JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 15, Number 4, 1071–1087, 2017

Symplectic circle actions with isolated fixed points

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Consider a symplectic circle action on a closed symplectic manifold with a non-empty discrete fixed point set. Associated to each fixed point, there are well-defined non-zero integers, called *weights*. We prove that the action is Hamiltonian if the sum of an odd number of weights is never equal to the sum of an even number of weights (the weights may be taken at different fixed points). Moreover, we show that if $\dim M = 6$, or if $\dim M = 2n \leq 10$ and each fixed point has weights $\{\pm a_1, \dots, \pm a_n\}$ for some positive integers a_i, the action is Hamiltonian if the sum of three weights is never equal to zero. As applications, we recover the results for semi-free actions, and for certain circle actions on six-dimensional manifolds.

1. Introduction

One important question in symplectic geometry is when a symplectic group action is Hamiltonian. In particular, let the circle act symplectically on a closed symplectic manifold. In some cases, the existence of a fixed point implies that the action is Hamiltonian. For instance, if the manifold is Kähler [F], if the dimension of the manifold is less than or equal to four [MD], if there are three fixed points [J], or if the circle action is semi-free with discrete fixed points [TW]. In [MD], D. McDuff constructs a non-Hamiltonian symplectic circle action on a six-dimensional symplectic manifold with fixed tori. Therefore, if the dimension of a manifold is greater than or equal to six, it requires more restrictions for a symplectic action to be Hamiltonian.

In this paper, we focus on symplectic circle actions on closed symplectic manifolds with discrete fixed point sets. Let the circle act symplectically on a $2n$-dimensional closed symplectic manifold and suppose that the fixed points are isolated. Associated to each fixed point p, there are well-defined non-zero integers w_p^i, called *weights*, $1 \leq i \leq n$. We prove that if the weights at the fixed points satisfy certain conditions, then the action is Hamiltonian. Consider a collection of weights among all the fixed points, counted with multiplicity, where, for each integer a, the multiplicity of a in the collection is precisely $\max_{p \in M^{S^1}} |\{i | a = w_p^i\}|$. For instance, if there are fixed points whose weights are $\{-1, -1, 1, 1\}$ and $\{-1, -1, -1, 2\}$, then the multiplicity of -1 and 1 in the collection is at least 3 and 2, respectively. First, we show that the symplectic action is Hamiltonian if the sum of an odd number of weights in the collection is never equal to the sum of an even number of weights in the collection.

Theorem 1.1. *Consider a symplectic circle action on a closed symplectic manifold with a non-empty discrete fixed point set. The action is Hamiltonian if the sum of an odd number of weights among all fixed points is never equal to the sum of an even number of weights.*

For instance, if the action is semi-free, all the weights are either $+1$ or -1. Therefore, the sum of an odd number of weights cannot equal the sum of an even number of weights and hence the action is Hamiltonian. In some cases, we only need to consider if the sum of three weights is never equal to zero.

Theorem 1.2. *Consider a symplectic circle action on a $2n$-dimensional closed symplectic manifold with a non-empty fixed point set, where weights are $\{\pm a_1, \pm a_2, \ldots, \pm a_n\}$ at all fixed points for some positive integers a_i, $1 \leq i \leq n$. Assume that $n \leq 5$ and $\pm a_i \pm a_j \pm a_k \neq 0$ for all $i < j < k$. Then the action is Hamiltonian.*

Theorem 1.3. *Consider a symplectic circle action on a six-dimensional closed symplectic manifold with a non-empty discrete fixed point set. The action is Hamiltonian if the sum of three weights among all fixed points is never equal to zero.*

The condition that the sum of three weights among all fixed points is never equal to zero, seems to play a certain role for a symplectic circle action to be Hamiltonian. If a symplectic circle action on a closed symplectic manifold M has two fixed points, then either M is the 2-sphere, or $\dim M = 6$ and the weights at the two fixed points are $\{-a-b, a, b\}$ and $\{-a, -b, a+b\}$ for some positive integers a and b [K], [PT]. If $\dim M = 6$, then the action cannot be Hamiltonian, since a compact Hamiltonian S^1-manifold M has at least $\frac{1}{2}\dim M + 1$ fixed points. Moreover, there are sums of three weights that are equal to zero. In fact, the first Chern class at each fixed point, which is the sum of weights at the fixed point, is equal to zero. However, to the author's knowledge, we do not know, whether such a manifold exists or not. Recently, S. Tolman constructs a non-Hamiltonian symplectic circle action on a 6-dimensional closed symplectic manifold with 32 fixed points, 16 of which have weights $\{-2, 1, 1\}$ and the other 16 of which have weights $\{-1, -1, 2\}$ [T], which seems to provide a further clue to the following question:

Question 1.4. *Let the circle act symplectically on a closed symplectic manifold with a non-empty discrete fixed point set. Suppose that the sum of three weights among all fixed points is never equal to zero. Then is the action Hamiltonian?*

2. Background and notation

A manifold M is called a **symplectic manifold**, if there is a closed, non-degenerate two-form ω on M, called a **symplectic form**. Let the circle act on a symplectic manifold (M, ω). The action is called **symplectic**, if it preserves the symplectic form ω. Let X_M be the vector field on M generated by the circle action. The action is called **Hamiltonian**, if there is a map $\mu : M \to \mathbb{R}$ such that $d\mu = \iota_{X_M}\omega$.

Let the circle act on a $2n$-dimensional almost complex manifold M and assume that the action preserves the almost complex structure on M. Suppose that p is an isolated fixed point. We can identify T_pM with $\mathbb{C}^n$ and the action of S^1 at p with $\lambda \cdot (z_1, \ldots, z_n) = (\lambda^{\xi_p^1} z_1, \ldots, \lambda^{\xi_p^n} z_n)$, where ξ_p^i are non-zero integers. These non-zero integers are called **weights** at the fixed point p. Let λ_p be twice of the number of negative weights at p. This is called the **index** of p. Denote by σ_i the elementary symmetric polynomial of degree i in n variables. Note that any symplectic manifold admits an almost complex structure, hence is an almost complex manifold. In [L], P. Li shows the following:

Theorem 2.1. *[L] Consider a circle action on a $2n$-dimensional closed almost complex manifold M. Suppose that the action preserves the almost complex structure and the fixed points are isolated. Then*

$$\chi^i(M) = \sum_{p \in M^{S^1}} \frac{\sigma_i(t^{\xi_p^1}, \ldots, t^{\xi_p^n})}{\prod_{j=1}^n (1 - t^{\xi_p^j})} = (-1)^i N^i = (-1)^{n-i} N^{n-i},$$

where N^i is the number of fixed points of index $2i$ and t is an indeterminate. In addition, suppose that M is a symplectic manifold and the action is symplectic. Then $\chi^0(M) = 1$ if the action is Hamiltonian, and $\chi^0(M) = 0$ if it is not Hamiltonian.

3. Symplectic circle actions with isolated fixed points

We begin with the proof of Theorem 1.1. Recall that for a symplectic circle action on a closed symplectic manifold M with a non-empty discrete fixed point set, we consider a collection of weights among all the fixed points, counted with multiplicity, where, for each integer a, the multiplicity of a in the collection is precisely $\max_{p\in M^{S^1}} |\{i|a = w_p^i\}|$.

Proof of Theorem 1.1. The main idea of the proof is to manipulate the formula in Theorem 2.1; we consider $\chi^0(M)$, make each exponent in the denominator positive, and clear up denominators by multiplying by the least common multiple of the denominators. In such a way each term has the exponent that is the sum of the absolute values of the weights and we derive the conclusion.

Assume, on the contrary, that the action is not Hamiltonian. By Theorem 2.1, $\chi^0(M) = \chi^n(M) = 0$ and there are no fixed points of index 0 and $2n$. Moreover,

$$\begin{aligned}\chi^0(M) &= \sum_{p\in M^{S^1}} \frac{1}{\prod_{m=1}^n(1-t^{\xi_p^m})} = \sum_{p\in M^{S^1}} \frac{1}{\prod_{m=1}^n(1-t^{\xi_p^m})}\frac{\prod_{\xi_p^m<0}(-t^{-\xi_p^m})}{\prod_{\xi_p^m<0}(-t^{-\xi_p^m})}\\ &= \sum_{p\in M^{S^1}} \frac{\prod_{\xi_p^m<0}(-t^{-\xi_p^m})}{\prod_{\xi_p^m>0}(1-t^{\xi_p^m})\prod_{\xi_p^m<0}\{(1-t^{\xi_p^m})(-t^{-\xi_p^m})\}}\\ &= \sum_{p\in M^{S^1}} \frac{(-1)^{\frac{\lambda_p}{2}}\prod_{\xi_p^m<0}t^{-\xi_p^m}}{\prod_{\xi_p^m>0}(1-t^{\xi_p^m})\prod_{\xi_p^m<0}(1-t^{-\xi_p^m})}\\ &= \sum_{p\in M^{S^1}} (-1)^{\frac{\lambda_p}{2}}\frac{t^{\sum_{\xi_p^m<0}(-\xi_p^m)}}{\prod_{m=1}^n(1-t^{|\xi_p^m|})} = 0.\end{aligned}$$

Denote by $A = \{a_1, a_2, \ldots, a_l\}$ the collection of all the absolute values of weights among all the fixed points, counted with multiplicity, where, for each positive integer $a \in A$, the multiplicity of a is precisely

$$\max_{p\in M^{S^1}} |\{i|a = |w_p^i|\}|.$$

Therefore, the least common multiple of the denominators is

$$\prod_{i=1}^l(1-t^{a_i}).$$

Denote by

$$B_p = A \setminus \{|w_p^1|, \ldots, |w_p^n|\} = \{b_p^1, b_p^2, \ldots, b_p^{l-n}\}$$

the set of elements in A minus the absolute values of the weights at p. We multiply the equation above by $\prod_{i=1}^{l}(1-t^{a_i})$ to get

$$\begin{aligned}
0 &= \sum_{p\in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \prod_{a\in B_p}(1-t^a)\\
&= \sum_{p\in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \left(1-\sum_j t^{b_p^j}+\sum_{j_1<j_2} t^{b_p^{j_1}+b_p^{j_2}}+\cdots\right)\\
&= \sum_{p\in M^{S^1}} \Bigg[(-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} - (-1)^{\frac{\lambda_p}{2}} \sum_j t^{b_p^j+\sum_{\xi_p^m<0}(-\xi_p^m)}\\
&\qquad + (-1)^{\frac{\lambda_p}{2}} \sum_{j_1<j_2} t^{\{\sum_{\xi_p^m<0}(-\xi_p^m)\}+(b_p^{j_1}+b_p^{j_2})} + \cdots\Bigg].
\end{aligned}$$

In the last equation, each summand of the exponent of a term, $-\xi_p^m$ or b_p^j, is a positive integer and is an element of A. Each term whose exponent is the sum of an odd number of elements in A has the coefficient -1 and each term whose exponent is the sum of an even number of elements in A has the coefficient 1.

Since $\chi^0(M)=0$, this implies that each term whose exponent is the sum of an odd number of elements must cancel out with another term whose exponent is the sum of an even number of elements.

Suppose that there is a fixed point p_0 of index $2i_0$, for some i_0 such that $0<i_0<n$. Then p_0 contributes a summand $(-1)^{i_0}t^{\sum_{\xi_{p_0}^m<0}(-\xi_{p_0}^m)}$, where $\xi_{p_0}^m<0$ are the negative weights at p_0. Since $\chi^0(M)=0$, this term must be cancelled out. The coefficient of the term is $(-1)^{i_0}$. Therefore, if the term is cancelled out by another term, then its exponent must be the sum of j_0-elements in A, where j_0 and i_0 have different parities. Suppose that the summand $(-1)^{i_0}t^{-\sum_{\xi_{p_0}^m<0}\xi_{p_0}^m}$ is cancelled out by another term, say $(-1)^{j_0}t^{d_1+d_2+\cdots+d_{j_0}}$. These d_i's form a subset of $\{-\xi_q^1,\ldots,-\xi_q^n\}\cup B_q$ for some fixed point q, i.e., $\{d_1,\ldots,d_{j_0}\}\subset\{-\xi_q^1,\ldots,-\xi_q^n\}\cup B_q$. Let us rewrite $-\sum_{\xi_{p_0}^m<0}\xi_{p_0}^m=c_1+c_2+\cdots+c_{i_0}$, i.e.,

$$c_1+c_2+\cdots+c_{i_0}=d_1+d_2+\cdots+d_{j_0}. \tag{3.1}$$

For each positive integer a, the multiplicity of a on each side does not exceed $\max_{p\in M^{S^1}}|\{i : a=|w_p^i|\}|$.

For the equation (3.1), we do the following: if $c_k=d_{k'}$ for some k and k', then we cancel these terms out on the equation. By performing these steps as many times as possible and by permuting c_i's and d_i's if necessary, assume that we have (3.1). For any positive integer a that appears in (3.1), either it appears only on the left hand side or only on the right hand side. Moreover, the multiplicity of a in the equation does not exceed $\max_{p\in M^{S^1}}|\{i : a = |w_p^i|\}|$.

Denote by C the collection of all possible equations with the LHS a sum of an odd number of weights and the RHS a sum of an even number of weights, where weights are taken among all the fixed points, counted with multiplicity. Consider an element of C. It is an equation of the form

$$w_1 + w_2 + \cdots + w_{i'} = w_{i'+1} + \cdots + w_{2j'+1}$$

for some i', j', where each w_k is a weight at some fixed point. For each integer a, the multiplicity of a on each side is at most $\max_{p\in M^{S^1}}|\{i|a = w_p^i\}|$. For each equation we do the following: if $w_k = -w_{k'}$ and they appear on the same side of the equation, we cancel out these terms. If $w_k = w_{k'}$ and they appear on the opposite side of the equation, we also cancel out these terms. If w_k is a negative weight, we move the term to the opposite side as $-w_k$. By performing these steps as many times as possible, assume that we have

$$e_1 + e_2 + \cdots + e_{i''} = f_1 + f_2 + \cdots + f_{2j''+1},$$

where all these e_k's and f_k's are positive. For any positive integer a that appears in the last equation, either it appears only on the left hand side or only on the right hand side. Moreover, the multiplicity of a in the equation does not exceed $\max_{p\in M^{S^1}}|\{i| - a = w_p^i\}| + \max_{p\in M^{S^1}}|\{i|a = w_p^i\}|$.

Note that for each positive integer a, we have

$$\begin{aligned}&\max_{p\in M^{S^1}}|\{i : a = |w_p^i|\}|\\ \leq\ &\max_{p\in M^{S^1}}|\{i| - a = w_p^i\}| + \max_{p\in M^{S^1}}|\{i|a = w_p^i\}|.\end{aligned}$$

Therefore, the equation (3.1) is an element of C. However, by the assumption that the sum of an odd number of weights is never equal to the sum of an even number of weights, C is an emptyset. Therefore, there are no fixed points of index $2i$ for all $0 < i < n$, which is a contradiction. □

We can generalize Theorem 1.1 further. Let the circle act symplectically on a closed symplectic manifold M with discrete fixed point set. As in the proof of Theorem 1.1, denote by $A = \{a_1, a_2, \ldots, a_l\}$ the collection of all the absolute values of weights among all the fixed points counted with multiplicity, where, for each positive integer a, the multiplicity of a in A

is precisely $\max_{p\in M^{S^1}} |\{i|a = |w_p^i|\}|$. Note that here we consider the collection of the absolute values of the weights, and hence it is different from the one in the introduction. For instance, with the fixed points of the same weights $\{-1,-1,1,1\}$ and $\{-1,-1,-1,2\}$ as before, the multiplicity of 1 in the collection is 4.

Theorem 3.1. *Let the circle act symplectically on a closed symplectic manifold M with a non-empty discrete fixed point set. Denote by $A = \{a_1, a_2, \ldots, a_l\}$ the collection of all the absolute values of weights among all the fixed points, counted with multiplicity, where, for each positive integer a, the multiplicity of a in A is precisely*

$$\max_{p\in M^{S^1}} |\{i|a = |w_p^i|\}|.$$

Denote by

$$A_i = \{a_{j_1} + a_{j_2} + \cdots + a_{j_i}\}_{a_{j_1}<a_{j_2}<\cdots<a_{j_i}}$$

the collection of sums of i-elements of A, for $1 \leq i \leq l$. If there exists $0 < i < n$ such that $A_i \cap A_j = \emptyset$ for all j such that $j \neq i \mod 2$, then the action is Hamiltonian.

Proof. The idea of the proof is similar to that of Theorem 1.1. However, we consider $\chi^i(M)$ for many i's.

Assume, on the contrary, that the action is not Hamiltonian. By Theorem 2.1, $\chi^0(M) = \chi^n(M) = 0$ and there are no fixed points of index 0 and $2n$. As in the proof of Theorem 1.1, we have

$$\chi^0(M) = \sum_{p\in M^{S^1}} \frac{1}{\prod_{m=1}^n (1 - t^{\xi_p^m})} = \sum_{p\in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} \frac{t^{\sum_{\xi_p^m<0}(-\xi_p^m)}}{\prod_{m=1}^n (1 - t^{|\xi_p^m|})} = 0.$$

For each fixed point p, denote by

$$B_p = A \setminus \{|w_p^1|, \ldots, |w_p^n|\} = \{b_p^1, b_p^2, \ldots, b_p^{l-n}\}$$

the set of elements in A minus the absolute values of the weights at p. We multiply the equation above by $\prod_{i=1}^l (1 - t^{a_i})$, the least common multiple of the denominators, to get

$$0 = \sum_{p\in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \prod_{a\in B_p} (1 - t^a)$$

$$= \sum_{p\in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \left(1 - \sum_j t^{b_p^j} + \sum_{j_1<j_2} t^{b_p^{j_1}+b_p^{j_2}} + \cdots \right)$$

$$= \sum_{p\in M^{S^1}} \Bigg[(-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} - (-1)^{\frac{\lambda_p}{2}} \sum_j t^{b_p^j + \sum_{\xi_p^m<0}(-\xi_p^m)} + (-1)^{\frac{\lambda_p}{2}} \sum_{j_1<j_2} t^{\{\sum_{\xi_p^m<0}(-\xi_p^m)\}+(b_p^{j_1}+b_p^{j_2})} + \cdots \Bigg].$$

In the equation, $-\xi_p^m \in A$ and $b_p^j \in A$ for all $-\xi_p^m, b_p^j$. Therefore, if a term has the exponent that is the sum of i-elements in A, then the exponent is an element of A_i. Each term whose exponent is the sum of an odd number of elements in A has the coefficient -1 and each term whose exponent is the sum of an even number of elements in A has the coefficient 1.

Since $\chi^0(M) = 0$, this implies that each term whose exponent is the sum of an odd number of elements must cancel out with another term whose exponent is the sum of an even number of elements.

Suppose that there is a fixed point p of index $2i$, $0 < i < n$. Recall that i is an integer such that $A_i \cap A_j = \emptyset$ for $j \neq i \mod 2$. Then p contributes a summand $(-1)^i t^{\sum_{\xi_p^m<0}(-\xi_p^m)}$, where $\xi_p^m < 0$ are the negative weights at p. Since $\chi^0(M) = 0$, this term must be cancelled out. The coefficient of the term is $(-1)^i$. Therefore, if the term is cancelled out by another term, then its exponent must be the sum of j-elements, where j and i have different parities. By the assumption that $A_i \cap A_j = \emptyset$ for $j \neq i \mod 2$, the summand $(-1)^i t^{\sum_{\xi_p^m<0}(-\xi_p^m)}$ cannot be cancelled out, which is a contradiction. Therefore, there are no fixed points of index $2i$.

From now on we seperate into several cases, depending on the parity of i and n and on whether $i \leq \frac{n}{2}$ or $i > \frac{n}{2}$. Each case is a slight variation of the other cases. If $i > \frac{n}{2}$, we use the symmetry that $N^j = N^{n-j}$ for all j where N_j is the number of fixed points of index $2j$, that follows from Theorem 2.1. With the symmetry, the case where $i > \frac{n}{2}$ is a slight variation of the case where $i \leq \frac{n}{2}$.

First, suppose that $i \leq \frac{n}{2}$ and i is odd. By Theorem 2.1,

$$\chi^i(M) = \sum_{p\in M^{S^1}} \frac{\sigma_i(t^{\xi_p^1}, \ldots, t^{\xi_p^n})}{\prod_{m=1}^n (1 - t^{\xi_p^m})} = 0.$$

As in the proof of Theorem 1.1, we make the exponent of each term in the denominators positive to get

$$\chi^i(M) = \sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} \frac{t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \sigma_i(t^{\xi_p^1}, \dots, t^{\xi_p^n})}{\prod_{m=1}^n (1 - t^{|\xi_p^m|})} = 0.$$

Multiplying the equation above by $\prod_{i=1}^l (1 - t^{a_i})$, we have

$$\sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \sigma_i(t^{\xi_p^1}, \dots, t^{\xi_p^n}) \prod_{a \in B_p} (1 - t^a) = 0.$$

Let us consider $t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \sigma_i(t^{\xi_p^1}, \dots, t^{\xi_p^n})$. When we expand the terms, the exponent of any term is the sum of positive integers that are in A. No negative integer appears in the exponent of any term when expanded. Therefore, when we multiply $t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \sigma_i(t^{\xi_p^1}, \dots, t^{\xi_p^n})$ by $\prod_{a \in B_p}(1 - t^a)$ and expand the terms, each summand in the equation has the exponent that belongs to A_i for some i. If it belongs to A_i, then it has the sign $(-1)^i$.

Suppose that a fixed point p has index $2k$, where k is even and $0 \leq k \leq 2i$. In the last equation, such a point contributes a summand whose exponent is the sum of i elements. By the assumption, such a term cannot be cancelled out. Hence there are no fixed points of index $0, 4, \dots, 4i$, i.e. $N^0 = N^4 = \cdots = N^{4i} = 0$ and thus $N^{2n} = N^{2n-4} = \cdots = N^{2n-4i} = 0$, by Theorem 2.1. In particular, $\chi^{i+1}(M) = (-1)^{i+1} N^{i+1} = 0$.

Next, we consider $\chi^{i+1}(M) = 0$. Using the same argument, one can show that there are no fixed points of index $2k$ where k is odd and $0 \leq k \leq i + 2$. And then we consider $\chi^{i+2}(M) = 0$ to conclude that there are no fixed points of index $2k$ where k is even and $0 \leq k \leq i + 3$. We continue this to conclude that there are no fixed points of any index, which is a contradiction.

Second, suppose that $i \leq \frac{n}{2}$ and i is even. Using the same argument as in the first case, by considering $\chi^i(M) = 0$, one can show that there are no fixed points of index $2k$, where k is even and $0 \leq k \leq 2i$. Next, consider $\chi^{i+2}(M) = 0$ and conclude that there are no fixed points of index $2k$ where k even and $k \leq i + 4$. And then we consider $\chi^{i+4}(M) = 0$ to conclude that there are no fixed points of index $2k$ where k is even and $k \leq i + 6$. We continue this until $\chi^{n-i}(M)$ if n is even and $\chi^{n-i-1}(M)$ if n is odd, to conclude that there are no fixed points of index that is a multiple of 4, which contradicts Corollary 3.7 below that there must be fixed points whose indices differ by 2.

Third, suppose that n is odd, $i > \frac{n}{2}$, and i is odd. Considering $\chi^i(M) = 0$, it follows that there are no fixed points of index $2k$ such that k is even and $0 \leq k \leq 2(n - i)$. By Theorem 2.1, since $N^j = N^{n-j}$ for all j, there are no fixed points of index $2k$, where k is odd and $n - (2n - 2i) = 2i - n \leq k \leq n$. In particular, there are no fixed points of index $i - 2$. Next, considering

$\chi^{i-2}(M) = 0$, we have that there are no fixed points of index $2k$, where k is even and $k \leq 2n - 2i + 2$. By the symmetry that $N^j = N^{n-j}$ for all j, there are no fixed points of index $2k$ such that k is odd and $2i - n - 2 \leq k \leq n$. We continue this to have that there are no fixed points of any index, which is a contradiction.

As a slight variation of the arguments above, the other cases, (4) n is odd, $i > \frac{n}{2}$, and i is even, (5) n is even, $i > \frac{n}{2}$, and i is odd, and (6) n is even, $i > \frac{n}{2}$, and i is even, are proved. □

Corollary 3.2. *[TW], [L] A semi-free, symplectic circle action on a closed symplectic manifold M with discrete fixed point set is Hamiltonian if and only if it has a fixed point.*

Proof. All the weights are either 1 or -1. Therefore, $A_i = \{i\}$ for each i and so the corollary follows. □

We show that, in certain cases, it is enough to look at sums of three weights. The first instance is the Theorem 1.2; here we give a proof of it.

Proof of Theorem 1.2. Assume, on the contrary, that the action in not Hamiltonian. By Theorem 2.1, $\chi^0(M) = \chi^n(M) = 0$ and there are no fixed points of index 0 and $2n$. Denote by

$$A = \{a_1, a_2, \dots, a_n\} \quad \text{and}$$
$$A_i = \{a_{j_1} + a_{j_2} + \dots + a_{j_i}\}_{a_{j_1} < a_{j_2} < \dots < a_{j_i}}$$

the collection of sums of i elements of A, where $1 \leq i \leq n$. Then the problem is equivalent to showing that if $A_1 \cap A_2 = \emptyset$, the action is Hamiltonian. We consider $A_i \cap A_j$ for all i, j such that $i \neq j \mod 2$, $1 \leq i \leq n-1$, $1 \leq j \leq n$.

First, assume that $n \leq 3$. Then $A_1 \cap A_2$ is the only intersection that we consider, so the result follows from Theorem 1.2.

Second, assume that $n = 4$. Then $A_1 \cap A_2$ and $A_2 \cap A_3$ are the only ones that we consider. However, $A_1 \cap A_2 = \emptyset$ if and only if $A_2 \cap A_3 = \emptyset$. Therefore the result follows from Theorem 1.2.

Finally, assume that $n = 5$. Since we assume that $A_1 \cap A_2 = \emptyset$, in order to use Theorem 1.2 it is enough to show that $A_1 \cap A_2 = \emptyset$ implies that either $A_1 \cap A_4 = \emptyset$ or $A_2 \cap A_3 = \emptyset$ (in the first case we take $i = 1$ in the statement on Theorem 1.2 while in the second case we take $i = 2$.) However, if $A_1 \cap A_4 \neq \emptyset$, then necessarily $A_2 \cap A_3 = \emptyset$ so the result follows. □

Another case where the condition that the sum of any three weights is never equal to zero guarantees that a symplectic circle action is Hamiltonian, is when the dimension of the manifold is six (Theorem 1.3). In fact, we prove a stronger result:

Theorem 3.3. *Consider a symplectic circle action on a six-dimensional closed symplectic manifold with a non-empty discrete fixed point set. Suppose that each negative weight at a fixed point of index 2 is never equal to the sum of the negative weights at a fixed point of index 4. Then the action is Hamiltonian.*

Proof. Assume, on the contrary, that the action is not Hamiltonian. By Theorem 2.1, $\chi^0(M) = \chi^3(M) = 0$ and there are no fixed points of index 0 and 6. Moreover, the number of fixed points of index 2 and that of 4 are equal. Suppose that there are k fixed points of index 2, and let p_i, q_i be the fixed points of index 2,4, respectively, for $1 \leq i \leq k$. Let

$$\Sigma_{p_i} = \{-b_{p_i}^1, b_{p_i}^2, b_{p_i}^3\}$$
$$\Sigma_{q_i} = \{-c_{q_i}^1, -c_{q_i}^2, c_{q_i}^3\}$$

be the weights at p_i, q_i, respectively, where $b_{p_i}^j, c_{q_i}^j$ are positive integers. By permuting p_i's and q_i's if neccesary, we may assume that

$$b_{p_1}^1 \leq b_{p_2}^1 \leq \cdots \leq b_{p_k}^1 \quad \text{and}$$
$$c_{q_1}^1 + c_{q_1}^1 \leq c_{q_2}^1 + c_{q_2}^1 \leq \cdots \leq c_{q_k}^1 + c_{q_k}^1.$$

By Theorem 2.1,

$$\begin{aligned}\chi^0(M) &= \sum_i \frac{1}{(1-t^{-b_{p_i}^1})(1-t^{b_{p_i}^2})(1-t^{b_{p_i}^3})} \\ &\quad + \sum_i \frac{1}{(1-t^{-c_{q_i}^1})(1-t^{-c_{q_i}^2})(1-t^{c_{q_i}^3})} \\ &= -\sum_i \frac{t^{b_{p_i}^1}}{(1-t^{b_{p_i}^1})(1-t^{b_{p_i}^2})(1-t^{b_{p_i}^3})} \\ &\quad + \sum_i \frac{t^{c_{q_i}^1+c_{q_i}^2}}{(1-t^{c_{q_i}^1})(1-t^{c_{q_i}^2})(1-t^{c_{q_i}^3})} = 0.\end{aligned}$$

Denote by $A = \{a_1, a_2, \dots, a_l\}$ the collection of all the absolute values of the weights over all the fixed points counted with multiplicity, where, for each positive integer a the multiplicity of a in A is precisely

$$\max_{p \in M^{S^1}} |\{i | a = |w_p^i|\}|.$$

Let

$$B_i = A \setminus \{b_{p_i}^1, b_{p_i}^2, b_{p_i}^3\} = \{d_{p_i}^1, \dots, d_{p_i}^{l-3}\},$$
$$C_i = A \setminus \{c_{q_i}^1, c_{q_i}^2, c_{q_i}^3\} = \{e_{q_i}^1, \dots, e_{q_i}^{l-3}\}$$

be the elements in A minus the absolute values of weights at p_i,q_i, respectively.

Multiplying the equation above by $\prod_{a\in A}(1-t^a)$, the least common multiple of the denomnators, we have

$$\begin{aligned} 0 &= -\sum_i t^{b_{p_i}^1} \prod_{a\in B_i}(1-t^a) + \sum_i t^{c_{q_i}^1 + c_{q_i}^2} \prod_{a\in C_i}(1-t^a) \\ &= \left\{ -\sum_i t^{b_{p_i}^1} + \sum_{i,j} t^{b_{p_i}^1 + d_{p_i}^j} - \sum_{i,j_1<j_2} t^{b_{p_i}^1 + d_{p_i}^{j_1} + d_{p_i}^{j_2}} + \cdots \right\} \\ &\quad + \left\{ \sum_i t^{c_{q_i}^1 + c_{q_i}^2} - \sum_{i,j} t^{c_{q_i}^1 + c_{q_i}^2 + e_{q_i}^j} + \sum_{i,j_1<j_2} t^{c_{q_i}^1 + c_{q_i}^2 + e_{q_i}^{j_1} + e_{q_i}^{j_2}} - \cdots \right\}. \end{aligned}$$

In the equation, each summand in the exponent of any term is an element of A. A term has coefficient -1 if its exponent is the sum of an odd number of elements of A and 1 if its exponent is the sum of an even number of elements of A. Consider $-t^{b_{p_1}^1}$. Since $b_{p_1}^1 \leq b_{p_i}^1$ for $i \geq 2$, this term cannot be cancelled out by any summand in

$$-\sum_i t^{b_{p_i}^1} + \sum_{i,j} t^{b_{p_i}^1 + d_{p_i}^j} - \sum_{i,j_1<j_2} t^{b_{p_i}^1 + d_{p_i}^{j_1} + d_{p_i}^{j_2}}.$$

Therefore, it must be cancelled out by another summand in

$$\sum_i t^{c_{q_i}^1 + c_{q_i}^2} - \sum_{i,j} t^{c_{q_i}^1 + c_{q_i}^2 + e_{q_i}^j} + \sum_{i,j_1<j_2} t^{c_{q_i}^1 + c_{q_i}^2 + e_{q_i}^{j_1} + e_{q_i}^{j_2}} - \cdots$$

for some i, whose exponent is the sum of even elements in A, where at least two elements of them are $c_{q_i}^1, c_{q_i}^2$. By the assumption, the exponent of such a summand cannot be $c_{q_i}^1 + c_{q_i}^2$. Hence, the exponent of the term must be the sum of at least four elements, say $b_{p_1}^1 = c_{q_i}^1 + c_{q_i}^2 + \alpha$. Next, consider $t^{c_{q_i}^1 + c_{q_i}^2}$. We have that

$$c_{q_i}^1 + c_{q_i}^2 < c_{q_i}^1 + c_{q_i}^2 + \alpha = b_{p_1}^1.$$

Since

$$c_{q_1}^1 + c_{q_1}^1 \leq c_{q_2}^1 + c_{q_2}^1 \leq \cdots \leq c_{q_k}^1 + c_{q_k}^1,$$

the term $t^{c^1_{q_i}+c^2_{q_i}}$ cannot be cancelled out by any term in

$$\sum_i t^{c^1_{q_i}+c^2_{q_i}} - \sum_{i,j} t^{c^1_{q_i}+c^2_{q_i}+e^j_{q_i}} + \sum_{i,j_1<j_2} t^{c^1_{q_i}+c^2_{q_i}+e^{j_1}_{q_i}+e^{j_2}_{q_i}} - \cdots .$$

On the other hand, $c^1_{q_i} + c^2_{q_i} < b^1_{p_1}$. Therefore, it cannot also be cancelled out by any term in

$$-\sum_i t^{b^1_{p_i}} + \sum_{i,j} t^{b^1_{p_i}+d^j_{p_i}} - \sum_{i,j_1<j_2} t^{b^1_{p_i}+d^{j_1}_{p_i}+d^{j_2}_{p_i}},$$

which is a contradiction. □

As a corollary, we recover the result by L. Godinho:

Corollary 3.4. *[G] Let the circle act symplectically on a six-dimensional closed symplectic manifold. Suppose that fixed points are isolated and their weights are $\{\pm a, \pm b, \pm c\}$, where $0 < a \leq b \leq c$ and $a + b \neq c$. If there is a fixed point, then the action is Hamiltonian.*

Proof. This follows from Theorem 1.1, Theorem 1.2, Theorem 1.3, or Theorem 3.3. □

For the rest of this section, we prove a technical lemma and its corollary, to complete the proof of Theorem 3.1. For this, consider a circle action on a closed almost complex manifold with a non-empty discrete fixed point set. Suppose that a fixed point p of index $2i$ has a positive weight w. If w is never equal to a sum of other positive weights, then w satisfies some special property; there must exist another fixed point q of index $2i + 2$ that has weight $-w$. In other words, for such a weight w, indices of fixed points that have the weights $\pm w$ are nearby. For a precise statement, we introduce a terminology.

Definition 3.5. Let the circle act on a closed almost complex manifold. Suppose that the action preserves the almost complex structure and the fixed points are isolated. Denote by $A = \{a_1, a_2, \ldots, a_l\}$ the collection of all the absolute values of weights among all the fixed points counted with multiplicity, where, for each positive integer $a \in A$, the multiplicity of a is precisely

$$\max_{p \in M^{S^1}} |\{i | a = |w^i_p|\}|.$$

Moreover, denote by

$$A_i = \{a_{j_1} + a_{j_2} + \cdots + a_{j_i}\}_{a_{j_1} < a_{j_2} < \cdots < a_{j_i}}$$

the collection of sums of i elements of A, for $1 \leq i \leq l$. A positive weight w is called **primitive**, if $w \notin A_i$ for $i \geq 2$, i.e. w is never equal to the sum of the absolute values of weights among all the fixed points, counted with multiplicity, other than w itself.

Note that the smallest positive weight is primitive. In [K], C. Kosniowski derives a certain formula for a holomorphic vector field on a complex manifold with only simple isolated zeros. We follow the idea of C. Kosniowski to find a restriction for a primitive weight of a circle action on a closed almost complex manifold with discrete fixed point set. For the smallest positive weight, the Lemma is already given in [JT] and the proof is almost identical, but we give a detailed proof.

Lemma 3.6. *Consider a circle action on a $2n$-dimensional closed almost complex manifold. Suppose that the action preserves the almost complex structure and the fixed points are isolated. For each primitive weight w, the number of times the weight $-w$ occurs at fixed points of index $2i$ is equal to the number of times the weight w occurs at fixed points of index $2i-2$, for all i.*

Proof. We first show that

$$\sum_{\lambda_p = 2i} [N_p(w) + N_p(-w)] = \sum_{\lambda_p = 2i-2} N_p(w) + \sum_{\lambda_p = 2i+2} N_p(-w), \tag{3.2}$$

where $N_p(w)$ is the number of times the weight w occurs at p. The basic idea is to manipulate $\chi^i(M)$ and compare the coefficients of t^w-terms. By Theorem 2.1,

$$\begin{aligned} \chi^i(M) &= \sum_{p \in M^{S^1}} \frac{\sigma_i(t^{\xi_p^1}, \ldots, t^{\xi_p^n})}{\prod_{m=1}^n (1 - t^{\xi_p^m})} \\ &= \sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} \frac{[\prod_{\xi_p^m < 0} t^{-\xi_p^m}] \sigma_i(t^{\xi_p^1}, \ldots, t^{\xi_p^n})}{\prod_{m=1}^n (1 - t^{|\xi_p^m|})}. \end{aligned} \tag{3.3}$$

Denote by

$$J_p = \left[\prod_{\xi_p^m<0} t^{-\xi_p^m}\right] \sigma_i(t^{\xi_p^1}, \ldots, t^{\xi_p^n}) \quad \text{and} \quad K_p = \prod_{m=1}^{n} (1 - t^{|\xi_p^m|}).$$

If $\lambda_p = 2i$, then $J_p = 1 + f_p(t)$, where $f_p(t)$ is a polynomial that does not have a constant term and t^w-term.

If $\lambda_p = 2i \pm 2$, then $J_p = N_p(\mp w)t^w + f_p(t)$, where $f_p(t)$ is a polynomial that does not have a constant term and t^w-term.

If $\lambda_p \neq 2i, 2i \pm 2$, then $J_p = f_p(t)$, where $f_p(t)$ is a polynomial that does not have a constant term and t^w-term.

Multiplying (3.3) by $\prod_{p\in M^{S^1}} K_p$ yields

$$\begin{aligned}
&\chi^i(M)[1 - \sum_p (N_p(-w) + N_p(w))t^w] + g_1(t) \\
&= \Bigg\{ (-1)^{i-1} \sum_{\lambda_p=2i-2} N_p(w) + (-1)^{i+1} \sum_{\lambda_p=2i+2} N_p(-w) \\
&\quad + (-1)^i \sum_{\lambda_p=2i} (N_p(w) + N_p(-w)) - \chi^i(M) \sum_p (N_p(w) + N_p(-w)) \Bigg\} t^w \\
&\quad + \sum_{\lambda_p=2i} (-1)^i + g_2(t),
\end{aligned}$$

where $g_i(t)$ are polynomials without constant terms and t^w-terms. Comparing the coefficients of t^w-terms, the claim follows.

Applying (3.2) for $i = 0$, we have

$$\sum_{\lambda_p=0} N_p(w) = \sum_{\lambda_p=2} N_p(-w).$$

Next, applying (3.2) for $i = 1$, we have

$$\sum_{\lambda_p=2} [N_p(-w) + N_p(w)] = \sum_{\lambda_p=0} N_p(w) + \sum_{\lambda_p=4} N_p(-w).$$

Since $\sum_{\lambda_p=0} N_p(w) = \sum_{\lambda_p=2} N_p(-w)$, it follows that

$$\sum_{\lambda_p=2} N_p(w) = \sum_{\lambda_p=4} N_p(-w).$$

Continuing this, the Lemma follows. □

As an application, there must be at least two fixed points whose indices are nearby. This is shown for a holomorphic vector field on a compact complex manifold with only simple isolated zeroes by C. Kosniowski [K].

Corollary 3.7. *Consider a circle action on a closed almost complex manifold. Suppose that the action preserves the almost complex structure and the fixed points are non-empty and isolated. Then there exist two fixed points whose indices differ by 2.*

Proof. Apply Lemma 3.6 to the smallest positive weight. □

Acknowledgements

This work was partially supported by Campus Research Board Awards — University of Illinois at Urbana-Champaign. The author would like to thank the anonymous referee for helping the author to improve the exposition of this paper.

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Received April 14, 2015
Accepted August 10, 2016