–Mumford stack of dimension $2k - 2$ and the map p is the forgetting morphism, which forgets the marked point. The following short exact sequence over $\mathbb{CP}^1$:

$$0 \to \mathcal{O}_{\mathbb{CP}^1} \to \mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}(1) \to \mathcal{O}_{\mathbb{CP}^1}(2) = T_{\mathbb{CP}^1} \to 0,$$

induces the short exact sequence

$$0 \to T_{\mathbb{CP}^1}^* = \mathcal{O}_{\mathbb{CP}^1}(-2) \to \mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathcal{O}_{\mathbb{CP}^1} \to 0.$$

Given a genus 0, 0–pointed, degree k stable map $u : C \to \mathbb{CP}^1$, we have a short exact sequence of vector bundles over the domain C:

$$(15) \qquad 0 \to u^*\mathcal{O}_{\mathbb{CP}^1}(-2) \to u^*\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus u^*\mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathcal{O}_C \to 0\,.$$

Since $H^0(C, u^*\mathcal{O}_{\mathbb{CP}^1}(-2)) = H^0(C, u^*\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus u^*\mathcal{O}_{\mathbb{CP}^1}(-1)) = \{0\}$, the long exact sequence in cohomology associated to (15) becomes

$$0 \to H^0(C, \mathcal{O}_C) \to H^1(C, u^*\mathcal{O}_{\mathbb{CP}^1}(-2))$$
$$\to H^1(C, u^*\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus u^*\mathcal{O}_{\mathbb{CP}^1}(-1)) \to 0,$$

where the complex dimension of $H^1(C, u^*\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus u^*\mathcal{O}_{\mathbb{CP}^1}(-1))$ and $H^1(C, u^*\mathcal{O}_{\mathbb{CP}^1}(-2))$ are respectively $2k - 2$ and $2k - 1$.

Next we define two bundles over $\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1, k)$:

$$E_k := p_*\mathrm{ev}^*\mathcal{O}_{\mathbb{CP}^1}(-2) \quad \text{and} \quad V_k := p_*\mathrm{ev}^*(\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1))\,.$$

The bundle E_k has rank $2k - 1$ and its fiber over $[u : C \to \mathbb{CP}^1]$ is given by $H^1(C, u^*\mathcal{O}_{\mathbb{CP}^1}(-2))$, while V_k has rank $2k - 2$ and fiber $H^1(C, u^*\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus u^*\mathcal{O}_{\mathbb{CP}^1}(-1))$. They belong to the following short exact sequence

$$0 \to \mathcal{O}_\mathcal{M} \to E_k \to V_k \to 0,$$

where $\mathcal{O}_\mathcal{M}$ is the trivial line bundle over $\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1, k)$. Therefore, the Euler and Chern classes of these bundles satisfy

$$(16) \qquad e(E_k) = c_{2k-1}(E_k) = 0, \quad e(V_k) = c_{2k-2}(V_k) = c_{2k-2}(E_k).$$

Finally, recall that $\int_{[\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1, k)]} e(V_k) = \frac{1}{k^3}$ (see Manin [30]).

Step 2. We now consider the case of a toric fibration $\pi : M_\Lambda \to \mathbb{CP}^1$ where the total space is a toric manifold of (complex) dimension 3 and each fiber is diffeomorphic to the toric surface M. Using the previous notation, we want to show that

$$\mathrm{GW}(kA_n) = \int_{[\overline{\mathcal{M}}_{0,0}(M_\Lambda, kA_n)]^{\mathrm{vir}}} 1 = -\frac{1}{k^3}.$$

We first introduce some notation. We have

$$H^*_{\mathbb{C}^*}(\mathrm{point}; \mathbb{Z}) = H^*(B\mathbb{C}^*; \mathbb{Z}) = H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[u],$$

where $u = c_1(\mathcal{O}_{\mathbb{CP}^\infty}(-1))$ is the first Chern class of the tautological line bundle over $B\mathbb{C}^* = \mathbb{CP}^\infty$. Let L_{mu} denote the $\mathbb{C}^*$–equivariant line bundle

over a point given by the 1–dimensional $\mathbb{C}^*$–representation $t \mapsto t^m$. Then

$$(c_1)_{\mathbb{C}^*}(L_{mu}) = mu \in H^2_{\mathbb{C}^*}(\text{point}; \mathbb{Z}) = \mathbb{Z}[u].$$

The action of $\mathbb{C}^*$ on $\mathbb{CP}^1$ by $t \cdot [x, y] = [tx, y]$ has two fixed points: $0 = [0 : 1]$ and $\infty = [1 : 0]$ and at these points

$$(c_1)_{\mathbb{C}^*}(T_0\mathbb{CP}^1) = u, \quad (c_1)_{\mathbb{C}^*}(T_\infty\mathbb{CP}^1) = -u.$$

There is a unique lift of this action to M_Λ which acts trivially on $\pi^{-1}(0)$. This lift induces a $\mathbb{C}^*$–action on $\overline{\mathcal{M}}_{0,0}(M_\Lambda, kA_n)$ and we have

$$\overline{\mathcal{M}}_{0,0}(M_\Lambda, kA_n)^{\mathbb{C}^*} = F_0 \cup F_\infty$$

where F_0 and F_∞ can be identified with $\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1, k)$ as moduli spaces of maps to $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$, respectively.

By virtual localization [20],

$$\int_{[\overline{\mathcal{M}}_{0,0}(M_\Lambda, kA_n)]^{\text{vir}}} 1 = \int_{[F_0]^{\text{vir}}} \frac{1}{e_{\mathbb{C}^*}(N^{\text{vir}}_{F_0})} + \int_{[F_\infty]^{\text{vir}}} \frac{1}{e_{\mathbb{C}^*}(N^{\text{vir}}_{F_\infty})}$$

where $N^{\text{vir}}_{F_0}$ and $N^{\text{vir}}_{F_\infty}$ are the virtual normal bundles to F_0 and F_∞, respectively.

Let $\xi = [u : C \to \mathbb{CP}^1] \in F_0$. As explained in [29], the tangent space T^1_ξ and the obstruction space T^2_ξ at the moduli point $\xi \in \overline{\mathcal{M}}_{0,0}(M_\Lambda, kA_n)$ fit in the *tangent-obstruction exact sequence*:

$$(17) \quad 0 \to \text{Ext}^0(\Omega_C, \mathcal{O}_C) \to H^0(C, u^*TM_\Lambda) \to T^1_\xi$$
$$\to \text{Ext}^1(\Omega_C, \mathcal{O}_C) \to H^1(C, u^*TM_\Lambda) \to T^2_\xi \to 0$$

where

- $\text{Ext}^0(\Omega_C, \mathcal{O}_C)$, respectively $\text{Ext}^1(\Omega_C, \mathcal{O}_C)$, is the space of infinitesimal automorphisms, respectively deformations, of the domain C,

- $H^0(C, u^*TM_\Lambda)$, respectively $H^1(C, u^*TM_\Lambda)$, is the space of infinitesimal deformations of, respectively obstructions to deforming, the map u.

Equivalently,

$$0 \to \mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) \to H^0(C, u^*T\mathbb{CP}^1) \oplus L_u \to T^1_\xi$$
$$\to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to H^1(C, u^*\mathcal{O}(-2)) \to T^2_\xi \to 0 \, .$$

Together with the fact that $e(E_k) = 0$, this leads to

$$\int_{[F_0]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_0})} = \int_{\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1, k)} \frac{e(E_k)}{e(L_u)} = 0 \, .$$

Suppose now that $\xi \in F_\infty$. In this case (17) is equivalent to

$$0 \to \mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) \to H^0(C, u^*T\mathbb{CP}^1) \oplus L_{-u} \to T^1_\xi$$
$$\to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to H^1(C, u^*\mathcal{O}(-2)) \otimes L_{-u} \to T^2_\xi \to 0$$

so that

$$\int_{[F_\infty]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\infty})} = \int_{\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1, k)} \frac{e(E_k \otimes L_{-u})}{u}$$

where

$$e(E_k \otimes L_{-u}) = \sum_{i=0}^{2k-1} (-u)^i c_{2k-1-i}(E_k) = -ue(V_k) + \sum_{i=2}^{2k-1} c_{2k-1-i}(E_k)(-u)^i$$

by (16). Together with the aforementioned result due to Manin, this now yields

$$\int_{[F_\infty]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\infty})} = -\int_{\overline{\mathcal{M}}_{0,0}(\mathbb{CP}^1, k)} e(V_k) = -\frac{1}{k^3} \, .$$

This proves that $\mathrm{GW}(kA_n) = -\frac{1}{k^3}$, which finishes the proof of the first case of the proposition. The second case follows by symmetry.

Step 3. For the third and fourth cases we adapt Steps 1 and 2 above to the case of genus 0, 1–pointed, stable maps $u : C \to \mathbb{CP}^1 \times \mathbb{CP}^1$ of degree k to the first sphere and of degree l to the second sphere. We denote the moduli space of such maps by $\overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1 \times \mathbb{CP}^1, (k, l))$, it is a Deligne–Mumford stack of dimension $2k + 2l$.

As above, we define the evaluation map $\mathrm{ev} : \overline{\mathcal{M}}_{0,2}(\mathbb{CP}^1 \times \mathbb{CP}^1, (k, l)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ and the forgetful map given by $p : \overline{\mathcal{M}}_{0,2}(\mathbb{CP}^1 \times \mathbb{CP}^1, (k, l)) \to$

$\overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1 \times \mathbb{CP}^1, (k,l))$ which forgets the second marked point, and we consider the following short exact sequence over $\mathbb{CP}^1 \times \mathbb{CP}^1$:

$$0 \to \mathcal{O}_{\mathbb{CP}^1}(-2) \times \mathcal{O}_{\mathbb{CP}^1}(-2)$$
$$\to (\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)) \times (\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1))$$
$$\to \mathcal{O}_{\mathbb{CP}^1} \times \mathcal{O}_{\mathbb{CP}^1} \to 0 \,.$$

Given $[u : C \to \mathbb{CP}^1 \times \mathbb{CP}^1] \in \overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1 \times \mathbb{CP}^1, (k,l))$, this exact sequence pulls-back to

$$0 \to u^*(\mathcal{O}_{\mathbb{CP}^1}(-2) \times \mathcal{O}_{\mathbb{CP}^1}(-2))$$
$$\to u^*((\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)) \times (\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)))$$
$$\to u^*(\mathcal{O}_{\mathbb{CP}^1} \times \mathcal{O}_{\mathbb{CP}^1}) \to 0 \,.$$

In a similar way to the previous case we define bundles

$$E_{k,l} := p_*\mathrm{ev}^*(\mathcal{O}_{\mathbb{CP}^1}(-2) \times \mathcal{O}_{\mathbb{CP}^1}(-2)) \quad \text{and}$$
$$V_{k,l} := p_*\mathrm{ev}^*((\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)) \times (\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)))$$

over $\overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1 \times \mathbb{CP}^1, (k,l))$. Now $E_{k,l}$ and $V_{k,l}$ have rank $2k + 2l - 2$ and $2k + 2l - 4$, respectively. In this case we have the following short exact sequence of bundles

$$0 \to \mathcal{O}_{\mathcal{M}} \to E_{k,l} \to V_{k,l} \to 0 \,,$$

where, again, $\mathcal{O}_{\mathcal{M}}$ is the trivial bundle. So relations (16) become in this case

$$e(E_{k,l}) = c_{2k+2l-2}(E_{k,l}) = 0, \quad e(V_{k,l}) = c_{2k+2l-4}(V_{k,l}) = c_{2k+2l-4}(E_{k,l}) \,.$$

We consider the same $\mathbb{C}^*$–action as above, with fixed points $0 = [0:1]$ and $\infty = [1:0]$, and its lift to M_Λ acting trivially on $\pi^{-1}(0)$. It induces a $\mathbb{C}^*$–action on $\overline{\mathcal{M}}_{0,0}(M_\Lambda, kA_n + A_1)$. Analogously to the first case we have

$$\overline{\mathcal{M}}_{0,0}(M_\Lambda, kA_n + lA_1)^{\mathbb{C}^*} = F_0 \cup F_\infty$$

where F_0 and F_∞ can now be identified with $\overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1 \times \mathbb{CP}^1, (k,l))$.

Again, by virtual localization [20],

$$\int_{[\overline{\mathcal{M}}_{0,0}(M_\Lambda, kA_n + lA_1)]^{\mathrm{vir}}} 1 = \int_{[F_0]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(N_{F_0}^{\mathrm{vir}})} + \int_{[F_\infty]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(N_{F_\infty}^{\mathrm{vir}})} \,.$$

However, in this case, since $\dim \overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1 \times \mathbb{CP}^1, (k,l)) = 2k + 2l$ and both Euler classes $e(E_{k,l})$ and $e(E_{k,l} \otimes L_{-u})$ have smaller degree than this

dimension we conclude that both integrals

$$\int_{[F_0]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(N_{F_0}^{\mathrm{vir}})} = \int_{\overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1\times\mathbb{CP}^1,(k,l))} \frac{e(E_k)}{e(L_u)} \qquad \text{and}$$

$$\int_{[F_\infty]^{\mathrm{vir}}} \frac{1}{e_{\mathbb{C}^*}(N_{F_\infty}^{\mathrm{vir}})} = \int_{\overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1\times\mathbb{CP}^1,(k,l))} \frac{e(E_k \otimes L_{-u})}{u}$$

vanish, unless $k = l$ when we can reduce the calculation of the Gromov–Witten invariant to the first case by considering curves in class $k(A_n + A_1)$. $\qquad\square$

4.4. Proof of Theorem 4.6

We are now ready to prove Theorem 4.6 which will conclude the proof of Theorem 4.4.

We use an induction argument. First notice that using the results from Spielberg recalled in Section 2.5, we can easily compute the value of the three Gromov–Witten invariants of Theorem 4.6 for the base cases $k = 0, 1$ and $l = 0, 1$ (see Lemma 3.3 for the computation of some of these invariants). Now we assume they hold for all values i, j such that $i \le k - 1$ and $j \le l - 1$ and we will prove they also hold for $i = k$ and $j = l$. Because $[M] \cdot [\sigma] = 1$ for any section class σ, the divisor axiom for Gromov–Witten invariants (see Proposition 2.2) implies that the 1–point invariant $\mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(c)$ equals the 3–point invariant $\mathrm{GW}^{M_\Lambda}_{A_{\max}+B,3}([M], [M], c)$. It follows easily, from the fan description of the manifold M_Λ in Section 3, that $\mathrm{PD}([M]) = Z_b$. Therefore we need to compute the Gromov–Witten invariants

$$\mathrm{GW}^{M_\Lambda}_{A_{\max}+B,3}(Z_b, Z_b, Z)$$

with $Z \in H^4(M; \mathbb{Z})$ since the degrees satisfy the equation $2 \deg Z_b + \deg Z = 2N + 2c_1(B_{k,l}) + 2m - 6$ where $\dim M_\Lambda = 2N = 6$, $\deg Z_b = 2$, $c_1(B_{k,l}) = 1$ and $m = 3$ is the number of marked points.

The main idea of the proof is to compute well-chosen Gromov–Witten invariants via the splitting axiom along two different partitions and then deduce relations from the two resulting expressions. Namely, we start with $\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1, Z_b, Z_b, Z_1; [\mathrm{pt}])$, from which we will deduce:

Lemma 4.9. $\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_b)$ and $\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2)$ satisfy the following equations:

$$(18) \quad (k - 2l)\,\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_b) + \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) = k - 2l\,, \qquad \text{if } k \geq l,$$

$$(19) \quad (k - 2l)\,\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_b) + \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) = 2k - 4l + 2\,, \quad \text{if } k < l.$$

Proof. **Step 1. We use the partition $S_0 = \{1, 2\}, S_1 = \{3, 4\}$ of the index set $\{1, 2, 3, 4\}$ and apply the splitting axiom so that we get:**

$$\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1, Z_b, Z_b, Z_1; [\mathrm{pt}]) =$$
$$\sum_{C_0 + C_1 = B_{k,l}} \mathrm{GW}^{M_\Lambda}_{C_0,3}(Z_1, Z_b, e_\nu)\, g^{\nu\mu}\, \mathrm{GW}^{M_\Lambda}_{C_1,3}(e_\mu, Z_b, Z_1)$$

where the sum runs over all C_0, C_1 such that

$$\begin{cases} C_0 = A_{\max} + k_0 A_n + l_0 A_1 \\ C_1 = k_1 A_n + l_1 A_1 \end{cases} \quad \text{or} \quad \begin{cases} C_0 = k_0 A_n + l_0 A_1 \\ C_1 = A_{\max} + k_1 A_n + l_1 A_1 \end{cases}$$

with $k_0 + k_1 = k$ and $l_0 + l_1 = l$. In order to ease the reading, we used in the equality above *as well as in the rest of this proof*, the Einstein summation convention with respect to the basis of the cohomology (and thus forgot $\sum_{\nu,\mu}$ from the notation).

This leads us to

$$\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1, Z_b, Z_b, Z_1; [\mathrm{pt}])$$
$$= \mathrm{GW}^{M_\Lambda}_{B_{k,l},3}(Z_1, Z_b, e_\nu)\, g^{\nu\mu}\, \mathrm{GW}^{M_\Lambda}_{0,3}(e_\mu, Z_b, Z_1)$$
$$+ \mathrm{GW}^{M_\Lambda}_{A_{\max},3}(Z_1, Z_b, e_\nu)\, g^{\nu\mu}\, \mathrm{GW}^{M_\Lambda}_{kA_n + lA_1,3}(e_\mu, Z_b, Z_1)$$
$$+ \sum_{1 \leq k_0 + l_0 \leq k + l - 1} \mathrm{GW}^{M_\Lambda}_{B_{k_0,l_0},3}(Z_1, Z_b, e_\nu)\, g^{\nu\mu}\, \mathrm{GW}^{M_\Lambda}_{k_1 A_n + l_1 A_1,3}(e_\mu, Z_b, Z_1)$$
$$+ \mathrm{GW}^{M_\Lambda}_{0,3}(Z_1, Z_b, e_\nu)\, g^{\nu\mu}\, \mathrm{GW}^{M_\Lambda}_{B_{k,l},3}(e_\mu, Z_b, Z_1)$$
$$+ \mathrm{GW}^{M_\Lambda}_{kA_n + lA_1,3}(Z_1, Z_b, e_\nu)\, g^{\nu\mu}\, \mathrm{GW}^{M_\Lambda}_{A_{\max},3}(e_\mu, Z_b, Z_1)$$
$$+ \sum_{1 \leq k_0 + l_0 \leq k + l - 1} \mathrm{GW}^{M_\Lambda}_{k_0 A_n + l_0 A_1,3}(Z_1, Z_b, e_\nu)\, g^{\nu\mu}\, \mathrm{GW}^{M_\Lambda}_{B_{k_1,l_1},3}(e_\mu, Z_b, Z_1)$$

Now, by using the divisor axiom (see Proposition 2.2) together with the fact that $\int_{A_n} Z_b = \int_{A_1} Z_b = 0$, we end up with:

$$\begin{aligned}(20) \qquad &\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1, Z_b, Z_b, Z_1; [\mathrm{pt}]) \\ &= 2\,\mathrm{GW}^{M_\Lambda}_{B_{k,l},3}(Z_1, Z_b, e_\nu)\, g^{\nu\mu}\,\mathrm{GW}^{M_\Lambda}_{0,3}(e_\mu, Z_b, Z_1)\,.\end{aligned}$$

Moreover, it is clear that $\int_{B_{k,l}} Z_1 = Z_1(A_{\max} + kA_n + lA_1) = k - 2l$ and $\int_{B_{k,l}} Z_b = 1$, and by the zero axiom (see Proposition 2.2):

$$\mathrm{GW}^{M_\Lambda}_{0,3}(e_\mu, Z_b, Z_1) = \int_{M_\Lambda} e_\mu \cup Z_b \cup Z_1 = \begin{cases} -2 & \text{if } e_\mu = Z_1, \\ 1 & \text{if } e_\mu = Z_2, \\ 0 & \text{otherwise.} \end{cases}$$

So one gets that (20) leads to

$$\begin{aligned}(21) \qquad &\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1, Z_b, Z_b, Z_1; [\mathrm{pt}]) \\ &= 2(k - 2l) \sum_{\nu\,:\,|e_\nu|=4} \mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(e_\nu)\,(g^{\nu n} - 2g^{\nu 1})\,.\end{aligned}$$

Remark 4.10. From the diagram of Figure 4, $\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(e_\nu) \neq 0$ only if the class e_ν is Poincaré dual to one of the following homology classes: A_{n-1}, A_n, A_1, A_2, $A_{\max}$, or $A_{\max} + A_1$, since the marked point should lie in one of the following cones: σ_2, σ_3, σ_4, σ_7, σ_8, or σ_9. Their Poincaré duals are the classes $Z_{n-1}Z_b$, Z_nZ_b, Z_1Z_b, Z_2Z_b, Z_1Z_n, and Z_1Z_2, respectively. Note that the only ones that belong to the basis of the cohomology are Z_1Z_b, Z_2Z_b, and Z_1Z_2. Therefore, at most three terms appear in the summation in Equation (21) above and the coefficients can be computed thanks to Lemma 3.2.

In the case of Equation (21), we end up with

$$(22) \qquad \mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1, Z_b, Z_b, Z_1; [\mathrm{pt}]) = 2(k-2l)\mathrm{GW}^{M_\Lambda}_{B_{k,l},1}(Z_1 Z_b)\,.$$

Step 2. We use the partition $S_0 = \{1,4\}$, $S_1 = \{2,3\}$.

The same Gromov–Witten invariant is given by the following expression

$$\begin{aligned}(23) \qquad &\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1, Z_b, Z_b, Z_1; [\mathrm{pt}]) \\ &= \mathrm{GW}^{M_\Lambda}_{0,3}(Z_1, Z_1, e_\nu)\, g^{\nu\mu}\,\mathrm{GW}^{M_\Lambda}_{B_{k,l},3}(e_\mu, Z_b, Z_b) \\ &\quad + \sum_{1 \leq k_0 + l_0 \leq k+l} \mathrm{GW}^{M_\Lambda}_{k_0 A_n + l_0 A_1, 3}(Z_1, Z_1, e_\nu)\, g^{\nu\mu}\,\mathrm{GW}^{M_\Lambda}_{B_{k_1,l_1},3}(e_\mu, Z_b, Z_b)\,.\end{aligned}$$

Since (by the zero axiom):

$$\text{GW}_{0,3}^{M_\Lambda}(Z_1, Z_1, e_\nu) = \int_{M_\Lambda} Z_1^2 \cup e_\nu$$

$$= \begin{cases} 4 & \text{if } e_\nu = Z_1 \\ -2 & \text{if } e_\nu = Z_2 \text{ or } e_\nu = Z_b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and} \quad \int_{k_0 A_n + l_0 A_1} e_\nu = \begin{cases} k_0 - 2l_0 & \text{if } e_\nu = Z_1, \\ l_0 & \text{if } e_\nu = Z_2, \\ 0 & \text{otherwise} \end{cases}$$

it follows from the divisor axiom that (23) is equal to

$$(24) \qquad = \sum_{\mu\,:\,|e_\mu|=4} (4g^{1\mu} - 2g^{2\mu} - 2g^{b\mu})\text{GW}_{B_{k,l},1}^{M_\Lambda}(e_\mu)$$

$$+ \sum_{\mu} \sum_{1 \le k_0 + l_0 \le k+l} \Big[(k_0 - 2l_0)^2 \text{GW}(k_0 A_n + l_0 A_1)$$

$$\big((k_0 - 2l_0)g^{1\mu} + l_0\, g^{2\mu}\big)\, \text{GW}_{B_{k_1,l_1},1}^{M_\Lambda}(e_\mu)\Big]$$

where $\text{GW}(k_0 A_n + l_0 A_1)$ denotes the 0–point invariant in class $k_0 A_n + l_0 A_1$. These were computed in Proposition 4.8. In order to simplify the expression, we will denote them by GW_0. We will also omit the index 1 indicating the number of marked points for the various 1–point Gromov–Witten invariants appearing in what remains of the proof.

   In view of Remark 4.10 above and Lemma 3.2, equation (24) actually reads

$$= -2\text{GW}_{B_{k,l}}^{M_\Lambda}(Z_1 Z_2)$$

$$+ \sum_{1 \le k_0 + l_0 \le k+l} (k_0 - 2l_0)^2 \text{GW}_0 \Big[(k_0\, g^{1,1b} + l_0)\text{GW}_{B_{k_1,l_1}}^{M_\Lambda}(Z_1 Z_b)$$

$$+ k_0\, g^{1,2b}\text{GW}_{B_{k_1,l_1}}^{M_\Lambda}(Z_2 Z_b) + k_0\, g^{1,12}\text{GW}_{B_{k_1,l_1}}^{M_\Lambda}(Z_1 Z_2))\Big].$$

Then, using Proposition 4.8, we separate the summation in three summations: $k_0 = 0$, $l_0 = 0$, and $k_0 = l_0$:

$$(25) \qquad = -2\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) + \sum_{l_0=1}^{l} 4l_0^3 \left(-\frac{1}{l_0^3}\right) \mathrm{GW}^{M_\Lambda}_{B_{k,l_1}}(Z_1 Z_b)$$

$$+ \sum_{k_0=1}^{k} k_0^3 \left(-\frac{1}{k_0^3}\right) \Big[g^{1,1b}\, \mathrm{GW}^{M_\Lambda}_{B_{k_1,l}}(Z_1 Z_b)$$

$$+ g^{1,2b}\mathrm{GW}^{M_\Lambda}_{B_{k_1,l}}(Z_2 Z_b) + g^{1,12}\mathrm{GW}^{M_\Lambda}_{B_{k_1,l}}(Z_1 Z_2)) \Big]$$

$$- \sum_{k_0=1}^{\min(k,l)} k_0^3 \left(\frac{1}{k_0^3}\right) \Big[(g^{1,1b}+1)\, \mathrm{GW}^{M_\Lambda}_{B_{k_1,l_1}}(Z_1 Z_b)$$

$$+ g^{1,2b}\mathrm{GW}^{M_\Lambda}_{B_{k_1,l_1}}(Z_2 Z_b) + g^{1,12}\mathrm{GW}^{M_\Lambda}_{B_{k_1,l_1}}(Z_1 Z_2)) \Big].$$

Applying the induction hypotheses and Lemma 3.2 we can simplify even further this expression. However we need to consider two different cases:

(a) If $k \geq l$ then (25) is equal to

$$(26) \qquad = -2\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) - 4l + \sum_{k_1=0}^{l-1}(-2g^{1,1b}+g^{1,2b}-2g^{1,12})$$

$$- \sum_{k_1=l}^{k-1} g^{1,1b} - \sum_{k_0=1}^{l}(g^{1,1b}+1)$$

$$= -2\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) + l\,(g^{1,2b}-2g^{1,12}) - (k+2l)\,g^{1,1b} - 5l$$

(b) If $k < l$ then (25) is equal to

$$(27) \qquad = -2\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) - 4\sum_{l_1=0}^{k} 1 - 4 \sum_{l_1=k+1}^{l-1} 2$$

$$- \sum_{k_0=1}^{k}(2g^{1,1b}-g^{1,2b}+2g^{1,12}) - \sum_{k_0=1}^{k}(g^{1,1b}-g^{1,2b}+2g^{1,12})$$

$$= -2\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) - 4(k+1) - 8(l-1-k) + 6k$$

$$- 2k(2g^{1,12}-g^{1,2b})$$

Step 3. We use the fact that the results of Steps 1 and 2 coincide, i.e. when $k \geq l$, (22)=(26) while when $k < l$, (22)=(27).

First we consider the case $k \geq l$, (22)=(26) leads to

$$2(k - 2l) \, \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_b) + 2 \, \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2)$$
$$= l \, (g^{1,2b} - 2g^{1,12}) - (k + 2l) \, g^{1,1b} - 5l.$$

In particular, when $k = 1, l = 0$ and $k = 1, l = 1$, it follows from the base cases (Lemma 3.3) that the matrix elements satisfy:

(28) $$g^{1,1b} = -2 \quad \text{and} \quad 2g^{1,12} - g^{1,2b} = 3,$$

respectively. Getting back to the general case, we finally deduce:

(a) For $k \geq l$, (22)=(26) together with (28) give

$$(k - 2l) \, \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_b) + \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) = k - 2l \,,$$

(b) and for $k < l$, (22)=(27) together with (28) give

$$(k - 2l) \, \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_b) + \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) = 2k - 4l + 2 \,.$$

This ends the proof of the lemma. $\qquad\square$

We now proceed along the same lines but for two other Gromov–Witten invariants, namely,

$$\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1, Z_b, Z_b, Z_2; [\mathrm{pt}]) \quad \text{and} \quad \mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_2, Z_b, Z_b, Z_2; [\mathrm{pt}]) \,.$$

Since the method is exactly the same, we leave the computation to the interested reader and we simply give the four resulting equations.

Lemma 4.11. *From* $\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_1, Z_b, Z_b, Z_2; [\mathrm{pt}])$, *we deduce: if* $k \geq l$

(29) $$l \, \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_b) + (k - 2l) \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_2 Z_b) - \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) = l \,;$$

and if $k < l$

(30) $$l \, \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_b) + (k - 2l) \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_2 Z_b) - \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2)$$
$$= 4l - k - 2 \,.$$

Lemma 4.12. *From* $\mathrm{GW}^{M_\Lambda}_{B_{k,l},4}(Z_2, Z_b, Z_b, Z_2; [\mathrm{pt}])$ *we deduce:*

$$(31) \quad (2l + 2d_2 - 1)\,\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_2 Z_b) + d_2 \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) = 0\,, \qquad \text{if } k \geq l,$$

$$(32) \quad (2l + 2d_2 - 1)\,\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_2 Z_b) + d_2 \mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2) = 1 - 2l\,, \quad \text{if } k < l,$$

where d_2 comes from the matrix G, see Table (13).

In order to conclude the proof of Theorem 4.6, we consider two linear systems:

- one given by the equations (18), (29), (31), corresponding to the case $k \geq l$ of Lemmas 4.9, 4.11, and 4.12 above,

- the other given by the equations (19), (30), (32) corresponding to the case $k < l$.

The unknowns of these linear systems are the Gromov–Witten invariants we are looking for, namely, $\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_b)$, $\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_1 Z_2)$, and $\mathrm{GW}^{M_\Lambda}_{B_{k,l}}(Z_2 Z_b)$. The unique solutions of these systems give us the desired result.

5. Applications and explicit examples

In this section we show some applications of our results and illustrate their relevance with some particular examples. More precisely, in Section 5.1 we show how to obtain an expression for the Landau–Ginzburg superpotential from the moment polytope of a NEF toric 4–manifold. In Section 5.2 we compute the Seidel elements, the quantum homology ring and the Landau–Ginzburg superpotential for two examples of NEF toric surfaces, namely $\mathbb{CP}^2$ blown–up at 4 or 5 points. Finally, in Section 5.3 we show how we can use the Fano and NEF computations to obtain explicit expressions of Seidel elements for some particular non-NEF manifolds, namely the Hirzebruch surfaces $\mathbb{F}_{2k}$ or $\mathbb{F}_{2k-1}$ with $k \geq 2$. As an example, we compute them explicitly for $\mathbb{F}_4$.

5.1. The Landau–Ginzburg potential

In this section we follow the works of McDuff–Tolman [32] and Ostrover–Tyomkin [33] which were themselves developments of original ideas due to Batyrev [5] and Givental [17, 18]. In particular, we will also use quantum cohomology. The definition is similar to quantum homology in Section 2.4,

except that the coefficient ring is $\check{\Pi} := \check{\Pi}^{\mathrm{univ}}[q, q^{-1}]$, with

$$\check{\Pi}^{\mathrm{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \mid r_\kappa \in \mathbb{Q}, \ \#\{\kappa < c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}$$

(compare with (5)) and that the product on $QH^*(M; \omega) = H^*(M; \mathbb{Q}) \otimes_\mathbb{Q} \check{\Pi}$ is Poincaré dual to the intersection product and is called the quantum cup product.

Let us recall some notation. Consider a torus T with Lie algebra $\mathfrak{t}$ and lattice $\mathfrak{t}_\mathbb{Z}$. Let (M, ω) be a smooth toric $2m$–manifold with moment map $\Phi : M \to \mathfrak{t}^*$ and with moment polytope P. Let $D_1, \ldots, D_n$ be the facets of P, inducing homology classes $A_i = [\Phi^{-1}(D_i)] \in H_2(M; \mathbb{Z})$, and let $v_1, \ldots, v_n$ denote the outward primitive integral vectors normal to the facets. The moment polytope is given by

$$P = \{x \in \mathfrak{t}^* \mid \langle x, v_j \rangle \leq \kappa_j, \ \text{for } j = 1, \ldots, n\}$$

where $\kappa_j \in \mathbb{R}$. Any face of P, given as the intersection of facets $D_{j_1}, \ldots, D_{j\ell}$, admits a *dual cone* consisting by definition of those elements in $\mathfrak{t}$ which are positive linear combinations of $v_{j_1}, \ldots, v_{j_\ell}$. As explained in [32, Section 5.1], any vector in $\mathfrak{t}$ lies in the dual cone of a unique face of P. Therefore, a subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots n\}$ determines a unique face of P whose dual cone contains $v_{i_1} + \cdots + v_{i_k}$. This face is given as the intersection of facets which we (still) denote by $D_{j_1}, \ldots, D_{j\ell}$ and there exist unique positive integers $c_1, \ldots, c_\ell$ so that $v_{i_1} + \cdots + v_{i_k} - c_1 v_{j_1} - \cdots - c_\ell v_{j_\ell} = 0$. Batyrev showed that if I is primitive, the sets I and $J = \{j_1, \ldots, j_\ell\}$ are disjoint. Moreover, if $\beta_I \in H_2(M; \mathbb{Z})$ is the class corresponding to the above relation (recall from Section 2.3 that $H_2(M; \mathbb{Z})$ is isomorphic to the set of $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $\sum a_i v_i = 0$), then by (3):

(33) $\quad c_1(\beta_I) = k - c_1 - \cdots - c_\ell$,

(34) $\quad \omega(\beta_I) = v_{i_1}(D_{i_1}) + \cdots + v_{i_k}(D_{i_k}) - c_1 v_{j_1}(D_{j_1}) - \cdots - c_\ell v_{j_\ell}(D_{j_\ell})$

$\qquad\qquad = \kappa_{i_1} + \cdots + \kappa_{i_k} - c_1 \kappa_{j_1} - \cdots - c_\ell \kappa_{j_\ell}$.

Denote by Λ_i the circle action corresponding to v_i, that is, Λ_i is the circle action whose moment map Φ_{Λ_i} is given by the composition of the moment map $\Phi : M \to \mathfrak{t}^*$ with the linear functional $v_i \in \mathfrak{t}$. Let

$$S^*(\Lambda_i) = y_i \otimes q^{-1} t^{-v_i(D_i)} \in QH^{\mathrm{ev}}(M, \omega)^\times$$

be the cohomological counterpart of the Seidel element. In [32] the authors show the following result.

Proposition 5.1. *Let $QH^*(M, \omega)$ denote the small quantum cohomology of the toric manifold (M, ω). The map Θ which sends Z_i to the Poincaré dual of $\Phi^{-1}(D_i)$ induces an isomorphism*

$$\mathbb{Q}[Z_1, \ldots, Z_n] \otimes \check{\Pi}/(\mathrm{Lin}(P) + SR_Y(P)) \cong QH^*(M, \omega),$$

where the ideal $\mathrm{Lin}(P)$ is generated by the linear relations

$$\mathrm{Lin}(P) = \Big\langle \sum (x, v_j) Z_j \mid x \in \mathfrak{t}_{\mathbb{Z}}^* \Big\rangle$$

and the ideal $SR_Y(P)$ is given by

(35)
$$SR_Y(P) = \Big\langle Y_{i_1} \cdots Y_{i_k} - Y_{j_1}^{c_1} \cdots Y_{j_\ell}^{c_\ell} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)}$$
$$\Big| I = \{i_1, \ldots, i_k\} \text{ is primitive} \Big\rangle,$$

where

(36)
$$Y_i = Z_i + \text{ higher order terms,}$$

is a lift of the Seidel element y_i in $\mathbb{Q}[Z_1, \ldots, Z_n] \otimes \check{\Pi}$, such that $\Theta(Y_i) = y_i$.

As McDuff and Tolman explain in [32], in general, it is not possible to find Y_i without prior knowledge of the ring structure on $QH^*(M, \omega)$ but, in special cases, we can indeed describe Y_i. In the Fano case the higher terms vanish and we may take $Y_i = Z_i$. In the NEF case there might be higher order terms in the Seidel elements y_i, however, from [32, Theorem 1.10] we know that the lifts Y_i of y_i are determined by some linear combination of the Z_i which is unique up to the additive relations $\mathrm{Lin}(P)$ (see [32, Example 5.4] for more details).

5.1.1. Fano case. In this case the Landau–Ginzburg superpotential is given by

$$W = \sum_{j=1}^{n} z^{v_j} t^{\kappa_j}$$

where for $v_j = (v_{j,1}, \ldots, v_{j,m}) \in \mathbb{Z}^m$ the term z^{v_j} represents the monomial $z_1^{v_{j,1}} \ldots z_m^{v_{j,m}}$.

We now recall a result obtained by Givental in [18] (which we illustrate with Ostrover–Tyomkin's formalism, see [33, Proposition 3.3]).

Theorem 5.2. *If (M, ω) is a symplectic Fano manifold, then*

$$QH^*(M, \omega) \cong \check{\Pi}[z_1^{\pm}, \ldots, z_m^{\pm}]/J_W \ as \ \check{\Pi}\text{–}algebras$$

and in particular

$$QH^0(M, \omega) \cong \check{\Pi}^{\mathrm{univ}}[z_1^{\pm}, \ldots, z_m^{\pm}]/J_W \ as \ \check{\Pi}^{\mathrm{univ}}\text{–}algebras$$

where J_W is the ideal generated by all partial derivatives of W.

In [33] the authors consider the natural homomorphism

$$\Psi : \mathbb{Q}[Z_1, \ldots, Z_n] \otimes \check{\Pi} \to \check{\Pi}[z_1^{\pm}, \ldots, z_m^{\pm}]$$

such that $SR_Y(P)$ is in the kernel of Ψ and the image of the additive relations gives the ideal J_W. In this case the homomorphism is defined by

$$\Psi(Z_j) = q z^{v_j} t^{\kappa_j}$$

and it is easy to see that Ψ satisfies the desired properties. Indeed, as we saw above, in the Fano case we may set $Y_i = Z_i$ hence

$$SR_Y(P) = \left\langle Z_{i_1} \cdots Z_{i_k} - Z_{j_1}^{c_1} \cdots Z_{j_\ell}^{c_\ell} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)} \right.$$
$$\left. \Big| \, I = \{i_1, \ldots, i_k\} \text{ is primitive} \right\rangle$$

and

$$\Psi(Z_{i_1} \cdots Z_{i_k} - Z_{j_1}^{c_1} \cdots Z_{j_\ell}^{c_\ell} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)})$$
$$= q^k z^{v_{i_1}} \cdots z^{v_{i_k}} t^{\kappa_{i_1} + \cdots + \kappa_{i_k}}$$
$$\quad - q^{c_1 + \cdots + c_\ell} z^{c_1 v_{j_1}} \cdots z^{c_\ell v_{j_\ell}} t^{c_1 \kappa_{j_1} + \cdots + c_\ell \kappa_{j_\ell}} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)}$$
$$= 0$$

by (33) and (34). Therefore the ideal $SR_Y(P)$ is in the kernel of Ψ.

The image of the additive relations is the following

$$\Psi \left(\sum_{j=1}^{n} (x, v_j) Z_j \right) = q \sum_{j=1}^{n} (x, v_j) z^{v_j} t^{\kappa_j}.$$

On the other hand, we have

$$q z_i \frac{\partial W}{\partial z_i} = q z_i \sum_{j=1}^{n} v_{j,i} \, z_1^{v_{j,1}} \cdots z_i^{v_{j,i}-1} \cdots z_m^{v_{j,m}} t^{\kappa_j} = q \sum_{j=1}^{n} v_{j,i} \, z^{v_j} t^{\kappa_j}.$$

Note that if $x = e_i$ is the i–th vector of the canonical base in $\mathbb{R}^n$ then $(x, v_j) = v_{j,i}$ and one obtains the desired result.

5.1.2. NEF case. In this subsection we give the explicit expression of the Landau–Ginzburg superpotential when M is a NEF 4–dimensional toric manifold for which at most 2 of the homology classes $A_i = [\Phi^{-1}(D_i)]$ of the pre-image of the facets D_i have vanishing first Chern number. It follows from the proof of the next proposition that the result generalizes to any number of classes (corresponding to facets of the polytope) with Chern number zero, but the expressions get more complicated as we increase the number of such classes. Moreover, Theorem 5.2 still holds for these cases.

Proposition 5.3. *If M is a NEF toric 4–manifold and $A_i = [\Phi^{-1}(D_i)]$ where D_i is a facet of the moment polytope then the Landau–Ginzburg superperpotential is given by the following expression:*

1) if c_1 vanishes only on the class A_k then

$$W = \sum_{j=1}^{n} z^{v_j} t^{\kappa_j} + z^{v_k} t^{\kappa_{k+1}+\kappa_{k-1}-\kappa_k},$$

2) if c_1 vanishes only on the classes A_{k-1} and A_k then

$$W = \sum_{j=1}^{n} z^{v_j} t^{\kappa_j} + z^{v_k} t^{\kappa_{k+1}+\kappa_{k-1}-\kappa_k} + z^{v_{k-1}} t^{\kappa_k+\kappa_{k-2}-\kappa_{k-1}}$$

$$+ z^{v_k} t^{\kappa_{k+1}+\kappa_{k-2}-\kappa_{k-1}} + z^{v_{k-1}} t^{\kappa_{k+1}+\kappa_{k-2}-\kappa_k}.$$

Proof. Case (1): in this case the Seidel elements are given by Theorem 4.5:

$$S(\Lambda_j) = A_j \otimes qt^{\kappa_j} \text{ if } j \neq k-1, k, k+1,$$

$$S(\Lambda_{k-1}) = A_{k-1} \otimes qt^{\kappa_{k-1}} - A_k \otimes q\frac{t^{\kappa_{k-1}-\omega(A_k)}}{1 - t^{-\omega(A_k)}},$$

$$S(\Lambda_k) = A_k \otimes q\frac{t^{\kappa_k}}{1 - t^{-\omega(A_k)}},$$

$$S(\Lambda_{k+1}) = A_{k+1} \otimes qt^{\kappa_{k+1}} - A_k \otimes q\frac{t^{\kappa_{k+1}-\omega(A_k)}}{1 - t^{-\omega(A_k)}}.$$

If S^* denotes the Seidel morphism in cohomology then we have

$$S^*(\Lambda_j) = Z_j \otimes q^{-1} t^{-\kappa_j} \ \text{ if } \ j \neq k-1, k, k+1,$$

$$S^*(\Lambda_{k-1}) = \left(Z_{k-1} - Z_k \otimes \frac{t^{\omega(A_k)}}{1 - t^{\omega(A_k)}} \right) \otimes q^{-1} t^{-\kappa_{k-1}},$$

$$S^*(\Lambda_k) = Z_k \otimes q^{-1} \frac{t^{-\kappa_k}}{1 - t^{\omega(A_k)}},$$

$$S^*(\Lambda_{k+1}) = \left(Z_{k+1} - Z_k \otimes \frac{t^{\omega(A_k)}}{1 - t^{\omega(A_k)}} \right) \otimes q^{-1} t^{-\kappa_{k+1}}.$$

Thus in equation (36) we may take

$$Y_j = Z_j \ \text{ if } \ j \neq k-1, k, k+1, \quad Y_k = Z_k \otimes \frac{1}{1 - t^{\omega(A_k)}},$$

$$Y_{k-1} = Z_{k-1} - Z_k \otimes \frac{t^{\omega(A_k)}}{1 - t^{\omega(A_k)}}, \quad Y_{k+1} = Z_{k+1} - Z_k \otimes \frac{t^{\omega(A_k)}}{1 - t^{\omega(A_k)}}$$

where $\omega(A_k) = \kappa_{k+1} + \kappa_{k-1} - 2\kappa_k$. In this case, the definition of the homomorphism Ψ is such that

(37) $$\forall\, 1 \leq j \leq n, \quad \Psi(Y_j) = q z^{v_j} t^{\kappa_j}$$

so one obtains

$$\Psi(Z_j) = q z^{v_j} t^{\kappa_j} \ \text{ if } \ j \neq k-1, k, k+1 \,,$$

$$\Psi(Z_{k-1}) = q z^{v_{k-1}} t^{\kappa_{k-1}} + q z^{v_k} t^{\kappa_{k+1} + \kappa_{k-1} - \kappa_k} \,,$$

$$\Psi(Z_k) = q z^{v_k} t^{\kappa_k} (1 - t^{\omega(A_k)}) = q z^{v_k} t^{\kappa_k} - q z^{v_k} t^{\kappa_{k+1} + \kappa_{k-1} - \kappa_k} \,,$$

$$\Psi(Z_{k+1}) = q z^{v_{k+1}} t^{\kappa_{k+1}} + q z^{v_k} t^{\kappa_{k+1} + \kappa_{k-1} - \kappa_k} \,.$$

It is clear, by definition of Ψ and the proof in the Fano case that $SR_Y(P)$ is in the kernel of the homomorphism. Computing the image of the additive relations gives

$$\Psi\left(\sum_{j=1}^n (x, v_j) Z_j \right) = q \sum_{j=1}^n (x, v_j) z^{v_j} t^{\kappa_j} - q(x, v_k) z^{v_k} t^{\kappa_{k+1} + \kappa_{k-1} - \kappa_k}$$

$$+ q(x, v_{k-1}) z^{v_k} t^{\kappa_{k+1} + \kappa_{k-1} - \kappa_k} + q(x, v_{k+1}) z^{v_k} t^{\kappa_{k+1} + \kappa_{k-1} - \kappa_k}.$$

In order to obtain the derivatives of the potential we need

$$(x, v_{k-1}) + (x, v_{k+1}) - (x, v_k) = (x, v_k), \ \text{ that is, } \ v_{k-1} + v_{k+1} = 2v_k,$$

which holds, if dim $M = 4$ and $c_1(A_k) = 0$, as noticed already in Section 3.1, Equation (8).

Case (2): In this case Theorem 4.5 gives the following:

$$Y_j = Z_j \ \text{ if } \ j \neq k-2, k-1, k, k+1\,,$$

$$Y_{k-2} = Z_{k-2} - Z_{k-1} \otimes \frac{t^{\omega(A_{k-1})+\omega(A_k)}}{(1 - t^{\omega(A_{k-1})})(1 - t^{\omega(A_{k-1})+\omega(A_k)})}$$

$$- Z_{k-1} \otimes \frac{t^{\omega(A_{k-1})}}{1 - t^{\omega(A_{k-1})}} + Z_k \otimes \frac{t^{\omega(A_{k-1})+2\omega(A_k)}}{(1 - t^{\omega(A_k)})(1 - t^{\omega(A_{k-1})+\omega(A_k)})}\,,$$

$$Y_{k-1} = \left(Z_{k-1} \otimes \frac{1}{1 - t^{\omega(A_{k-1})}} - Z_k \otimes \frac{t^{\omega(A_k)}}{1 - t^{\omega(A_k)}} \right) \frac{1}{1 - t^{\omega(A_{k-1})+\omega(A_k)}}\,,$$

$$Y_k = \left(Z_k \otimes \frac{1}{1 - t^{\omega(A_k)}} - Z_{k-1} \otimes \frac{t^{\omega(A_{k-1})}}{1 - t^{\omega(A_{k-1})}} \right) \frac{1}{1 - t^{\omega(A_{k-1})+\omega(A_k)}}\,,$$

$$Y_{k+1} = Z_{k+1} - Z_k \otimes \frac{t^{\omega(A_k)}}{1 - t^{\omega(A_k)}} - Z_k \otimes \frac{t^{\omega(A_{k-1})+\omega(A_k)}}{(1 - t^{\omega(A_k)})(1 - t^{\omega(A_{k-1})+\omega(A_k)})}$$

$$+ Z_{k-1} \otimes \frac{t^{\omega(A_k)+2\omega(A_{k-1})}}{(1 - t^{\omega(A_{k-1})})(1 - t^{\omega(A_{k-1})+\omega(A_k)})}\,.$$

Therefore, as above, if we define Ψ such that it satisfies (37) then we obtain

$$\Psi(Z_j) = q z^{v_j} t^{\kappa_j} \ \text{ if } \ j \neq k-2, k-1, k, k+1,$$

$$\Psi(Z_{k-2}) = q z^{v_{k-2}} t^{\kappa_{k-2}} + q z^{v_k} t^{\kappa_{k+1,k-1}+\kappa_{k-2}} + q z^{v_{k-1}} t^{\kappa_{k-2}} \left(t^{\kappa_{k,k-1}} + t^{\kappa_{k+1,k}} \right),$$

$$\Psi(Z_{k-1}) = q z^{v_{k-1}} t^{\kappa_{k-1}} \left(1 - t^{\kappa_{k,k-1}+\kappa_{k-2,k-1}} \right) + q z^{v_k} t^{\kappa_{k+1}} \left(t^{\kappa_{k-1,k}} - t^{\kappa_{k-2,k-1}} \right),$$

$$\Psi(Z_k) = q z^{v_k} t^{\kappa_k} \left(1 - t^{\kappa_{k+1,k}+\kappa_{k-1,k}} \right) + q z^{v_{k-1}} t^{\kappa_{k-2}} \left(t^{\kappa_{k,k-1}} - t^{\kappa_{k+1,k}} \right),$$

$$\Psi(Z_{k+1}) = q z^{v_{k+1}} t^{\kappa_{k+1}} + q z^{v_{k-1}} t^{\kappa_{k+1,k}+\kappa_{k-2}} + q z^{v_k} t^{\kappa_{k+1}} \left(t^{\kappa_{k-1,k}} + t^{\kappa_{k-2,k-1}} \right)$$

where $\kappa_{i,j} = \kappa_i - \kappa_j$. Again, it is clear that $SR_Y(P)$ is in the kernel of the homomorphism and it is not hard to check that the image of the additive relations gives the derivatives of the superpotential, under the assumptions that $v_{k-1} + v_{k+1} = 2v_k$ and $v_{k-2} + v_k = 2v_{k-1}$. $\qquad\square$

Remark 5.4. Note that in the NEF case we know by, for example, the previous works of [8] and [19] that the Seidel elements completely determine the potential so we cannot have constant terms (terms containing only the variable t) in the potential. In general, in the non-NEF case, we do not know if that holds. Moreover, we might have an infinite number of (constant) terms in the potential, as explained to us by K. Chan. See also Remark 5.6.

5.2. NEF examples: The case of a blow–up of $\mathbb{CP}^2$ at 4 or 5 points

In this section, as an application of our results, we compute explicitly the small quantum cohomology (and homology) of the manifold obtained from $\mathbb{CP}^2$ by performing 4 and 5 blow-ups, $\mathbb{X}_4$ and $\mathbb{X}_5$ respectively. Note that these manifolds admit NEF almost complex structures, but no Fano ones. Since the computations are similar, we show the full computations for $\mathbb{X}_4$ and only give the final result for $\mathbb{X}_5$. As already noticed in Example 1.2, $\mathbb{X}_4$ is symplectomorphic to the 3–point blow-up of $S^2 \times S^2$ endowed with the split symplectic form ω_μ for which the symplectic area of the first factor is μ and the area of the second factor is 1 (see [3, Section 2.1] for more details). Let c_i, $i = 1, \ldots, 4$ be the capacities of the blow-ups. Let B, $F \in H_2(\mathbb{X}_4; \mathbb{Z})$ be the homology classes defined by $B = [S^2 \times \{p\}]$, $F = [\{p\} \times S^2]$ and let $E_i \in H_2(\mathbb{X}_4; \mathbb{Z})$ be the exceptional class corresponding to the blow-up of capacity c_i. Consider $\mathbb{X}_4$ endowed with the standard action of the torus $T = S^1 \times S^1$ for which the moment polytope is given by

$$P = \big\{ (x_1, x_2) \in \mathbb{R}^2 \,|\, 0 \leq x_2 \leq \mu, \, x_2 + x_1 \leq \mu - c_3,$$
$$-1 \leq x_1 \leq 0, \, c_1 \leq x_2 - x_1 \leq \mu + 1 - c_2 \big\}$$

so the primitive outward normals to P are as follows:

$$v_1 = (0, 1), \quad v_2 = (1, 1), \quad v_3 = (1, 0), \quad v_4 = (1, -1), \quad v_5 = (0, -1),$$
$$v_6 = (-1, 0), \quad \text{and} \quad v_7 = (-1, 1).$$

The normalised moment map $\Phi : \mathbb{X}_4 \to \mathbb{R}^2$ is given by

$$\Phi(z_1, \ldots, z_7) = \left(-\frac{1}{2}|z_3|^2 + \epsilon_1, -\frac{1}{2}|z_1|^2 + \mu - \epsilon_2 \right),$$

where

$$\epsilon_1 = \frac{c_1^3 + 3c_2^2 - c_2^3 + c_3^3 - 3\mu}{3(c_1^2 + c_2^2 + c_3^2 - 2\mu)} \quad \text{and} \quad \epsilon_2 = \frac{c_1^3 - c_2^3 - c_3^3 + 3c_2^2\mu + 3c_3^2\mu - 3\mu^2}{3(c_1^2 + c_2^2 + c_3^2 - 2\mu)}.$$

Moreover, the homology classes $A_i = [\Phi^{-1}(D_i)]$ of the pre-images of the corresponding facets D_i are: $A_1 = F - E_2 - E_3$, $A_2 = E_3$, $A_3 = B - E_1 - E_3$, $A_4 = E_1$, $A_5 = F - E_1$, $A_6 = B - E_2$, and $A_7 = E_2$. Let Λ_i be the circle action associated to v_i. Since the complex structure on $\mathbb{X}_4$ is NEF and T–invariant, it follows from Theorem 4.5 that the Seidel elements associated

to these actions are given by the following expressions

$$S(\Lambda_1) = (F - E_2 - E_3) \otimes q\frac{t^{\mu-\epsilon_2}}{1 - t^{c_2+c_3-1}},$$

$$S(\Lambda_2) = E_3 \otimes qt^{\mu-c_3+\epsilon_1-\epsilon_2} - (F - E_2 - E_3) \otimes q\frac{t^{\mu+c_2-1+\epsilon_1-\epsilon_2}}{1 - t^{c_2+c_3-1}}$$

$$- (B - E_1 - E_3) \otimes q\frac{t^{c_1+\epsilon_1-\epsilon_2}}{1 - t^{c_1+c_3-\mu}},$$

$$S(\Lambda_3) = (B - E_1 - E_3) \otimes q\frac{t^{\epsilon_1}}{1 - t^{c_1+c_3-\mu}},$$

$$S(\Lambda_4) = E_1 \otimes qt^{\epsilon_1+\epsilon_2-c_1} - (B - E_1 - E_3) \otimes q\frac{t^{\epsilon_1+\epsilon_2+c_3-\mu}}{1 - t^{c_1+c_3-\mu}},$$

$$S(\Lambda_5) = (F - E_1) \otimes qt^{\epsilon_2}, \qquad S(\Lambda_6) = (B - E_2) \otimes qt^{1-\epsilon_1},$$

$$S(\Lambda_7) = E_2 \otimes qt^{\mu+1-c_2-\epsilon_1-\epsilon_2} - (F - E_2 - E_3) \otimes q\frac{t^{\mu+c_3-\epsilon_1-\epsilon_2}}{1 - t^{c_2+c_3-1}}.$$

Therefore we have

$$S^*(\Lambda_1) = Z_1 \otimes q^{-1}\frac{t^{\epsilon_2-\mu}}{1 - t^{1-c_2-c_3}},$$

$$S^*(\Lambda_2) = Z_2 \otimes q^{-1}t^{c_3-\mu-\epsilon_1+\epsilon_2} - Z_1 \otimes q^{-1}\frac{t^{1-\mu-c_2-\epsilon_1+\epsilon_2}}{1 - t^{1-c_2-c_3}}$$

$$- Z_3 \otimes q^{-1}\frac{t^{\epsilon_2-\epsilon_1-c_1}}{1 - t^{\mu-c_1-c_3}},$$

$$S^*(\Lambda_3) = Z_3 \otimes q^{-1}\frac{t^{-\epsilon_1}}{1 - t^{\mu-c_1-c_3}},$$

$$S^*(\Lambda_4) = Z_4 \otimes q^{-1}t^{c_1-\epsilon_1-\epsilon_2} - Z_3 \otimes q^{-1}\frac{t^{\mu-c_3-\epsilon_1-\epsilon_2}}{1 - t^{\mu-c_1-c_3}},$$

$$S^*(\Lambda_5) = Z_5 \otimes q^{-1}t^{-\epsilon_2}, \qquad S^*(\Lambda_6) = Z_6 \otimes q^{-1}t^{\epsilon_1-1},$$

$$S^*(\Lambda_7) = Z_7 \otimes q^{-1}t^{c_2-\mu-1+\epsilon_1+\epsilon_2} - Z_1 \otimes q^{-1}\frac{t^{\epsilon_1+\epsilon_2-\mu-c_3}}{1 - t^{1-c_2-c_3}}.$$

Thus in equation (36) we may take

$$Y_1 = Z_1 \otimes \frac{1}{1 - t^{1-c_2-c_3}},$$

$$Y_2 = Z_2 - Z_1 \otimes \frac{t^{1-c_2-c_3}}{1 - t^{1-c_2-c_3}} - Z_3 \otimes \frac{t^{\mu-c_1-c_3}}{1 - t^{\mu-c_1-c_3}},$$

$$Y_3 = Z_3 \otimes \frac{1}{1 - t^{\mu - c_1 - c_3}}, \quad Y_4 = Z_4 - Z_3 \otimes \frac{t^{\mu - c_1 - c_3}}{1 - t^{\mu - c_1 - c_3}}, \quad Y_5 = Z_5,$$

$$Y_6 = Z_6, \quad \text{and} \quad Y_7 = Z_7 - Z_1 \otimes \frac{t^{1 - c_2 - c_3}}{1 - t^{1 - c_2 - c_3}}.$$

There are fourteen primitive sets: $\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,4\}, \{2,5\},$ $\{2,6\}, \{2,7\}, \{3,5\}, \{3,6\}, \{3,7\}, \{4,6\}, \{4,7\}, \{5,7\}$.

Let $t_{2,3} = 1 - t^{1 - c_2 - c_3}$ and $t_{1,3} = 1 - t^{\mu - c_1 - c_3}$. The corresponding multiplicative relations for $QH^*(\mathbb{X}_4, \omega_\mu)$, that is, the generators of the ideal $SR_Y(P)$ defined in Proposition 5.1, can be written as follows

(38)
$$Z_1 Z_3 = Z_2 \otimes qt^{c_3} t_{2,3} t_{1,3} - Z_1 \otimes qt^{1 - c_2} t_{1,3} - Z_3 \otimes qt^{\mu - c_1} t_{2,3},$$

$$Z_1 Z_4 t_{1,3} = Z_1 Z_3 \otimes t^{\mu - c_1 - c_3} + Z_3 \otimes qt^{\mu - c_1} t_{2,3},$$

$$Z_1 Z_5 = \mathbb{1} \otimes q^2 t^\mu t_{2,3},$$

$$Z_1 Z_6 = Z_7 \otimes qt^{c_2} t_{2,3} - Z_1 \otimes qt^{1 - c_3},$$

$$Z_2 Z_4 t_{2,3} t_{1,3} = Z_3(Z_2 + Z_3 + Z_4) \otimes t^{\mu - c_1 - c_3} t_{2,3} + Z_1 Z_4 \otimes t^{1 - c_2 - c_3} t_{1,3}$$
$$- Z_1 Z_3 \otimes t^{1 + \mu - c_1 - c_2 - 2c_3},$$

$$Z_2 Z_5 t_{2,3} t_{1,3} = Z_1 Z_5 \otimes t^{1 - c_2 - c_3} t_{1,3} + Z_3 Z_5 \otimes t^{\mu - c_1 - c_3} t_{2,3} + Z_3 \otimes qt^{\mu - c_3} t_{2,3},$$

$$Z_2 Z_6 t_{2,3} t_{1,3} = Z_1 Z_6 \otimes t^{1 - c_2 - c_3} t_{1,3} + Z_3 Z_6 \otimes t^{\mu - c_1 - c_3} t_{2,3} + Z_1 \otimes qt^{1 - c_3} t_{1,3},$$

$$Z_2 Z_7 t_{2,3} t_{1,3} = Z_1(Z_1 + Z_2 + Z_7) \otimes t^{1 - c_2 - c_3} t_{1,3} + Z_3 Z_7 \otimes t^{\mu - c_1 - c_3} t_{2,3}$$
$$- Z_1 Z_3 \otimes t^{1 + \mu - c_1 - c_2 - 2c_3},$$

$$Z_3 Z_5 = Z_4 \otimes qt^{c_1} t_{1,3} - Z_3 \otimes qt^{\mu - c_3},$$

$$Z_3 Z_6 = \mathbb{1} \otimes q^2 tt_{1,3},$$

$$Z_3 Z_7 t_{2,3} = Z_1 Z_3 \otimes t^{1 - c_2 - c_3} + Z_1 \otimes qt^{1 - c_2} t_{1,3},$$

$$Z_4 Z_6 t_{1,3} = Z_5 \otimes qt^{1 - c_1} t_{1,3} + Z_3 Z_6 \otimes t^{\mu - c_1 - c_3},$$

$$Z_4 Z_7 t_{2,3} t_{1,3} = Z_1 Z_4 \otimes t^{1 - c_2 - c_3} t_{1,3} + Z_3 Z_7 \otimes t^{\mu - c_1 - c_3} t_{2,3}$$
$$- Z_3 Z_1 \otimes qt^{1 + \mu - c_1 - c_2 - 2c_3} + \mathbb{1} \otimes q^2 t^{\mu + 1 - c_1 - c_2} t_{2,3} t_{1,3},$$

$$Z_5 Z_7 t_{2,3} = Z_1 Z_5 \otimes t^{1 - c_2 - c_3} + Z_6 \otimes qt^{\mu - c_2} t_{2,3}$$

where we should also take in account the additive relations $Z_6 = Z_1 + 2Z_2 + Z_3 - Z_5$ and $Z_7 = -Z_1 - Z_2 + Z_4 + Z_5$. It follows from Proposition 5.1 that $QH^*(\mathbb{X}_4, \omega_\mu)$ is isomorphic as a ring to $\mathbb{Q}[Z_1, \ldots, Z_n] \otimes \check{\Pi}/I$ where I is the ideal generated by the relations above. We can describe the result also in terms of homology. For that consider the homology classes $A_i = [\Phi^{-1}(D_i)] \in H_2(\mathbb{X}_4; \mathbb{Z})$. They are additive generators of $H_2(\mathbb{X}_4; \mathbb{Z})$ and multiplicative generators of $QH_*(\mathbb{X}_4, \omega_\mu)$. Moreover $QH_4(\mathbb{X}_4, \omega_\mu)$ is generated, as a subring

of $QH_*(\mathbb{X}_4, \omega_\mu)$, by the elements qA_i. These generators are $E_i \otimes q$, where $i = 1, 2, 3$, $(F - E_1) \otimes q$, $(B - E_2) \otimes q$, $(F - E_2 - E_3) \otimes q$, and $(B - E_1 - E_3) \otimes q$. In what follows in order to simplify notation we shall drop the sign $*$ for the quantum product. The multiplicative relations (38) translated to homology together with the additive relations give a complete description of the Π^{univ}–algebra $QH_4(\mathbb{X}_4, \omega_\mu)$. More precisely, we obtain

$$QH_4(\mathbb{X}_4, \omega_\mu) \cong \Pi^{\text{univ}}[u, v]/J$$

where

$$u = (F - E_2 - E_3) \otimes q \frac{1}{1 - t^{c_2 + c_3 - 1}}, \quad v = (B - E_1 - E_3) \otimes q \frac{1}{1 - t^{c_1 + c_3 - \mu}}$$

and J is the ideal generated by the two following relations:

(39) $$v(1 + vt^{c_1}) = u^2 t^\mu (v + t^{c_2 - 1})(1 + vt^{c_3}),$$

(40) $$u(1 + ut^{c_2}) = v^2 t(u + t^{c_1 - \mu})(1 + ut^{c_3}).$$

It follows from Proposition 5.3 (1) that the Landau–Ginzburg superpotential is given in this example by

(41) $$W = z_2 t^\mu + z_1 z_2 t^{\mu - c_3} + z_1 + z_1 z_2^{-1} t^{-c_1} + z_2^{-1} + z_1^{-1} t + z_1^{-1} z_2 t^{\mu + 1 - c_2}$$
$$+ z_1 t^{\mu - c_1 - c_3} + z_2 t^{\mu + 1 - c_2 - c_3}.$$

Therefore we have

$$\frac{\partial W}{\partial z_1} = z_2 t^{\mu - c_3} + 1 + z_2^{-1} t^{-c_1} - z_1^{-2} t - z_1^{-2} z_2 t^{\mu + 1 - c_2} + t^{\mu - c_1 - c_3},$$

$$\frac{\partial W}{\partial z_2} = t^\mu + z_1 t^{\mu - c_3} - z_1 z_2^{-2} t^{-c_1} - z_2^{-2} + z_1^{-1} t^{\mu + 1 - c_2} + t^{\mu + 1 - c_2 - c_3}.$$

Passing to homology, simplifying the expressions and setting $u = z_2^{-1} t^{-\mu}$ and $v = z_1^{-1}$ we obtain relations (39), as we wish.

Similar arguments give an explicit description of the quantum homology algebra $QH_4(\mathbb{X}_5, \omega_\mu)$. Moreover, we have

$$QH_4(\mathbb{X}_5, \omega_\mu) \cong \Pi^{\text{univ}}[u, v]/J$$

where again $u = (F - E_2 - E_3) \otimes q(1 - t^{c_2 + c_3 - 1})^{-1}$, $v = (B - E_1 - E_3) \otimes q(1 - t^{c_1 + c_3 - \mu})^{-1}$ and J is now the ideal generated by the two following

relations:

$$u^2 t^\mu (v + t^{c_2 - 1})(1 + vt^{c_3}) = (1 + vt^{c_1})(v + t^{c_4 - 1}),$$
$$v^2 t(u + t^{c_1 - \mu})(1 + ut^{c_3}) = (1 + ut^{c_2})(u + t^{c_4 - \mu}).$$

In this case the Landau–Ginzburg superpotential is given by

$$W = z_2 t^\mu + z_1 z_2 t^{\mu - c_3} + z_1 + z_1 z_2^{-1} t^{-c_1} + z_2^{-1} + z_1^{-1} z_2^{-1} t^{1 - c_4} + z_1^{-1} t$$
$$+ z_1^{-1} z_2 t^{\mu + 1 - c_2} + z_1 t^{\mu - c_1 - c_3} + z_2 t^{\mu + 1 - c_2 - c_3}$$
$$+ z_1^{-1} t^{\mu + 1 - c_2 - c_4} + z_2^{-1} t^{1 - c_1 - c_4}.$$

Remark 5.5. Note that these results agree with the results of Chan and Lau. The manifolds $\mathbb{X}_4$ and $\mathbb{X}_5$ coincide with the surfaces X_7 and X_{10}, respectively, described in [7, Appendix A]. We obtain the same expressions for the potential after changes of variable: replacing z_2 by $z_1 z_2^{-1} t^{-c_1}$, keeping the variable z_1 and letting $q_1 = t^{\mu - c_1 - c_3}$, $q_2 = t^{\mu - c_2}$, $q_3 = t^{c_2}$, $q_4 = t^{1 - c_2 - c_3}$ and $q_5 = t^{c_3}$ in the potential for X_7 leads to (41) above. Similarly, making the same change of variable for X_{10} and letting $q_1 = t^{\mu - c_1 - c_3}$, $q_2 = t^{c_4}$, $q_3 = t^{\mu - c_2 - c_4}$, $q_4 = t^{c_2}$, $q_5 = t^{1 - c_2 - c_3}$ and $q_6 = t^{c_3}$ we see again that the two expressions for the potential agree.

5.3. Non-NEF examples

Particularly interesting examples which are relevant for our study are the Hirzebruch surfaces. We use the conventions and the description adopted in [3] for these surfaces. We recall that the toric "even" Hirzebruch surfaces $(\mathbb{F}_{2k}, \omega_\mu)$, $0 \le k \le \ell$ with $\ell \in \mathbb{N}$ and $\ell < \mu \le \ell + 1$, can be identified with the symplectic manifolds $(S^2 \times S^2, \omega_\mu)$ where ω_μ is the split symplectic form with area $\mu \ge 1$ for the first S^2–factor, and with area 1 for the second factor. The moment polytope of $\mathbb{F}_{2k}$ is

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1, \ x_2 + kx_1 \ge 0, \ x_2 - kx_1 \le \mu - k \right\}$$

and its primitive outward normals are

$$v_1 = (1, 0), \ v_2 = (-k, -1), \ v_3 = (-1, 0), \ \text{and} \ v_4 = (-k, 1).$$

Let $\Lambda_{e_1}^{2k}$ and $\Lambda_{e_2}^{2k}$ represent the circle actions whose moment maps are, respectively, the first and the second component of the moment map associated to the torus action T_{2k} acting on $\mathbb{F}_{2k}$. We will also denote by $\Lambda_{e_1}^{2k}, \Lambda_{e_2}^{2k}$

the generators in $\pi_1(T_{2k})$. It follows from the classification of 4–dimensional Hamiltonian S^1–spaces given by Karshon in [26] that $\Lambda^{2k}_{e_1}, \Lambda^{2k}_{e_2}$ satisfy the relations $\Lambda^{2k}_{e_1} = k\Lambda^2_{e_1} + (k-1)\Lambda^0_{e_1}$ and $\Lambda^{2k}_{e_2} = k\Lambda^0_{e_1} + \Lambda^0_{e_2}$. Since $\mathbb{F}_0$ is Fano and $\mathbb{F}_2$ is NEF we can obtain from our results the Seidel elements associated to $\Lambda^0_{e_1}$, $\Lambda^0_{e_2}$, and $\Lambda^2_{e_1}$, and thus the ones associated to the circle actions of $\mathbb{F}_{2k}$ even though for all $k \geq 2$, $\mathbb{F}_{2k}$ is non-NEF.

In particular, we can give explicit expressions for the Seidel elements associated to $\mathbb{F}_4$ which admits a pseudo-holomorphic sphere with negative first Chern number, representing the class $B - 2F$ where $B = [S^2 \times \{p\}]$, and $F = [\{p\} \times S^2]$. Since $\mathbb{F}_0$ is Fano it is easy to check that the Seidel elements associated to the circle actions $\Lambda^0_{e_1}$ and $\Lambda^0_{e_2}$ are given by $S(\Lambda^0_{e_1}) = B \otimes qt^{\frac{1}{2}}$ and $S(\Lambda^0_{e_2}) = F \otimes qt^{\frac{\mu}{2}}$ (see [32, Example 5.7]). From this case we can also obtain the following products in the quantum homology ring: $F * F = \mathbb{1} \otimes q^{-2}t^{-\mu}$, $B * B = \mathbb{1} \otimes q^{-2}t^{-1}$, $F * B = p$ and deduce the remaining products from these ones.

For the toric manifold $\mathbb{F}_2$ the normalised moment map is given by

$$\Phi(z_1, z_2, z_3, z_4) = \left(-\frac{1}{2}|z_1|^2 + \frac{1}{2} - \epsilon, -\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_4|^2 + \frac{\mu + 1}{2} \right),$$

where $\epsilon = \frac{1}{6\mu}$. Let $\Lambda^{2k}_{v_i}$ denote the circle action associated to the normal vector v_i to the polytope of the surface $\mathbb{F}_{2k}$. Then Theorem 4.5 implies that, in the case of $\mathbb{F}_2$, the Seidel elements associated to these actions are given by

$$S(\Lambda^2_{v_1}) = (B + F) \otimes qt^{\frac{1}{2} - \epsilon}, \ S(\Lambda^2_{v_3}) = (B - F) \otimes q\frac{t^{\frac{1}{2} + \epsilon}}{1 - t^{1 - \mu}} \quad \text{and}$$

$$S(\Lambda^2_{v_2}) = S(\Lambda^2_{v_4}) = F \otimes qt^{\frac{\mu}{2} + \epsilon} - (B - F) \otimes q\frac{t^{1 - \frac{\mu}{2} + \epsilon}}{1 - t^{1 - \mu}}.$$

Since $\Lambda^2_{e_1} = \Lambda^2_{v_1}$, $S(\Lambda^2_{e_1}) = S(\Lambda^2_{v_1})$ and it follows that for the non-NEF toric manifold $\mathbb{F}_4$ the Seidel elements associated to the circle actions $\Lambda^4_{e_1}$ and $\Lambda^4_{e_2}$ are given by

$$S(\Lambda^4_{e_1}) = S(\Lambda^2_{e_1})^2 * S(\Lambda^0_{e_1}) = (B + 2F) \otimes qt^{\frac{1}{2} - 2\epsilon} + B \otimes qt^{\frac{3}{2} - \mu - 2\epsilon},$$
$$S(\Lambda^4_{e_2}) = S(\Lambda^0_{e_1})^2 * S(\Lambda^0_{e_2}) = S(\Lambda^0_{e_2}) = F \otimes qt^{\frac{\mu}{2}},$$

because $S(\Lambda^0_{e_1})^2 = \mathbb{1}$. Therefore in this case, since $\Lambda^4_{e_1} = \Lambda^4_{v_1}$, it follows that

$$S(\Lambda^4_{v_1}) = qt^{\frac{1}{2} - 2\epsilon} \otimes (B + 2F + B \otimes t^{1 - \mu}).$$

Since $v_1 + v_3 = 0$ we have $S(\Lambda_{v_3}^4) = S(\Lambda_{e_1}^4)^{-1} = \left(S(\Lambda_{e_1}^2)^{-1}\right)^2 * S(\Lambda_{e_1}^0)^{-1}$ and since

$$S(\Lambda_{e_1}^2)^{-1} = (B - F) \otimes q \frac{t^{\frac{1}{2}+\epsilon}}{1 - t^{1-\mu}}$$

we obtain

$$S(\Lambda_{v_3}^4) = \frac{qt^{\frac{1}{2}+2\epsilon}}{(1 - t^{1-\mu})^2} \otimes \left[B - 2F + B \otimes t^{1-\mu}\right].$$

Finally, since $v_4 = 2v_3 + (0, 1)$, $v_2 = 2v_3 + (0, -1)$, and $S(\Lambda_{e_2}^4) = S(\Lambda_{e_2}^4)^{-1}$ it follows that

$$S(\Lambda_{v_2}^4) = S(\Lambda_{v_4}^4) = S(\Lambda_{v_3}^4)^2 * S(\Lambda_{e_2}^4)$$
$$= \frac{qt^{\frac{\mu}{2}+4\epsilon}}{(1 - t^{1-\mu})^4} \left[F \otimes (1 - t^{1-\mu})^2 - 4t^{1-\mu}(B - 2F + B \otimes t^{1-\mu})\right].$$

It follows that in equation (36) we may take

$$Y_1 = Z_1 + (Z_3 + Z_2 + Z_4) \otimes t^{\mu-1}, \quad Y_2 = \frac{1}{(1 - t^{\mu-1})^2}(Z_2 - 4t^{\mu-1}Y_3)$$

$$Y_3 = \frac{1}{(1 - t^{\mu-1})^2}(Z_3 + (Z_3 + Z_2 + Z_4) \otimes t^{\mu-1}),$$

$$Y_4 = \frac{1}{(1 - t^{\mu-1})^2}(Z_4 - 4t^{\mu-1}Y_3).$$

Since the ring structure on the quantum homology is known we can check that this choice of Y_i satisfies the equations induced by the primitive relations, that is,

$$Y_1 Y_3 - \mathbb{1} \otimes q^2 t \quad \text{and} \quad Y_2 Y_4 - (Y_3)^4 \otimes q^{-2} t^{\mu-2}$$

are generators of the ideal $SR_Y(P)$. In order to have a potential W such that the isomorphism in Theorem 5.2 holds we need that the homomorphism Ψ, inducing the isomorphism, satisfies equations (37). Recall that the generators of the ideal $SR_Y(P)$ should be in the kernel of Ψ and the image of the additive relations gives the derivatives of the potential.

(42)
$$\Psi(Y_1) = qz_1 t \iff \Psi(Z_1) + \Psi(Z_2 + Z_3 + Z_4)t^{\mu-1} = qz_1 t$$
$$\Psi(Y_2) = qz_1^{-2} z_2^{-1} \iff \Psi(Z_2) - 4t^{\mu-1}\Psi(Y_3) = qz_1^{-2} z_2^{-1}(1 - t^{\mu-1})^2$$
$$\Psi(Y_3) = qz_1^{-1} \iff \Psi(Z_3) + \Psi(Z_2 + Z_3 + Z_4)t^{\mu-1} = qz_1^{-1}(1 - t^{\mu-1})^2$$
$$\Psi(Y_4) = qz_1^{-2} z_2 t^{\mu-2} \iff \Psi(Z_4) - 4t^{\mu-1}\Psi(Y_3) = qz_1^{-2} z_2 t^{\mu-2}(1 - t^{\mu-1})^2$$

Since the additive relations are $Z_1 - Z_3 - 2Z_2 - 2Z_4 = 0$ and $Z_4 - Z_2 = 0$ it follows from equations (42) that the derivatives of the potential W are given by the following expressions:

$$qz_1 \frac{\partial W}{\partial z_1} = \Psi(Z_1) - \Psi(Z_3) - 2\Psi(Z_2) - 2\Psi(Z_4) =$$

$$qz_1 t - qz_1^{-1}(1 - t^{\mu-1})^2 - 16qz_1^{-1}t^{\mu-1} - 2(qz_1^{-2}z_2^{-1} + qz_1^{-2}z_2 t^{\mu-2})(1 - t^{\mu-1})^2$$

$$qz_2 \frac{\partial W}{\partial z_2} = \Psi(Z_4) - \Psi(Z_2) = (qz_1^{-2}z_2 t^{\mu-2} - qz_1^{-2}z_2^{-1})(1 - t^{\mu-1})^2.$$

Therefore the potential is given by

$$(43) \qquad W = z_1 t + (z_1^{-1} + z_1^{-2}z_2^{-1} + z_1^{-2}z_2 t^{\mu-2})(1 - t^{\mu-1})^2 + 16z_1^{-1}t^{\mu-1},$$

up to constant terms, as we pointed out in Remark 5.4.

Remark 5.6. In this non-NEF example we see that the number of terms corresponding to the quantum corrections in the Landau–Ginzburg super-potential is still finite. In the formalism of [7] and [8] the primitive rays of the fan (or the interior normal vectors of the polytope) are given by $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, -4)$, and $v_4 = (0, -1)$ and the polytope is defined by the following inequalities

$$x_1 \geq 0, \ x_2 \geq 0, \ 4t_1 + t_2 - x_1 - 4x_2 \geq 0 \text{ and } t_1 - x_2 \geq 0,$$

where the t_l's are positive numbers. Let $q_l = \exp(-t_l)$ be the Kähler parameters. Then, in their formalism, the potential is given by

$$W = z_1(1 - 2q_1 q_2 + q_1^2 q_2^2) + z_2 + \frac{q_1^4 q_2}{z_1 z_2^4}(1 - 2q_1 q_2 + q_1^2 q_2^2)$$

$$+ \frac{q_1}{z_2}(1 + 14q_1 q_2 + q_1^2 q_2^2).$$

In this expression z_1 and z_2 correspond to $z_1^{-2}z_2^{-1}$ and $z_1 t$, respectively, in equation (43) while $q_1 = t$ and $q_2 = t^{\mu-2}$. Moreover, if [8, Conjecture 6.7] holds then we can obtain some of the open Gromov–Witten invariants of $\mathbb{F}_4$ from our computation of the potential. In particular we see that there must be some negative open Gromov–Witten invariants, phenomenon which does never happen in the NEF case. Note that if this conjecture holds it implies that the Seidel elements only determine the non-constant terms of the potential (see Remark 5.4). Indeed it is natural to expect the potential

of this example to contain an infinite number of terms, as explained to us by K. Chan, so these only contain the variable t.

We conclude that, even in this non-NEF example, although there are infinitely many contributions to the Seidel elements associated to the Hamiltonian circle actions, these quantum classes can still be expressed by explicit closed formulas. It is clear that as we increase the value of k the expressions for the Seidel elements corresponding to the circle actions $\Lambda_{e_1}^{2k}, \Lambda_{e_2}^{2k}$ in $\mathbb{F}_{2k}$ are going to be harder to write explicitly. However, from the work of Abreu and McDuff in [1] we know that the generators of the fundamental group of the symplectomorphism group of $(S^2 \times S^2, \omega_\mu)$ are given by $\Lambda_{e_1}^0, \Lambda_{e_2}^0$ and $\Lambda_{e_1}^2$, so our computations allow us to give a complete description of the Seidel representation for these manifolds (regardless of the value of μ provided that $\mu \geq 1$).

Remark 5.7. The "Odd" Hirzebruch surfaces $(\mathbb{F}_{2k-1}, \omega_\mu')$, $1 \leq k \leq \ell$ with $\ell \in \mathbb{N}$ and $\ell < \mu \leq \ell + 1$, can be identified with the symplectic manifolds $(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, \omega_\mu')$ where the symplectic area of the exceptional divisor is $\mu > 0$ and the area of the projective line is $\mu + 1$. Its moment polytope is

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 + x_2 \leq 1, \ x_2(k-1) + kx_1 \geq 0, \right.$$
$$\left. kx_2 + (k-1)x_1 \geq k - \mu - 1 \right\}.$$

Similar computations can be made for $\mathbb{F}_{2k-1}$, since $\mathbb{F}_1$ is Fano and we can show that $\Lambda_{e_1}^{2k-1} = \Lambda_{e_2}^{2k-1} = (2k-1)\Lambda_{e_1}^1$, using Karshon's classification of Hamiltonian circle actions.

Appendix A. Additional computations of Seidel's elements

We gather here results of computations of Seidel's elements in the case when the number of facets, in the vicinity of D_n, corresponding to spheres in M with vanishing first chern number is 3 (this is complementary to Theorem 4.5, see Figure 1). In order to ease the reading, we denote the weights $\omega(A_i)$ by ω_i.

(2c) If $c_1(A_n) = c_1(A_1) = c_1(A_2) = 0$ but $c_1(A_{n-1})$ and $c_1(A_3)$ are non-zero, then

$$S(\Lambda) = \left[\left(A_n \otimes q \frac{t^{\Phi_{\max}}}{1 - t^{-\omega_n}} - A_1 \otimes q \frac{t^{\Phi_{\max} - \omega_1}}{1 - t^{-\omega_1}} \right) \cdot \frac{1}{1 - t^{-\omega_n - \omega_1}} \right.$$
$$\left. - \left(A_1 \otimes q \frac{t^{\Phi_{\max}}}{1 - t^{-\omega_1}} - A_2 \otimes q \frac{t^{\Phi_{\max} - \omega_2}}{1 - t^{-\omega_2}} \right) \cdot \frac{t^{-\omega_1 - \omega_2}}{1 - t^{-\omega_1 - \omega_2}} \right] \cdot \frac{1}{1 - t^{-\omega_n - \omega_1 - \omega_2}}$$

(2d) If $c_1(A_n) = c_1(A_{n-1}) = c_1(A_1) = 0$ but $c_1(A_2)$ and $c_1(A_{n-2})$ are non-zero, then

$$
\begin{aligned}
S(\Lambda) = \Bigg[&\left(A_n \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega_n}} - A_{n-1} \otimes q\, \frac{t^{\Phi_{\max}-\omega_{n-1}}}{1 - t^{-\omega_{n-1}}} \right) \cdot \frac{1}{1 - t^{-\omega_n - \omega_{n-1}}} \\
&+ \left(A_n \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega_n}} - A_1 \otimes q\, \frac{t^{\Phi_{\max}-\omega_1}}{1 - t^{-\omega_1}} \right) \cdot \frac{1}{1 - t^{-\omega_n - \omega_1}} \\
&- A_n \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega_n}} \Bigg] \cdot \frac{1}{1 - t^{-\omega_n - \omega_{n-1} - \omega_1}}
\end{aligned}
$$

(3d) If $c_1(A_1) = c_1(A_2) = c_1(A_3) = 0$ but $c_1(A_n)$, $c_1(A_4)$ and $c_1(A_{n-1})$ are non-zero, then

$$
\begin{aligned}
S(\Lambda) = &A_n \otimes qt^{\Phi_{\max}} - A_1 \otimes q\, \frac{t^{\Phi_{\max}-\omega_1}}{1 - t^{-\omega_1}} \\
&- \left(A_1 \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega_1}} - A_2 \otimes q\, \frac{t^{\Phi_{\max}-\omega_2}}{1 - t^{-\omega_2}} \right) \cdot \frac{t^{-\omega_1 - \omega_2}}{1 - t^{-\omega_1 - \omega_2}} \\
&- \left(A_1 \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega_1}} - A_2 \otimes q\, \frac{t^{\Phi_{\max}-\omega_2}}{1 - t^{-\omega_2}} \right) \cdot \frac{t^{-\omega_1 - \omega_2 - \omega_3}}{(1 - t^{-\omega_1 - \omega_2 - \omega_3})(1 - t^{-\omega_1 - \omega_2})} \\
&+ \left(A_2 \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega_2}} - A_3 \otimes q\, \frac{t^{\Phi_{\max}-\omega_3}}{1 - t^{-\omega_3}} \right) \cdot \frac{t^{-\omega_1 - 2\omega_2 - 2\omega_3}}{(1 - t^{-\omega_2 - \omega_3})(1 - t^{-\omega_1 - \omega_2 - \omega_3})}
\end{aligned}
$$

(3e) If $c_1(A_{n-1}) = c_1(A_1) = c_1(A_2) = 0$ but $c_1(A_n)$, $c_1(A_3)$ and $c_1(A_{n-2})$ are non-zero, then

$$
\begin{aligned}
S(\Lambda) = &A_n \otimes qt^{\Phi_{\max}} - A_{n-1} \otimes q\, \frac{t^{\Phi_{\max}-\omega_{n-1}}}{1 - t^{-\omega_{n-1}}} - A_1 \otimes q\, \frac{t^{\Phi_{\max}-\omega_1}}{1 - t^{-\omega_1}} \\
&- \left(A_1 \otimes q\, \frac{t^{\Phi_{\max}}}{1 - t^{-\omega_1}} - A_2 \otimes q\, \frac{t^{\Phi_{\max}-\omega_2}}{1 - t^{-\omega_2}} \right) \cdot \frac{t^{-\omega_1 - \omega_2}}{1 - t^{-\omega_1 - \omega_2}}
\end{aligned}
$$

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Center for Mathematical Analysis, Geometry and Dynamical Systems
Departamento de Matemática, Instituto Superior Técnico
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail address: `sanjos@math.ist.utl.pt`

Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS
Université Paris-Saclay
91405 Orsay, France
E-mail address: `remi.leclercq@math.u-psud.fr`