JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 15, Number 4, 1089–1128, 2017

Distinguishing symplectic blowups of the complex projective plane

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A symplectic manifold that is obtained from $\mathbb{CP}^2$ by k blowups is encoded by $k+1$ parameters: the size of the initial $\mathbb{CP}^2$, and the sizes of the blowups. We determine which values of these parameters yield symplectomorphic manifolds.

1. Introduction

A symplectic manifold that is obtained from $\mathbb{CP}^2$ by k blowups is encoded by $k+1$ parameters: the size λ of the initial $\mathbb{CP}^2$, and the sizes $\delta_1, \ldots, \delta_k$ of the blowups. In this paper we answer the following question:

Which values of the parameters yield symplectomorphic manifolds?

Example 1.1. For each of the vectors $(\lambda; \delta_1, \delta_2, \delta_3)$ in the table below, consider the manifold that is obtained from a $\mathbb{CP}^2$ of size λ by blowups of sizes δ_1, δ_2, δ_3. These three manifolds have the same classical invariants: the symplectic volume, which is proportional to $\lambda^2 - \sum_{j=1}^{3} {\delta_j}^2$; the pairing of the symplectic form with the first Chern class, which is proportional to $3\lambda - \sum_{j=1}^{3} \delta_j$; and the set of values that the symplectic form takes on $H_2(M)$, which is proportional to $\mathbb{Z}\lambda + \mathbb{Z}\delta_1 + \mathbb{Z}\delta_2 + \mathbb{Z}\delta_3$. The first two manifolds are symplectomorphic, but the third is not symplectomorphic to the first two. This follows from Lemma 2.16 and Theorem 1.8.

λ	δ_1	δ_2	δ_3
15	9	5	4
12	6	2	1
11	4	1	1

To compare different blowups, it is convenient to fix the underlying manifold, as in McDuff and Polterovich [22]. Once and for all, we fix a sequence

$$p_1, p_2, p_3, \ldots$$

of distinct points on the complex projective plane $\mathbb{CP}^2$, and we denote by M_k the manifold that is obtained from $\mathbb{CP}^2$ by complex blowups at $p_1, \ldots, p_k$. We have a decomposition

$$H_2(M_k) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k$$

where L is the image of the homology class of a line $\mathbb{CP}^1$ in $\mathbb{CP}^2$ under the inclusion map $H_2(\mathbb{CP}^2) \to H_2(M_k)$ and where $E_1, \ldots, E_k$ are the homology classes of the exceptional divisors. A **blowup form** on M_k is a symplectic form for which there exist pairwise disjoint embedded symplectic spheres in the classes $L, E_1, \ldots, E_k$. (The terminology "blowup form" was suggested to us by Dusa McDuff.)

Lemma 1.2. *The set of blowup forms on M_k is an equivalence class under the following equivalence relation: symplectic forms ω and ω' on M_k are equivalent iff there exists a diffeomorphism $f\colon M_k \to M_k$ that acts trivially on homology and such that $f^*\omega$ and ω' are homotopic through symplectic forms.*

Lemma 1.3. *Any two cohomologous blowup forms on M_k are diffeomorphic through a diffeomorphism that acts trivially on homology.*

Sketch of proof of Lemmas 1.2 and 1.3. These two lemmas follow from work of Gromov and McDuff. Lemma 1.2 follows from results of Gromov [4, 2.4.A', 2.4.A1'], McDuff [19] and McDuff-Salamon [23, Proposition 7.21]; the deduction of Lemma 1.3 from Lemma 1.2 is by a result of McDuff [20] using the "inflation" technique. We give more details in [10]. For Lemmas 1.2 and 1.3 in the context of uniqueness questions for symplectic structures, see [28, Examples 3.8, 3.9, 3.10]. When $k = 0$, Lemma 1.3 is Gromov's result [4, 2.4 B_2' and 2.4 B_3'], □

Our convention is that the **size** of $\mathbb{CP}^2$ equipped with a symplectic form is $1/2\pi$ times the symplectic area of a line $\mathbb{CP}^1 \subset \mathbb{CP}^2$ and the **size** of a blowup is $1/2\pi$ times the symplectic area of the exceptional divisor.

Definition 1.4. Fix a non-negative integer k. Let $\langle \cdot, \cdot \rangle$ denote the pairing between cohomology and homology on M_k. A vector $(\lambda; \delta_1, \ldots, \delta_k)$ in $\mathbb{R}^{1+k}$ **encodes** a cohomology class $\Omega \in H^2(M_k; \mathbb{R})$ if $\frac{1}{2\pi} \langle \Omega, L \rangle = \lambda$ and $\frac{1}{2\pi} \langle \Omega, E_j \rangle = \delta_j$ for $j = 1, \ldots, k$.

By Lemma 1.3, a blowup form on M_k whose cohomology class is encoded by the vector $(\lambda; \delta_1, \ldots, \delta_k)$ is unique up to a diffeomorphism that acts trivially on the homology. We denote any of these symplectic manifolds by

$$(M_k, \omega_{\lambda;\delta_1,\ldots,\delta_k}).$$

Remark 1.5. Suppose that the vector $(\lambda; \delta_1, \ldots, \delta_k)$ encodes the cohomology class of a blowup form ω on M_k. Then the numbers $\lambda, \delta_1, \ldots, \delta_k$ are positive (from the definition of "blowup form"), they satisfy $\delta_i + \delta_j < \lambda$ for all $i \neq j$ ("the Gromov inequality", see [4, 0.3.B]), and they satisfy $\lambda^2 - \delta_1^2 - \cdots - \delta_k^2 > 0$ ("the volume inequality"). In particular, if $\delta_1 = \cdots = \delta_k = \lambda/3$, then $k \leq 8$.

Definition 1.6. Let $k \geq 3$, and let $\lambda, \delta_1, \ldots, \delta_k$ be real numbers. The vector $(\lambda; \delta_1, \ldots, \delta_k)$ is **reduced** if

$$\delta_1 \geq \cdots \geq \delta_k \quad \text{and} \quad \delta_1 + \delta_2 + \delta_3 \leq \lambda. \tag{1.7}$$

We now state our main theorem.

Theorem 1.8. *Let $k \geq 3$. Given a blowup form $\omega_{\lambda';\delta_1',\ldots,\delta_k'}$ on M_k, there exists a unique reduced vector $(\lambda; \delta_1, \ldots, \delta_k)$ that encodes a blowup form $\omega_{\lambda;\delta_1,\ldots,\delta_k}$ on M_k that is diffeomorphic to the given form:*

$$(M_k, \omega_{\lambda';\delta_1',\ldots,\delta_k'}) \cong (M_k, \omega_{\lambda;\delta_1,\ldots,\delta_k}).$$

The "existence" part of Theorem 1.8 is stated as Proposition 2.1 and proved in Section 2. The "uniqueness" part of Theorem 1.8 is stated as Theorem 5.1 and proved in Section 5.

Given a vector $v = (\lambda'; \delta'_1, \ldots, \delta'_k)$ that encodes the cohomology class of a blowup form, iterations of the "standard Cremona move" yield the corresponding reduced vector $v_{\text{red}} = (\lambda; \delta_1, \ldots, \delta_k)$. Given two blowup forms, to determine whether they are diffeomorphic, examine the resulting reduced vectors v_{red} and v'_{red}; the blowup forms are diffeomorphic if and only if these reduced vectors are equal. See the algorithms of Paragraph 2.17 and Paragraph 5.11. Moreover, the map $v \mapsto v_{\text{red}}$ is continuous. See Lemma 2.18.

In order to prove the "uniqueness" part of Theorem 1.8, for every blowup form whose cohomology class is represented by a reduced vector we give the complete list of exceptional homology classes with minimal symplectic area. See Theorem 3.12 and Remark 3.15. The list always contains the smallest exceptional divisor E_k and generically contains only it. We give two proofs of this result, one in Section 3, and one in Section 4 that uses a beautiful argument of McDuff.

Theorem 1.8, combined with work of Li-Li [16], further leads to the following characterization of blowup forms, which we prove in Section 6.

Theorem 1.9. *Let $k \geq 3$. Let $(\lambda; \delta_1, \ldots, \delta_k)$ be a vector with positive entries that is reduced and that satisfies the volume inequality $\lambda^2 - {\delta_1}^2 - \cdots - {\delta_k}^2 > 0$. Then there exists a blowup form $\omega_{\lambda;\delta_1,\ldots,\delta_k}$ whose cohomology class is encoded by this vector. This defines a bijection between the set of vectors with positive entries that are reduced and satisfy the volume inequality and the set of blowup forms modulo diffeomorphism.*

Given a cohomology class, to determine whether or not it contains a blowup form, first check if the corresponding vector v has positive entries and satisfies the volume inequality. If it does, apply iterations of the "standard Cremona move"; the cohomology class then contains a blowup form if and only if the entries of the resulting vector v_{red} are positive. See the algorithm of Paragraph 6.3.

For completeness, we also describe now the cases $0 \leq k \leq 2$, whose proofs we give in Section 5:

Lemma 1.10. ***The case*** $\mathbf{k = 2}$***:*** *A vector $(\lambda; \delta_1, \delta_2)$ encodes the cohomology class of a blowup form exactly if its entries are positive and satisfy* *the Gromov inequality* $\delta_1 + \delta_2 < \lambda$*. Blowup forms that correspond to vectors* $(\lambda; \delta_1, \delta_2)$ *and* $(\lambda'; \delta_1', \delta_2')$ *are diffeomorphic if and only if* $\lambda' = \lambda$ *and* $\{\delta_1', \delta_2'\} = \{\delta_1, \delta_2\}$.

The case $\mathbf{k = 1}$***:*** *A vector* $(\lambda; \delta_1)$ *encodes the cohomology class of a blowup form exactly if its entries are positive and satisfy* $\delta_1 < \lambda$*. Two blowup forms are diffeomorphic if and only if their cohomology classes are represented by the same vector.*

The case $\mathbf{k = 0}$***:*** *A vector* (λ) *encodes the cohomology class of a blowup form exactly if* $\lambda > 0$*. Two blowup forms are diffeomorphic if and only if they have the same size* λ.

In this paper, we rely on facts that are rather standard in the symplectic topology community but whose precise statements in the form that we need are not always explicit in the literature. We spell out more detailed justifications of these statements in an accompanying manuscript [10], which studies different toric actions on a fixed symplectic four-manifold.

Throughout this paper, unless we say otherwise, homology is taken with integer coefficients and cohomology is taken with real coefficients.

Acknowledgements. This paper branched off from a joint project with Martin Pinsonnault. We are grateful to Martin for his collaboration. We are also grateful for stimulating discussions with Paul Biran, Dusa McDuff, Dietmar Salamon, and Jake Solomon. In particular, our communication with Dusa McDuff has helped clarify Theorem 1.9, and our communication with Jake Solomon has helped clarify Lemma 2.18. The second author would also like to acknowledge support from Tamar Ziegler of the Technion, Israel Institute of Technology.

The first author is partially supported by the Natural Science and Engineering Research Council of Canada. The second author was partially supported by the Center for Absorption in Science, Ministry of Immigrant Absorption, State of Israel, and by the Israel Science Foundation, Grant 557/08.

2. Existence of reduced form

In this section we prove the "existence" part of Theorem 1.8:

Proposition 2.1 (Existence of reduced form). *Let* $k \geq 3$*. Let* ω *be a blowup form on* M_k*. Then there exists a blowup form on* M_k *that is diffeomorphic to* ω *and whose cohomology class is encoded by a reduced vector.*

Moreover, in Paragraph 2.17 we give an algorithm that associates to every vector v that encodes the cohomology class of a blowup form ω a reduced vector v_{red} that encodes the cohomology class of a blowup form that is diffeomorphic to ω, and in Lemma 2.18 we show that the map $v \mapsto v_{\text{red}}$ is continuous.

We begin with some algebraic preliminaries.

We will consider the $\mathbb{Z}$-module ("the lattice") with basis elements $L, E_1, \ldots, E_k$:

$$\mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k \quad \left(\cong \mathbb{Z}^{1+k}\right),$$

with the bilinear form ("the intersection form") that is given by

$$L \cdot L = 1, \quad E_i \cdot E_i = -1, \quad E_i \cdot E_j = 0 \text{ if } i \neq j, \quad L \cdot E_j = 0.$$

2.2. We identify the element $\Omega = (\lambda; \delta_1, \ldots, \delta_k)$ of $\mathbb{R}^{1+k}$ with the homomorphism from the lattice $\mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k$ to $\mathbb{R}$ that satisfies $\lambda = \frac{1}{2\pi}\Omega(L)$ and $\delta_j = \frac{1}{2\pi}\Omega(E_j)$ for all $1 \leq j \leq k$. (Of course, we think of each lattice element as a homology class in $H_2(M_k)$ and of each vector in $\mathbb{R}^{1+k}$ as the cohomology class in $H^2(M_k; \mathbb{R})$ that it encodes.)

We will use the following fact, which we learned from Martin Pinsonnault. This fact was also a crucial ingredient in our previous work [9].

Lemma 2.3. *Let $\Omega := (\lambda; \delta_1, \ldots, \delta_k)$ be a vector in $\mathbb{R}^{1+k}$ that satisfies the volume inequality $\lambda^2 - \delta_1^2 - \cdots - \delta_k^2 > 0$. Let*

$$\mathcal{H}_{-1} = \{E \in \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k \mid E \cdot E = -1\}.$$

Then the map $E \mapsto \Omega(E)$ from $\mathcal{H}_{-1}$ to $\mathbb{R}$ is proper. That is, for each bounded closed interval $I \subset \mathbb{R}$, the set $\{E \in \mathcal{H}_{-1} \mid \Omega(E) \in I\}$ is compact (hence finite).

Proof. We will refer to the Lorentzian inner product on $\mathbb{R}^{1+k}$:

$$\langle u, v \rangle = u_0 v_0 - u_1 v_1 - \cdots - u_k v_k$$

for $u = (u_0; u_1, \ldots, u_k)$ and $v = (v_0; v_1, \ldots, v_k)$. Then $\mathcal{H}_{-1}$ consists of exactly those elements E in the lattice that have the form

$$E = aL - b_1 E_1 - \cdots - b_k E_k$$

with $u := (a; b_1, \ldots, b_k) \in \mathbb{Z}^{1+k}$ and $\langle u, u \rangle = -1$. (Thinking of E as a homology class, the vector u encodes its Poincaré dual.) For such an E, we have

$$\frac{1}{2\pi}\Omega(E) = \langle \Omega, u \rangle.$$

Because $\mathbb{Z}^{1+k}$ is closed in $\mathbb{R}^{1+k}$, it is enough to show that the map

$$u \mapsto \langle \Omega, u \rangle$$

from $\mathcal{H}^{\mathbb{R}}_{-1} := \{u \in \mathbb{R}^{1+k} \mid \langle u, u \rangle = -1\}$ to $\mathbb{R}$ is proper.

Recall that $\langle \Omega, \Omega \rangle > 0$ (by the volume inequality); by rescaling, we assume without loss of generality that $\langle \Omega, \Omega \rangle = 1$. Setting $\epsilon_0 := \Omega$, by the Gram-Schmidt procedure there exist $\epsilon_1, \ldots, \epsilon_k$ such that $\langle \epsilon_0, \epsilon_0 \rangle = 1$, $\langle \epsilon_j, \epsilon_j \rangle = -1$ for $1 \le j \le k$, and $\langle \epsilon_i, \epsilon_j \rangle = 0$ for $i \neq j$. In this basis, the bilinear form $\langle , \rangle$ and hence the set $\mathcal{H}^{\mathbb{R}}_{-1}$ remain unchanged, Ω is represented by the vector $(1; 0, \ldots, 0)$, and the map $u \mapsto \langle \Omega, u \rangle$ becomes $(u_0; u_1, \ldots, u_k) \mapsto u_0$. It is enough to show that the preimage in $\mathcal{H}^{\mathbb{R}}_{-1}$ of the interval $[-N, N]$ is compact for each $N > 0$. This preimage consists of the set of those $(u_0; u_1, \ldots, u_k)$ that satisfy the conditions $u_0^2 - u_1^2 - \cdots - u_k^2 = -1$ and $u_0 \in [-N, N]$. This set is compact because it is closed and bounded. □

Definition 2.4. Let $k \ge 3$. For any vector $v = (\lambda; \delta_1, \ldots, \delta_k)$, define

$$\text{defect}(v) = \delta_1 + \delta_2 + \delta_3 - \lambda,$$

and define the Cremona transformation by

$$\text{cremona}(v) = (\lambda'; \delta'_1, \ldots, \delta'_k),$$

where

$$\begin{aligned} \lambda' &= \lambda - \text{defect}(v) \\ \delta'_j &= \begin{cases} \delta_j - \text{defect}(v) & \text{if } 1 \le j \le 3 \\ \delta_j & \text{if } 4 \le j \le k. \end{cases} \end{aligned}$$

Lemma 2.5. *Let $\Omega = (\lambda; \delta_1, \ldots, \delta_k)$ be a vector that satisfies the volume inequality $\lambda^2 - \delta_1^2 - \cdots \delta_k^2 > 0$. Then the set of real numbers δ'_i that occur among the last k entries in vectors $\Omega' = (\lambda'; \delta'_1, \ldots, \delta'_k)$ that can be obtained from $(\lambda; \delta_1, \ldots, \delta_k)$ by iterations of the Cremona transformation (Definition 2.4) and permutations of the last k entries has no accumulation points.*

Proof. Identifying $\mathbb{R}^{1+k}$ with the set of homomorphisms from the lattice $\mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k$ to $\mathbb{R}$ as in Paragraph 2.2, the Cremona transformation of $\mathbb{R}^{1+k}$ is induced by the transformation of the lattice that is given by

$$\begin{array}{lll} L & \mapsto & 2L - E_1 - E_2 - E_3 \\ E_1 & \mapsto & L - E_2 - E_3 \\ E_2 & \mapsto & L - E_3 - E_1 \\ E_3 & \mapsto & L - E_1 - E_2 \\ E_j & \mapsto & E_j \quad \text{if } 4 \leq j \leq k. \end{array}$$

Similarly, the permutations of $\delta_1, \ldots, \delta_k$ are induced from the transformations of the lattice that preserve L and permute $E_1, \ldots, E_k$. Thus, if $\Omega' = (\lambda'; \delta'_1, \ldots, \delta'_k)$ is obtained from $\Omega = (\lambda; \delta_1, \ldots, \delta_k)$ by iterations of the Cremona transformation and permutations of the last k entries, then each $\delta'_j = \frac{1}{2\pi}\Omega'(E_j)$ is equal to $\frac{1}{2\pi}\Omega(E)$ where E is obtained from E_j by the corresponding transformations of the lattice. Because these transformations preserve the intersection form on the lattice, we conclude that, for each j, the entry δ'_j belongs to the set $\{\frac{1}{2\pi}\Omega(E) \mid E \cdot E = -1\}$. By Lemma 2.3, this set has no accumulation points. □

Definition 2.6. Let $k \geq 3$. The **standard Cremona move** on $\mathbb{R}^{1+k}$ (cf. McDuff and Schlenk [25]) is the composition of the following two maps, in this order:

(i) The map $(\lambda; \delta_1, \ldots, \delta_k) \mapsto (\lambda; \delta_{i_1}, \ldots, \delta_{i_k})$ that permutes the last k entries such that $\delta_{i_1} \geq \cdots \geq \delta_{i_k}$.

(ii) The map $v \mapsto \begin{cases} \text{cremona}(v) & \text{if defect}(v) \geq 0 \\ v & \text{otherwise.} \end{cases}$

Lemma 2.7. 1) *The standard Cremona move is a piecewise linear continuous map from $\mathbb{R}^{1+k}$ to $\mathbb{R}^{1+k}$.*

2) *The standard Cremona move preserves the* ***forward positive cone***

$$\left\{(\lambda; \delta_1, \ldots, \delta_k) \mid \lambda^2 - \delta_1^2 - \cdots - \delta_k^2 > 0 \text{ and } \lambda > 0\right\}.$$

3) *If $v' = (\lambda'; \delta'_1, \ldots, \delta'_k)$ is obtained from $v = (\lambda; \delta_1, \ldots, \delta_k)$ by the standard Cremona move but is not equal to v, then*
a) *$\delta'_i \leq \delta_i$ for all i, and for at least one i we have $\delta'_i < \delta_i$; and*
b) *$\lambda' < \lambda$.*

4) *The vectors that are fixed by the standard Cremona move are exactly the reduced vectors (see Definition 1.6).*

We leave the proof of Lemma 2.7 as an exercise to the reader.

Remark 2.8. Consider the group of transformations of $\mathbb{R}^{1+k}$ that is generated by the Cremona transformation (Definition 2.4) and by permutations of the last k entries. The standard Cremona move is not an element of this group, but on each vector v it acts through some element of this group (that depends on v).

Lemma 2.9. *Let $k \geq 3$. For every vector v in the forward positive cone in $\mathbb{R}^{1+k}$ there exists a positive integer m such that applying m iterations of the standard Cremona move to v yields a reduced vector in the forward positive cone.*

Proof. Let $v = (\lambda; \delta_1, \dots, \delta_k)$ be a vector in the forward positive cone, and let $v^{(n)} = (\lambda^{(n)}, \delta_1^{(n)}, \dots, \delta_k^{(n)})$ be the vector that is obtained from v by applying n iterations of the standard Cremona move. By Lemma 2.7, for all n

- $\lambda^{(n)} > 0$
- $(\lambda^{(n)})^2 - (\delta_1^{(n)})^2 - \cdots - (\delta_1^{(n)})^2 > 0$
- $\lambda^{(n)} \leq \lambda$.

The second inequality implies that $(\delta_i^{(n)})^2 < (\lambda^{(n)})^2$. The first and third inequalities imply that $(\lambda^{(n)})^2 \leq \lambda^2$. So the numbers $\delta_i^{(n)}$ all lie in the bounded interval $(-\lambda, \lambda)$. By Lemma 2.5 and Remark 2.8, the set of numbers $\{\delta_i^{(n)}\}_{1 \leq i \leq k,\ n \in \mathbb{N}}$ is finite. The third and fourth items of Lemma 2.7 then imply that the sequence of vectors $v^{(n)}$ is eventually constant and hence reduced. □

Example 2.10. Let $k = 6$ and $\frac{1}{3} < \delta < \frac{2}{5}$. Then the vector $(1; \delta, \delta, \delta, \delta, \delta, \delta)$ is not reduced. Applying the Cremona transformation, we get the vector $(2 - 3\delta; 1 - 2\delta, 1 - 2\delta, 1 - 2\delta, \delta, \delta, \delta)$. Permuting, we get $(2 - 3\delta; \delta, \delta, \delta, 1 - 2\delta, 1 - 2\delta, 1 - 2\delta)$. Applying the Cremona transformation again, we get $(4 - 9\delta; 2 - 5\delta, 2 - 5\delta, 2 - 5\delta, 1 - 2\delta, 1 - 2\delta, 1 - 2\delta)$; permuting again, we get $(4 - 9\delta; 1 - 2\delta, 1 - 2\delta, 1 - 2\delta, 2 - 5\delta, 2 - 5\delta, 2 - 5\delta)$. Applying the Cremona transformation a third time, we get $(5 - 12\delta; 2 - 5\delta, 2 - 5\delta, 2 - 5\delta, 2 - 5\delta, 2 - 5\delta, 2 - 5\delta)$, which has positive entries and is reduced.

Remark 2.11. In Lemma 2.9, if the entries of v are integers, then applying iterations of the standard Cremona move eventually yields a reduced vector by a simpler reason: $\lambda^{(n)}$ is then a strictly decreasing sequence of positive integers, so it must be finite. A similar argument was used in [14, Proposition 1] and again in [16, Lemma 3.4], [18, Lemma 4.7], and [33, Prop. 2.3].

We will refer to the genus zero Gromov Witten invariant with point constraints,

$$\mathrm{GW}\colon H_2(M_k) \to \mathbb{Z}.$$

For the precise definition of this invariant, see [24]. Fixing a blowup form ω, if $\mathrm{GW}(A) \neq 0$ then for generic ω-tamed almost complex structure J there exists a J-holomorphic sphere in the class A. (We recall that J is ω-tame if $\omega(u, Ju) > 0$ for all nonzero tangent vectors u.)

The Gromov-Witten invariant is the same for all blowup forms; this follows from Lemma 1.2. Lemma 1.2 also implies that the first Chern class $c_1(TM_k) \in H^2(M_k)$ is the same for all blowup forms. Moreover, the first Chern class and the Gromov Witten invariant are consistent under the natural inclusion maps $H_2(M_k) \to H_2(M_{k+1})$ and the natural projection maps $H^2(M_{k+1}) \to H^2(M_k)$; see [6, Theorem 1.4], [13, Proposition 3.5], and the explanation in [10, Appendix].

Lemma 2.12 (Characterization of exceptional classes). *For a homology class E in $H_2(M_k)$, the following conditions are equivalent.*

(a) *There exists a blowup form ω such that the class E is represented by an embedded ω-symplectic sphere with self intersection -1.*

(b) (i) $c_1(TM)(E) = 1$,
 (ii) $E \cdot E = -1$, *and*
 (iii) $\mathrm{GW}(E) \neq 0$.

(c) *For every blowup form ω, the class E is represented by an embedded ω-symplectic sphere with self intersection -1.*

Lemma 2.12 follows from McDuff's "C_1 lemma" [19, Lemma 3.1], Gromov's compactness theorem [4, 1.5.B], and the adjunction formula [24, Corollary 1.7]. We give more details in [10].

Definition 2.13 (Definition of exceptional classes). A homology class E in $H_2(M_k)$ is **exceptional** if it satisfies the conditions (a), (b), (c) of Lemma 2.12.

Remark 2.14 (Examples of exceptional classes). The classes $E_1, \ldots, E_k$ are all exceptional, and so are the classes $L - E_i - E_j$ for all $1 \leq i < j \leq k$. The first fact is by the definition of a blowup form. The second fact is since $L - E_i - E_j$ contains the proper transform in the complex blowup M_k of the unique complex line in $\mathbb{CP}^2$ through the points p_i and p_j; this proper transform is an embedded complex sphere in M_k, hence an embedded ω-symplectic sphere with respect to a Kähler blowup form ω on M_k.

The following lemma is well known. It partially follows from Lemma 2.12 and Remark 2.14. We give details in [10].

Lemma 2.15. *Each of the following homology classes has non-zero Gromov Witten invariant:*

$$L, \quad E_1, \quad \ldots, \quad E_k, \quad L - E_i, \quad L - E_i - E_j \quad \text{for } i \neq j,$$
$$\text{and} \quad 2L - E_1 - E_2 - E_3.$$

Lemma 2.16. *Let $k \geq 3$. Let $v \in \mathbb{R}^{1+k}$ be a vector in the forward positive cone. Let v' be the vector that is obtained from v by the standard Cremona move. Then there exists a blowup form ω on M whose cohomology class is encoded by v if and only if there exists a blowup form ω' on M whose cohomology class is encoded by v'. Moreover, every such ω and ω' are diffeomorphic.*

Proof. By Remark 2.8, the vectors v and v' differ by the Cremona transformation (Definition 2.4) or by a transformation that permutes the last k entries or by the composition of these two maps.

Identifying $\mathbb{R}^{1+k}$ with $H^2(M_k;\mathbb{R})$ as in Definition 1.4, each of these transformations is induced by a diffeomorphism of M_k. Indeed, the Cremona transformation is induced by a diffeomorphism according to Wall [32]. As for the permutations, they are induced by diffeomorphisms of M_k that are obtained from diffeomorphisms of $\mathbb{CP}^2$ that permute the marked points $p_1, \ldots, p_k$ and are biholomorphic on neighbourhoods of these points.

Each of these diffeomorphisms takes $L, E_1, \ldots, E_k$ to homology classes with non-zero Gromov Witten invariants (see Lemma 2.15); this implies that these diffeomorphisms pull back blowup forms to blowup forms. This and Lemma 1.3 imply the last part of the statement. □

(As we will note in Section 6, by results of Tian-Jun Li, Bang-He Li, and Ai-Ko Liu, a reduced vector encodes a blowup form if and only if it is contained in the forward positive cone and its entries are positive.)

2.17 (Algorithm to obtain a reduced form). Let $k \geq 3$. Let v be a vector in the forward positive cone in $\mathbb{R}^{1+k}$.

Step 1: If v is reduced, declare $v_{\text{red}} = v$ and stop.

Step 2: If v is not reduced, replace it by its image under the standard Cremona move and return to Step 1.

By Lemma 2.9, this algorithm terminates, and it produces a reduced vector v_{red} in the forward positive cone. Moreover, by Lemma 2.16, if v encodes the cohomology class of a blowup form ω, then v_{red} encodes the cohomology class of a blowup form that is diffeomorphic to ω.

Proof of Proposition 2.1. The proposition follows immediately from Paragraph 2.17 because a vector that encodes the cohomology class of a blowup form must lie in the forward positive cone. □

Lemma 2.18. *The function $v \mapsto v_{\text{red}}$ of Paragraph 2.17, from the forward positive cone to the intersection of the forward positive cone with the cone of reduced vectors, is continuous.*

2.19. As before, we consider $\mathbb{R}^{1+k}$ with its Lorentzian inner product $\langle u, v\rangle = u_0v_0 - u_1v_1 - \cdots - u_kv_k$ for $u = (u_0; u_1, \ldots, u_k)$ and $v = (v_0; v_1, \ldots, v_k)$. The *null cone* is the set of x in $\mathbb{R}^{1+k}$ such that $\langle x, x\rangle = 0$, the *positive cone* is the set of x in $\mathbb{R}^{1+k}$ such that $\langle x, x\rangle > 0$, and, as already noted, the *forward positive cone* is the set of $x = (x_0; \ldots, x_k)$ such that $\langle x, x\rangle > 0$ and $x_0 > 0$.

For every nonzero vector e, its Lorentzian orthocomplement $e^\perp$ is a hyperplane in $\mathbb{R}^{1+k}$; the hyperplane $e^\perp$ determines the vector e up to scalar; every hyperplane is obtained in this way.

- If $\langle e, e\rangle < 0$, then the hyperplane $e^\perp$ meets the positive cone, and the restriction to $e^\perp$ of the Lorentzian metric on $\mathbb{R}^{1+k}$ is also a Lorentzian metric, of type $(1, k-1)$.
- If $\langle e, e\rangle > 0$, then the hyperplane $e^\perp$ does not meet the positive cone, it meets the null cone only at the origin, and the restriction to $e^\perp$ of the Lorentzian metric on $\mathbb{R}^{1+k}$ is negative definite.
- If $\langle e, e\rangle = 0$, then the hyperplane $e^\perp$ does not meet the positive cone, it meets the null cone along the line $\mathbb{R}e$, and the restriction to $e^\perp$ of the Lorentzian metric on $\mathbb{R}^{1+k}$ is negative semi-definite with null space $\mathbb{R}e$.

For a vector $e \in \mathbb{R}^{1+k}$ with $\langle e, e\rangle \neq 0$, the reflection $\tau_e(v) = v - 2\frac{\langle v,e\rangle}{\langle e,e\rangle}e$ is a Lorentzian isometry that fixes the hyperplane $e^\perp$. We call such a map a **Lorentzian reflection**. This reflection preserves the forward positive cone if and only if $\langle e, e\rangle < 0$. The map $e^\perp \mapsto \tau_e$, for e such that $\langle e, e\rangle < 0$, embeds the space of Lorentzian hyperplanes (with the topology induced from the Grassmannian) into the space of Lorentzian isometries.

The Cremona transformation is the Lorentzian reflection τ_e that corresponds to the vector $e = (1; 1, 1, 1, 0, \ldots, 0)$. The transposition that switches δ_i and δ_{i+1} is the Lorentzian reflection τ_e that corresponds to the vector $e = (0; 0, \ldots, -1, 1, 0, \ldots, 0)$ with $\delta_i = -1$, $\delta_{i+1} = 1$, and other entries $= 0$. In both of these types of reflections, the vector e has integer entries and satisfies $\langle e, e\rangle = -2$.

The following lemma is a slight reformulation of an argument of Jake Solomon [29].

Lemma 2.20. *Every compact subset of the forward positive cone in* $\mathbb{R}^{1+k}$ *meets only finitely many hyperplanes of the form* $e^\perp$ *for* $e \in \mathbb{Z}^{1+k}$ *with* $\langle e, e\rangle = -2$.

Proof. If e has integer entries and satisfies $\langle e, e\rangle = -2$, then the $(1+k) \times (1+k)$ matrix that represents the reflection τ_e has integer entries. Because the set of matrices with integer entries is a discrete subset of the set of all matrices, the set of hyperplanes of the form $e^\perp$ for $e \in \mathbb{Z}^{1+k}$ with $\langle e, e\rangle = -2$ is discrete in the set of all Lorentzian hyperplanes in $\mathbb{R}^{1+k}$. So a hyperplane that occurs as an accumulation point of such hyperplanes (in the Grassmannian) cannot be Lorentzian; in particular, it cannot meet the forward positive cone. (In fact, such a hyperplane must be tangent to the null cone.) The lemma then follows from the compactness of the Grassmannian. □

The hyperplanes of Lemma 2.20 divide the forward positive cone into *chambers*: the intersections of the forward positive cone with the closures of the connected components of the complements of these hyperplanes. Note that the Lorentzian isometries τ_e, for $e \in \mathbb{Z}^{1+k}$ with $\langle e, e\rangle = -2$, take chambers to chambers.

Lemma 2.21. *The restriction of the standard Cremona move to each chamber coincides with a Lorentzian isometry that takes the chamber to another chamber.*

Proof. Applying the standard Cremona move to a vector $(\lambda; \delta_1, \ldots, \delta_k)$ is achieved by iterations of the following single step:

- If the vector is reduced, then stop.

- Otherwise, if $\delta_1, \ldots, \delta_k$ are not in weakly decreasing order, let $i \in \{1, \ldots, k-1\}$ be the smallest index such that $\delta_i < \delta_{i+1}$, and switch δ_i and δ_{i+1}.
- Otherwise, apply the Cremona transformation (Definition 2.4).

Let S_0 denote the cone of reduced vectors. The hyperplanes that are spanned by its facets are the fixed point set $\{\lambda = \delta_1 + \delta_2 + \delta_3\}$ of the Cremona transformation and the fixed point set $\{\delta_i = \delta_{i+1}\}$ of the transposition that switches δ_i and δ_{i+1} for $i \in \{1, \ldots, k-1\}$. These hyperplanes divide $\mathbb{R}^{1+k}$ into "big chambers": the closures of the connected components of the complements of these hyperplanes. The above single step, restricted to a "big chamber", coincides with a Lorentzian reflection. The lemma then follows from the facts that every chamber is contained in a "big chamber" and that the Lorentzian reflections τ_e, for $e \in \mathbb{Z}^{1+k}$ with $\langle e, e \rangle = -2$, take chambers to chambers. □

Proof of Lemma 2.18. Let x be a point in the forward positive cone. By Lemma 2.20, there exists a neighbourhood U of x that meets only finitely many chambers. For every chamber there exists a positive integer m such that, on the chamber, the function $v \mapsto v_{\text{red}}$ is obtained by applying m iterations of the standard Cremona move; this follows from Lemma 2.21 and from the fact that the intersection of the set of reduced vectors with the forward positive cone is a union of chambers. We conclude that there exists a positive integer m such that, on the neighbourhood U of x, the function $v \mapsto v_{\text{red}}$ is obtained by applying m iterations of the standard Cremona move. Because the standard Cremona move is continuous, the function $v \mapsto v_{\text{red}}$ is continuous near x. Because x was arbitrary, the function is continuous on the entire forward positive cone. □

Remark 2.22. We can now give another proof of Lemma 2.9. Let v be a vector in the forward positive cone. Let v' be a vector in the same chamber as v and whose entries are rational. By Lemma 2.21, and since the intersection of the set of reduced vectors with the forward positive cone is a union of chambers, it is enough to show that there exists a positive integer m such that applying m iterations of the standard Cremona move to v' yields a reduced vector. This, in turn, follows by applying the argument of Remark 2.11 to Nv' where N is a positive integer such that Nv' has integer entries.

Remark 2.23. Let S_m denote the set of vectors in $\mathbb{R}^{1+k}$ that are brought to reduced form after m or fewer iterations of the standard Cremona move (but are not necessarily in the forward positive cone). Let S denote the (increasing) union of the sets S_m. (By Lemma 2.9, the forward positive cone is contained in S.)

By applying iterations of the standard Cremona move until we reach a reduced vector, we obtain a function $v \mapsto v_{\text{red}}$ that assigns to each vector in the set S a reduced vector, that is, a vector in S_0. The restriction of this function to each S_m, being the composition of m continuous functions, is continuous.

In particular, S_0 is the set of reduced vectors, given by the linear inequalities (1.7). and the spans of its facets are the fixed point sets of the Cremona transformation (Definition 2.4) and of the $k-1$ transpositions of consecutive elements δ_i, δ_{i+1} for $1 \leq i \leq k-1$.

Consider those cones that can be obtained from S_0 by Lorentzian reflections through the hyperplanes that are spanned by its facets. Continue recursively; at each stage we have a collection of convex polyhedral cones and we add those cones that can be obtained from them by Lorentzian reflections through the hyperplanes that are spanned by the facets of S_0. The set S_m is a finite union of finite intersections of such convex polyhedral cones. This implies that the union of the interiors of the sets S_m is open and dense in S. Because the function $v \mapsto v_{\text{red}}$ is continuous on the interior of each S_m, we conclude that this function is continuous on an open and dense subset of S. We don't know if this function is continuous on S (or even on the interior of S).

Remark 2.24. Other authors [14–16, 18, 33] define "reduced" by the slightly different conditions $\delta_1 + \delta_2 + \delta_3 \leq \lambda$ and $\delta_1 \geq \cdots \geq \delta_k \geq 0$, and they allow transformations that flip the signs of the δ_i.

3. Exceptional classes of minimal area

Let ω be a blowup form on M_k. Lemma 2.12 and Definition 2.13 imply that the set of exceptional classes of minimal ω-area only depends on the vector $v = (\lambda; \delta_1, \ldots, \delta_k)$ that encodes the cohomology class $[\omega]$. We denote this set by

$$\mathcal{E}^v_{\min}.$$

In this section we identify all the possibilities for the set $\mathcal{E}^v_{\min}$; see Theorem 3.12, Remark 3.14, and Remark 3.15.

More generally, let Ω be a cohomology class in $H^2(M_k; \mathbb{R})$ that is encoded by a vector $v = (\lambda; \delta_1, \ldots, \delta_k)$, and assume that the set of values

$\{\langle \Omega, E \rangle \mid E$ is an exceptional class$\}$ is bounded from below. Denote by $\mathcal{E}^v_{\min}$ the set of exceptional classes E for which $\frac{1}{2\pi} \langle \Omega, E \rangle$ is minimal. If v satisfies the volume inequality $\lambda^2 - \delta_1^2 - \cdots - \delta_k^2 > 0$, then this set is non-empty and finite, by Lemma 2.3.

The following lemma is well known, and is deduced from the positivity of intersections of J-holomorphic curves in four dimensional manifolds [24, Appendix E and Proposition 2.4.4], the Hofer-Lizan-Sikorav regularity criterion [5] (see also [24, Lemma 3.3.3]), and the implicit function theorem, see [24, Chapter 3]. We give more details in [10].

Lemma 3.1 (Positivity of intersections). *Let A and B be homology classes in $H_2(M_k)$ that are linearly independent (over $\mathbb{R}$). Suppose that $\mathrm{GW}(B) \neq 0$, that $c_1(TM_k)(A) \geq 1$, and that A is represented by a J holomorphic sphere for some almost complex structure J that is tamed by some blowup form on M_k. Then the intersection number $A \cdot B$ is non-negative.*

In particular, if E is an exceptional class and B is a class that is not a multiple of E and with $\mathrm{GW}(B) \neq 0$, then $E \cdot B$ is non-negative.

We recall that

$$c_1(TM_k)(L) = 3 \quad \text{and} \quad c_1(TM_k)(E_1) = \cdots = c_1(TM_k)(E_k) = 1.$$

We have the following easy technical lemma. Suppose $k \geq 3$. Recall that a vector $(\lambda; \delta_1, \ldots, \delta_k)$ with positive entries is *reduced* if $\delta_1 + \delta_2 + \delta_3 \leq \lambda$ and $\delta_1 \geq \cdots \geq \delta_k$. Denote

$$F := L - E_1, \quad B := L - E_2, \quad E_{12} := L - E_1 - E_2.$$

Lemma 3.2. *Let $k \geq 3$. Let Ω be a cohomology class in $H^2(M_k; \mathbb{R})$ that is encoded by a vector $v = (\lambda; \delta_1, \ldots, \delta_k)$ with positive entries that is reduced. Let A be a class in $H_2(M_k)$. Suppose that A is a multiple of one of the classes in the set*

$$\{L,\ E_1,\ \ldots,\ E_k,\ F,\ B,\ E_{12}\}, \tag{3.3}$$

and suppose that $c_1(TM_k)(A) \geq 1$. Then

$$\frac{1}{2\pi} \langle \Omega, A \rangle \geq \delta_k. \tag{3.4}$$

Moreover, equality in (3.4) holds if and only if either $A = E_\ell$ and $\delta_\ell = \delta_k$, or $A = E_{12}$ and $\lambda - \delta_1 - \delta_2 = \delta_3 = \delta_k$.

In this lemma, A is a homology class over the *integers*, and a-priori it is a *real* multiple of one of the classes in the set (3.3).

Proof. Because $(\lambda; \delta_1, \ldots, \delta_k)$ is a reduced vector,

$$\begin{aligned} \min_{C \in \{L, E_1, \ldots, E_k, F, B, E_{12}\}} \frac{1}{2\pi} \langle \Omega, C \rangle &= \min\{\lambda,\ \delta_1,\ \ldots,\ \delta_k,\ \lambda - \delta_1,\ \lambda - \delta_2,\ \lambda - \delta_1 - \delta_2\} \\ &= \delta_k. \end{aligned} \tag{3.5}$$

Moreover, the minimum is attained on a subset of $\{E_1, \ldots, E_k, E_{12}\}$ that contains E_ℓ if and only if $\delta_\ell = \delta_k$ and that contains E_{12} if and only if $\lambda - \delta_1 - \delta_2 = \delta_3 = \delta_k$.

Also note that $c_1(TM_k)(C)$ is positive for every $C \in \{L, E_1, \ldots, E_k, F, B, E_{12}\}$.

Each of the sets

$$\{L, E_1, \ldots, E_k\},\ \{F, E_1, \ldots, E_k\},\ \{F, B, E_{12}, E_3, \ldots, E_k\},$$

is a basis of $H_2(M_k)$ over $\mathbb{Z}$. Therefore, the assumption that A is a multiple of a class C in $\{L, E_1, \ldots, E_k, F, B, E_{12}\}$ and that $c_1(TM_k)(A) > 0$ is equivalent to

$$A = \gamma C \quad \text{for an integer } \gamma \geq 1.$$

The lemma then follows from (3.5). □

Lemma 3.6. *Let $k \geq 3$. Let Ω be a cohomology class in $H^2(M_k; \mathbb{R})$ that is encoded by a vector $v = (\lambda; \delta_1, \ldots, \delta_k)$ with positive entries that is reduced. Let A be a homology class in $H_2(M_k)$. Suppose that $c_1(TM_k)(A) \geq 1$, and suppose that A is represented by a J-holomorphic sphere for some almost complex structure J that is tamed by some blowup form on M_k. Then*

$$\frac{1}{2\pi} \langle \Omega, A \rangle \geq \delta_k. \tag{3.7}$$

Remark 3.8. Equality in (3.7) implies that $c_1(TM_k)(A) = 1$; we show this in our proof. We note that, for a class A of a J-holomorphic sphere, if $c_1(TM_k)(A) = 1$ then either A is an exceptional class or $A \cdot A \geq 0$; this is a consequence of the adjunction formula.

Remark 3.9. In [10] and [11], to count toric actions on blowups of $\mathbb{CP}^2$, we use the following "indecomposability of minimal exceptional classes", which follows from Lemma 3.6: if ω is a blowup form and E is an exceptional homology class with minimal ω-symplectic area, then, for every ω-tame almost complex structure J, there exists an embedded J-holomorphic sphere in the class E. This result was also obtained by Pinsonnault [26], for more general four-manifolds, using Seiberg-Witten-Taubes theory.

Lemma 3.10. *Let $k \geq 3$. Let ω be a blowup form whose cohomology class is encoded by a vector that is reduced. Then, for every exceptional class E in $H_2(M_k)$,*

$$\frac{1}{2\pi}\langle[\omega], E\rangle \geq \frac{1}{2\pi}\langle[\omega], E_k\rangle .$$

Moreover, let Ω be a cohomology class in $H^2(M_k;\mathbb{R})$ that is encoded by a vector $(\lambda;\delta_1,\ldots,\delta_k)$ with positive entries that is reduced. Then, for every exceptional class E in $H_2(M_k)$,

$$\frac{1}{2\pi}\langle\Omega, E\rangle \geq \delta_k.$$

Lemma 3.10 follows from Lemma 3.6. We prove these lemmas together with the next theorem, in which we identify the set $\mathcal{E}^v_{\min}$ of exceptional classes of minimal area. In the theorem we refer to the following cases. If $(\lambda;\delta_1,\ldots,\delta_k)$ is a vector with positive entries that is reduced, then exactly one of the following cases occurs, where

$$\lambda_F := \lambda - \delta_1, \quad \text{and} \quad \delta_{E_{1\ell}} := \lambda - \delta_1 - \delta_\ell \quad \text{for } \ell \neq 1.$$

1) $\delta_1 \leq \lambda_F/2$ (equivalently, $\delta_1 \leq \lambda/3$), and
   - a) $\delta_k < \lambda/3$.
   - b) $\delta_k = \lambda/3$.

2) $\delta_1 > \lambda_F/2$, $\delta_2 \leq \lambda_F/2$, and
   - a) $\delta_k < \lambda_F/2$.
   - b) $\delta_k = \lambda_F/2$.

3) $\delta_1 > \lambda_F/2$, $\delta_2 > \lambda_F/2$, and
   - a) $\delta_k < \delta_{E_{12}}$.
   - b) $\delta_k = \delta_{E_{12}}$.

Remark 3.11. Let $k \geq 3$. Let $v = (\lambda;\delta_1,\ldots,\delta_k)$ be a vector with positive entries that is reduced.

- If v is in case (1b), then $v = (\lambda; \lambda/3, \ldots, \lambda/3)$ and $k \leq 8$.
- If v is in case (2b), then $v = (\lambda; \delta_1, \lambda_F/2, \ldots, \lambda_F/2)$ and $\delta_1 > \lambda_F/2$.
- If v is in case (3b), then $v = (\lambda; \delta_1, \delta_2, \delta_{E_{12}}, \ldots, \delta_{E_{12}})$ and $\delta_2 > \delta_{E_{12}}$.

Theorem 3.12 (Exceptional classes with minimal area). *Let $k \geq 3$. Let $v = (\lambda; \delta_1, \ldots, \delta_k)$ be a vector with positive entries that is reduced; let Ω be a cohomology class in $H^2(M_k; \mathbb{R})$ that is encoded by this vector. Suppose also that Ω satisfies the volume inequality*

$$\lambda^2 - \delta_1^2 - \cdots - \delta_k^2 > 0. \tag{3.13}$$

Let j be the smallest non-negative integer for which $\delta_{j+1} = \cdots = \delta_k$.

- *If v is in one of the cases (1a), (2a), or (3a), then*

$$\mathcal{E}^v_{\min} = \{E_{j+1}, \ldots, E_k\}.$$

- *If v is in case (1b), then $k \leq 8$, and, by Demazure [3], the exceptional classes are those classes that can be written as $aL - b_1E_1 - \cdots - b_kE_k$ with $(a; b_1, \ldots, b_k)$ a multi-set of one of the following types: $(0; -1, 0^{k-1})$, $(1; 1^2, 0^{k-2})$, $(2; 1^5, 0^{k-5})$, $(3; 2, 1^6, 0^{k-7})$, $(4; 2^3, 1^5)$, $(5; 2^6, 1^2)$, $(6; 3, 2^7)$. In this case, $\mathcal{E}^v_{\min}$ contains* all *the exceptional classes.*
- *If v is in case (2b), then*

$$\mathcal{E}^v_{\min} = \{E_2, \ldots, E_k, E_{12}, \ldots, E_{1k}\}.$$

- *If v is in case (3b), then*

$$\mathcal{E}^v_{\min} = \{E_{12}, E_3, \ldots, E_k\}.$$

Remark 3.14. Combining Theorem 3.12 with the algorithm of Section 2, we obtain the list of exceptional classes with minimal ω-area in $H_2(M_k)$ for *any* blowup form ω on M_k, even if its cohomology class is not represented by a reduced vector. Indeed, let v be the vector that encodes the cohomology class $[\omega]$. The algorithm of Paragraph 2.17 and Definition 2.6 of the standard Cremona move give maps $\gamma_1, \ldots, \gamma_N$ of $\mathbb{R}^{1+k}$ such that each γ_i is a permutation of the last k entries and such that $v_{\text{red}} := (\text{cremona} \circ \gamma_N \circ \cdots \circ \text{cremona} \circ \gamma_1)(v)$ is reduced. Let $\overline{\text{cremona}}$, $\overline{\gamma_1}, \ldots, \overline{\gamma_N}$ be the transformations of $H_2(M_k)$ such that, for every homology class A, identifying every vector $v' \in \mathbb{R}^{1+k}$ with the cohomology class in $H^2(M_k;\mathbb{R})$ that it encodes, we have $\langle \text{cremona}(v'), A\rangle = \langle v', \overline{\text{cremona}}(A)\rangle$ and $\langle \gamma_i(v'), A\rangle = \langle v', \overline{\gamma_i}(A)\rangle$ for $i = 1,\dots,N$. Then $\mathcal{E}^v_{\text{min}} = (\overline{\gamma}_1 \circ \overline{\text{cremona}} \circ \cdots \circ \overline{\gamma}_N \circ \overline{\text{cremona}})\mathcal{E}^{v_{\text{red}}}_{\text{min}}$.

Remark 3.15 (Exceptional classes with minimal area when $\mathbf{k} = \mathbf{0}, \mathbf{1}, \mathbf{2}$). If $k = 0$, there are no exceptional classes, so $\mathcal{E}^v_{\text{min}} = \emptyset$.

If $k = 1$, then $\mathcal{E}^v_{\text{min}} = \{E_1\}$. In fact, in this case E_1 is the only exceptional class. Indeed, suppose that $E = aL - b_1E_1 \in H_2(M_1)$ is exceptional. The equality $E \cdot E = -1$ gives $a^2 - {b_1}^2 = -1$, and the equality $c_1(TM_k)(E) = 1$ (see Lemma 2.12) gives $3a - b_1 = 1$. Because a and b_1 are integers, we deduce that $a = 0$ and $b_1 = -1$, i.e., $E = E_1$.

Suppose now that $k = 2$. Then, by Demazure [3], the set of exceptional classes is $\{E_1, E_2, E_{12}\}$. We have $\delta_2 < \lambda_F/2$ exactly if $\delta_2 < \delta_{E_{12}}$ and $\delta_2 = \lambda_F/2$ exactly if $\delta_2 = \delta_{E_{12}}$.

- If $\delta_2 < \lambda_F/2$ and $\delta_2 < \delta_1$, then $\mathcal{E}^v_{\text{min}} = \{E_2\}$.
- If $\delta_2 < \lambda_F/2$ and $\delta_2 = \delta_1$, then $\mathcal{E}^v_{\text{min}} = \{E_1, E_2\}$.
- If $\delta_2 = \lambda_F/2$ and $\delta_2 < \delta_1$, then $\mathcal{E}^v_{\text{min}} = \{E_2, E_{12}\}$.
- If $\delta_2 = \lambda_F/2$ and $\delta_2 = \delta_1$, then $\mathcal{E}^v_{\text{min}} = \{E_1, E_2, E_{12}\}$.
- If $\delta_2 > \lambda_F/2$, then $\mathcal{E}^v_{\text{min}} = \{E_{12}\}$.

Proof of Lemmas 3.6 and 3.10 and Theorem 3.12. Lemma 3.10 follows from Lemma 3.6: because $c_1(TM_k)(E) = 1$ and $\text{GW}(E) \neq 0$ (by Lemma 2.12 and Definition 2.13), we can apply Lemma 3.6 to E.

Because v is reduced (see also Remark 3.11), in each of the cases in Theorem 3.12, each of the listed classes is exceptional and has size δ_k.

It remains to prove the following results. Let A be a homology class in $H_2(M_k)$. Suppose that $c_1(TM_k)(A) \geq 1$, and suppose that A is represented by a J-holomorphic sphere for some almost complex structure J that is tamed by some blowup form on M_k. Then $\frac{1}{2\pi}\langle \Omega, A\rangle \geq \delta_k$. If, moreover, A is an exceptional class with minimal area and v satisfies the volume inequality (3.13), then A is one of the classes that are listed in Theorem 3.12, according to the case of v.

Case 1: when $\delta_1 \leq \lambda_F/2$; equivalently, $\delta_1 \leq \lambda/3$.

First, suppose that A is not a multiple of any of the classes $L, E_1, \dots, E_k$. Write

$$A = aL - b_1E_1 - \cdots - b_kE_k.$$

By Lemma 2.15, $\mathrm{GW}(L), \mathrm{GW}(E_1), \ldots, \mathrm{GW}(E_k)$ are nonzero; by Lemma 3.1 and by the assumptions on A, the coefficients

$$a = A \cdot L, \quad b_1 = A \cdot E_1, \quad \ldots, \quad b_k = A \cdot E_k$$

are nonnegative. We have

$$c_1(TM_k)(A) = 3a - b_1 - \cdots - b_k;$$

by assumption, this number is ≥ 1. Also in this case, $0 < \delta_i \leq \lambda/3$ for $i = 1, \ldots, k$. Thus,

$$\begin{aligned} \frac{1}{2\pi} \langle \Omega, A \rangle &= a\lambda - b_1\delta_1 - \cdots - b_k\delta_k \\ &\geq a\lambda - b_1\lambda/3 - \cdots - b_k\lambda/3 \\ &= (3a - b_1 - \cdots - b_k)\,\lambda/3 \\ &\overset{(\star)}{\geq} \lambda/3 \\ &\geq \delta_1 \\ &\overset{(\star\star)}{\geq} \delta_k. \end{aligned} \tag{3.16}$$

(Moreover, equality in $(\star)$ implies that $c_1(TM_k)(A) = 1$.)

Suppose moreover that A is an exceptional class with minimal area. The last inequality of (3.16) being an equality implies that we are in case (1b). The class A is then in the set of listed classes because this set contains *all* the exceptional classes.

Now, suppose that A is a multiple of one of the classes $L, E_1, \ldots, E_k$. Then $\frac{1}{2\pi} \langle \Omega, A \rangle \geq \delta_k$, with equality only if A is one of the classes $E_{j+1}, \ldots, E_k$, as in Lemma 3.2. These classes are among those that are listed in all the cases, and in particular in the cases (1a) and (1b).

Case 2: when $\delta_1 > \lambda_F/2$ and $\delta_2 \leq \lambda_F/2$.

First, suppose that A is not a multiple of any of the classes $F, E_1, E_2, \ldots, E_k$. Write

$$A = a_L L + a_F F - b_2 E_2 - \cdots - b_k E_k.$$

By Lemma 2.15, $\mathrm{GW}(F)$, $\mathrm{GW}(E_1)$, $\mathrm{GW}(E_2)$, …, $\mathrm{GW}(E_k)$ are nonzero; by Lemma 3.1 and by the assumptions on A, the coefficients

$$a_L = A \cdot F, \quad a_F = A \cdot E_1, \quad b_2 = A \cdot E_2, \quad \ldots, \quad b_k = A \cdot E_k$$

are nonnegative. We have

$$c_1(TM_k)(A) = 3a_L + 2a_F - b_2 - \cdots - b_k;$$

by assumption, this number is ≥ 1. The assumption $\delta_1 > \lambda_F/2$ implies that $\lambda > \frac{3}{2}\lambda_F$. Also, $0 < \delta_j \leq \lambda_F/2$ for all $2 \leq j \leq k$. Thus,

$$\begin{aligned} \frac{1}{2\pi}\langle\Omega, A\rangle &= a_L\lambda + a_F\lambda_F - b_2\delta_2 - \cdots - b_k\delta_k \\ &\geq a_L\frac{3}{2}\lambda_F + a_F\lambda_F - b_2\lambda_F/2 - \cdots - b_k\lambda_F/2 \\ &= (3a_L + 2a_F - b_2 - \cdots - b_k)\,\lambda_F/2 \\ &\overset{(\star)}{\geq} \lambda_F/2 \\ &\geq \delta_2 \\ &\geq \delta_k. \end{aligned} \tag{3.17}$$

(Moreover, equality in $(\star)$ implies that $c_1(TM_k)(A) = 1$.)

Suppose moreover that A is an exceptional class of minimal area. The first inequality in (3.17) being an equality implies that the coefficient a_L is zero. So $-b_2^2 - \cdots - b_k^2 = A \cdot A = -1$ and $2a_F - b_2 - \cdots - b_k = c_1(TM_k)(A) = 1$. From this we deduce that A is one of the classes $E_{12}, \ldots,$ E_{1k}. The last two inequalities of (3.17) being equalities implies that we are in case (2b), so A is among the listed classes.

Now suppose that A is a multiple of one of the classes $F, E_1, E_2, \ldots, E_k$. Then $\frac{1}{2\pi}\langle\Omega, A\rangle \geq \delta_k$, with equality only if A is one of the classes $E_{j+1}, \ldots, E_k$, as in Lemma 3.2. These classes are among those that are listed in all the cases, and in particular in the cases (2a) and (2b).

Case 3: when $\delta_1 > \lambda_F/2$ and $\delta_2 > \lambda_F/2$.

First, suppose that A is not a multiple of any of the classes $F, B, E_{12},$ $E_3, \ldots, E_k$. Write

$$A = a_B B + a_F F - b_{12}E_{12} - b_3E_3 - \cdots - b_kE_k.$$

By Lemma 2.15, GW(F), GW(B), GW(E_{12}), GW(E_1),…, GW(E_k) are nonzero; by Lemma 3.1, the coefficients

$$\begin{aligned} &a_B = A \cdot F, \quad a_F = A \cdot B, \\ &b_{12} = A \cdot E_{12}, \quad b_3 = A \cdot E_3, \quad \ldots, \quad b_k = A \cdot E_k \end{aligned}$$

are nonnegative. We have

$$c_1(TM)(A) = 2a_B + 2a_F - b_{12} - b_3 - \cdots - b_k;$$

by assumption, this number is ≥ 1. Let

$$\lambda_B = \lambda - \delta_2.$$

The assumption $\delta_1 \geq \delta_2$ implies that $\lambda_B \geq \lambda_F$. The assumption $\delta_2 > \lambda_F/2$ implies that $\delta_{E_{12}} < \lambda_F/2$. Because $(\lambda; \delta_1, \ldots, \delta_k)$ is reduced, $\delta_k \leq \delta_3 \leq \lambda_F - \delta_2$, which also implies that $0 < \delta_j < \lambda_F/2$ for $j = 3, \ldots, k$. Thus,

$$\begin{aligned} \frac{1}{2\pi} \langle \Omega, A \rangle &= a_B \lambda_B + a_F \lambda_F - b_{12} \delta_{E_{12}} - b_3 \delta_3 - \cdots - b_k \lambda_k \\ &\geq a_B \lambda_F + a_F \lambda_F - b_{12} \lambda_F/2 - b_3 \lambda_F/2 - \cdots - b_k \lambda_F/2 \\ &= (2a_B + 2a_F - b_{12} - b_3 - \cdots - b_k)\, \lambda_F/2 \\ &\overset{(\star)}{\geq} \lambda_F/2 \\ &> \delta_3 \\ &\geq \delta_k. \end{aligned} \tag{3.18}$$

(Moreover, equality in $(\star)$ implies that $c_1(TM_k)(A) = 1$.)

The first inequality in (3.18) being an equality implies that $b_{12} = b_3 = \cdots = b_k = 0$, which cannot occur when A is exceptional.

Now, suppose that A is a multiple of one of the classes $F, B, E_{12}, E_3, \ldots, E_k$. Then $\frac{1}{2\pi} \langle \Omega, A \rangle \geq \delta_k$. In case (3a), equality holds only if $A \in \{E_{j+1}, \ldots, E_k\}$. In case (3b), equality holds only if $A \in \{E_{12}, E_{j+1}, \ldots, E_k\}$. See Lemma 3.2. In each of these cases, A belongs to the set of listed classes. □

Corollary 3.19. *Let $k \geq 3$. Let ω be a blowup form on M_k whose cohomology class is encoded by a reduced vector $v = (\lambda; \delta_1, \ldots, \delta_k)$. Then one of the following four possibilities (A), (B), (C), (D) occurs for the set $\mathcal{E}^v_{\min}$ of exceptional classes with minimal area.*

(A) $$\mathcal{E}^v_{\min} \supseteq \{E_1, E_2, \ldots, E_k, E_{12}\}.$$
In this case, $v = (\lambda, \lambda/3, \ldots, \lambda/3)$.

(B) $$\mathcal{E}^v_{\min} = \{E_2, \ldots, E_k, E_{12}, \ldots, E_{1k}\}.$$
In this case, $v = (\lambda; \delta_1, \lambda_F/2, \ldots, \lambda_F/2)$, and $\delta_1 > \lambda_F/2$.

(C) $$\mathcal{E}^v_{\min} = \{E_{12}, E_3, \dots, E_k\}.$$
In this case, $v = (\lambda; \delta_1, \delta_2, \delta_{E_{12}}, \dots, \delta_{E_{12}})$ *and* $\delta_2 > \delta_{E_{12}}$.

(D) $$\mathcal{E}^v_{\min} = \{E_{j+1}, \dots, E_k\}\,,$$
where j *is the smallest non-negative integer for which* $\delta_{j+1} = \cdots = \delta_k$.

4. McDuff's arguments

In Theorem 3.12 we give the complete list of exceptional homology classes with minimal symplectic area for a blowup form whose cohomology class is encoded by a reduced vector. McDuff has shown us a different proof approach, which uses the "reduced" assumption in such a beautiful way that we feel compelled to include it.

The following lemma and corollary are a slight variation of results that were communicated to us by Dusa McDuff. Their origin is in [21, Lemma 3.4], which is attributed to [16, Lemma 3.4].

Lemma 4.1. *Let* $k \geq 3$. *Let* A *be a homology class in* $H_2(M_k)$. *Write*

$$A = aL - b_1E_1 - \cdots - b_kE_k.$$

(1) *Suppose that* $a \geq 0$ *and* $b_\ell \geq 0$ *for all* ℓ, *and that*

$$A \cdot A + c_1(TM_k)(A) \geq 0.$$

Then $0 \leq b_\ell \leq a$ *for all* ℓ. *If, additionally,*

$$A \cdot A \geq -1 \quad \text{and} \quad A \neq 0,$$

then there exists $\ell \in \{1, \dots, k\}$ *such that* $b_\ell < a$.

(2) *Suppose that* $a \geq 0$ *and* $0 \leq b_\ell \leq a$ *for all* ℓ, *and that*

$$c_1(TM_k)(A) \geq 0.$$

Let Ω *be a cohomology class in* $H^2(M_k; \mathbb{R})$ *that is encoded by a vector* $(\lambda; \delta_1, \dots, \delta_k)$ *with positive entries and that is reduced. Then*

$$\langle \Omega, A \rangle \geq 0.$$

Proof of (1). Suppose otherwise. Then $b_{\ell_0} = a + \eta$ for some ℓ_0 and for some $\eta \geq 1$. Then

$$b_{\ell_0}^2 + b_{\ell_0} = (a+\eta)^2 + (a+\eta) = a^2 + \underbrace{(2\eta+1)}_{\geq 3}a + \underbrace{\eta^2+\eta}_{>0} > a^2 + 3a,$$

and so

$$\begin{aligned} A \cdot A + c_1(TM_k)(A) &= (a^2 - \sum b_\ell^2) + (3a - \sum b_\ell) \\ &= \underbrace{(a^2+3a) - (b_{\ell_0}^2 + b_{\ell_0})}_{<0} - \sum_{\ell \neq \ell_0} \underbrace{(b_\ell^2 + b_\ell)}_{\geq 0} \\ &< 0, \end{aligned}$$

contradicting our assumption on A. If there does not exist an ℓ such that $b_\ell < a$, then $b_\ell = a$ for all ℓ, and $A \cdot A = a^2(1-k)$, which is ≤ -2 if $A \neq 0$. □

Proof of (2). The assumption $c_1(TM_k)(A) \geq 0$ implies that $3a - \sum b_\ell \geq 0$. We can then write

$$\begin{aligned} \frac{1}{2\pi}\langle \Omega, A\rangle &= a\lambda - b_1\delta_1 - \cdots - b_k\delta_k \\ &= \underbrace{\lambda + \cdots + \lambda}_{a \text{ times}} \\ &\quad - (\underbrace{\delta_1 + \cdots + \delta_1}_{b_1 \text{ times}} + \cdots + \underbrace{\delta_k + \cdots + \delta_k}_{b_k \text{ times}} + \underbrace{0 + \cdots + 0}_{3a - \sum b_\ell \text{ times}}). \end{aligned} \tag{4.2}$$

We set $\delta_{k+1} = 0$.

We label the list of $3a$ indices

$$\underbrace{1,\ldots,1}_{b_1 \text{ times}}, \quad \ldots, \quad \underbrace{k,\ldots,k}_{b_k \text{ times}}, \quad \underbrace{k+1,\ldots,k+1}_{3a - \sum b_\ell \text{ times}}$$

as

$$j_{11},\ j_{21},\ \ldots, j_{a1}, \quad j_{12},\ j_{22},\ \ldots, j_{a2}, \quad j_{13},\ j_{23},\ \ldots, j_{a3}.$$

Because $0 \leq b_\ell \leq a$ for all ℓ, for each $1 \leq i \leq a$ those of the three indices j_{i1}, j_{i2}, j_{i3} that are different from the artificially-added index $k+1$ are distinct. The right hand side of (4.2) then becomes

$$\sum_{i=1}^{a} (\lambda - (\delta_{j_{i1}} + \delta_{j_{i2}} + \delta_{j_{i3}})) \tag{4.3}$$

where at each summand, $(\delta_{j_{i1}} + \delta_{j_{i2}} + \delta_{j_{i3}})$ is the sum of at most three of $\delta_1, \dots, \delta_k$. Because $(\lambda; \delta_1, \dots, \delta_k)$ is reduced, the sum (4.3) is ≥ 0. □

Corollary 4.4. *Let $k \geq 3$. Let Ω be a cohomology class in $H^2(M_k; \mathbb{R})$ that is encoded by a vector $(\lambda; \delta_1, \dots, \delta_k)$ with positive entries that is reduced. Let A be a class in $H_2(M_k)$ such that $c_1(TM_k)(A) \geq 1$ and such that A is represented by a J holomorphic sphere for some almost complex structure J that is tamed by some blowup form on M_k. Then*

$$\frac{1}{2\pi} \langle \Omega, A \rangle \geq \delta_k.$$

In particular, let E be an exceptional class in $H_2(M_k)$; then

$$\frac{1}{2\pi} \langle \Omega, E \rangle \geq \delta_k.$$

Moreover,

- *if E is not one of the classes $E_1, \dots, E_k$ nor $L - E_1 - E_\ell$ for $\ell \neq 1$, then*
$$\frac{1}{2\pi} \langle \Omega, E \rangle \geq \delta_1;$$
- *if E is not one of the classes $E_1, \dots, E_k$, then*

$$\frac{1}{2\pi} \langle \Omega, E \rangle \geq \lambda - \delta_1 - \delta_2.$$

Proof. Assume that A is not a multiple of any of the classes $L, E_1, \dots, E_k$; otherwise, the result is clearly true. By positivity of intersections (Lemma 3.1) we can write

$$E = aL - b_1 E_1 - \cdots - b_k E_k$$

where $a, b_1, \dots, b_k$ are nonnegative. By the adjunction formula,

$$\begin{aligned} A \cdot A &\geq c_1(TM_k)(A) - 2 \\ &\geq -1. \end{aligned}$$

By part (1) of Lemma 4.1, we have $0 \leq b_\ell \leq a$ for all ℓ, and there exists an ℓ such that $b_\ell < a$. We can then apply part (2) of Lemma 4.1 to $A - E_\ell$ and conclude that

$$\frac{1}{2\pi}\langle\Omega, A\rangle \geq \delta_\ell.$$

Now, let E be an exceptional class in $H_2(M_k)$ that is not one of the classes $E_1, \ldots, E_k$. We need to show that

$$\frac{1}{2\pi}\langle\Omega, E\rangle \geq \lambda - \delta_1 - \delta_2,$$

and that, if E is not equal to $L - E_1 - E_\ell$ for any $\ell \neq 1$, then

$$\frac{1}{2\pi}\langle\Omega, E\rangle \geq \delta_1.$$

Because E is not one of the classes $E_1, \ldots, E_k$ and is exceptional, E is not a multiple of any of the classes $L, E_1, \ldots, E_k$, and by positivity of intersections (Lemma 3.1) we can write

$$E = aL - b_1E_1 - \cdots - b_kE_k$$

where $a, b_1, \ldots, b_k$ are nonnegative. By part (1) of Lemma 4.1, we have $0 \leq b_\ell \leq a$ for all ℓ.

First, suppose that $b_i = a$ for some $1 \leq i \leq k$. The properties $E \cdot E = -1$ and $c_1(TM_k)(E) = 1$ then imply that $E = L - E_i - E_s$ for some $s \neq i$. Similarly, if $a = 1$, then again $E = L - E_i - E_s$ for $s \neq i$. In all these cases

$$\frac{1}{2\pi}\langle\Omega, E\rangle \geq \min_{1\leq i<s\leq k}\{\lambda - \delta_i - \delta_s\} = \lambda - \delta_1 - \delta_2,$$

and if i and s are both different from 1, then

$$\frac{1}{2\pi}\langle\Omega, E\rangle = \lambda - \delta_i - \delta_s \geq \delta_1$$

because $(\lambda; \delta_1, \ldots, \delta_k)$ is reduced.

It remains to consider the case that $a > 1$ and $0 \leq b_\ell < a$ for all ℓ. Because $E \cdot E = -1$, there exist two different indices i, s such that $b_i > 0$ and $b_s > 0$. We can then apply part (2) of Lemma 4.1 to $A := E - (L - E_i - E_s)$ and conclude that

$$\frac{1}{2\pi}\langle\Omega, E\rangle \geq \lambda - \delta_i - \delta_s. \tag{4.5}$$

The right hand side of (4.5) is $\geq \lambda - \delta_1 - \delta_2$, and it is $\geq \delta_1$ if i and s are both different from 1, since $(\lambda; \delta_1, \ldots, \delta_k)$ is reduced. $\square$

We now give an alternative proof to Theorem 3.12, using Corollary 4.4.

Proof of Lemmas 3.6 and 3.10 and Theorem 3.12. Lemmas 3.6 and 3.10 follow from Corollary 4.4.

Because v is reduced (see also Remark 3.11), in each of the cases in Theorem 3.12, each of the listed classes is exceptional and has size δ_k.

Now, let E be an exceptional class in $H_2(M_k)$. By Corollary 4.4, E is in $\mathcal{E}^v_{\min}$ if and only if $\frac{1}{2\pi}\langle\Omega, E\rangle = \delta_k$. If E is one of the classes $E_1, \dots, E_k$, then $\frac{1}{2\pi}\langle\Omega, E\rangle = \delta_k$ implies that E is in the set $\{E_{j+1}, \dots, E_k\}$, which is contained in all the sets of classes that are listed in Theorem 3.12.

We now assume that $\frac{1}{2\pi}\langle\Omega, E\rangle = \delta_k$ and E is not one of the classes $E_1, \dots, E_k$. It remains to prove that E is one of the classes that are listed in Theorem 3.12, according to the case of v.

Case 1: when $\delta_1 \leq \lambda/3$.

$$\begin{aligned}\frac{1}{2\pi}\langle\Omega, E\rangle &\geq \lambda - \delta_1 - \delta_2 \quad \text{by Corollary 4.4}\\ &\geq \lambda/3\\ &\geq \delta_1\\ &\geq \delta_k.\end{aligned}$$

Equality implies that we are in case (1b); the class E is then in the set of listed classes because this set contains *all* the exceptional classes.

Cases 2 and 3: when $\delta_1 > \lambda/3$.

Since v is reduced, we get

$$\delta_k \leq \delta_3 \leq \frac{\delta_1 + \delta_2 + \delta_3}{3} \leq \lambda/3,$$

and so

$$\delta_k \leq \lambda/3 < \delta_1.$$

By Corollary 4.4, E is one of the classes $E_{1\ell}$ for $\ell > 1$. It remains to show that this can hold only if v is in case (2b) or if $\ell = 2$ and v is in case (3b).

Indeed, we now rule out the remaining cases.

If v is in case (2a), for all $\ell > 1$ we have

$$\begin{aligned}\frac{1}{2\pi}\langle \Omega, E_{1\ell}\rangle &= \lambda - \delta_1 - \delta_\ell \\ &\geq \lambda - \delta_1 - \delta_2 \\ &= \lambda_F - \delta_2 \\ &\geq \lambda_F/2 \\ &> \delta_k.\end{aligned}$$

If v is case (3a), for all $\ell > 1$, we have

$$\begin{aligned}\frac{1}{2\pi}\langle \Omega, E_{1\ell}\rangle &= \lambda - \delta_1 - \delta_\ell \\ &\geq \lambda - \delta_1 - \delta_2 \\ &= \delta_{E_{12}} \\ &> \delta_k.\end{aligned}$$

If v is in case (3b) then by Remark 3.11 $\delta_2 > \delta_3 = \cdots = \delta_k = \delta_{E_{12}}$, and for $\ell > 2$,

$$\begin{aligned}\frac{1}{2\pi}\langle \Omega, E_{1\ell}\rangle &= \lambda - \delta_1 - \delta_\ell \\ &\geq \lambda - \delta_1 - \delta_3 \\ &\geq \delta_2 \\ &> \delta_k.\end{aligned}$$

Thus, we have shown that in the cases (2a), (3a), (3b) the class $E_{1\ell}$ cannot be minimal for any $2 \leq \ell \leq k$. □

5. Uniqueness of reduced form

Our goal in this section is to prove the following theorem, which is the "uniqueness" part of Theorem 1.8.

Theorem 5.1. *Let $k \geq 3$. Let ω and ω' be blowup forms on M_k whose cohomology classes are encoded by the vectors*

$$v = (\lambda; \delta_1, \ldots, \delta_k) \quad \text{and} \quad v' = (\lambda'; \delta'_1, \ldots, \delta'_k).$$

Suppose that v and v' are reduced. Suppose that (M_k, ω) and (M_k, ω') are symplectomorphic. Then $v = v'$.

Let (M, ω) be a closed symplectic four-manifold and C an embedded symplectic sphere of self intersection -1. We recall that a choice of Weinstein tubular neighbourhood of C determines a *symplectic blow-down* $(\overline{M}, \overline{\omega})$ of (M, ω) along C, and that we have a natural splitting

$$H_2(M) = H_2(\overline{M}) \oplus \mathbb{Z}[C]. \tag{5.2}$$

We also recall the "uniqueness of blow downs": if C_1 and C_2 are two spheres as above and are in the same homology class, and if $(\overline{M}_1, \overline{\omega}_1)$ and $(\overline{M}_2, \overline{\omega}_2)$ are blow-downs of (M, ω) with respect to some choices of Weinstein tubular neighbourhoods of C_1 and C_2, then there is a symplectomorphism between $(\overline{M}_1, \overline{\omega}_1)$ and $(\overline{M}_2, \overline{\omega}_2)$ that induces the identity map on the second homology with respect to the decompositions (5.2). An argument for this was given by McDuff in [19, §3]; for details, see [12, Lemma A.1].

Finally, suppose that $(\overline{M}, \overline{\omega})$ is obtained from (M, ω) by a symplectic blowdown along a sphere C with respect to some Weinstein neighbourhood of C, and let $\psi\colon (M, \omega) \to (M', \omega')$ be a symplectomorphism. Then ψ descends to a symplectomorphism from $(\overline{M}, \overline{\omega})$ to the manifold $(\overline{M}', \overline{\omega}')$ that is obtained from (M', ω') by a symplectic blowdown along $C' := \psi(C)$ with respect to the Weinstein tubular neighbourhood that is determined by ψ.

Lemma 5.3. *Let ω be a blowup form on M_k. Let $(\lambda; \delta_1, \ldots, \delta_k)$ be the vector that encodes the cohomology class $[\omega]$. Then there exists an embedded ω-symplectic sphere in the class E_k. For every such sphere, blowing down along it yields a symplectic manifold that is symplectomorphic to $(M_{k-1}, \overline{\omega})$, where $\overline{\omega}$ is a blowup form, and where the cohomology class $[\overline{\omega}]$ is encoded by the vector $(\lambda; \delta_1, \ldots, \delta_{k-1})$.*

We give details in [10].

To proceed, we will need to identify the two-point blowup M_2 of $\mathbb{CP}^2$ with the one-point blowup of $S^2 \times S^2$. We have a decomposition

$$H_2(S^2 \times S^2) = \mathbb{Z}B \oplus \mathbb{Z}F$$

where $B = [S^2 \times \{\text{point}\}]$ is the "base class" and $F = [\{\text{point}\} \times S^2]$ is the "fibre class". For positive real numbers a, b we consider the split symplectic form

$$\omega_{a,b} = a\tau_{S^2} \oplus b\tau_{S^2}$$

where τ_{S^2} is the rotation invariant area form on S^2, normalized such that $\frac{1}{2\pi}\int_{S^2} \tau_{S^2} = 1$.

Lemma 5.4. *Suppose that* $a \geq b > 0$ *and* $a' \geq b' > 0$. *Suppose that* $(S^2 \times S^2, \omega_{a,b})$ *and* $(S^2 \times S^2, \omega_{a',b'})$ *are symplectomorphic. Then* $a = a'$ *and* $b = b'$.

Proof. The forms $\omega_{a,b}$ and $\omega_{a',b'}$ induce the same orientation. An orientation preserving diffeomorphism on $S^2 \times S^2$ acts on $H^2(S^2 \times S^2) = \mathbb{Z}^2$ by a 2×2 matrix of integers, with determinant ± 1, which preserves the intersection form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The matrices with this property are $\pm \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\pm \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. These cannot take $\omega_{a,b}$ to $\omega_{a',b'}$, with $a \geq b > 0$ and $a' \geq b' > 0$, unless $(a,b) = (a',b')$. □

Lemma 5.5. *Let* ω *be a blowup form on* M_2. *Then there exists an embedded* ω*-symplectic sphere in the class* $E_{12} := L - E_1 - E_2$. *Moreover, for every such sphere, blowing down along it yields a symplectic manifold that is symplectomorphic to* $(S^2 \times S^2, \omega_{a,b})$, *with* $a = \lambda - \delta_2$ *and* $b = \lambda - \delta_1$, *where* $(\lambda; \delta_1, \delta_2)$ *is the vector that encodes the cohomology class* $[\omega]$.

We give details in [10].

We will also use the following observations on symplectomorphisms between blow ups of $\mathbb{CP}^2$. We say that homology classes are *disjoint* if their intersection product is zero.

Lemma 5.6. *Let* ω *and* ω' *be blowup forms on* M_k *whose cohomology classes are encoded by the vectors* $(\lambda; \delta_1, \ldots, \delta_k)$ *and* $(\lambda'; \delta'_1, \ldots, \delta'_k)$. *Let* $\varphi\colon (M_k, \omega) \to (M_k, \omega')$ *be a symplectomorphism, and let* $\varphi_*\colon H_2(M_k) \to H_2(M_k)$ *be the induced map on homology.*

1) *The isomorphism* φ_* *preserves the set of exceptional classes.*
2) *The isomorphism* φ_* *sends disjoint homology classes to disjoint homology classes.*
3) *The isomorphism* φ_* *restricts to a bijection from the set of minimal exceptional classes in* (M_k, ω) *to the set of minimal exceptional classes in* (M_k, ω'), *and*

$$\delta_k = \delta'_k. \tag{5.7}$$

4) *We have*

$$3\lambda - \sum_{i=1}^{k} \delta_i = 3\lambda' - \sum_{i=1}^{k} \delta'_i. \tag{5.8}$$

Proof. We give details in [10]. We note that (5.7) in the case $k \geq 3$ follows from Lemma 3.10 and that (5.8) follows from $\frac{1}{2\pi}\int_{M_k} \omega \wedge c_1(TM_k) = \frac{1}{2\pi}\int_{M_k} \omega' \wedge c_1(TM_k)$. □

The properties of a symplectomorphism listed in Lemma 5.6 and the identification of exceptional classes when $k = 1, 2$ yield the characterization of the blowup forms when $k \leq 2$ that was stated in Lemma 1.10.

Proof of Lemma 1.10. By Remark 1.5, a vector that encodes the cohomology class of a blowup form satisfies the conditions listed in the lemma. The fact that these conditions are sufficient for the cohomology class encoded by the vector to contain a blowup form can be shown by toric constructions, see e.g., [8, 10].

By Lemma 1.3, any two blowup forms whose cohomology classes are encoded by the same vector are diffeomorphic. When $k = 2$, switching δ_1 and δ_2 can be realized by a diffeomorphism. It remains to show that if two blowup forms are cohomologous, then the vectors that encode their cohomology classes are equal or (when $k = 2$) differ by switching δ_1 and δ_2.

Suppose that $\mathbf{k = 2}$. Suppose that there exists a symplectomorphism from $(M_2, \omega_{\lambda;\delta_1,\delta_2})$ to $(M_2, \omega_{\lambda';\delta_1',\delta_2'})$. By Demazure [3], the set of exceptional classes in M_2 is $\{E_1, E_2, E_{12}\}$, and the only pair of disjoint exceptional classes is $\{E_1, E_2\}$. Because a symplectomorphism takes disjoint exceptional classes to disjoint exceptional classes, $\{\delta_1, \delta_2\} = \{\delta_1', \delta_2'\}$. Because a symplectomorphism preserves the pairing of the symplectic form with the first Chern class, $3\lambda - \delta_1 - \delta_2 = 3\lambda' - \delta_1' - \delta_2'$, which further implies that $\lambda = \lambda'$. Thus, $(\lambda', \delta_1', \delta_2')$ is equal to either $(\lambda, \delta_1, \delta_2)$ or to $(\lambda, \delta_2, \delta_1)$. Conversely, these two vectors correspond to symplectomorphic manifolds. We give more details in [10].

Suppose that $\mathbf{k = 1}$. Suppose that there exists a symplectomorphism from $(M_2, \omega_{\lambda;\delta_1})$ to $(M_2, \omega_{\lambda';\delta_1'})$. As noted in Remark 3.15, in this case E_1 is the only exceptional class. Because a symplectomorphism must take an exceptional class to an exceptional class, the symplectomorphism $(M_k, \omega) \to (M_k, \omega')$ takes the set $\{E_1\}$ to itself. Thus, $\delta_1 = \delta_1'$. Because a symplectomorphism preserves the pairing of the symplectic form with the first Chern class, $3\lambda - \delta_1 = 3\lambda' - \delta_1'$, which further implies that $\lambda = \lambda'$.

Suppose that $\mathbf{k = 0}$. On $\mathbb{CP}^2$, if two blowup forms are diffeomorphic then they take the same value on the generator of $H_2(\mathbb{CP}^2)$ on which this value is positive. So they must have the same size. □

Proof of Theorem 5.1. Corollary 3.19 implies that exactly one of the following possibilities for the vector $v = (\lambda; \delta_1, \ldots, \delta_k)$ occurs. A similar list of possibilities holds for the vector $v' = (\lambda'; \delta'_1, \ldots, \delta'_k)$.

(A) Not every two minimal exceptional classes are disjoint, and there exist k pairwise disjoint minimal exceptional classes.
In this case, $v = (\lambda; \lambda/3, \ldots, \lambda/3)$.

(B) Not every two minimal exceptional classes are disjoint, and there do not exist k pairwise disjoint minimal exceptional classes.
In this case, $v = (\lambda; \delta_1, \lambda_F/2, \ldots, \lambda_F/2)$ and $\delta_1 > \lambda_F/2$.

(C) Every two minimal exceptional classes are disjoint, and the blowdown of (M, ω) along all the minimal exceptional classes yields a manifold that is symplectomorphic to $S^2 \times S^2$ with some split symplectic form $\omega_{a,b}$ with $a \geq b > 0$.
In this case, $v = (\lambda; \delta_1, \delta_2, \delta_{E_{12}}, \ldots, \delta_{E_{12}})$, with $\delta_2 > \delta_{E_{12}}$, and the parameters a, b are given by $a = \lambda - \delta_2$ and $b = \lambda - \delta_1$.

(D) Every two minimal exceptional classes are disjoint, and the blowdown of (M, ω) along all the minimal exceptional classes yields a manifold that is symplectomorphic to $(M_j, \overline{\omega})$ for some $0 \leq j < k$, where $\overline{\omega}$ is a blowup class.
In this case, the cohomology class $[\overline{\omega}]$ is encoded in the vector $(\lambda; \delta_1, \ldots, \delta_j)$.

By items (2) and (3) of Lemma 5.6, either (M_k, ω) and (M_k, ω') are both in the case (A), or they are both in the case (B), or they are both in the cases (C) or (D).

In the cases (C) or (D), because a symplectomorphism between (M_k, ω) and (M_k, ω') descends to a symplectomorphism between the blowdowns along the minimal exceptional spheres, and because $S^2 \times S^2$ is not symplectomorphic (or even homeomorphic) to any M_j, either both (M_k, ω) and (M_k, ω') are in the case (C) or they are both in the case (D).

Suppose v and v' are in case (A). This means that $v = (\lambda; \lambda/3, \ldots, \lambda/3)$ and $v' = (\lambda'; \lambda'/3, \ldots, \lambda'/3)$. Substituting in (5.7), the resulting equation implies that $\lambda = \lambda'$, and thus $v = v'$.

Suppose v and v' are in case (B). This means that $v = (\lambda; \delta_1, \lambda_F/2, \ldots, \lambda_F/2)$ and $v' = (\lambda'; \delta'_1, \lambda'_F/2, \ldots, \lambda'_F/2)$. Substituting in (5.7) and in (5.8), and recalling that $\lambda_F = \lambda - \delta_1$ and $\lambda'_F = \lambda' - \delta'_1$, we get two linearly independent equations that imply that $\lambda = \lambda'$ and $\delta_1 = \delta'_1$, and thus $v = v'$.

Suppose v and v' are in case (C). Then $v = (\lambda; \delta_1, \delta_2, \delta_{E_{12}}, \ldots, \delta_{E_{12}})$ and $v' = (\lambda'; \delta'_1, \delta'_2, \delta'_{E_{12}}, \ldots, \delta'_{E_{12}})$. By (5.7), we get

$$\delta_{E_{12}} = \delta'_{E_{12}}. \tag{5.9}$$

Because the symplectomorphism descends to a symplectomorphism between the blowdowns along the minimal exceptional spheres, and by Lemma 5.4, we obtain that

$$\delta_1 + \delta_{E_{12}} = \delta'_1 + \delta'_{E_{12}} \quad \text{and} \quad \delta_2 + \delta_{E_{12}} = \delta'_2 + \delta'_{E_{12}}.$$

By this and (5.9), we get that

$$\delta_1 = \delta'_1 \quad \text{and } \delta_2 = \delta'_2. \tag{5.10}$$

Substituting in (5.8), we get that $\lambda = \lambda'$. Thus, $v = v'$.

Suppose v and v' are in case (D). Because the symplectomorphism descends to a symplectomorphism between the blowdowns along the minimal exceptional spheres, we obtain a symplectomorphism between $(M_j, \overline{\omega})$ and $(M_j, \overline{\omega}')$, where $[\overline{\omega}]$ is encoded in the vector $\overline{v} = (\lambda, \delta_1, \ldots, \delta_j)$ and $[\overline{\omega}']$ is encoded in the vector $\overline{v}' = (\lambda', \delta'_1, \ldots, \delta'_j)$. Because the vectors $\overline{v}$ and $\overline{v'}$ are reduced, we can continue by induction. □

5.11 (Algorithm to determine whether two blowup forms are diffeomorphic). Suppose that $k \geq 3$. Let ω and ω' be blowup forms on M_k, and let v and v' be the vectors that encode their cohomology classes. Apply to each of v and v' the algorithm of Paragraph 2.17 to obtain reduced vectors v_{red} and v'_{red}. Then ω and ω' are diffeomorphic if and only if $v_{\text{red}} = v'_{\text{red}}$.

Indeed, as noted in Paragraph 2.17, the vectors v_{red} and v'_{red} encode cohomology classes of blowup forms ω_{red} and ω'_{red} that are, respectively, diffeomorphic to ω and to ω'. If ω and ω' are diffeomorphic, then so are ω_{red} and ω'_{red}, and, by Theorem 5.1, we conclude that $v_{\text{red}} = v'_{\text{red}}$. Conversely, if $v_{\text{red}} = v'_{\text{red}}$, then ω_{red} and ω'_{red} are diffeomorphic by Lemma 1.3, and then ω and ω' are diffeomorphic.

If $k = 0$, $k = 1$, or $k = 2$, two blowup forms on M_k are diffeomorphic if and only if the vectors that encode their cohomology classes are equal or (in the case $k = 2$) differ by switching δ_1 and δ_2. This follows from Lemma 1.10.

Remark 5.12. Zhao, Gao, and Qiu gave another version of "uniqueness of reduced form" [33]. They only refer to integral classes. They work with the slightly different notion of "reduced form" that we described in Remark 2.24.

They identify the group that is generated by the relevant Lorentzian reflections with the Weyl group of a certain Kac-Moody Lie algebra, and they rely on properties of such Weyl groups.

6. Characterization of blowup classes

In this section we give an algorithm that determines if a cohomology class contains a blowup form. The cone of classes of blowup forms on M_k is described by Li-Li [16] and Li-Liu [18], following the work of Biran [1, 2] and McDuff [20], and is explained in McDuff-Schlenk [25, §1.2]. We rely on the following two facts.

1) A cohomology class $\Omega \in H^2(M_k;\mathbb{R})$ is the cohomology class of a blowup form on M_k if and only if Ω is encoded by a vector in the forward positive cone and $\langle \Omega, E \rangle > 0$ for every exceptional class E on M_k.
2) Every exceptional class E on M_k can be obtained from E_1 by a sequence of applications of the transformations on $H_2(M_k)$ that induce the Cremona transformation and the permutations of the δ_js.

Remark 6.1. The fact that the cohomology class of every blowup form satisfies the conditions in (1) follows from our definition of "exceptional class" (Definition 2.13 and Lemma 2.12, which, in turn, relies on Lemma 1.2).

In the works that we quote above, the authors consider symplectic forms with a standard canonical class, that is, for which the first Chern class $c_1(TM)$ is the same as for blowup forms; in our notation (Definition 1.4), this class is encoded by the vector $(3; 1, \ldots, 1)$. And by "exceptional class", they refer to a homology class E that is represented by a smoothly embedded sphere with self intersection -1 and such that $c_1(TM)(E) = 1$. These authors show that a cohomology class Ω contains a symplectic form with standard canonical class if and only if it satisfies the two conditions that we listed in (1) with their interpretation of "exceptional class".

To use their work, we need to note that every homology class that is "exceptional" in their sense is also exceptional in our sense, and that every symplectic form with standard canonical class is a blowup form.

These facts follow from results that are given in Part 2 of Lemma 3.5 of [18]: let ω be a symplectic form with standard canonical class.

– If E is an exceptional class in the sense of Li-Li-Liu, then E is represented by an embedded ω-symplectic sphere.

- Every finite set of exceptional classes in the sense of Li-Li-Liu that are pairwise disjoint (with respect to the intersection form) is represented by a finite set of embedded ω-symplectic spheres that are pairwise disjoint (as sets).

The first of these results also appeared as the "-1 curve theorem" in Theorem A of [17], which implies that, for every symplectic form on M, if E is an exceptional class in the sense of Li-Li-Liu and its pairing with $c_1(TM)$ is positive then either E or $-E$ can be represented by an embedded symplectic sphere. Li and Liu prove this result using a method of Taubes [30].

Given a finite set of exceptional classes in the sense of Li-Li-Liu that are pairwise disjoint with respect to the intersection form, there exists an ω-tamed almost complex structure J for which there exists an embedded J-holomorphic sphere in each of the classes in the set. This follows from the first result above, together with the Hofer-Lizan-Sikorav regularity criterion [5] (see also [24, Lemma 3.3.3]) and the implicit function theorem, see [24, Chapter 3]. These spheres are disjoint, as follows from the positivity of intersections of J-holomorphic spheres in four-dimensional manifolds, see [24, Appendix E and Proposition 2.4.4], and the fact that the classes in the given set are pairwise disjoint. This yields the second result above.

In particular, the classes $E_1, \ldots, E_k$ of the exceptional divisors are represented by disjoint embedded ω-symplectic spheres. Blowing down along k disjoint embedded ω-symplectic spheres in the classes $E_1, \ldots, E_k$ yields a symplectic manifold that is diffeomorphic to $\mathbb{CP}^2$. By a result of Gromov [4, 2.4 B_2' and 2.4 B_3'] and a theorem of Taubes, which uses Seiberg-Witten invariants to guarantee the existence of a symplectically embedded two-sphere [31], this resulting manifold is symplectomorphic to $\mathbb{CP}^2$ with a multiple of the Fubini-Study form and L is represented by a symplectically embedded sphere. See [28, Example 3.4]. We conclude that ω is a blowup form. Then Lemma 2.12 and the first result above show that every exceptional class in the sense of Li-Li-Liu is also exceptional in our sense.

Lemma 6.2. *Let $k \geq 3$. Let Ω be a cohomology class that is encoded by a vector $(\lambda; \delta_1, \ldots, \delta_k)$ with positive entries that is reduced. Suppose that Ω has positive square. Then Ω contains a blowup form.*

Proof. By Lemma 3.10, for every exceptional class E in $H_2(M_k)$, we have $\frac{1}{2\pi} \langle \Omega, E \rangle \geq \delta_k$, and in particular $\langle \Omega, E \rangle > 0$. The result then follows from the above fact (1). □

Proof of Theorem 1.9. Theorem 1.9 follows from Lemma 2.9, Lemma 2.16, Lemma 6.2, and Remark 1.5. □

6.3 (Algorithm that, given a cohomology class in $\mathbf{H^2(M_k;\mathbb{R})}$, determines whether or not it contains a blowup form). The cases $k = 0, 1, 2$ have been addressed in Lemma 1.10. Suppose that $k \geq 3$.

Let v denote the vector that encodes the cohomology class. If v is not in the forward positive cone then the cohomology class does not contain any blowup form. If v is in the forward positive cone, apply the algorithm of Paragraph 2.17 to obtain v_{red}. If the entries of v_{red} are all positive, then the given cohomology class contains a blowup form. Otherwise, it does not.

Indeed, by the definition of a blowup form, a vector that encodes the cohomology class of a blowup form must be in the forward positive cone. As noted in Paragraph 2.17, if v is in the forward positive cone, so is v_{red} and v encodes the cohomology class of some blowup form if and only if v_{red} does. If the entries of v_{red} are all positive, then by Lemma 6.2, the cohomology class encoded by v_{red} contains a blowup form. If the entries of v_{red} are not all positive then, by the definition of a blowup form, it cannot encode the cohomology class of a blowup form.

References

[1] P. Biran, *Symplectic packing in dimension* 4, Geom. Funct. Anal. **7** (1997), no. 3, 420–437.

[2] P. Biran, *Constructing new ample divisors out of old ones*, Duke Math. J. **98** (1999), no. 1, 113–135.

[3] M. Demazure, *Surfaces de Del Pezzo II*, in: Séminaire sur les Singularités des Surfaces, Ed.: M. Demazure, H. C. Pinkham, and B. Teissier, Lecture Notes in Mathematics **777**, Springer, Berlin, 1980, pp. 23–35.

[4] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Inv. Math. **82** (1985), 307–347.

[5] H. Hofer, V. Lizan, and J-C. Sikorav, *On genericity for holomorphic curves in four-dimensional almost-complex manifolds*, J. Geom. Anal. **7** (1997), no. 1, 149–159.

[6] Jianxun Hu, *Gromov-Witten invariants of blow-ups along points and curves*, Math. Z. **233** (2000), 709–739.

[7] Y. Karshon, *Maximal tori in the symplectomorphism groups of Hirzebruch surfaces*, Math. Research Letters **10** (2003), no. 1, 125–132.

[8] Y. Karshon and L. Kessler, *Circle and torus actions on equal symplectic blow-ups of* $\mathbb{CP}^2$, Mathematical Research Letters **14** (2007), no. 5, 807–823.

[9] Y. Karshon, L. Kessler, and M. Pinsonnault, *A compact symplectic four-manifold admits only finitely many inequivalent toric actions*, J. Sympl. Geom. Volume **5** (2007), no. 2, 139–166.

[10] Y. Karshon, L. Kessler, and M. Pinsonnault, *Symplectic blowups of the complex projective plane and counting toric actions.* A preliminary version is posted on the authors' websites.

[11] Y. Karshon, L. Kessler, and M. Pinsonnault, *Counting toric actions on symplectic four-manifolds*, Comptes Rendus Mathématique, Rep. Acad. Sci. Canada Vol **37** (2015), no. 1, 33–40.

[12] L. Kessler, *Torus actions on small blow ups of* $\mathbb{CP}^2$, Pacific J. of Math. **244** (2010), no. 1, 133–154.

[13] F. Lalonde and M. Pinsonnault, *The topology of the space of symplectic balls in rational* 4*-manifolds*, Duke Math. J. **122** (2004), no. 2, 233–422.

[14] Bang-He Li, *Representing nonnegative homology classes of* $\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2}$ *by minimal genus smooth embeddings*, Trans. Amer. Math. Soc. **352** (1999), no. 9, 4155–4169.

[15] Bang-He Li and Tian-Jun Li, *Minimal genus smooth embeddings in* $S^2 \times S^2$ *and* $\mathbb{CP}^2\sharp n\overline{\mathbb{CP}^2}$ *with* $n \leq 8$, Topology **37** (1998), no. 3, 575–594.

[16] Bang-He Li and Tian-Jun Li, *Symplectic genus, minimal genus and diffeomorphisms*, Asian J. Math. **6** (2002), 123–144.

[17] Tian-Jun Li and Ai-Ko Liu, *Symplectic structure on ruled surfaces and a generalized adjunction formula*, Math. Res. Lett. **2** (1995), no. 4, 453–471.

[18] Tian-Jun Li and Ai-Ko Liu, *Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with* $B^+ = 1$, J. Diff. Geom. **58** (2001), 331–370.

[19] D. McDuff, *The structure of rational and ruled symplectic manifolds*, J. Amer. Math. Soc., **3** (1990), no. 3, 679–712.

[20] D. McDuff, *From symplectic deformation to isotopy*, in: Topics in symplectic 4-manifolds (Irvine, CA, 1996), 85–99, First int. Press Lect. Ser., I, internat. Press, Cambridge, MA, 1998.

[21] D. McDuff, *The Hofer conjecture on embedding symplectic ellipsoids*, J. Differ. Geom. **88** (2011), no. 3, 379–552.

[22] D. McDuff and L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. **115** (1994), 405–434.

[23] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Second Edition, Oxford University Press, 1998.

[24] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, Amer. Math. Soc., 2004.

[25] D. McDuff and F. Schlenk, *The embedding capacity of* 4*-dimensional symplectic ellipsoids*, Ann. Math. **175** (2012), 1191–1282.

[26] M. Pinsonnault, *Maximal compact tori in the Hamiltonian groups of* 4*-dimensional symplectic manifolds*, J. Modern Dynamics **2** (2008), no. 3, 431–455.

[27] M. Pinsonnault, *Symplectomorphism groups and embeddings of balls into rational ruled* 4*-manifolds*, Compos. Math. **144** (2008), no. 3, 787–810.

[28] D. Salamon, *Uniqueness of symplectic structures*, `arXiv:1211.2940v5 [math.SG]`.

[29] J. Solomon, *Period map image density*, Appendix to Systolic Geometry and Topology, Mikhail G. Katz, Amer. Math. Soc. Math. Surveys and Monographs, 2007.

[30] C. H. Taubes, *The Seiberg-Witten and Gromov invariants*, Math. Res. Lett. **2** (1995), no. 2, 221–238

[31] C. H. Taubes, *GR=SW: counting curves and connections*, J. Diff. Geom. **52** (1999), no. 3, 453–609.

[32] C. T. C. Wall, *Diffeomorphisms of* 4*-manifolds*, J. London Math. Soc. **39** (1964), 131–140.

[33] Xu'an Zhao, Hongzhu Gao, and Huaidong Qiu, *The minimal genus problem in rational surfaces* $\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2}$, Science in China Ser. A: Mathematics **49** (2006), no. 9, 1275–1283.

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Received July 20, 2015
Accepted March 6, 2016