

Inequalities for moment cones of finite-dimensional representations

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We give a general description of the moment cone associated with an arbitrary finite-dimensional unitary representation of a compact, connected Lie group in terms of finitely many linear inequalities. Our method is based on combining differential-geometric arguments with a variant of Ressayre’s notion of a dominant pair. As applications, we obtain generalizations of Horn’s inequalities to arbitrary representations, new inequalities for the one-body quantum marginal problem in physics, which concerns the asymptotic support of the Kronecker coefficients of the symmetric group, and a geometric interpretation of the Howe-Lee-Tan-Willenbring invariants for the tensor product algebra.

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1. Introduction

The study of the convexity properties of the moment map and of its image has a long history in mathematics, starting from Schur and Horn's observation that the diagonal entries of a $d \times d$ Hermitian matrix are always contained in the convex hull of the spectrum [1, 2]; cf. [3]. More generally, Atiyah and Guillemin-Sternberg have shown that, for any torus action on a compact, connected Hamiltonian manifold, the image of the moment map is a convex polytope, called the *moment polytope* [4, 5]. It can be explicitly computed as the convex hull of the images of torus fixed points. For non-abelian groups, the image of the moment map is no longer convex. Instead, Kirwan's celebrated convexity theorem asserts that, for the action of an arbitrary compact, connected Lie group on a compact, connected Hamiltonian manifold, the intersection of the image of the moment map with a positive Weyl chamber is a convex polytope [6]. This is the correct generalization of the moment polytope to non-abelian group actions. Mumford has given a different proof in the case of projective subvarieties, which relies on a concrete description in terms of the decomposition of the homogeneous coordinate ring into irreducible representations [7]; cf. [8, 9]. However, no effective general methods are known for the computation of these polytopes (in contrast to the case of torus actions).

In this article, we are concerned with the moment polytope of the complex projective space $\mathbb{P}(M)$ associated with a unitary representation Π of a compact, connected Lie group K on a finite-dimensional Hilbert space M . Equivalently, we shall study the moment cone of the latter, which is the convex cone spanned by the moment polytope (see Section 2 below). Even in this geometrically straightforward situation, computing the moment cone can be remarkably challenging. For example, the classical problem of characterizing the eigenvalues λ, μ, ν of triples of $d \times d$ Hermitian matrices that add up to zero, $A + B + C = 0$, is equivalent to determining the moment cone associated with the representation of $K = \mathrm{SU}(d) \times \mathrm{SU}(d) \times \mathrm{SU}(d)$ on $M = \mathfrak{gl}(d) \oplus \mathfrak{gl}(d)$ given by $(g, h, k) \cdot (a, b) = (gak^{-1}, hbk^{-1})$, also known as the *Horn cone*. While this observation is mathematically straightforward, a concrete description of the Horn cone in terms of finitely many linear inequalities has only been achieved in [10] and was an important step towards the proof of Horn's conjecture [11, 12]. Similarly, the *one-body quantum marginal problem* in quantum physics amounts to the determination of the moment cone for the action of $K = \mathrm{SU}(a) \times \mathrm{SU}(b) \times \mathrm{SU}(c)$ on $M = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ by tensor products [13, 14]. The underlying difficulty in both problems is in the representation theory rather than the geometry: For example, it is well-known

that the Horn cone describes the asymptotic support of the Littlewood-Richardson coefficients [15], while the solution to the one-body quantum marginal problem is given by the asymptotic support of the Kronecker coefficients of the symmetric group [13, 16–18]. Just as for the representation-theoretic coefficients [19, 20], the former problem can be understood as special case of the latter [21, 22]. Both problems can also be phrased in terms of projections of coadjoint orbits [23–25]. We remark that, locally, moment cones of arbitrary Hamiltonian K -manifolds can be described in terms of unitary representations [26, 27] and therefore fall into the scenario discussed in this paper.

Our main contribution is a clean algebraic description of the moment cone in terms of finitely many linear inequalities. To state the result, let \mathfrak{t} denote the Lie algebra of a maximal torus of T , it_+^* a positive Weyl chamber, and π the (complex) Lie algebra representation induced by Π (see Section 2 below for precise definitions). We shall say that H is a *Ressayre element* if 1) the hyperplane $(-, H) = 0$ is spanned by weights of M and 2) there exists a vector $\psi \in M$ annihilated by $\pi(H)$ such that the “tangent map” at ψ ,

$$(1.1) \quad \mathfrak{n}_-(H < 0) \rightarrow M(H < 0), \quad X \mapsto \pi(X)\psi,$$

is an isomorphism; here, $M(H < 0)$ denotes the direct sum of all negative eigenspaces of $\pi(H)$ and $\mathfrak{n}_-(H < 0)$ the sum of all root spaces for negative roots α such that $(\alpha, H) < 0$ (Definition 3.14). Note that there are only finitely many Ressayre elements for any given representation π . In Section 3, we will prove the following result:

Theorem 1.1. *The moment cone for the K -action on M is given by*

$$C_K = \{\lambda \in it_+^* : (H, \lambda) \geq 0 \text{ for all Ressayre elements } H\}.$$

To prove Theorem 1.1, we show that any facet corresponds to a Ressayre element by studying the moment map, which is quadratic, locally up to second order. To show that, conversely, any Ressayre element determines a valid inequality, we use Mumford’s description of the moment cone as in [25]. Indeed, our notion of a Ressayre element is closely related to Ressayre’s notion of a dominant pair. We note that the description in Theorem 1.1 will typically contain redundancies. Thus our result differs from [28], where the non-trivial or “general” faces of the moment polytope are characterized precisely at the cost of requiring a recursive strategy for their computation.

If H is a Ressayre element then the domain and codomain of the tangent map (1.1) necessarily have the same dimension, i.e., $\dim \mathfrak{n}_-(H < 0) =$

$\dim M(H < 0)$. We call this the *trace condition*. Moreover, note that the determinant δ_H of (1.1) (with respect to any fixed pair of bases) is a non-zero polynomial in $\psi \in M(H = 0)$, the zero eigenspace of $\pi(H)$. In fact, δ_H is a canonical (up to scalar multiplication) lowest weight vector for the action of $K(H = 0)$, the centralizer of the torus generated by H , on the space of polynomials on $M(H = 0)$. This implies the following result in Section 4, which we call the *Horn condition*:

Proposition 1.2 (Horn condition). *For any Ressayre element H ,*

$$\kappa_H := \text{Tr } \pi(-)|_{M(H < 0)} - \text{Tr } \text{ad}(-)|_{\mathfrak{n}_-(H < 0)} = \sum_{\omega \in \Omega: \langle \omega, H \rangle < 0} \omega - \sum_{\alpha \in R_{G,-}: \langle \alpha, H \rangle < 0} \alpha$$

is an element of the moment cone $C_{K(H=0)}(M(H = 0))$. In fact, δ_H is a lowest weight vector of weight $-\kappa_H$.

Here, Ω denotes the set of weights of M and $R_{G,-}$ denotes the set of negative roots of G . By applying Theorem 1.1 to the lower-dimensional scenario, the Horn condition can be explicitly stated as a set of linear inequalities that have to be satisfied by κ_H .

Tangent maps and their determinants have been studied in great generality by Ressayre and Belkale from an algebro-geometric point of view [25, 29, 30], and our Theorem 1.1 and Proposition 1.2 can also be deduced from their results. In these works, the non-vanishing of the determinant has been in turn been translated into a cohomological condition. In contrast, we propose that, for the purposes of computing moment cones explicitly, it can be useful to instead test the non-vanishing of the determinant directly—either symbolically, which is easily possible in small dimensions, or numerically by using fast algorithms for polynomial identity testing, as we discuss in Section 5 below. The challenge imposed by higher dimensions is rather in finding additional *a priori* constraints on the facets of the moment cone.

We apply our approach to the two paradigmatic examples mentioned above. In Section 6, we use it to give a new solution to the one-body quantum marginal problem (equivalently, a description of the asymptotic support of the Kronecker coefficients of the symmetric group for triples of Young diagrams with bounded numbers of rows). In particular, our method allows us to compute the moment polytope for $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, which was out of reach with previous methods. In Section 7, we revisit the classical Horn inequalities from the perspective of our work. We find that they are instances of the trace and Horn conditions derived above, which justifies our terminology. In other words, our trace condition and Horn condition can be understood as

generalizations of Horn’s inequalities to arbitrary representations. We also give a geometric explanation of the invariants constructed in [31, 32]: they can be obtained directly from the determinant polynomial δ_H . In both cases, we find that the description of Theorem 1.1 can be readily refined to the mathematical scenario at hand. We hope that our method similarly provides a useful tool for the study of moment cones in other mathematical applications. A preliminary version of this work has appeared in the thesis of MW [21].

2. Moment cones of finite-dimensional representations

Let K be a compact, connected Lie group with Lie algebra \mathfrak{k} . Let G be its complexification, which is a connected reductive algebraic group with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ and denote its exponential map by $\exp: \mathfrak{g} \rightarrow G$. Let $T \subseteq K$ be a maximal torus with Lie algebra $\mathfrak{t} \subseteq \mathfrak{k}$ and $W_K = N_K(T)/T$ the Weyl group; we write $r_K := \dim T$ for the rank of K . The complexification of T is a maximal abelian subgroup $T_{\mathbb{C}} \subseteq G$ with Lie algebra $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$. In view of $\mathfrak{u}(1) = i\mathbb{R}$ we consider the weight lattice P_G as a subset of $i\mathfrak{t}^* \cong \{\omega \in \mathfrak{h}^* : \omega(i\mathfrak{t}) \in \mathbb{R}\}$. We denote the set of roots by $R_G \subseteq P_G$ and write the root space decomposition as $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_G} \mathfrak{g}_{\alpha}$. For each root α , we can find basis vectors $E_{\alpha} \in \mathfrak{g}_{\alpha}$ and “co-roots” $H_{\alpha} \in i\mathfrak{t}$ that satisfy the commutation relations of $\mathfrak{sl}(2)$,

$$(2.1) \quad [E_{\alpha}, E_{-\alpha}] = H_{\alpha} \quad \text{and} \quad [H_{\alpha}, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}.$$

Let $R_{G,+}$ denote a choice of positive roots. Correspondingly, we get the negative roots $R_{G,-} = -R_{G,+}$, nilpotent Lie algebras $\mathfrak{n}_{\pm} = \bigoplus_{\alpha \in R_{G,\pm}} \mathfrak{g}_{\alpha}$, maximal unipotent subgroups $N_{\pm} \subseteq G$ and a positive Weyl chamber, which we take to be the convex polyhedral cone $i\mathfrak{t}_{+}^* = \{\lambda \in i\mathfrak{t}^* : (H_{\alpha}, \lambda) \geq 0 \ \forall \alpha \in R_{G,+}\} \subseteq i\mathfrak{t}^*$ with relative interior $i\mathfrak{t}_{>0}^* = \{\lambda \in i\mathfrak{t}^* : (H_{\alpha}, \lambda) > 0 \ \forall \alpha \in R_{G,+}\}$. We may consider $\mathfrak{h}^* \subseteq \mathfrak{g}^*$ and $i\mathfrak{t}^* \subseteq i\mathfrak{k}^*$ by extending each functional by zero on the root spaces \mathfrak{g}_{α} and $\mathfrak{k}_{\alpha} = (\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$, respectively. The finite-dimensional irreducible representations of G are parametrized by their highest weight λ in $P_{G,+} = P_G \cap i\mathfrak{t}_{+}^*$; the corresponding representation will be denoted by $V_{G,\lambda}$. Then $V_{G,\lambda}^* \cong V_{G,\lambda^*}$ for $\lambda^* := -w_0\lambda$; here and throughout this paper, w_0 denotes the longest Weyl group element, which exchanges the positive and negative roots (the group will always be clear from the context).

For $SU(d)$, whose complexification is $SL(d)$, we will always use the maximal torus $T(d)$ that consists of the diagonal unitary matrices of unit determinant. Its Lie algebra will be denoted by $\mathfrak{t}(d)$. We will use as positive roots the $\alpha_{i,j}(H) := H_{ii} - H_{jj}$ with $i < j$, and abbreviate the (positive) roots by

R_d and $R_{d,+}$, respectively. Finally, we write \mathcal{O}_λ^d for the coadjoint $SU(d)$ orbit through a highest weight λ .

Now let $\Pi: G \rightarrow GL(M)$ a representation of G on a finite-dimensional Hilbert space M that is equipped with a K -invariant Hermitian inner product $\langle - | - \rangle$, which we take to be antilinear in the *first* argument. We will oftentimes use Dirac's notation $\langle \phi | A | \psi \rangle := \langle \phi | A \psi \rangle$ for $\phi, \psi \in M$ and $A \in \mathfrak{gl}(M)$. We denote by $\pi: \mathfrak{g} \rightarrow \mathfrak{gl}(M)$ the induced Lie algebra representation, by $\Omega \subseteq P_G$ the set of weights and write the weight space decomposition as $M = \bigoplus_{\omega \in \Omega} M_\omega$. The K -action on M admits a canonical (up to conventions) *moment map*, defined by

$$(2.2) \quad \mu_K: M \rightarrow i\mathfrak{k}^*, \quad (\mu_K(\psi), X) = \langle \psi | \pi(X) | \psi \rangle$$

for all $\psi \in M$ and $X \in i\mathfrak{k}$. Here and in the following, we write $(\varphi, X) = \varphi(X)$ for the duality pairing. The map μ_K is indeed a moment map in the sense of symplectic geometry: it is K -invariant and satisfies the basic identity

$$(2.3) \quad d(\mu_K, iX)|_\psi = \omega_M(\pi(X)\psi, -)$$

for all $X \in \mathfrak{k}$, where $\omega_M(\phi, \psi) = 2 \operatorname{Im} \langle \phi | \psi \rangle$ denotes the symplectic form that we will use for M . The *moment cone* then is defined as intersection of the moment map image with the positive Weyl chamber,

$$C_K = \mu_K(M) \cap i\mathfrak{k}_+^* = \{ \lambda \in i\mathfrak{k}_+^* : \lambda \in \mu_K(M) \}.$$

The representation of G also induces an action on the complex projective space $\mathbb{P}(M)$, $g \cdot [\psi] = [g\psi]$, with corresponding moment map $\tilde{\mu}_K: \mathbb{P}(M) \rightarrow i\mathfrak{k}^*$, $(\tilde{\mu}_K([\psi]), X) = \langle \psi | \pi(X) | \psi \rangle / \langle \psi | \psi \rangle$ for all $[\psi] \in \mathbb{P}(M)$ and $X \in i\mathfrak{k}$ [7, 33]. Kirwan's convexity theorem [6] (or Mumford's version [7]) asserts that the *moment polytope* $\Delta_K = \tilde{\mu}_K(\mathbb{P}(M)) \cap i\mathfrak{k}_+^*$ is indeed a convex polytope. It is plain from the definitions that $C_K = \mathbb{R}_+ \Delta_K$. Therefore, the moment cone C_K is a polyhedral cone, and we have the following representation-theoretic description [7, 9],

$$(2.4) \quad C_K = \operatorname{cone} \{ \lambda \in P_{G,+} : V_{G,\lambda}^* \subseteq R(M) \},$$

where $R(M) = \operatorname{Sym}(M)^*$ denotes the space of polynomials on M . A representation $V_{G,\lambda}^*$ occurs in $R(M)$ if and only if there exists a polynomial $P \in R(M)$ of weight $-\lambda$ that is invariant under the action of the lower unipotent subgroup N_- . We shall call such a polynomial a *lowest weight vector* in $R(M)$.

Conversely, suppose that the action of G contains the multiplication by scalars, generated by some $J \in \mathfrak{it}$ such that $\pi(J) = \mathbb{1}$ (this can always be arranged for by adding a \mathbb{C}^* -factor to G). Then the moment polytope can be recovered from the moment cone by

$$\Delta_K = \{\lambda \in C_K : (\lambda, J) = 1\}$$

(and C_K is a pointed cone with base Δ_K). Thus we may equivalently study moment cones of representations or moment polytopes of the corresponding projective spaces.

Throughout this paper, we shall always work with moment cones (but see [21] for an exposition from the projective point of view). We shall moreover assume that the moment cone is of maximal dimension, i.e., $\dim C_K = \dim \mathfrak{it}^*$. This is the case if and only if there exists a vector with finite stabilizer.

3. Facets of the moment cone

Like any polyhedral cone, the moment cone can be described by finitely many linear inequalities $(-, H) \geq 0$. Since we have assumed that C_K is of maximal dimension, its facets are of codimension one in \mathfrak{it}^* and their inward-pointing normal vectors may be identified with the defining linear inequalities $(-, H) \geq 0$ of the moment cone. Since the moment cone is obtained by intersecting $\mu_K(M)$ with the positive Weyl chamber, which itself is a maximal-dimensional polyhedral cone, some of the facets of C_K can be subsets of facets of \mathfrak{it}_+^* , and we shall call those the trivial facets of the moment cone:

Definition 3.1. A facet of the moment cone is *trivial* if it corresponds to an inequality of the form $(-, H_\alpha) \geq 0$ for some positive root $\alpha \in R_{G,+}$. Otherwise, the facet is called *non-trivial*.

Non-trivial facets have also been called “general” in the literature [28]. We record the following straightforward observation:

Lemma 3.2. *Any non-trivial facet of C_K meets the relative interior $\mathfrak{it}_{>0}^*$ of the positive Weyl chamber.*

3.1. Admissibility

We first consider the moment map μ_T , defined as in (2.2) for the action of the maximal torus $T \subseteq K$. Let $M = \bigoplus_{\omega \in \Omega} M_\omega$ be the decomposition of M

into weight spaces and let $\psi \in M$ be a vector decomposed accordingly as $\psi = \sum_{\omega} \psi_{\omega} v_{\omega}$. Then μ_T has the following concrete description:

$$(3.1) \quad \mu_T(\psi) = \sum_{\omega} |\psi_{\omega}|^2 \omega$$

Observe that $\mu_T(\psi)$ is a conic combination of weights. It follows that the “abelian” moment cone C_T of M is precisely equal to the conical hull of the set of weights; it is maximal-dimensional since it contains C_K . More generally, if $\Omega' \subseteq \Omega$ is a subset of weights and $M_{\Omega'} := \bigoplus_{\omega \in \Omega'} M_{\omega}$ then $C_T(M_{\Omega'}) = \text{cone } \Omega'$. For the next lemma recall that a critical point of a smooth map $f: M \rightarrow M'$ is a point $m \in M$ where the differential $df|_m$ is not surjective; a critical value is the image $f(m)$ of a critical point. Then the following is well-known (e.g., [18, Remark 4.14]):

Lemma 3.3. *The set of critical values of μ_T is equal to the union of the codimension-one conic hulls of subsets of weights.*

Proof. Let $\psi \in M$ with weight decomposition $\psi = \sum_{\omega} \psi_{\omega} v_{\omega}$. By (3.1), $\mu_T(\psi)$ is a conic combination of weights in $\Omega_{\psi} := \{\omega \in \Omega : \psi_{\omega} \neq 0\}$. By the moment map property (2.3) and non-degeneracy of the symplectic form, ψ is a critical point if and only if there exists $0 \neq X \in \mathfrak{t}$ such that $\pi(X)\psi = 0$ [5, Lemma 2.1], i.e., if and only if $\omega(X) = 0$ for all $\omega \in \Omega_{\psi}$. It follows that ψ is a critical point if and only if the conic hull of Ω_{ψ} is of positive codimension.

In particular, any critical value is contained in a codimension-one conic hull of weights, since we may always add additional weights. Conversely, if $\Omega' \subseteq \Omega$ is a subset of weights that spans a conic hull of codimension one then $C_T(M_{\Omega'}) = \text{cone } \Omega'$ consists of critical values. \square

We now derive a basic necessary condition that cuts down the defining inequalities of the moment cone to a finite set of candidates (cf. [18, Remark 3.6]).

Definition 3.4. An element $H \in \mathfrak{t}$ is called *admissible* if the linear hyperplane $(-, H) = 0$ is spanned by a subset of weights in Ω .

The notion of admissibility is invariant under the action of the Weyl group W_K .

Lemma 3.5 (Admissibility condition). *Let $(-, H) \geq 0$ be an inequality corresponding to a non-trivial facet of the moment cone. Then H is admissible.*

Proof. By Lemma 3.2, the intersection of $(-, H) = 0$ with the interior of the positive Weyl chamber $i\mathfrak{t}_{>0}^*$ is non-empty. Each point in this intersection is a critical value for $(\mu_K, H) = (\mu_T, H)$, hence of μ_T , and therefore according to Lemma 3.3 contained in a linear hyperplane spanned by a subset of weights. Since this is true for all points in the intersection, which contains the relative interior of the facet, it follows that the facet is in fact contained in a single such hyperplane. □

3.2. Description of the moment cone by Ressayre elements

Let us now fix an inequality $(-, H) \geq 0$ corresponding to a non-trivial facet $(-, H) = 0$ of the moment cone. Let $\psi \in M$ be a preimage of a point $\mu_K(\psi) \in i\mathfrak{t}_{>0}^*$ on the facet $(-, H) = 0$. In the proof of Lemma 3.5, we have used that ψ is a critical point of (μ_K, H) (equivalently, that $\pi(H)\psi = 0$) to gain information on the set of possible facets. To study the function (μ_K, H) in the vicinity of such a critical point ψ it is natural to consider the Hessian, which is the quadratic form

$$(3.2) \quad Q(V, V) = 2 \langle V | \pi(H) | V \rangle .$$

For tangent vectors generated by the infinitesimal action of $X, Y \in \mathfrak{k}$, we have the formula

$$(3.3) \quad \begin{aligned} Q(\pi(X)\psi, \pi(Y)\psi) &= -2 \operatorname{Re} \langle \psi | \pi(X)\pi(H)\pi(Y) | \psi \rangle \\ &= \langle \psi | \pi([[H, X], Y]) | \psi \rangle = (\mu_K(\psi), [[H, X], Y]), \end{aligned}$$

where we have used that $\pi(H)\psi = 0$. We now decompose

$$(3.4) \quad M = M(H < 0) \oplus M(H = 0) \oplus M(H > 0),$$

where $M(H < 0) = \bigoplus_{\omega: (\omega, H) < 0} M_\omega$ is the sum of the eigenspaces of the Hermitian operator $\pi(H)$ with eigenvalue less than 0, etc. Then it is plain from (3.2) that the index of the Hessian Q , i.e., the dimension of a maximal subspace on which the quadratic form is negative definite, is equal to twice the complex dimension of $M(H < 0)$.

We will now deduce a second formula for the index by observing that the Hessian is necessarily positive semidefinite on the subspace of those tangent vectors V that are mapped to $i\mathfrak{t}^*$ by the differential of the moment map. To see this, consider a curve ψ_t with $\psi_0 = \psi$, $\dot{\psi}_0 = V$ and $\mu_K(\psi_t) \in i\mathfrak{t}_{>0}^*$ for all $t \in (-\varepsilon, \varepsilon)$ (such a curve can always be constructed by using the

symplectic cross section [34, Theorem 26.7]). Then, since $(\mu_K(\psi), H) = 0$ and $d(\mu_K, H)|_\psi \equiv 0$,

$$(\mu_K(\psi_t), H) = \frac{t^2}{2}Q(V, V) + O(t^3).$$

But $(\mu_K(\psi_t), H) \geq 0$, since $\mu_K(\psi_t) \in C_K$ and $(-, H) \geq 0$ is a valid inequality for the moment cone. Together, this shows that, indeed, $Q(V, V) \geq 0$.

The subspace of all such V can be computed in a different way. For this, let $\mathfrak{r} = \bigoplus_{\alpha \in R_{G,+}} \mathfrak{k}_\alpha$ denote the sum of the root spaces of the compact Lie algebra. It follows from the moment map property (2.3) that

$$d\mu_K|_\psi(V) \in i\mathfrak{t}^* \Leftrightarrow 0 = d(\mu_K, iR)|_\psi(V) = -\omega_M(V, \pi(R)\psi) \quad (\forall R \in \mathfrak{r}).$$

We thus obtain the following lemma.

Lemma 3.6. *The Hessian is positive semidefinite on the symplectic complement*

$$\begin{aligned} M_0^{\omega_M} &= \{V \in M : \omega_M(V, W) = 0 \ (\forall W \in M_0)\} \\ &= \{V \in M : d\mu_K|_\psi(V) \in i\mathfrak{t}^*\} \end{aligned}$$

of $M_0 := \pi(\mathfrak{r})\psi \subseteq M$.

In fact, it is well-known that M_0 is a symplectic subspace of M (see, e.g., [5, Lemma 6.7]). To see this, note that the restriction of the symplectic form to M_0 is given by

$$\omega_M(\pi(X)\psi, \pi(Y)\psi) = (\mu_K(\psi), [X, Y])$$

and can therefore be identified with the Kirillov-Kostant-Souriau symplectic form on the coadjoint orbit through $\mu_K(\psi) \in i\mathfrak{t}_{>0}^*$. As a consequence, we have the decomposition $M = M_0 \oplus M_0^{\omega_M}$.

Lemma 3.7. *The Hessian is block-diagonal with respect to the decomposition $M = M_0 \oplus M_0^{\omega_M}$.*

Proof. For all $\pi(R)\psi \in M_0$ and $V \in M_0^{\omega_M}$ we have that

$$\begin{aligned} Q(\pi(R)\psi, V) &= -2 \operatorname{Re} \langle \psi | \pi(R)\pi(H) | V \rangle = 2 \operatorname{Re} \langle \psi | \pi([H, R]) | V \rangle \\ &= \omega_M(\pi([-iH, R])\psi, V) = 0, \end{aligned}$$

since $[-iH, R] \in \mathfrak{r}$ and therefore $\pi([-iH, R])\psi \in M_0$. □

Lemma 3.8. *The tangent map $\mathfrak{r} \rightarrow M, R \mapsto \pi(R)\psi$ is injective.*

Proof. The stabilizer of the coadjoint action of K at any $\lambda \in i\mathfrak{t}_{>0}^*$ is T , while $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{r}$. As $\mu_K(\psi) \in i\mathfrak{t}_{>0}^*$, the claim follows from this and K -equivariance of the moment map. □

Lemma 3.9. *The index of the Hessian is equal to twice the number of positive roots $\alpha \in R_{G,+}$ such that $(\alpha, H) > 0$.*

Proof. Since Q is positive semidefinite on $M_0^{\omega_M}$ (Lemma 3.6) and block-diagonal with respect to the decomposition $M = M_0 \oplus M_0^{\omega_M}$ (Lemma 3.7), it suffices to compute the index of Q on $M_0 = \{\pi(R)\psi : R \in \mathfrak{r}\}$. For this, recall from (3.3) that $Q(\pi(R)\psi, \pi(S)\psi) = (\mu_K(\psi), [[H, R], S])$ for all $R, S \in \mathfrak{r}$. Since the tangent map $R \mapsto \pi(R)\psi$ is injective (Lemma 3.8), we may instead consider the form

$$\tilde{Q}(R, S) := (\mu_K(\psi), [[H, R], S])$$

on \mathfrak{r} . Now observe that \tilde{Q} is block-diagonal with respect to $\mathfrak{r} = \bigoplus_{\alpha \in R_{G,+}} \mathfrak{k}_\alpha$, since for all $R \in \mathfrak{k}_\alpha$ and $S \in \mathfrak{k}_\beta$, $[[H, R], S] \in i\mathfrak{k}_{\alpha \pm \beta}$, while $\mu_K(\psi) \in i\mathfrak{t}^*$. Therefore, it suffices to compute the index on a single root space \mathfrak{k}_α . For this, define the ‘‘Pauli matrices’’ $X_\alpha := E_\alpha + E_{-\alpha}$ and $Y_\alpha := i(E_{-\alpha} - E_\alpha)$, which satisfy the commutation relations $[X_\alpha, Y_\alpha] = 2iH_\alpha$ etc. Then iX_α and iY_α form a basis of \mathfrak{k}_α and

$$\begin{aligned} \tilde{Q}(iX_\alpha, iX_\alpha) &= -(\mu_K(\psi), [[H, X_\alpha], X_\alpha]) = -i(\alpha, H) (\mu_K(\psi), [Y_\alpha, X_\alpha]) \\ &= -2(\alpha, H) (\mu_K(\psi), H_\alpha). \end{aligned}$$

Likewise, $\tilde{Q}(iY_\alpha, iY_\alpha) = -2(\alpha, H)(\mu_K(\psi), H_\alpha)$, while $\tilde{Q}(iX_\alpha, iY_\alpha) = 0$. Since $(\mu_K(\psi), H_\alpha) > 0$, we conclude that the index of Q is equal to twice the number of positive roots α with $(\alpha, H) > 0$. □

Since we have already seen above that the index of the Hessian is also equal to twice the complex dimension of $M(H < 0)$, we obtain the following result, which can also be extracted from [28, Theorem 2]:

Corollary 3.10. *If $(-, H) \geq 0$ defines a non-trivial facet of the moment cone then*

$$\dim_{\mathbb{C}} M(H < 0) = \#\{\alpha \in R_{G,+} : (\alpha, H) > 0\} = \#\{\alpha \in R_{G,-} : (\alpha, H) < 0\}.$$

We now study the complexified group action. To this end, we consider the Lie algebra $\mathfrak{n}_- = \bigoplus_{\alpha \in R_{G,-}} \mathfrak{g}_\alpha$ of the negative unipotent subgroup, which

plays a role analogous to \mathfrak{r} for vectors ψ that are mapped into the positive Weyl chamber (compare the following with Lemma 3.8).

Lemma 3.11. *The tangent map $\mathfrak{n}_- \rightarrow M, X \rightarrow \pi(X)\psi$ is injective.*

Proof. Let $E_- := \sum_{\alpha \in R_{G,+}} z_\alpha E_{-\alpha}$ be an arbitrary element in \mathfrak{n}_- . Since $\pi(E_{\pm\alpha})^\dagger = \pi(E_{\mp\alpha})$, we find that $\pi(E_-)^\dagger = \pi(E_+)$ with $E_+ := \sum_{\alpha \in R_{G,+}} \bar{z}_\alpha E_\alpha$ (the Cartan involution of E_-). Therefore,

$$\begin{aligned} \|\pi(E_-)\psi\|^2 &= \langle \psi | \pi(E_+) \pi(E_-) | \psi \rangle = \langle \psi | \pi([E_+, E_-]) | \psi \rangle + \|\pi(E_+)\psi\|^2 \\ &\geq \langle \psi | \pi([E_+, E_-]) | \psi \rangle. \end{aligned}$$

But (2.1) implies that

$$[E_+, E_-] - \sum_{\alpha \in R_{G,+}} |z_\alpha|^2 H_\alpha \in i\mathfrak{r},$$

so that by using $\mu_K(\psi) \in i\mathfrak{t}_{>0}^*$ we find that

$$\|\pi(E_-)\psi\|^2 \geq \langle \psi | \pi([E_+, E_-]) | \psi \rangle = \sum_{\alpha \in R_{G,+}} |z_\alpha|^2 (\mu_K(\psi), H_\alpha) > 0.$$

□

In contrast to Lemma 3.8, which continues to hold true if $\mu_K(\psi)$ is mapped to the relative interior of a different Weyl chamber, it is important in Lemma 3.11 to choose the negative unipotent subgroup (relative to the choice of positive Weyl chamber). For example, consider an irreducible G -representation $M = V_{G,\lambda}$ with highest weight $\lambda \in i\mathfrak{t}_{>0}^*$ and highest weight vector v_λ . Then $\mu_K(v_\lambda) = \lambda \in i\mathfrak{t}_{>0}^*$ and the “lowering operators” in \mathfrak{n}_- indeed act injectively. On the other hand, the “raising operators” in the positive nilpotent Lie algebra \mathfrak{n}_+ annihilate the highest weight vector (by definition).

We now decompose the Lie algebra \mathfrak{n}_- similarly to (3.4),

$$\mathfrak{n}_- = \mathfrak{n}_-(H < 0) \oplus \mathfrak{n}_-(H = 0) \oplus \mathfrak{n}_-(H > 0),$$

where $\mathfrak{n}_-(H < 0) = \bigoplus_{\alpha \in R_{G,-}: (\alpha, H) < 0} \mathfrak{g}_\alpha$ is the sum of the complex root spaces with negative H -weight $(\alpha, H) < 0$, etc. We observe that Corollary 3.10 can be equivalently stated as

$$(3.5) \quad \dim_{\mathbb{C}} M(H < 0) = \dim_{\mathbb{C}} \mathfrak{n}_-(H < 0).$$

Note that $\pi(\mathfrak{n}_-(H < 0))M(H = 0) \subseteq M(H < 0)$. Thus we obtain the following important result:

Proposition 3.12. *Let $\psi \in M$ such that $\mu_K(\psi) \in \mathfrak{it}_{>0}^*$ is a point on a non-trivial facet of the moment cone corresponding to the inequality $(-, H) \geq 0$. Then $\psi \in M(H = 0)$ and the tangent map restricts to an isomorphism*

$$\mathfrak{n}_-(H < 0) \rightarrow M(H < 0), \quad X \mapsto \pi(X)\psi.$$

Proof. The fact that $\psi \in M(H = 0)$ is just a reformulation of $\pi(H)\psi = 0$. By the preceding discussion, the tangent map is well-defined as a map from $\mathfrak{n}_-(H < 0)$ to $M(H < 0)$; it is injective by Lemma 3.11 and surjective since the dimensions agree according to (3.5). □

We now prove a partial converse to Proposition 3.12, inspired by the argument of Ressayre [25].

Proposition 3.13. *Suppose there exists $\psi \in M(H = 0)$ such that the tangent map*

$$\mathfrak{n}_-(H < 0) \rightarrow M(H < 0), \quad X \mapsto \pi(X)\psi$$

is surjective. Then $(-, H) \geq 0$ is a valid inequality for the moment cone.

Proof. Consider the smooth map

$$N_- \times M(H \geq 0) \rightarrow M, \quad (g, \phi) \mapsto \Pi(g)\phi.$$

Its differential at $(1, \psi)$ is the linear map

$$\mathfrak{n}_- \oplus M(H \geq 0) \rightarrow M, \quad (X, V) \mapsto \pi(X)\psi + V.$$

The assumption implies that this map is surjective. It follows that $\Pi(N_-)M(H \geq 0) \subseteq M$ contains a small Euclidean ball around ψ . In particular, any N_- -invariant polynomial that is zero on $M(H \geq 0)$ is automatically zero everywhere on M .

We now prove the inequality. By the description of the moment cone in (2.4), it suffices to show that $(\lambda, H) \geq 0$ for all highest weights λ such that $V_{G,\lambda}^* \subseteq R(M)$. Recall that the highest weight of $V_{G,\lambda}^*$ is $\lambda^* = -w_0\lambda$, where w_0 is the longest Weyl group element that flips the positive and negative roots. Consider a lowest weight vector, i.e., a polynomial $P \in R(M)$ that is a weight vector of weight $-\lambda$ and invariant under the action of N_- . Then $\pi(H)P = -(\lambda, H)P$ and the restriction of P to $M(H \geq 0)$ is non-zero by our discussion above. But this restriction is an element of $R(M(H \geq 0)) = \text{Sym}(M(H \geq 0))^*$, the space of polynomials on $M(H \geq 0)$. Since all H -weights in $R(M(H \geq 0))$ are non-positive, it follows that $-(\lambda, H) \leq 0$, as we set out to prove. □

We remark that Proposition 3.13 holds unconditionally without any assumption on the dimension of the moment cone C_K . We summarize our findings in the following definition and theorem that we had already advertised in the introduction.

Definition 3.14. An element $H \in \mathfrak{it}$ is called a *Ressayre element* if

- 1) H is admissible, i.e., the linear hyperplane $(-, H) = 0$ is spanned by a subset of weights in Ω , and
- 2) there exists $\psi \in M(H = 0)$ such that the map

$$(3.6) \quad \mathfrak{n}_-(H < 0) \rightarrow M(H < 0), \quad X \mapsto \pi(X)\psi$$

is an isomorphism.

Theorem 1.1. The moment cone is given by

$$C_K = \{\lambda \in \mathfrak{it}_+^* : (H, \lambda) \geq 0 \text{ for all Ressayre elements } H\}.$$

Proof. This follows directly from Lemma 3.5 and Propositions 3.12 and 3.13. \square

Theorem 1.1 gives a complete description of the moment cone of an arbitrary finite-dimensional K -representation M (under the assumption that C_K is of maximal dimension). The set of inequalities thus obtained may still be redundant (i.e., not all inequalities necessarily correspond to facets of the moment cone). In contrast, Ressayre's *well-covering pairs* [25] characterize the facets of the moment polytope precisely. In our language, his condition amounts to requiring that the generic fiber of the map $N_- \times_{N_-(H \geq 0)} M(H \geq 0) \rightarrow M$ is a point (as opposed to only requiring that the map be dominant). Our characterization is also related to [28, Theorem 2], which uses algebraic geometry to characterize non-trivial faces of arbitrary codimension. Unlike Proposition 3.13, it relies on an assumption about lower-dimensional moment polytopes, which can in principle be obtained recursively.

4. Generalized trace and Horn condition

We now extract two useful necessary conditions that have to hold for any Ressayre element H , and therefore for any non-trivial facet of the moment polytope. We will see in Subsection 7.1 below that they are a generalization of the classical Horn inequalities, which justifies our terminology.

The first condition, which we call the *trace condition* is the observation that the domain and range of the tangent map (3.6) necessarily have to agree, i.e.,

$$(4.1) \quad \dim \mathfrak{n}_-(H < 0) = \dim M(H < 0).$$

We note that the right-hand side of (4.1) is invariant under the action of the Weyl group. This suggests that we first compute the *dominant* admissible H_0 and then determine those $H \in W_K \cdot H_0$ which satisfy the trace condition (4.1). For this, we will need the following well-known lemma.

Lemma 4.1. *Let $H_0 \in \mathfrak{it}_+ := \{H_0 \in \mathfrak{it} : (\alpha, H_0) \geq 0 \ \forall \alpha \in R_{G,+}\}$ and $H \in W_K \cdot H_0$. Let $w \in W_K$ be the unique Weyl group element such that $H = w \cdot H_0$ and $w_0 w \cdot \alpha \in R_{G,+}$ for all $\alpha \in R_{G,+}$ with $(\alpha, H_0) = 0$. Then,*

$$\ell(w_0 w) = \#\{\alpha \in R_{G,+} : (\alpha, H) > 0\}.$$

Proof. Fix any $\rho_K \in \mathfrak{it}_+$ such that $(\alpha, \rho_K) > 0$ for all positive roots $\alpha \in R_{G,+}$, and set $H^\varepsilon := H - \varepsilon \rho_K$. Let $w \in W_K$ be a Weyl group element such that $H = w \cdot H_0$. For all positive roots $\alpha \in R_{G,+}$ and $\varepsilon > 0$ small enough,

$$(\alpha, w^{-1} \cdot H^\varepsilon) = (\alpha, H_0) - \varepsilon(w \cdot \alpha, \rho_K) > 0$$

if and only if $(\alpha, H_0) = 0$ implies that $w \cdot \alpha \in R_{G,-}$. In other words, $H_0^\varepsilon := w^{-1} \cdot H^\varepsilon \in \mathfrak{it}_+$ if and only if $w_0 w \cdot \alpha \in R_{G,+}$ for all $\alpha \in R_{G,+}$ with $(\alpha, H_0) = 0$. Since H^ε is regular, this immediately shows that such Weyl group elements w exist and are unique. What is more, regularity also implies that

$$\begin{aligned} \ell(w_0 w) &= \ell(w_0 w^{-1}) = \#\{\alpha \in R_{G,+} : w^{-1} \cdot \alpha \in R_{G,+}\} \\ &= \#\{\alpha \in R_{G,+} : (w^{-1} \cdot \alpha, H_0^\varepsilon) > 0\} = \#\{\alpha \in R_{G,+} : (\alpha, w \cdot H_0^\varepsilon) > 0\} \\ &= \#\{\alpha \in R_{G,+} : (\alpha, H^\varepsilon) > 0\} = \#\{\alpha \in R_{G,+} : (\alpha, H) > 0\}. \end{aligned}$$

□

We obtain the following useful corollary:

Corollary 4.2. *Let $H_0 \in \mathfrak{it}_+$ and $H \in W_K \cdot H_0$. Let $w \in W_K$ be the unique Weyl group element such that $H = w \cdot H_0$ and $w_0 w \cdot \alpha \in R_{G,+}$ for all $\alpha \in R_{G,+}$ with $(\alpha, H_0) = 0$. Then H satisfies the trace condition (4.1) if and only if $\ell(w_0 w) = \dim_{\mathbb{C}} M(H_0 < 0)$.*

The second condition, called the *Horn condition*, is based on the observation that for any Ressayre element H the *determinant polynomial*

$$(4.2) \quad \delta_H: \begin{cases} M(H = 0) \rightarrow \mathbb{C} \\ \psi \mapsto \det \left(\mathfrak{n}_-(H < 0) \ni X \mapsto \pi(X)\psi \in M(H < 0) \right) \end{cases}$$

is non-zero (we take the determinant with respect to any fixed pair of bases). This can be understood to imply a statement about a smaller moment cone. To see this, let $G(H = 0)$ denote the identity component of the centralizer of the torus generated by H , i.e., the connected subgroup of G with Lie algebra $\mathfrak{g}(H = 0) = \mathfrak{h} \oplus \bigoplus_{\alpha: \langle \alpha, H \rangle = 0} \mathfrak{g}_\alpha$; denote by $N_-(H = 0)$ the corresponding negative unipotent subgroup and by $K(H = 0) = G(H = 0) \cap K$ the maximal compact subgroup. Then $G(H = 0)$ acts on $M(H = 0)$ and thus on the space $R(M(H = 0)) = \text{Sym}(M(H = 0))^*$ of polynomial functions.

Proposition 1.2 (Horn condition). For any Ressayre element H ,

$$\kappa_H := \text{Tr } \pi(-)|_{M(H < 0)} - \text{Tr } \text{ad}(-)|_{\mathfrak{n}_-(H < 0)} = \sum_{\omega \in \Omega: \langle \omega, H \rangle < 0} \omega - \sum_{\alpha \in R_G, -: \langle \alpha, H \rangle < 0} \alpha$$

is an element of the moment cone $C_{K(H=0)}(M(H = 0))$. In fact, δ_H is a lowest weight vector of weight $-\kappa_H$.

Proof. In view of (2.4), it suffices to argue that the determinant polynomial δ_H is a lowest weight vector in $R(M(H = 0))$ of weight $-\kappa_H$.

To see this, fix a basis ψ_1, \dots, ψ_k of $M(H < 0)$ such that each ψ_j is a weight vector of weight ω_j , and denote by $\alpha_1, \dots, \alpha_k$ the negative roots with $\langle \alpha, H \rangle < 0$, so that $E_k := E_{\alpha_k}$ is a basis of $\mathfrak{n}_-(H < 0)$. Then the determinant polynomial δ_H with respect to this basis can be written as

$$\delta_H(\psi) = \langle \psi_1 \wedge \dots \wedge \psi_k | \Lambda^k(\pi(-)\psi) | E_1 \wedge \dots \wedge E_k \rangle$$

where we write $\Lambda^k A$ for the canonical homomorphism $\Lambda^k V \rightarrow \Lambda^k W$ induced by a linear map $A: V \rightarrow W$. It follows that for any $g \in G$ and $\psi \in M(H = 0)$,

$$\begin{aligned} (g \cdot \delta_H)(\psi) &= \delta_H(\Pi(g^{-1})\psi) \\ &= \langle \psi_1 \wedge \dots \wedge \psi_k | \Lambda^k(\Pi(g^{-1})) \Lambda^k(\pi(-)\psi) \Lambda^k(\text{Ad}(g)) | E_1 \wedge \dots \wedge E_k \rangle. \end{aligned}$$

Both $\Lambda^k(\mathfrak{n}_-(H < 0)) = \mathbb{C}E_1 \wedge \dots \wedge E_k$ and $\Lambda^k(M(H < 0)) = \mathbb{C}\psi_1 \wedge \dots \wedge \psi_k$ are one-dimensional representations of $K(H = 0)$. In particular, they are invariant under the action of $N_-(H = 0)$, and it is immediate that their

weight is given by $\sum_k \alpha_k$ and $\sum_k \omega_k$, respectively. It follows at once that δ_H is a lowest weight vector of weight $-\kappa_H$. \square

5. Computation of moment cones

Theorem 1.1 reduces the computation of the moment cone of an arbitrary finite-dimensional representation to an enumeration of all Ressayre elements, which in principle is straightforward: Since there are only finitely many weights, the admissibility condition cuts down the number of possible inequalities down to a finite list of candidates, and for each such candidate $(-, H) \geq 0$, the isomorphism condition can be easily checked. Indeed, we only need to verify the trace condition (4.1) and that the determinant polynomial (4.2) is non-zero. In this way, we obtain a deterministic algorithm to compute the moment cone for an arbitrary representation that can easily be implemented in a computer program [35]. We remark that checking whether the determinant polynomial is non-zero can be sped up by using a fast *probabilistic* algorithm for polynomial identity testing, e.g., based on the Schwartz-Zippel lemma.

In practice, naively enumerating all admissible hyperplanes by considering all $(r_K - 1)$ -element subsets of Ω quickly becomes infeasible as one considers representations of larger dimensions. In this case, it is useful to first determine the admissible elements *up to the Weyl group* and to impose further necessary conditions by a more refined analysis of the representation at hand. Then Corollary 4.2 can be used to obtain directly only those candidates that satisfy the trace condition. In this way, we may often cut down the number of candidates substantially for which we need to check that δ_H is non-zero. In Section 6 below, we illustrate this analysis in the case of the one-body quantum marginal problem.

6. Quantum marginal problem and Kronecker cone

The state of a quantum system is specified by a unit vector ψ in a Hilbert space M , which we will always assume to be finite-dimensional. Quantum systems composed of several distinguishable particles are described by the tensor product of the Hilbert spaces of their constituents, $M = \bigotimes_{k=1}^n M_k$. The state of the k -th subsystem can be described by the *reduced density matrix* ρ_k , which is the unique positive semi-definite operator on M_k such that

$$(6.1) \quad \text{Tr } \rho_k X_k = \langle \psi | \mathbb{1}^{\otimes(k-1)} \otimes X_k \otimes \mathbb{1}^{\otimes(n-k)} | \psi \rangle$$

for all Hermitian operators X_k on M_k (it is also called the *partial trace* of ψ). The fundamental *one-body quantum marginal problem* asks which ρ_1, \dots, ρ_n can arise as the reduced density matrices of a quantum state $\psi \in M$ [13, 14, 16, 17, 21]. Equivalently, it asks for the compatibility conditions that the ρ_k have to satisfy in order for there to exist a global state ψ . For two subsystems, it is a straightforward consequence of the singular value decomposition that ρ_1 and ρ_2 are compatible if and only if they have the same non-zero eigenvalues (including multiplicities). In general, however, the problem is much more involved; it can be shown that it is a strict generalization of the problem of computing the Horn cone of Section 7 [13, 21, 22] and it has been solved in [13, 14] by using similar methods [10, 24]. However, a concrete description akin to the Horn inequalities is still elusive.

In this section, we shall consider the case of three subsystems, to which the general case can always be reduced [21]. By comparing (6.1) and (2.2), it is not hard to see that solving the one-body quantum marginal problem amounts to computing the moment cone for the action of $K = \text{SU}(a) \times \text{SU}(b) \times \text{SU}(c) \times \text{U}(1)$ on $M = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$. The rank of K is $r = (a - 1) + (b - 1) + (c - 1) + 1 = a + b + c - 2$. The elements of \mathfrak{it} can be identified with quadruples $H = (H_A, H_B, H_C, z) \in \mathbb{R}^{a+b+c+1}$ with $\sum_i H_{A,i} = \sum_j H_{B,j} = \sum_k H_{C,k} = 0$. It will be convenient to identify the elements of the dual space \mathfrak{it}^* with triples $\lambda = (\lambda_A, \lambda_B, \lambda_C) \in \mathbb{R}^{a+b+c}$ with $|\lambda| := \sum_i \lambda_{A,i} = \sum_j \lambda_{B,j} = \sum_k \lambda_{C,k}$. The natural pairing of elements $H \in \mathfrak{it}$ and $\lambda \in \mathfrak{it}^*$ is then given by

$$(H, \lambda) = (H_A, \lambda_A) + (H_B, \lambda_B) + (H_C, \lambda_C) + z|\lambda|.$$

Thus the set of weights of M is $\Omega = \{(e_{A,i}, e_{B,j}, e_{C,k}) : i = 1, \dots, a, j = 1, \dots, b, k = 1, \dots, c\}$, where we write $e_{A,i}$ for the vector in \mathbb{R}^a with i -th entry equal to 1 and all other entries equal to zero, etc. Throughout this section, we will denote the moment cone by $C(a, b, c)$.

By Schur-Weyl duality and Mumford’s description (2.4), the moment cone $C(a, b, c)$ can equivalently be defined in terms of the representation theory of the symmetric group [13, 16–18]: We have

$$C(a, b, c) = \text{cone}\{(\alpha, \beta, \gamma) : g_{\alpha, \beta, \gamma} \neq 0\},$$

where $\alpha, \beta,$ and γ vary over the set of Young diagrams with the same number k of boxes and no more than $a, b,$ and c rows, respectively, and where $g_{\alpha, \beta, \gamma}$ denotes the *Kronecker coefficient of the symmetric group*, i.e., the multiplicity of the invariant subspace in the corresponding triple tensor product of irreducible S_k –representations. We will henceforth refer to $C(a, b, c)$ as the *Kronecker cone*.

It will be convenient to assume without loss of generality that $1 < a \leq b \leq c \leq ab$. In this case, $C(a, b, c)$ is maximal-dimensional (see Corollary A.2 in the Appendix) and our method is directly applicable. In fact, we shall see below that all other cases can be reduced to the case $c = ab$. According to Theorem 1.1, the moment cone $C(a, b, c)$ is thus cut out by those H which are Ressayre elements, i.e., which are admissible and satisfy the isomorphism condition. Naively determining the admissible H by enumerating all subsets of Ω with cardinality $r - 1$, determining whether they span a hyperplane and computing the normal vector amounts to considering $\binom{|\Omega|}{r-1} = \binom{abc}{a+b+c-3}$ subsets, which rapidly becomes infeasible (e.g., for $a = b = c = 4$, there are over 27 billion such subsets). We therefore need to derive additional constraints to make this approach computationally feasible.

6.1. Candidates

As a first step, we recall that the bipartite variant of the problem has a straight-forward solution: The moment cone of $SU(a) \times SU(b) \times U(1)$ acting on $\mathbb{C}^a \otimes \mathbb{C}^b$ is given by

$$(6.2) \quad C(a, b) = \{(\lambda_A, \lambda_B) \in \mathbb{R}^{a+b} : \lambda_{A,1} \geq \dots \geq \lambda_{A,a} \geq 0, \lambda_B = (\lambda_A, 0, \dots, 0)\},$$

where we have used the same conventions as above and assumed that $a \leq b$. This can be easily proved directly; it also follows from Mumford’s description of the moment polytope, since $\text{Sym}^k(\mathbb{C}^a \otimes \mathbb{C}^b) = \bigoplus_{\lambda} V_{\lambda}^a \otimes V_{\lambda}^b$ by Schur-Weyl duality, where the direct sum is over all Young diagrams λ with k boxes and no more than $a = \min(a, b)$ rows. In quantum-mechanical terms, (6.2) amounts to the well-known assertion that the reduced density matrices of a bipartite pure state are isospectral.

By considering $SU(a) \times SU(b) \times SU(c) \subseteq SU(ab) \times SU(c)$, the moment cone for the tripartite problem can now be written in terms of the bipartite cone $C(ab, c)$ and the moment polytopes $\Delta(a, b | \lambda_{AB})$ of the action of $SU(a) \times SU(b)$ on the coadjoint $SU(ab)$ -orbits $\mathcal{O}_{\lambda_{AB}}^{ab}$:

$$(6.3) \quad C(a, b, c) = \{(\lambda_A, \lambda_B, \lambda_C) : \exists \lambda_{AB} \text{ s.th. } (\lambda_{AB}, \lambda_C) \in C(ab, c), (\lambda_A, \lambda_B) \in \Delta(a, b | \lambda_{AB})\}$$

A first consequence is that the case $c \neq ab$ can always be reduced to $c = ab$. Indeed, (6.2) and (6.3) imply that

$$C(a, b, c) = \{(\lambda_A, \lambda_B, (\lambda_{AB}, 0, \dots, 0)) : (\lambda_A, \lambda_B, \lambda_{AB}) \in C(a, b, ab)\}$$

if $c > ab$, and

$$(6.4) \quad C(a, b, c) = \{(\lambda_A, \lambda_B, \lambda_C) : (\lambda_A, \lambda_B, (\lambda_C, 0, \dots, 0)) \in C(a, b, ab)\}$$

if $c < ab$. Thus the moment cone for $c > ab$ is isometric to $C(a, b, ab)$, while for $c < ab$ it is obtained as a projection of the latter. In the case where $c = ab$, it was observed in [13, 36] that any normal vector of the moment cone $C(a, b, ab)$ necessarily has a rather special form. We state this result and give a succinct alternative proof that does not rely on the results of [13, 24]:

Lemma 6.1. *Let $H = (H_A, H_B, H_{AB}, z)$ be the normal vector of a non-trivial facet of the moment cone $C(a, b, ab)$ such that $(H_A, H_B) \neq 0$. Then:*

- 1) (H_A, H_B) is determined by a maximal number of equations of the form $H_{A,i} + H_{B,j} = H_{A,k} + H_{B,l}$,
- 2) the components of H_{AB} are precisely all possible partial sums $-H_{A,i} - H_{B,j}$, and
- 3) $z = 0$.

Proof. By Lemma 3.2, there exists a regular dominant point $\lambda = (\lambda_A, \lambda_B, \lambda_{AB}) \in \text{int}_{>0}^*$ in the interior of this facet. In a neighborhood of λ , the moment cone locally looks like a half-space, so that

$$(6.5) \quad (H_A, \mu_A) + (H_B, \mu_B) \geq z|\lambda| - (H_{AB}, \lambda_{AB})$$

is not only a valid inequality that holds for all $(\mu_A, \mu_B) \in \Delta(a, b|\lambda_{AB})$, as follows from (6.3), but in fact a facet of the moment polytope $\Delta(a, b|\lambda_{AB})$, since we have assumed that $(H_A, H_B) \neq 0$.

While determining the moment polytopes $\Delta(a, b|\lambda_{AB})$ is just as hard as determining the cone $C(a, b, ab)$, this reformulation gives us an additional insight: Recall that the Duistermaat-Heckman measure for the action of the maximal torus $T(a) \times T(b)$ of $\text{SU}(a) \times \text{SU}(b)$ on the coadjoint $\text{SU}(ab)$ -orbit $\mathcal{O}_{\lambda_{AB}}^{ab}$ is a measure on the abelian moment polytope. Now let π denote the projection $\text{it}(ab)^* \rightarrow \text{it}(a)^* \oplus \text{it}(b)^*$. The affine hyperplanes through some $\pi(w\lambda_{AB})$ spanned by subsets of the *restricted roots* $\pi(\alpha)$, $\alpha \in R_{ab,+}$, partition the abelian moment polytope into a finite number of polyhedral chambers, and it as an immediate consequence of the Heckman formula [23, 37] that the measure has a polynomial density function on each chamber (cf. [38]). The Duistermaat-Heckman measure for the action of $\text{SU}(a) \times \text{SU}(b)$ can be recovered by applying a number of partial derivatives to the measure for

$T(a) \times T(b)$ (e.g., [18]). It follows that the moment polytope $\Delta(a, b | \lambda_{AB})$, which is the support of the latter measure, is equal to a finite union of chambers; in particular, its non-trivial facets are contained in hyperplanes of the form just described.

Applied to the facet (6.5), it follows that its normal vector (H_A, H_B) is defined by a maximal number of equations of the form

$$\begin{aligned} ((H_A, H_B), \pi(\alpha)) &= (H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B, \alpha) \\ &= H_{A,i} + H_{B,j} - H_{A,k} - H_{B,l} = 0 \end{aligned}$$

for some indices i, j, k, l . This shows the first assertion. Moreover, since the facet contains $\pi(w\lambda_{AB})$ for some $w \in S_{ab}$ we obtain that

$$\begin{aligned} ((H_A, H_B), \pi(w\lambda_{AB})) &= (w^{-1}(H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B), \lambda_{AB}) \\ &= z|\lambda| - (H_{AB}, \lambda_{AB}). \end{aligned}$$

A priori, the permutation w will depend on the choice of λ_{AB} . However, there are only finitely many permutations w , while we may vary λ_{AB} arbitrarily in a small neighborhood. It follows that $w^{-1}(H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B) = -H_{AB}$ and $z = 0$ (since $H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B$ is traceless). This shows the second and third assertion. □

Following Klyachko, we shall call any (H_A, H_B) that satisfies condition 1 of Lemma 6.1 an *extremal edge* if it is in addition dominant and primitive (in the dual of the root lattice). There are only finitely many extremal edges and we shall denote them by $\mathcal{E}_+(a, b)$. The extremal edges span the extreme rays of the *cubicles*, which are the full-dimensional convex cones of elements (H_A, H_B) cut out by a maximal set of inequalities of the form $H_{A,i} + H_{B,j} \geq H_{A,k} + H_{B,l}$. In other words, a cubicle is defined as a set of (H_A, H_B) with fixed order of the $H_{A,i} + H_{B,j}$ and can therefore be encoded by a standard Young tableaux of rectangular shape $a \times b$ [13]. This gives a straightforward way of computationally determining all extremal edges (see Table 1).

We remark that standard tableaux that correspond to cubicles have also been called *additive* in the literature [39, 40]. Manivel has shown that they can be associated with minimal regular faces of the moment cone for which the corresponding Kronecker coefficients stabilize [36, 40]. The corresponding extremal edges determine non-trivial facets of $C(a, b, ab)$ incident to this face [13, 36]. However, not all non-trivial facets can be obtained in this way.

(a, b)	$(2, 2)$ [13]	$(3, 3)$ [13]	$(4, 4)$
Tableaux	2	42	24024
Cubicles	2	36	6660
Extremal edges ($ \mathcal{E}_+ $)	3 (2)	17 (10)	457 (233)

Table 1: Bipartite candidates (counts in parentheses are with permutations removed).

Corollary 6.2. *Let $H = (H_A, H_B, H_C, z)$ be the normal vector of a non-trivial facet of the moment cone $C(a, b, c)$, where $c \leq ab$. Then $(H_A, H_B) = 0$ or (H_A, H_B) is proportional to an element in the $S_a \times S_b$ -orbit of $\mathcal{E}_+(a, b)$ (i.e., an extremal edge up to $S_a \times S_b$ and rescaling).*

Proof. In view of (6.4), we may obtain finite and complete set of inequalities for $C(a, b, c)$ by taking any facet $H = (H_A, H_B, H_{AB}, 0)$ for $C(a, b, ab)$ and restricting H_{AB} to its first c components (that is, set H_C to be the traceless part of $(H_{AB,1}, \dots, H_{AB,c})$ and define z accordingly). Such an inequality will not necessarily define a facet of $C(a, b, c)$, but all facets arise in this way. If we started with a non-trivial inequality then the claim follows from Lemma 6.1. If we started with a trivial inequality then we either obtain a trivial inequality or $(H_A, H_B) = 0$. □

Lemma 6.1 is efficient in finding candidates for facets of $C(a, b, ab)$, but not necessarily so for $C(a, b, c)$ with $c < ab$. Indeed, while we know from the proof of Corollary 6.2 that any facet of the latter can be obtained by “restriction” of a facet of the former, for each given facet there are many possible restrictions, as we need to pick a subset of c components out of the ab components of the given H_{AB} . If $c \ll ab$ then it can be more efficient to apply Corollary 6.2 to all three of (H_A, H_B) , (H_A, H_C) and (H_B, H_C) (e.g., in the case of $a = b = c = 4$). For this we will use the following lemma:

Lemma 6.3. *Let (H_A, H_B) be an extremal edge. Then H_A and H_B are each either zero or primitive in the dual of the root lattice.*

Proof. By symmetry, it suffices to show assertion for H_A . Let $\Omega(a, b) := \{(\alpha, \beta) \neq 0 : \alpha \in R_a \cup \{0\}, \beta \in R_b \cup \{0\}\}$ and denote by $S \subseteq \Omega(a, b)$ the subset of all elements that are orthogonal to $H := (H_A, H_B)$. Note that the linear hyperplane H^\perp is always spanned by the set of S (this is just a reformulation

of the defining property of an extremal edge). Define $S_B := \{\beta : (\alpha, \beta) \in S\}$ and consider the subspace $\mathfrak{m} := \text{span}_{\mathbb{R}} S_B \subseteq \text{it}(b)$.

Case 1: $\mathfrak{m} \subsetneq \text{it}(b)$. Then $H^\perp = \text{span}_{\mathbb{R}} S \subseteq \text{it}(a) \oplus \mathfrak{m}$. By comparing dimensions, it follows that we have in fact equality, $H^\perp = \text{it}(a) \oplus \mathfrak{m}$. We conclude that $H_A = 0$.

Case 2: $\mathfrak{m} = \text{it}(b)$. Since the matrix with columns the roots of A_{b-1} is totally unimodular (e.g., [41, p. 274, (18)]), it follows that S_B spans the root lattice. To find a contradiction, suppose that H_A is neither zero nor primitive. Then we can write $H_A = nH'_A$ for some $n > 1$ and a non-zero element H'_A in the dual of the root lattice. But then,

$$(H_B, \beta) = -(H_A, \alpha) = -n(H'_A, \alpha_i) \in n\mathbb{Z}$$

for all $(\alpha, \beta) \in S$. Since the S_B spans the root lattice, we conclude that also H_B/n is an element of the dual of the root lattice. But then $H = (H_A, H_B)$ is not primitive, which is the desired contradiction. □

Lemma 6.4. *Let $H = (H_A, H_B, H_C, z)$ be the normal vector of a non-trivial facet of the moment cone $C(a, b, c)$, where $c \leq ab$. Then (H_A, H_B, H_C) is proportional to an element in the $S_a \times S_b \times S_c$ -orbit of*

$$\begin{aligned} \mathcal{E}_+(a, b, c) := \{ & (H_A, H_B, H_C) \neq 0 : (H_A, H_B) \in \mathcal{E}_+(a, b) \cup \{0\}, \\ & (H_A, H_C) \in \mathcal{E}_+(b, c) \cup \{0\}, (H_B, H_C) \in \mathcal{E}_+(a, c) \cup \{0\}\}. \end{aligned}$$

Proof. Since the following argument is equivariant under the Weyl group $S_a \times S_b \times S_c$, we may without loss of generality assume that (H_A, H_B, H_C) is dominant

Case 1: Only one component is non-zero, say, $H_C \neq 0$ and therefore $(H_A, H_B) = 0$. We may rescale H such that H_C is primitive. Then (H_A, H_C) and (H_B, H_C) are also primitive and therefore extremal edges by Corollary 6.2.

Case 2: At least two components are non-zero, say, H_A and H_B . We may rescale H such that (H_A, H_B) is primitive. Since H_A and H_B are non-zero, Lemma 6.3 shows that both H_A and H_B are individually primitive. It follows that three of (H_A, H_B) , (H_A, H_C) and (H_B, H_C) are primitive and therefore extremal edges by Corollary 6.2. □

By Theorem 1.1, any non-trivial facet is necessarily admissible. As admissibility is a Weyl group-invariant property, we immediately obtain the following corollary:

Corollary 6.5. *Let H be the normal vector of a non-trivial facet of the moment cone $C(a, b, c)$, where $c \leq ab$. Then H is proportional to an element in the $S_a \times S_b \times S_c$ -orbit of*

$$\mathcal{E}_{+,adm}(a, b, c) := \{H = (H_A, H_B, H_C, z) \neq 0 : (H_A, H_B, H_C) \in \mathcal{E}_+(a, b, c), \\ H \text{ admissible}\}.$$

For any given $(H_A, H_B, H_C) \in \mathcal{E}_+(a, b, c)$, there are in general several z such that $H = (H_A, H_B, H_C, z)$ is admissible (or no such z at all). The sets $\mathcal{E}_+(a, b, c)$ and $\mathcal{E}_{+,adm}$ consist of dominant, primitive elements, and they can be easily obtained algorithmically from the sets of extremal edges.

As a final step we implement the trace condition (4.1). For this, recall that the *length* of a permutation π is equal to the number of inversions, $\ell(\pi) = \#\{(i, j) : i < j, \pi(i) > \pi(j)\}$. A permutation π is called a *shuffle* with respect to a dominant vector $x \in \mathbb{R}^d$ if $x_i = x_j$ for $i < j$ implies that $\pi(i) < \pi(j)$. Then the following is an immediate consequence of Corollaries 4.2 and 6.5:

Corollary 6.6. *Let H be the normal vector of a non-trivial facet of the moment cone $C(a, b, c)$, where $c \leq ab$. Then H is proportional to an element in*

$$\mathcal{E}(a, b, c) := \{H = w \cdot H_0 : H_0 \in \mathcal{E}_{+,adm}, w \in W(H_0)\}$$

where $W(H_0)$ denotes the set of triples $w = (w_A, w_B, w_C) \in S_a \times S_b \times S_c$ such that $(w_0 w_A, w_0 w_B, w_0 w_C)$ is a triple of shuffles with respect to the components of H_0 whose lengths sum up to $\dim M(H_0) < 0$.

We remark that it is straightforward to computationally generate all shuffles of a given length by adapting the algorithm of Effler and Ruskey [42]. The upshot of Corollary 6.6 then is that the moment cone $C(a, b, c)$ is cut out by those candidates in $\mathcal{E}(a, b, c)$ that are also Ressayre elements (together with the trivial inequalities):

$$C(a, b, c) = \{\lambda \in i\mathfrak{t}_+^* : (H, \lambda) \geq 0 \text{ for all Ressayre elements } H \in \mathcal{E}(a, b, c)\}.$$

We conclude this section with an illustrative example of the method.

Example 6.7. Consider the moment cone $C(d, d, d)$ for any $d > 1$. It is not hard to verify that all pairs formed from $(1, \dots, 1, 1 - d)$ and $(d - 1, -1, \dots, -1)$ are extremal edges (formally, we should divide by d to work with primitive elements, but we refrain from doing so in the interest of

readability). Therefore,

$$((1, \dots, 1, 1 - d), (1, \dots, 1, 1 - d), (d - 1, -1, \dots, -1)) \in \mathcal{E}_+(d, d, d).$$

There are two possible values of z which extend the above to an admissible element $H_0 \in \mathcal{E}_{+, \text{adm}}(d, d, d)$. The first option is $z = -1$. However, the dimension of $M(H_0 < 0)$ is $2(d - 1)^2 + d$, and therefore strictly larger than the dimension of \mathfrak{n}_- . Thus the trace condition (4.1) can never be satisfied!

The second option is $z = d - 1$. Here we find that $\dim M(H_0 < 0) = d - 1$. Consider the Weyl group element w such that $w_0 w = (\mathbb{1}, \mathbb{1}, \sigma)$ where σ is the permutation that sends $1 \mapsto d$ and all other $k \mapsto k - 1$. Observe that w is an element in the set $W(H_0)$ defined in Corollary 6.6, so that we obtain the following candidate:

$$H = w \cdot H_0 = ((1 - d, 1, \dots, 1), (1 - d, 1, \dots, 1), (d - 1, -1, \dots, -1), d - 1) \in \mathcal{E}(d, d, d).$$

We now verify that H is a Ressayre element. For this, observe that we have

$$\begin{aligned} \mathfrak{n}_-(H < 0) &= \text{span}\{(0, 0, E_{\alpha_{21}}), \dots, (0, 0, E_{\alpha_{d1}})\} \\ M(H = 0) &= \text{span}\{e_{111}, e_{i1k}, e_{1jk} : i, j, k = 2, \dots, d\} \\ M(H < 0) &= \text{span}\{e_{112}, \dots, e_{11d}\}, \end{aligned}$$

where $e_{ijk} := e_i \otimes e_j \otimes e_k$, with e_1, \dots, e_d the standard basis in \mathbb{C}^d . We note that all basis vectors in $M(H = 0)$ are annihilated by $\mathfrak{n}_-(H < 0)$ except for e_{111} , which is sent by the lowering operator $(0, 0, E_{\alpha_{k1}})$ to e_{11k} ($k = 2, \dots, d$). Thus the tangent map (3.6) is diagonal with respect to the basis given above and the determinant polynomial (4.2) is given by

$$\delta_H(\psi) = \det \begin{pmatrix} \psi_{111} & & \\ & \ddots & \\ & & \psi_{111} \end{pmatrix} = \psi_{111}^{d-1} \neq 0,$$

for any $\psi = \sum_{ijk} \psi_{ijk} e_{ijk} \in M(H = 0)$. It follows at once from Theorem 1.1 that $(H, \lambda) \geq 0$ is a valid inequality for the moment polytope. We have thus obtained the well-known *polygonal inequality* [43]

$$\lambda_{A,1} + \lambda_{B,1} \leq |\lambda| + \lambda_{C,1},$$

in a completely mechanical fashion. By symmetry, the two other inequalities obtained by permuting the subsystems A, B , and C are also valid. We remark

(a, b, c)	$(2, 2, 2)$ [43]	$(3, 3, 3)$ [45]	$(4, 4, 4)$
$ \mathcal{E}_+ $	7 (3)	51 (17)	3027 (600)
$ \mathcal{E}_{+, \text{adm}} $	11 (5)	67 (25)	2231 (484)
$ \mathcal{E} $	9 (3)	192 (41)	32406 (5633)
Inequalities	9 (3)	114 (25)	1749 (323)
Facets	6 (2)	45 (10)	270 (50)
Extreme Rays	5 (3)	33 (11)	328 (65)

Table 2: Tripartite candidates (counts in parentheses are with permutations removed).

that the moment cone $C_{K(H=0)}(M(H=0))$ is closely related to the Horn cone for $U(d-1)$ discussed in Section 7 below [19–22].

6.2. Computational results

To verify that a given H is a Ressayre element, we have implemented a computer program that works for arbitrary representations. To compute the set of candidates $\mathcal{E}(a, b, c)$ in the case of the one-body quantum marginal problem, we have used the strategy explained above. In Table 2 we list some results obtained by our program in the symmetric scenario $a = b = c$, which corresponds to three quantum particles with the same number of degrees of freedom. While the moment cones $C(2, 2, 2)$ and $C(3, 3, 3)$ had already been computed in [43–46] using different methods, the cone $C(4, 4, 4)$ had been out of reach using current methods. In contrast, our method allows us the computation of $C(4, 4, 4)$ in a few minutes, since it does not rely on an intermediate computation of the higher-dimensional cone $C(4, 4, 16)$.

In Tables 3 and 4, we list the extreme rays and facets of $C(4, 4, 4)$ up to permutations of the subsystems. Facet No. 21 is an instance of the polygonal inequality discussed in Example 6.7. Illustrating a pattern observed in [45] for $C(3, 3, 3)$, there are several facets that contain neither the highest weight $((1, 0, 0, 0), (1, 0, 0, 0), (1, 0, 0, 0))$ nor the “origin” (τ_4, τ_4, τ_4) , where $\tau_d = \mathbb{1}/d \in \mathbb{R}^d$. Moreover, the only facets that contain the origin are the Weyl chamber inequalities. This is an instance of a general fact that might be of independent interest (Lemma A.3 in the Appendix).

We refer to [35] for a complete list of our computational results. In particular, we have verified the inequalities proposed previously by Klyachko

#	V_A	V_B	V_C
1	(1/4, 1/4, 1/4, 1/4)	(1/4, 1/4, 1/4, 1/4)	(1/4, 1/4, 1/4, 1/4)
2	(1/4, 1/4, 1/4, 1/4)	(1/4, 1/4, 1/4, 1/4)	(1/3, 1/3, 1/3, 0)
3	(1/4, 1/4, 1/4, 1/4)	(1/4, 1/4, 1/4, 1/4)	(1/2, 1/2, 0, 0)
4	(1/4, 1/4, 1/4, 1/4)	(1/4, 1/4, 1/4, 1/4)	(1, 0, 0, 0)
5	(1/4, 1/4, 1/4, 1/4)	(1/3, 1/3, 1/3, 0)	(1/3, 1/3, 1/3, 0)
6	(1/4, 1/4, 1/4, 1/4)	(1/3, 1/3, 1/3, 0)	(1/2, 1/2, 0, 0)
7	(1/4, 1/4, 1/4, 1/4)	(1/3, 1/3, 1/3, 0)	(2/3, 1/6, 1/6, 0)
8	(1/4, 1/4, 1/4, 1/4)	(1/3, 1/3, 1/3, 0)	(2/3, 1/4, 1/12, 0)
9	(1/4, 1/4, 1/4, 1/4)	(1/3, 1/3, 1/3, 0)	(3/4, 1/12, 1/12, 1/12)
10	(1/4, 1/4, 1/4, 1/4)	(3/8, 3/8, 1/4, 0)	(5/8, 3/8, 0, 0)
11	(1/4, 1/4, 1/4, 1/4)	(3/8, 3/8, 1/4, 0)	(3/4, 1/8, 1/8, 0)
12	(1/4, 1/4, 1/4, 1/4)	(2/5, 3/10, 3/10, 0)	(7/10, 3/20, 3/20, 0)
13	(1/4, 1/4, 1/4, 1/4)	(5/12, 5/12, 1/6, 0)	(2/3, 1/6, 1/12, 1/12)
14	(1/4, 1/4, 1/4, 1/4)	(1/2, 1/6, 1/6, 1/6)	(1/2, 1/2, 0, 0)
15	(1/4, 1/4, 1/4, 1/4)	(1/2, 1/4, 1/8, 1/8)	(5/8, 3/8, 0, 0)
16	(1/4, 1/4, 1/4, 1/4)	(1/2, 1/4, 1/4, 0)	(1/2, 1/2, 0, 0)
17	(1/4, 1/4, 1/4, 1/4)	(1/2, 1/4, 1/4, 0)	(2/3, 1/6, 1/6, 0)
18	(1/4, 1/4, 1/4, 1/4)	(1/2, 1/4, 1/4, 0)	(3/4, 1/4, 0, 0)
19	(1/4, 1/4, 1/4, 1/4)	(1/2, 3/8, 1/8, 0)	(5/8, 1/8, 1/8, 1/8)
20	(1/4, 1/4, 1/4, 1/4)	(1/2, 1/2, 0, 0)	(1/2, 1/2, 0, 0)
21	(2/7, 2/7, 2/7, 1/7)	(4/7, 1/7, 1/7, 1/7)	(4/7, 3/7, 0, 0)
22	(7/24, 7/24, 5/24, 5/24)	(1/3, 1/3, 1/3, 0)	(3/4, 1/8, 1/8, 0)
23	(3/10, 3/10, 1/5, 1/5)	(2/5, 3/10, 3/10, 0)	(4/5, 1/10, 1/10, 0)
24	(3/10, 3/10, 3/10, 1/10)	(1/2, 1/2, 0, 0)	(11/20, 3/20, 3/20, 3/20)
25	(3/10, 3/10, 3/10, 1/10)	(1/2, 1/2, 0, 0)	(3/5, 1/5, 1/10, 1/10)
26	(1/3, 2/9, 2/9, 2/9)	(1/3, 1/3, 1/3, 0)	(2/3, 1/3, 0, 0)
27	(1/3, 2/9, 2/9, 2/9)	(1/3, 1/3, 1/3, 0)	(7/9, 1/9, 1/9, 0)
28	(1/3, 2/9, 2/9, 2/9)	(4/9, 4/9, 1/9, 0)	(2/3, 1/9, 1/9, 1/9)
29	(1/3, 1/3, 1/6, 1/6)	(1/3, 1/3, 1/3, 0)	(7/9, 1/9, 1/9, 0)
30	(1/3, 1/3, 1/6, 1/6)	(1/3, 1/3, 1/3, 0)	(5/6, 1/6, 0, 0)
31	(1/3, 1/3, 1/6, 1/6)	(1/2, 1/4, 1/4, 0)	(3/4, 1/12, 1/12, 1/12)
32	(1/3, 1/3, 1/6, 1/6)	(2/3, 1/6, 1/6, 0)	(2/3, 1/6, 1/6, 0)
33	(1/3, 1/3, 1/3, 0)	(1/3, 1/3, 1/3, 0)	(1/3, 1/3, 1/3, 0)
34	(1/3, 1/3, 1/3, 0)	(1/3, 1/3, 1/3, 0)	(1/2, 1/2, 0, 0)
35	(1/3, 1/3, 1/3, 0)	(1/3, 1/3, 1/3, 0)	(1, 0, 0, 0)
36	(1/3, 1/3, 1/3, 0)	(2/5, 1/5, 1/5, 1/5)	(11/15, 2/15, 1/15, 1/15)
37	(1/3, 1/3, 1/3, 0)	(5/12, 1/4, 1/6, 1/6)	(3/4, 1/12, 1/12, 1/12)
38	(1/3, 1/3, 1/3, 0)	(5/12, 5/12, 1/12, 1/12)	(3/4, 1/12, 1/12, 1/12)
39	(1/3, 1/3, 1/3, 0)	(4/9, 1/3, 1/9, 1/9)	(7/9, 1/9, 1/9, 0)
40	(1/3, 1/3, 1/3, 0)	(1/2, 1/6, 1/6, 1/6)	(1/2, 1/2, 0, 0)
41	(1/3, 1/3, 1/3, 0)	(1/2, 1/6, 1/6, 1/6)	(2/3, 1/9, 1/9, 1/9)
42	(1/3, 1/3, 1/3, 0)	(1/2, 1/6, 1/6, 1/6)	(2/3, 1/3, 0, 0)
43	(1/3, 1/3, 1/3, 0)	(1/2, 1/2, 0, 0)	(1/2, 1/2, 0, 0)
44	(1/3, 1/3, 1/3, 0)	(1/2, 1/2, 0, 0)	(7/12, 1/4, 1/12, 1/12)
45	(1/3, 1/3, 1/3, 0)	(1/2, 1/2, 0, 0)	(2/3, 1/6, 1/6, 0)
46	(1/3, 1/3, 1/3, 0)	(5/9, 2/9, 1/9, 1/9)	(2/3, 1/9, 1/9, 1/9)
47	(1/3, 1/3, 1/3, 0)	(2/3, 1/3, 0, 0)	(2/3, 1/3, 0, 0)
48	(5/14, 5/14, 1/7, 1/7)	(3/7, 2/7, 2/7, 0)	(11/14, 1/14, 1/14, 1/14)
49	(4/11, 4/11, 3/11, 0)	(5/11, 2/11, 2/11, 2/11)	(8/11, 1/11, 1/11, 1/11)
50	(3/8, 1/4, 1/4, 1/8)	(1/2, 1/2, 0, 0)	(5/8, 1/8, 1/8, 1/8)
51	(3/8, 3/8, 1/4, 0)	(5/8, 1/8, 1/8, 1/8)	(5/8, 1/8, 1/8, 1/8)
52	(2/5, 1/5, 1/5, 1/5)	(2/5, 2/5, 1/5, 0)	(4/5, 1/5, 0, 0)
53	(2/5, 1/5, 1/5, 1/5)	(1/2, 1/2, 0, 0)	(3/5, 1/5, 1/10, 1/10)
54	(2/5, 3/10, 3/10, 0)	(2/5, 2/5, 1/10, 1/10)	(4/5, 1/10, 1/10, 0)
55	(2/5, 2/5, 1/10, 1/10)	(3/5, 1/5, 1/5, 0)	(7/10, 1/10, 1/10, 1/10)
56	(5/12, 5/12, 1/12, 1/12)	(1/2, 1/4, 1/4, 0)	(3/4, 1/12, 1/12, 1/12)
57	(3/7, 3/7, 1/7, 0)	(4/7, 1/7, 1/7, 1/7)	(5/7, 1/7, 1/7, 0)
58	(1/2, 1/6, 1/6, 1/6)	(1/2, 1/2, 0, 0)	(2/3, 1/6, 1/6, 0)
59	(1/2, 1/4, 1/4, 0)	(1/2, 1/2, 0, 0)	(5/8, 1/8, 1/8, 1/8)
60	(1/2, 1/4, 1/4, 0)	(1/2, 1/2, 0, 0)	(3/4, 1/4, 0, 0)
61	(1/2, 1/2, 0, 0)	(1/2, 1/2, 0, 0)	(1/2, 1/2, 0, 0)
62	(1/2, 1/2, 0, 0)	(1/2, 1/2, 0, 0)	(1, 0, 0, 0)
63	(1/2, 1/2, 0, 0)	(5/8, 1/8, 1/8, 1/8)	(5/8, 1/8, 1/8, 1/8)
64	(1/2, 1/2, 0, 0)	(2/3, 1/6, 1/6, 0)	(2/3, 1/6, 1/6, 0)
65	(1, 0, 0, 0)	(1, 0, 0, 0)	(1, 0, 0, 0)

Table 3: Extreme rays (V_A, V_B, V_C) of the moment cone for $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ (with permutations removed).

#	H_A	H_B	H_C	z	Remarks
1	(-5, -1, 3, 3)	(-5, 3, 3, -1)	(5, 1, -3, -3)	5	*
2	(-5, -1, 3, 3)	(1, -3, -3, 5)	(3, 3, -1, -5)	5	-
3	(-5, 3, -1, 3)	(-5, 3, -1, 3)	(5, 1, -3, -3)	5	*
4	(-5, 3, -1, 3)	(-5, 3, 3, -1)	(5, -3, 1, -3)	5	*
5	(-5, 3, -1, 3)	(-3, 1, -3, 5)	(3, 3, -1, -5)	5	*
6	(-5, 3, -1, 3)	(-3, 5, 1, -3)	(3, -5, 3, -1)	5	*
7	(-5, 3, -1, 3)	(1, -3, -3, 5)	(3, -1, 3, -5)	5	-
8	(-5, 3, -1, 3)	(1, -3, 5, -3)	(3, -1, -5, 3)	5	-
9	(-5, 3, 3, -1)	(-5, 3, 3, -1)	(5, -3, -3, 1)	5	*
10	(-5, 3, 3, -1)	(-3, -3, 1, 5)	(3, 3, -1, -5)	5	*
11	(-5, 3, 3, -1)	(-3, -3, 5, 1)	(3, 3, -5, -1)	5	*
12	(-5, 3, 3, -1)	(-3, 1, -3, 5)	(3, -1, 3, -5)	5	*
13	(-5, 3, 3, -1)	(-3, 1, 5, -3)	(3, -1, -5, 3)	5	*
14	(-5, 3, 3, -1)	(-3, 5, -3, 1)	(3, -5, 3, -1)	5	*
15	(-5, 3, 3, -1)	(-3, 5, 1, -3)	(3, -5, -1, 3)	5	*
16	(-5, 3, 3, -1)	(-1, -5, 3, 3)	(1, 5, -3, -3)	5	*
17	(-5, 3, 3, -1)	(-1, 3, -5, 3)	(1, -3, 5, -3)	5	*
18	(-5, 3, 3, -1)	(-1, 3, 3, -5)	(1, -3, -3, 5)	5	*
19	(-3, -1, 3, 1)	(-3, 3, 1, -1)	(3, 1, -1, -3)	3	*
20	(-3, -1, 3, 1)	(1, -1, -3, 3)	(3, 1, -1, -3)	3	-
21	(-3, 1, 1, 1)	(-3, 1, 1, 1)	(3, -1, -1, -1)	3	*
22	(-3, 1, 1, 1)	(-2, -2, 2, 2)	(2, 2, -2, -2)	3	*
23	(-3, 1, 1, 1)	(-2, 2, -2, 2)	(2, -2, 2, -2)	3	*
24	(-3, 1, 1, 1)	(-2, 2, 2, -2)	(2, -2, -2, 2)	3	*
25	(-3, 1, 1, 1)	(-1, -1, -1, 3)	(1, 1, 1, -3)	3	*
26	(-3, 1, 1, 1)	(-1, -1, 3, -1)	(1, 1, -3, 1)	3	*
27	(-3, 1, 1, 1)	(-1, 3, -1, -1)	(1, -3, 1, 1)	3	*
28	(-3, 3, 1, -1)	(-3, 3, 1, -1)	(3, -1, -3, 1)	3	*
29	(-3, 3, 1, -1)	(-1, -3, 1, 3)	(3, 1, -1, -3)	3	-
30	(-3, 3, 1, -1)	(-1, -3, 3, 1)	(1, 3, -1, -3)	3	*
31	(-3, 3, 1, -1)	(-1, -3, 3, 1)	(3, 1, -3, -1)	3	-
32	(-3, 3, 1, -1)	(-1, 3, 1, -3)	(1, -1, -3, 3)	3	*
33	(-2, -2, 2, 2)	(-2, 2, 2, -2)	(1, 1, -3, 1)	3	*
34	(-2, 2, -2, 2)	(-2, 2, 2, -2)	(1, -3, 1, 1)	3	*
35	(-1, -1, -1, 3)	(0, 0, 0, 0)	(0, 0, 0, 0)	1	*
36	(-1, 0, 0, 1)	(-1, 1, 0, 0)	(1, 0, 0, -1)	1	*
37	(-1, 0, 0, 1)	(0, 0, -1, 1)	(1, 0, 0, -1)	1	-
38	(-1, 0, 1, 0)	(-1, 0, 1, 0)	(1, 0, 0, -1)	1	*
39	(-1, 0, 1, 0)	(-1, 1, 0, 0)	(1, 0, -1, 0)	1	*
40	(-1, 0, 1, 0)	(0, -1, 0, 1)	(1, 0, 0, -1)	1	-
41	(-1, 0, 1, 0)	(0, -1, 1, 0)	(0, 1, 0, -1)	1	*
42	(-1, 0, 1, 0)	(0, -1, 1, 0)	(1, 0, -1, 0)	1	-
43	(-1, 0, 1, 0)	(0, 0, -1, 1)	(0, 1, 0, -1)	1	*
44	(-1, 1, 0, 0)	(-1, 1, 0, 0)	(1, -1, 0, 0)	1	*
45	(-1, 1, 0, 0)	(0, -1, 0, 1)	(0, 1, 0, -1)	1	*
46	(-1, 1, 0, 0)	(0, -1, 1, 0)	(0, 1, -1, 0)	1	*
47	(-1, 1, 0, 0)	(0, 0, -1, 1)	(0, 0, 1, -1)	1	*
48	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 1, -1)	0	†,*
49	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 1, -1, 0)	0	†,*
50	(0, 0, 0, 0)	(0, 0, 0, 0)	(1, -1, 0, 0)	0	†

Table 4: Normal vectors (H_A, H_B, H_C, z) of the facets of the moment cone for $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ (with permutations removed). The last column states whether the facet includes the origin (†) or the highest weight (*).

for $a, b \leq 3$ [13]. For $M = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^{12}$, which can be treated by a variant of the technique described above, we found that some of the proposed inequalities do *not* correspond to extremal edges and are likely typographic mistakes (e.g, the second block on [13, p. 43]). This can also be deduced from the fact that they do not agree with Bravyi’s inequalities when restricted to $\lambda_D = (1, 0, \dots, 0)$. We remark that the main challenge to obtaining concrete computational results in higher dimension is to find a tractable set of candidates beyond simple admissibility.

There are further variants of the one-body quantum marginal problem, such as for fermionic systems [47, 48], where the facets amount to strengthenings of the classical *Pauli exclusion principle*. Our theory is also applicable to these scenarios, and it would be interesting to undertake a similar analysis of the facets that would allow the computation of the corresponding moment cones beyond what has been possible in the literature [49]. We refer to [21, §3] for initial investigations, where a family of fermionic inequalities has been proved for all local dimensions by using our method. We remark that we have verified numerically that the pure-state fermionic inequalities list in [49] are correct (but not their sufficiency).

7. Horn cone and Howe-Lee-Tan-Willenbring invariants

In this section, we consider the representation of $G = \text{GL}(d) \times \text{GL}(d) \times \text{GL}(d)$ on $M = \mathfrak{gl}(d) \oplus \mathfrak{gl}(d)$ given by

$$\Pi(g, h, k)(a, b) = (gak^{-1}, hbk^{-1}),$$

where $\mathfrak{gl}(d)$ is the Hilbert space of complex $d \times d$ matrices equipped with the trace inner product $\langle a|b \rangle := \text{Tr } a^\dagger b$. We choose $K = \text{U}(d) \times \text{U}(d) \times \text{U}(d)$ and take the maximal torus T to consist of triples of unitary diagonal matrices, so that it can be identified with $\mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}^d$. We furthermore identify $i\mathfrak{k} \cong i\mathfrak{k}^*$ and $it \cong it^*$ by using the trace inner product and choose the usual positive roots $\alpha_{i,j}$, $i < j$, of each $\text{GL}(d)$, such that the positive Weyl chamber it^*_+ can be identified with triples of vectors with non-increasing entries each. Finally, we set $|x| := \sum_i x_i$ and $x^* := (-x_d, \dots, -x_1)$ for $x \in \mathbb{R}^d$. Using these conventions, the moment map for the K -action can be written as

$$\mu_K(a, b) = (aa^\dagger, bb^\dagger, -a^\dagger a - b^\dagger b).$$

Any non-negative Hermitian matrix can be written in the form $a^\dagger a$; since the spectra of aa^\dagger and $a^\dagger a$ are equal, the moment cone is equal to

$$C(d) := \{(\text{spec } X, \text{spec } Y, \text{spec } Z) : X, Y \geq 0, Z \leq 0, X + Y + Z = 0\},$$

where $\text{spec } X$ denotes the eigenvalues of a Hermitian matrix X , ordered non-increasingly. We will call $C(d)$ the *Horn cone* in d dimensions. As proved in [10, 12], cf. [50, 51], $C(d)$ is cut out by the Horn inequalities, which we will recall below.

We remark that the Horn cone as defined thus is not maximal-dimensional but rather supported in the linear subspace $\{(x, y, z) : |x| + |y| + |z| = 0\}$. Dually, any two normal vectors that differ by $(\mathbb{1}, \mathbb{1}, \mathbb{1})$ determine the same facet. This can be avoided by working with the subgroup $G' = \text{GL}(d) \times \text{GL}(d) \times \text{SL}(d) \subseteq G$, but we will not discuss this point any further.

For any subset $I \subseteq [d] := \{1, \dots, d\}$, we define $\mathbb{C}^I := \bigoplus_{i \in I} \mathbb{C}e_i \subseteq \mathbb{C}^d$. We define E_I to be the orthogonal projection onto \mathbb{C}^I and abbreviate $E_r := E_{[r]}$ for any $r \leq d$. It is easy to see (e.g., by using an argument involving restricted roots as in Subsection 6.1) that apart from positivity ($x_d \geq 0$, etc.), any non-trivial facet of the Horn cone is of the form $(H_{IJK}, -) \geq 0$, where $H_{IJK} := (E_I, E_J, E_K)$ for subsets $I, J, K \subseteq [d]$ with $r := |I| = |J| = |K| < d$.

7.1. Trace and Horn condition

In the following, we will show that the trace condition and the Horn condition assume their familiar form in the context of Horn’s problem (e.g., [50]), thereby justifying our terminology. For any $I = \{i_1 < \dots < i_r\} \subseteq [d]$ of cardinality r , let w_I denote the permutation that sends $[r]$ to I and $[r]^c$ to the complement $I^c = \{i_1^c < \dots < i_{d-r}^c\}$ while preserving the order of each block. Then $E_I = w_I \cdot E_r = w_I E_r w_I^{-1}$ (we identify w_I with a permutation matrix). However, w_I does *not* satisfy the conditions of Corollary 4.2. Instead, we shall write $E_I = \tilde{w}_I \cdot E_r$ with $\tilde{w}_I := w_{I^c} w_0$, which reverses the order of each block. Then we find that H_{IJK} satisfies the trace condition if and only if

$$\ell(w_{I^c}) + \ell(w_{J^c}) + \ell(w_{K^c}) = \dim M(H_r < 0),$$

where $H_r := (E_r, E_r, E_r)$. For the left-hand side, we compute

$$\begin{aligned} \ell(w_{I^c}) &= \#\{(b, a) : b < a, w_{I^c}(b) > w_{I^c}(a)\} = \sum_{a=1}^r \#\{j \in I^c : i_a < j\} \\ &= \sum_{a=1}^r ((d - r) - (i_a - a)) = |\lambda_I|, \end{aligned}$$

where we have defined $\lambda_I := (d - r - (i_a - a))_{a=1}^r$, which is a sequence of non-increasing non-negative integers. For the right-hand side, observe that $M(H_r < 0)$ consists of pairs of block matrices of the form $\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$, hence is isomorphic to $M_{d-r,r}^2$, the vector space of pairs of complex $(d - r) \times r$ -matrix. We conclude that the trace condition for H_{IJK} amounts to

$$(7.1) \quad |\lambda_I| + |\lambda_J| + |\lambda_K| = 2(d - r)r.$$

It is not hard to see that (7.1) is precisely equivalent to Horn’s trace condition.

To analyze the Horn condition, we note that the centralizer of H_{IJK} is $K(H_{IJK} = 0) = (U(I) \times U(I^c)) \times (U(J) \times U(J^c)) \times (U(K) \times U(K^c))$ where we denote by $U(I)$ the subgroup of unitaries that act non-trivially only on $\mathbb{C}^I \subseteq \mathbb{C}^d$, etc. It will be convenient to use the three isomorphisms

$$(7.2) \quad \begin{aligned} U(r)^3 \times U(d - r)^3 &\rightarrow K(H_{IJK} = 0), \\ (U, V, W, U', V', W') &\mapsto (w_I \begin{pmatrix} U & 0 \\ 0 & U' \end{pmatrix} w_I^{-1}, w_J \begin{pmatrix} V & 0 \\ 0 & V' \end{pmatrix} w_J^{-1}, w_K \begin{pmatrix} W & 0 \\ 0 & W' \end{pmatrix} w_K^{-1}), \\ M_{d-r,r}^2 &\rightarrow M(H_{IJK} < 0), \\ (A, B) &\mapsto (w_I \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} w_K^{-1}, w_J \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} w_K^{-1}), \\ T_I \times T_J \times T_K &\rightarrow \mathfrak{n}_-(H_{IJK} < 0), \\ (X, Y, Z) &\mapsto (w_I \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} w_I^{-1}, w_J \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} w_J^{-1}, w_K \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} w_K^{-1}), \end{aligned}$$

where we have defined $T_I := \{X \in M_{d-r,r} : \langle b|X|a \rangle = 0 \text{ if } i_a > i_b^c\}$ etc. A short computation then reveals that the element $\kappa_{H_{IJK}}$ of Proposition 1.2 identifies with the highest weight

$$(7.3) \quad \kappa_{IJK} := (\lambda_I, \lambda_J, \lambda_K - 2(d - r)\chi_r) \oplus (\lambda_{I^c}, \lambda_{J^c}, \lambda_{K^c} - r\chi_{d-r})$$

of $U(r)^3 \times U(d - r)^3$, where χ_r denotes the weight of the determinant representation of $U(r)$. On the other hand, by using the isomorphism

$$(7.4) \quad \begin{cases} \mathfrak{gl}(r)^2 \oplus \mathfrak{gl}(d - r)^2 \rightarrow M(H_{IJK} = 0) \\ (a, b, a', b') \mapsto (w_I \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} w_I^{-1}, w_J \begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix} w_J^{-1}) \end{cases}$$

the moment cone $K(H_{IJK} = 0)$ on $M(H_{IJK} = 0)$ gets likewise identified with the direct product of the Horn cones $C(r) \times C(d - r)$. Thus the Horn condition (Proposition 1.2) asserts that $(\lambda_I, \lambda_J, \lambda_K - 2(d - r)\chi_r) \in C(r)$ as well as $(\lambda_{I^c}, \lambda_{J^c}, \lambda_{K^c} - r\chi_{d-r}) \in C(d - r)$. These two conditions are in fact equivalent by a well-known duality of the Littlewood-Richardson coefficients.

Therefore, we arrive at a single condition

$$(7.5) \quad (\lambda_I, \lambda_J, \lambda_K - 2(d - r)\chi_r) \in C(r).$$

This condition is not only necessary for H_{IJK} to be a facet, but it is also sufficient for H_{IJK} to be a valid inequality, as is well-known (e.g., [50]). We will show in the next section how in the context of our work this can be deduced from the saturation conjecture and Schubert calculus (in fact, we shall see that (7.5) implies that H_{IJK} is a Ressayre element). In particular, we obtain the familiar recursive definition of Horn’s inequalities: Define $\text{Horn}(d, r)$ to be the set of all triples $I, J, K \subseteq [d]$ of cardinality $r < d$ that satisfy the trace condition (7.1) as well as

$$\sum_{a \in A} i_a + \sum_{b \in B} j_b + \sum_{c \in C} k_c \leq s(d + 1) + \frac{s(s + 1)}{2}$$

for all $s < r$ and all triples $(A, B, C) \in \text{Horn}(r, s)$. Then (7.5) implies that H_{IJK} satisfies the Horn condition if and only if $(I, J, K) \in \text{Horn}(d, r)$. Therefore, our trace and Horn conditions are indeed a generalization of the classical conditions due to Horn.

7.2. The determinant polynomial

In this section, we will show that any H_{IJK} that satisfies the trace and Horn condition is automatically a Ressayre element, i.e., that the determinant polynomial $\delta_{IJK} := \delta_{H_{IJK}}$ is non-zero. To start, we note that by the saturation property of the Littlewood-Richardson coefficients [12] (cf. [51]), the Horn condition (7.5) implies that the following space of $\text{GL}(r)$ -invariants is non-zero:

$$(7.6) \quad (V_{\lambda_I} \otimes V_{\lambda_J} \otimes V_{\lambda_K - 2(d-r)\chi_r})^{\text{GL}(r)} \neq 0$$

A natural candidate is certainly the determinant polynomial itself (from which we had obtained the Horn condition), but it is not obvious that the Horn condition should imply that $\delta_{IJK} \neq 0$. We will give two alternative arguments that show that this is indeed the case.

The first proof follows an argument of Belkale [51]. We start with the observation that

$$\begin{aligned} \dim (V_{\lambda_I} \otimes V_{\lambda_J} \otimes V_{\lambda_K - 2(d-r)\chi_r})^{\text{GL}(r)} &= \dim (V_{\lambda_I} \otimes V_{\lambda_J} \otimes V_{\lambda_K})^{\text{SL}(r)} \\ &= \dim (V_{\lambda_{I_-}} \otimes V_{\lambda_{J_-}} \otimes V_{\lambda_{K_-}})^{\text{SL}(r)} \end{aligned}$$

where $I_- := \{d + 1 - i : i \in I\}$, etc.; the second equality follows from $\lambda_{I_-} = (d - r)\chi_r + \lambda_{I_-}^*$. Now consider the Grassmannian $\text{Gr}(r, d)$, which consists of the r -dimensional subspaces of \mathbb{C}^d . It is a homogeneous $\text{GL}(d)$ -manifold of real dimension $2r(d - r)$. Let B denote the standard (upper-triangular) Borel and denote by V_I the point in $\text{Gr}(r, d)$ corresponding to the subspace $\mathbb{C}^I \subseteq \mathbb{C}^d$; set $V_r := V_{[r]}$. For any $w \in S_d$, we define Schubert cell $\Omega^0(w) := Bw \cdot V_r$ and denote by $[\Omega(w)]$ the cohomology class determined by its closure. We now use the classical result that the Littlewood-Richardson coefficients can be computed as an intersection number of Schubert cells [52, §9.4]:

$$\dim (V_{\lambda_{I_-}} \otimes V_{\lambda_{J_-}} \otimes V_{\lambda_{K_-}})^{\text{SL}(r)} = [\Omega(w_{I_-})] \cap [\Omega(w_{J_-})] \cap [\Omega(w_{K_-})].$$

Using that $\Omega(w_{I_-}) = \Omega(w_0 w_I)$, we recognize that (7.6) is equivalent to the cohomological condition

$$(7.7) \quad [\Omega(w_0 w_I)] \cap [\Omega(w_0 w_J)] \cap [\Omega(w_0 w_K)] \neq 0.$$

Now consider the “action map”

$$(7.8) \quad N_-(H_{IJK} < 0) \times M(H_{IJK} \geq 0) \rightarrow M, (g, \phi) \mapsto \Pi(g)\phi,$$

a close variant of what we had analyzed in the proof of Proposition 3.13. We will use the parametrizations

$$\begin{aligned} T_I \times T_J \times T_K &\rightarrow N_-(H_{IJK} < 0), & (x, y, z) &\mapsto (w_I \tilde{x} w_I^{-1}, w_J \tilde{y} w_J^{-1}, w_K \tilde{z} w_K^{-1}) \\ (M_{d-r,r}^\perp)^2 &\rightarrow M(H_{IJK} \geq 0), & (p, q) &\mapsto (w_I p w_K^{-1}, w_J q w_K^{-1}), \end{aligned}$$

where $\tilde{x} := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ etc. and $M_{d-r,r}^\perp := \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \}$. In explicit coordinates $p = \begin{pmatrix} a & a'' \\ 0 & a' \end{pmatrix}$ and $q = \begin{pmatrix} b & b'' \\ 0 & b' \end{pmatrix}$, the map (7.8) becomes

$$(7.9) \quad R_{IJK} : \begin{cases} T_I \times T_J \times T_K \times (M_{d-r,r}^\perp)^2 \rightarrow \mathfrak{gl}(d)^2 \\ (x, y, z, p, q) \mapsto (\tilde{x} p \tilde{z}^{-1}, \tilde{y} q \tilde{z}^{-1}) = \left(\begin{pmatrix} a-a''z & a'' \\ xa-a'z-xa''z & xa''+a' \\ b-b''z & b'' \\ yb-b'z-yb''z & yb''+b' \end{pmatrix} \right) \end{cases}$$

Lemma 7.1. *The map R_{IJK} is dominant.*

Proof. We show that the fibers of R_{IJK} are generically non-empty. Thus let $(g, h) \in \text{GL}(d)^2 \subseteq \mathfrak{gl}(d)^2$. Then (x, y, z, p, q) is an element of the fiber

$R_{IJK}^{-1}(g, h)$ if and only if

$$g^{-1}\tilde{x}p = h^{-1}\tilde{y}q = \tilde{z}.$$

Any such p and q is automatically invertible, and therefore a general element in the stabilizer group of $V_r \in \text{Gr}(r, d)$. Thus the fiber is non-empty if and only if we can find (x, y, z) such that

$$g^{-1}\tilde{x} \cdot V_r = h^{-1}\tilde{y} \cdot V_r = \tilde{z} \cdot V_r.$$

Since $\{\tilde{x} \cdot V_r : x \in T_I\} = (w_0w_I)^{-1}\Omega(w_0w_I)$ etc., this is the case if and only if

$$g^{-1}(w_0w_I)^{-1}\Omega(w_0w_I) \cap h^{-1}(w_0w_J)^{-1}\Omega(w_0w_J) \cap (w_0w_I)^{-1}\Omega(w_0w_I) \neq \emptyset.$$

By Kleiman’s transversality theorem, the cohomological condition (7.7) ensures that this is the case for generic (g, h) . □

Now observe that using the identifications (7.2) and (7.4), the tangent map (3.6) at some base point $(a, b, a', b') \in \mathfrak{gl}(r)^2 \oplus \mathfrak{gl}(d - r)^2$ reads

$$(7.10) \quad V_{IJK}(a, b, a', b') : \begin{cases} T_I \oplus T_J \oplus T_K \rightarrow M_{d-r,r}^2 \\ (X, Y, Z) \mapsto (Xa - a'Z, Yb - b'Z) \end{cases}$$

Corollary 7.2. *The determinant polynomial $\delta_{IJK} = \det V_{IJK}$ is non-zero, i.e., H_{IJK} is a Ressayre element.*

Proof. By Sard’s theorem and Lemma 7.1, there exists a point where the differential of R_{IJK} is surjective. By writing the differential of (7.9) in coordinates and comparing with (7.10), it is not hard to see that surjectivity of the former at some point (x, y, z, p, q) implies surjectivity of the tangent map at (a, b, a', b') , where $p = \begin{pmatrix} a & a'' \\ 0 & a' \end{pmatrix}$ and $q = \begin{pmatrix} b & b'' \\ 0 & b' \end{pmatrix}$. Thus H_{IJK} is a Ressayre element. □

Example 7.3. Let $d = 6, r = 3$, and $I = J = K = \{1 < 3 < 5\}$, so that $I^c = J^c = K^c = \{2 < 4 < 6\}$. Then $\lambda_I = \lambda_J = \lambda_K = (3, 2, 1)$ and $\lambda_{I^c} = \lambda_{J^c} = \lambda_{K^c} = (2, 1, 0)$, and the associated Littlewood-Richardson coefficients are

Generators	Weight
$f_1(x) = x_{1,3}$	$((-1, 0, 0), (0, 0, 0), (0, 0, 1))$
$f_2(x) = \det \begin{pmatrix} x_{1,2} & x_{2,3} \\ x_{2,2} & x_{2,3} \end{pmatrix}$	$((-1, -1, 0), (0, 0, 0), (0, 1, 1))$
$f_3(x) = \det x$	$((-1, -1, -1), (0, 0, 0), (1, 1, 1))$
$g(x, y) = x_{1,2}y_{1,3} - y_{1,2}x_{1,3}$	$((-1, 0, 0), (-1, 0, 0), (0, 1, 1))$
$h(x, y) = x_{1,1} \det \begin{pmatrix} y_{1,2} & y_{1,3} \\ y_{2,2} & y_{2,3} \end{pmatrix}$	$((-1, 0, 0), (-1, -1, 0), (1, 1, 1))$
$-x_{1,2} \det \begin{pmatrix} y_{1,1} & y_{1,3} \\ y_{2,1} & y_{2,3} \end{pmatrix} + x_{1,3} \det \begin{pmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{pmatrix}$	$((-1, 0, 0), (-1, -1, 0), (1, 1, 1))$

Table 5: A set of homogeneous generators of the algebra of $N_-(3)^3$ -invariant polynomials on $\mathfrak{gl}(3)^2$ together with their weights (up to permutation $x \leftrightarrow y$).

given by

$$\begin{aligned} & \dim (V_{(3,2,1)} \otimes V_{(3,2,1)} \otimes V_{(-3,-4,-5)})^{\text{GL}(3)} \\ &= \dim (V_{(2,1,0)} \otimes V_{(2,1,0)} \otimes V_{(-1,-2,-3)})^{\text{GL}(3)} = 2. \end{aligned}$$

In Table 5 we list a set of homogeneous generators of the algebra of lowest weight vectors in $R(\mathfrak{gl}(3)^2)$. Note that $F_1(x, y) := f_1(x)f_2(y)h(y, x)$ and $F_2(y, x) := F_1(x, y)$ span the two-dimensional subspace of weight $((-2, -1, 0), (-2, -1, 0), (1, 2, 3))$. An explicit calculation best left to a computer algebra system [53] verifies that

$$\delta_{IJK}(a, b, a', b') = -f_3(a)f_3(b) (F_1(a, b)F_2(a', b') - F_2(a, b)F_1(a', b')) .$$

Therefore, in agreement with (7.3) δ_{IJK} is indeed a lowest weight vector of weight

$$\begin{aligned} -\kappa_{IJK} &= ((-3, -2, -1), (-3, -2, -1), (3, 4, 5)) \\ &\oplus ((-2, -1, 0), (-2, -1, 0), (1, 2, 3)). \end{aligned}$$

Note that if we consider a' and b' as coefficients rather than indeterminates, $\delta_{IJK}(-, a', b')$ spans the two-dimensional subspace of lowest weight vectors of weight $((-3, -2, -1), (-3, -2, -1), (3, 4, 5))$ as we vary a' and b' over $\mathfrak{gl}(3)$. Likewise, $\delta_{IJK}(a, b, -)$ spans the subspace of lowest weight vectors of weight $((-2, -1, 0), (-2, -1, 0), (1, 2, 3))$ if we instead vary a and b .

The phenomenon just observed in Example 7.3 is in fact of a general nature. In [31, 32] for any triple of Young diagrams (D, E, F) a matrix determinant

$\Delta_{(D,E,F),(A,B)}(X, Y)$ had been constructed that likewise depends on two pairs of matrices (cf. [54, 55]). For each choice of A and B , this determinant is a highest weight vector of weight (D^T, E^T, F^T) , where the superscript denotes the transpose diagrams. Moreover, as one varies A and B , the $\Delta_{(D,E,F),(A,B)}$ span the corresponding subspace of highest weight vectors in the *tensor product algebra* of $\text{GL}(r)$, whose dimension is equal to the corresponding Littlewood-Richardson coefficient $\dim(V_{D^T} \otimes V_{E^T} \otimes V_{F^T})^{\text{GL}(r)}$. In fact, their construction is related by a trivial transform to our determinant polynomial:

$$(7.11) \quad \Delta_{(D,E,F),(A,B)}(X, Y) = \delta_{IJK}(w_0 X^{-1}, w_0 Y^{-1}, -A^T w_0, -B^T w_0) (\det XY)^{d-r}$$

where $(D, E, F) = (\lambda_{I^c}, \lambda_{J^c}, \lambda_{K^c}^* + r\chi_{d-r})$. Equation (7.11) can be shown by manual inspection, relating the matrix elements of the tangent map (3.6) with the matrix constructed by Howe et al. This gives a geometric interpretation of the invariants constructed in [31] – namely, as the determinant of the tangent map (7.10) associated with a Ressayre element $-$, and it also serves as an alternative, second proof that $\delta_{IJK} \neq 0$ is implied by the Horn condition via (7.6).

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Appendix A. On the dimension of the Kronecker cone

Lemma A.1. *Let $1 < a \leq b \leq c \leq ab$. Then there exists an operator $M: \mathbb{C}^a \otimes \mathbb{C}^b \rightarrow \mathbb{C}^c$ such that the following is true: For any $U_A \in \text{U}(a)$, $U_B \in \text{U}(b)$, and $U_C \in \text{U}(c)$,*

$$(A.1) \quad U_C M(U_A \otimes U_B) = M$$

implies that $U_A, U_B,$ and U_C are scalars (i.e., proportional to the identity matrix).

Proof. Let e_i denote the standard basis vectors in any \mathbb{C}^d . We define the following b orthonormal vectors in $\mathbb{C}^a \otimes \mathbb{C}^b$:

$$\begin{aligned} v_1 &:= e_1 \otimes e_1, \dots, v_{a-1} := e_{a-1} \otimes e_{a-1}, \\ v_a &:= e_{a-1} \otimes e_a, \dots, v_{b-1} := e_{a-1} \otimes e_{b-1}, \\ v_b &:= \sum_{\text{all other}} e_i \otimes e_j. \end{aligned}$$

Set $V := \text{span}_{\mathbb{C}}\{v_j\}$. Define an operator M with $Mv_j = \alpha_j e_j$ for some non-zero coefficients α_j to be determined later and which sends V^\perp surjectively to $\text{span}\{e_{b+1}, \dots, e_c\}$. This is always possible since $\dim V^\perp = ab - b \geq c - b$, and the resulting operator M is surjective. Note that $M^\dagger M$ is the direct sum of two Hermitian submatrices; the first acts on V , sending $v_j \mapsto |\alpha_j|^2 v_j$, the second acts on V^\perp and is independent of our choice of α_j . Thus we may choose the α_j such that their absolute values squared are pairwise distinct and also distinct from the eigenvalues of the second submatrix.

Now suppose that $U_A, U_B,$ and U_C are unitaries such that (A.1) holds. Then,

$$M^\dagger M = (U_A \otimes U_B)^\dagger M^\dagger M (U_A \otimes U_B),$$

i.e., $M^\dagger M$ commutes with $U_A \otimes U_B$. But then our choice of α_j implies that $U_A \otimes U_B$ leaves each of the eigenspaces $\mathbb{C}v_1, \dots, \mathbb{C}v_b$ stable. From the first $b - 1$ eigenvectors, which are tensor products, it follows that $U_A e_1 \in \mathbb{C}e_1, \dots, U_A e_{a-1} \in \mathbb{C}e_{a-1}$ as well as $U_B e_1 \in \mathbb{C}e_1, \dots, U_B e_{b-1} \in \mathbb{C}e_{b-1}$ (this requires $a > 1$). This implies that U_A and U_B are diagonal, since they are unitaries. Thus the fact that v_b is also an eigenvector shows that U_A and U_B in fact act by scalars. Finally, observe that this in turn implies that $U_C M \in \mathbb{C}M$. Since M is surjective, we conclude that also U_C acts by a scalar. □

Corollary A.2. *Let $1 < a \leq b \leq c \leq ab$. Then the Kronecker cone $C(a, b, c)$ is maximal-dimensional.*

In the case where $a = b = c$ the following lemma strengthens Corollary A.2:

Lemma A.3. *Let $\tau_d = \mathbb{1}/d \in \mathbb{R}^d$. The only facets of $C(d, d, d)$ that contain the point (τ_d, τ_d, τ_d) are the trivial facets.*

Proof. For any $\lambda_A \in \mathfrak{it}^*$ with $\lambda_d \geq 0$ and $|\lambda| = 1$, there exists a unit vector $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ whose one-body reduced density matrices have spectra $(\lambda_A, \tau_d, \tau_d)$ [56]. It follows that $(\tau_d + \varepsilon_A, \tau_d, \tau_d) \in C(d, d, d)$ for any perturbation $\varepsilon_A \in \mathfrak{it}(d)_+^*$ whose components are small enough in absolute value. By permutation symmetry and convexity, we obtain that

$$(\tau_d + \varepsilon_A, \tau_d + \varepsilon_B, \tau_d + \varepsilon_C) \in C(d, d, d)$$

for any triple of small perturbations $\varepsilon_A, \varepsilon_B, \varepsilon_C$. Therefore, the only constraints in the vicinity of (τ_d, τ_d, τ_d) are the Weyl chamber inequalities. \square

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