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KCH representations, augmentations, and A-polynomials

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We describe a correspondence between augmentations of knot contact homology and certain representations of the knot group. The correspondence makes the 2-variable augmentation polynomial into a generalization of the classical A-polynomial. It also associates to an augmentation a rank, which is bounded by the bridge number and shares its behavior under connect sums. We also study augmentations with rank equal to the braid index.

1. Introduction

Let K be a knot in $\mathbb{R}^3$ and denote by π_K the fundamental group of the complement $\overline{\mathbb{R}^3 \setminus n(K)}$. Define an element of π_K to be a *meridian of* K if it may be represented by the boundary of a disk D that is embedded in $\mathbb{R}^3$ and intersects K at one point in the interior of D. Fix a field $\mathbb{F}$.

Definition 1.1. If V is an $\mathbb{F}$-vector space, a homomorphism $\rho : \pi_K \to \mathrm{GL}(V)$ is a *KCH representation of* π_K if for a meridian m of K, $\rho(m)$ is diagonalizable and has an eigenvalue of 1 with multiplicity $\dim V - 1$. We call ρ a *KCH irrep* if it is irreducible as a representation.

Describing the *knot contact homology* $HC_*(K)$ is somewhat involved and we delay its definition until Section 2. However, we remark here that $HC_*(K)$ is a non-commutative graded algebra over $\mathbb{Z}[U^{\pm1}, \lambda^{\pm1}, \mu^{\pm1}]$, and is defined as the homology of a certain differential graded algebra $(\mathcal{A}_K, \partial_K)$. An *augmentation* is a graded algebra map $\epsilon : \mathcal{A}_K \to \mathbb{F}$ such that $\epsilon \circ \partial = 0$ (and $\epsilon(1) = 1$), where $\mathbb{F}$ has grading 0. This paper deals with the specialization of $HC_*(K)$ to an $\mathbb{Z}[\lambda^{\pm1}, \mu^{\pm1}]$-algebra where we set $U = 1$. In this case $HC_0(K)$ is isomorphic to the cord algebra $\mathcal{C}_K$ introduced in [19], and we may view augmentations as homomorphisms $\epsilon : \mathcal{C}_K \to \mathbb{F}$.

It is discussed in [21] how to associate an augmentation to a KCH representation, giving a correspondence

$$\begin{aligned} \{\rho : \pi_K \to \mathrm{GL}(V) \mid \rho \text{ is a KCH irrep}\} &\to \{\epsilon : \mathcal{C}_K \to \mathbb{F} \mid \epsilon(\mu) \neq 1\} \\ \rho &\mapsto \epsilon_\rho \end{aligned} \tag{1.1}$$

If ρ, ρ' are conjugate then $\epsilon_\rho = \epsilon_{\rho'}$ (see Remark 2.13). Our primary result shows this correspondence to be surjective.

Theorem 1.2. *Let $\epsilon : \mathcal{C}_K \to \mathbb{F}$ be an augmentation such that $\epsilon(\mu) \neq 1$. Then a KCH irrep $\rho : \pi_K \to GL(V)$ can be constructed explicitly from ϵ with the property that $\epsilon_\rho = \epsilon$. Moreover, any KCH irrep that induces ϵ is isomorphic to (V, ρ).*

Remark 1.3. Augmentations are geometrically motivated from the fact that Legendrian DGA's are functorial under (exact Lagrangian) cobordisms, and a cobordism to a Legendrian from the empty set induces an augmentation. See Section 2.3 for how the correspondence $\rho \mapsto \epsilon_\rho$ may be viewed from this perspective.

Remark 1.4. We state Theorem 1.2 for a field $\mathbb{F}$ to discuss irreducible representations and for the relationship to the A-polynomial. However, if we drop irreducibility and choose a ring with unity S, any augmentation $\epsilon : \mathcal{C}_K \to S$ (which sends $1 - \mu$ to a unit) is induced from a representation $\rho : \pi_K \to \mathrm{Aut}_S(V)$ (see Theorem 3.5 and Corollary 3.7).

Every KCH representation ρ has an eigenvalue $\mu_0 \neq 1$ of $\rho(m)$. An eigenvector corresponding to μ_0 is also an eigenvector for $\rho(\ell)$, where ℓ is the preferred (Seifert-framed) longitude. Let λ_0 be the corresponding eigenvalue of $\rho(\ell)$.

Let $\mathbb{F} = \mathbb{C}$ and write $\mathbb{C}^*$ for $\mathbb{C} \setminus \{0\}$ and define the following sets in $(\mathbb{C}^*)^2$:

$$U_K = \{(\lambda_0, \mu_0) \mid \rho : \pi_K \to \mathrm{GL}(V) \text{ is a KCH irrep}\};$$
$$V_K = \{(\epsilon(\lambda), \epsilon(\mu)) \mid \epsilon : \mathcal{C}_K \to \mathbb{C} \text{ is an augmentation}\} \setminus (\mathbb{C}^* \times \{1\}).$$

It is conjectured that, for any K, the maximum dimensional part of the Zariski closure of U_K (resp. that of V_K) is a complex curve. If so a polynomial in $\mathbb{Z}[\lambda, \mu]$ exists with zero locus the closure of U_K (resp. V_K). This polynomial is unique (up to a sign) once repeated factors and extraneous powers of λ, μ are removed and coefficients are made to be coprime.

The polynomial for V_K, if multiplied by $1 - \mu$, is called the *2-variable augmentation polynomial* $\mathrm{Aug}_K(\lambda, \mu)$. However, as this paper only considers augmentations sending U to 1, we will not encounter its 3-variable analogue and so we refer to $\mathrm{Aug}_K(\lambda, \mu)$ simply as the *augmentation polynomial* (see [1] for an interesting conjecture that relates the 3-variable polynomial to HOMFLY-PT polynomials).

The polynomial for U_K, studied in [7], is called the *stable A-polynomial* $\widetilde{A}_K(\lambda, \mu)$. The terminology "stable" is motivated by the bound of Theorem 1.5 below. The reason for "A-polynomial" is explained as follows. A KCH representation ρ may be modified to an $\mathrm{SL}(V)$ representation by multiplying ρ by some 1-dimensional representation determined by $m \mapsto (\mu_0)^{-1/d}$, $d = \dim V$. The 2-dimensional KCH representations then determine the original A-polynomial.

While no *a priori* restriction is placed on $\dim V$ in the definition of U_K, it was shown in [7] that K itself imposes a restriction.

Theorem 1.5 ([7]). *Let $\{g_1, \ldots, g_r\}$ be a set of meridians that generate π_K. If $\rho : \pi_K \to GL(V)$ is a KCH irrep of π_K then* $\dim V \leq r$.

The reason for the bound above is that each meridian has a distinguished 1-dimensional eigenspace and the sum of these eigenspaces is an invariant subspace. This will play an important role in the proof of Theorem 1.2 (see Section 3.4).

Theorem 1.2 implies that $U_K = V_K$, giving us the following corollary.

Corollary 1.6. *Given $K \subset \mathbb{R}^3$ a knot, $Aug_K(\lambda, \mu) = (1 - \mu)\widetilde{A}_K(\lambda, \mu)$ holds up to a sign.*

The set U_K was computed in [7] for torus knots. We may now view this as a computation of the augmentation polynomial of torus knots.

Corollary 1.7. *Given $0 < p < q$ relatively prime, let $T(p,q)$ denote the (p,q)–torus knot. Then*

$$Aug_{T(p,q)}(\lambda,\mu) = (1-\mu)(\lambda\mu^{(p-1)q} + (-1)^p)\prod_{n=1}^{p-1}(\lambda^n\mu^{(n-1)pq} - 1).$$

In the proof of Theorem 1.2 we construct a matrix from ϵ with rank equal to $\dim V$, where the KCH irrep corresponding to ϵ has image in $\mathrm{GL}(V)$.

Definition 1.8. The *rank* of an augmentation $\epsilon : \mathcal{C}_K \to \mathbb{F}$, with the property $\epsilon(\mu) \neq 1$, is the dimension of any KCH irrep that induces ϵ. The *augmentation rank*, $\mathrm{ar}(K,\mathbb{F})$, of a knot K is the maximal rank of an augmentation to $\mathbb{F}$.

Let $\mathrm{mr}(K)$ denote the *meridional rank* of K, i.e. the minimal size of a generating set of meridians for π_K. It is well-known that $\mathrm{mr}(K)$ is at most the bridge number $b(K)$. Recalling Theorem 1.5 we have,

$$\mathrm{ar}(K,\mathbb{F}) \leq \mathrm{mr}(K) \leq b(K). \tag{1.2}$$

Problem 1.11 in [14], a question of Cappell and Shaneson that remains open, asks whether $\mathrm{mr}(K) = b(K)$. The following result implies that, similar to bridge number, $\mathrm{ar}(K,\mathbb{F}) - 1$ is additive under connect sums.

Theorem 1.9. *Let $K_1, K_2 \subset \mathbb{R}^3$ be oriented knots and suppose $\mu_0 \in \mathbb{F}^*$ is such that for $n = 1, 2$ there is an augmentation $\epsilon_n : \mathcal{C}_{K_n} \to \mathbb{F}$ with rank d_n and so that $\epsilon_n(\mu) = \mu_0 \neq 1$. Then $K_1\#K_2$ has an augmentation with rank $d_1 + d_2 - 1$. Furthermore, $ar(K_1\#K_2,\mathbb{F}) = ar(K_1,\mathbb{F}) + ar(K_2,\mathbb{F}) - 1$.*

Since $\mathcal{C}_K$ is isomorphic to $HC_0(K)|_{U=1}$ a study of augmentations can carried out in this setting, where the algebra is described from a closed braid representing K (see Section 2.2). This formulation allows us to obtain the following result.

Theorem 1.10. *Suppose that K is the closure of $B \in B_n$, and that $\epsilon : \mathcal{C}_K \to \mathbb{C}$ is an augmentation of K with rank n and $\epsilon(\mu) = \mu_0$. Then $\epsilon(\lambda) = (-\mu_0)^{-w(B)}$, where $w(B)$ is the writhe (or algebraic length) of B. Furthermore, there is a curve of rank n augmentations in the closure of V_K that corresponds to a factor $\lambda\mu^{w(B)} - (-1)^{w(B)}$ of $Aug_K(\lambda,\mu)$.*

Corollary 1.11. *If K is the closure of a 3-braid then*

$$Aug_K(\lambda, \mu^2) = (1-\mu^2)A_K(\lambda,\mu)B_K(\lambda,\mu^2)$$

where $A_K(\lambda,\mu)$ is the A-polynomial and $B_K(\lambda,\mu)$ is either 1 or $(\lambda\mu^{w(B)} \pm 1)$.

Proof. Follows from Theorem 1.5, Corollary 1.6, and Theorem 1.10. □

The hypothesis of Theorem 1.10 can only possibly hold if K has a braid representative with (necessarily minimal) index equal to the bridge number of K. In this setting the number $w(B)$ is, in fact, an invariant of K by independent work in [9] and [16], where the Jones Conjecture is proved.

The proof of Theorem 1.10 also supplies us techniques to find knots for which the left-hand inequality in (1.2) is strict (see Theorem 5.3).

Corollary 1.12. *If K is one of the knots $\{8_{16}, 8_{17}, 10_{91}, 10_{94}\}$ then $2 = ar(K,\mathbb{C}) < mr(K) = 3$.*

It would be very interesting if there were a coherent (or even geometric!) way to understand the absence of augmentations with rank $\mathrm{mr}(K)$. Theorem 1.10 does apply to the following family of knots. Many 3-braid closures that admit a positive or negative flype fit into this family [15].

Theorem 1.13. *If $|u|, |w| \geq 2$, $|v| \geq 3$, and $\delta = \pm 1$ and a knot K is the closure of $b = \sigma_1^w \sigma_2^\delta \sigma_1^u \sigma_2^v$, then the closure of V_K contains a curve of rank 3 augmentations.*

The paper is organized as follows. In Section 2 we review the background on knot contact homology in our setting, particularly the cord algebra and KCH representations. Section 3 is dedicated to determining KCH representations from augmentations and the proof of Theorem 1.2. In Section 4 we discuss how to build an augmentation from basic data and how augmentation rank behaves under connect sum, proving Theorem 1.9. Finally, Section 5 studies augmentations (particularly those of highest possible rank) from the view of a braid closure. In this section Theorem 1.10 and Theorem 1.13 are proved.

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2. Background

We begin by reviewing the definition of the cord algebra $\mathcal{C}_K$ introduced in [19]. We also discuss two alternate constructions of the cord algebra.

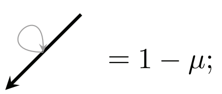
The first construction (Section 2.2) is the (degree zero) framed knot contact homology defined in [19], which will be needed in Section 5. This version is the degree zero homology of the ($U = 1$) combinatorial knot DGA, which is a computation of the Legendrian DGA of the conormal lift of K to the unit cotangent bundle [10]. The conventions we use in the definition here match those from [21]. We use the notation $HC_0(K)|_{U=1}$ to highlight when we work with the combinatorial knot DGA construction.

A second construction we review (Section 2.3) works with the set of elements in π_K and is our starting point for the correspondence in (1.1). In the current section and those that follow, we write $\mathcal{P}_K$ to refer to this incarnation of the cord algebra.

2.1. The cord algebra

Let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. Given a knot $K \subset S^3$ with a basepoint $*$ on K, define a *cord* of $(K, *)$ to be a path $\gamma : [0, 1] \to S^3$ such that $\gamma^{-1}(K) = \{0, 1\}$ and $* \notin \gamma([0, 1])$.

Definition 2.1. Consider the noncommutative unital algebra over R_0 freely generated by homotopy classes of cords of $(K, *)$ for some choice of basepoint. (Here, homotopy of cords allows endpoints to move, but not past the basepoint.) The *cord algebra* $\mathcal{C}_K$ is the quotient of this algebra by the ideal generated by relations

(1)


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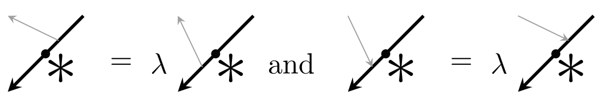

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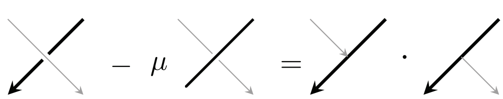

In the above definition, relations are between any cords that differ only locally, as shown. The knot K is depicted more thickly. Also the figures are understood to be 3-dimensional, rather than depicting relations on planar diagrams.

2.2. Framed knot contact homology

We review the construction of $HC_0(K)|_{U=1}$ from the combinatorial knot DGA viewpoint. We content ourselves with only defining the algebra that arises as the degree zero part of the knot DGA. For more details see [21].

Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by $n(n-1)$ elements a_{ij}, $1 \leq i \neq j \leq n$. Let B_n denote the braid group on n strands. If σ_k is one of the standard generators (twisting strands in a right-handed manner), then define $\phi : B_n \to \text{Aut } \mathcal{A}_n$ by defining it on each generator as

$$\phi_{\sigma_k} : \begin{cases} a_{ij} \mapsto a_{ij}, & i,j \neq k, k+1 \\ a_{k+1,i} \mapsto a_{ki}, & i \neq k, k+1 \\ a_{i,k+1} \mapsto a_{ik}, & i \neq k, k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k}, & \\ a_{k+1,k} \mapsto -a_{k,k+1}, & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}a_{ki} & i \neq k, k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}a_{k,k+1} & i \neq k, k+1 \end{cases}$$

Include $\iota : B_n \hookrightarrow B_{n+1}$ so that the $(n+1)^{st}$ strand does not interact, and for $B \in B_n$ let $\phi^*_B = \phi_{\iota(B)} \in \text{Aut } \mathcal{A}_{n+1}$. Define matrices $\Phi^L_B, \Phi^R_B \in \text{Mat}_{n\times n}(\mathcal{A}_n)$ by

$$\phi^*_B(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi^L_B)_{ij} a_{j,n+1}, \qquad \phi^*_B(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi^R_B)_{ji}.$$

Define an involution $x \mapsto \overline{x}$ on $\mathcal{A}_n$ as follows: first $\overline{a_{ij}} = a_{ji}$; then, for any $x, y \in \mathcal{A}_n$, $\overline{xy} = \overline{y}\,\overline{x}$ and extend the operation linearly to $\mathcal{A}_n$.

Proposition 2.2 ([17], Prop. 6.2). *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $B \in B_n$, Φ^R_B is the transpose of $\overline{\Phi^L_B}$.*

Finally, define a matrix $\mathbf{A}$ by

$$\mathbf{A}_{ij} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1-\mu & i = j \end{cases}, \tag{2.1}$$

and, given $B \in B_n$, the diagonal matrix $\mathbf{\Lambda} = \mathrm{diag}[\lambda\mu^w, 1, \ldots, 1]$ where w is the writhe (algebraic length) of B. Extend the map ϕ_B to $\mathcal{A}_n \otimes R_0$ so that it fixes λ, μ.

Definition 2.3. Let $K = \widehat{B}$ be the (braid) closure of $B \in B_n$ and let $\mathcal{I}_B$ be the ideal in $\mathcal{A}_n \otimes R_0$ generated by entries in the matrices $\mathbf{A} - \mathbf{\Lambda} \cdot \phi_B(\mathbf{A}) \cdot \mathbf{\Lambda}^{-1}, \mathbf{A} - \mathbf{\Lambda} \cdot \Phi_B^L \cdot \mathbf{A}$, and $\mathbf{A} - \mathbf{A} \cdot \Phi_B^R \cdot \mathbf{\Lambda}^{-1}$. The algebra $(\mathcal{A}_n \otimes R_0)/\mathcal{I}_B$ is the degree zero homology of the combinatorial knot DGA, denoted $HC_0(K)|_{U=1}$.

In Definition 2.3 (and throughout the paper), given a homomorphism f defined on the entries of a matrix M we use $f(M)$ for the matrix obtained by applying f to the entries. The following was proved in [19] (see also [21, §3, 4]).

Theorem 2.4. *There is an isomorphism $F_{HC} : \mathcal{C}_K \to HC_0(K)|_{U=1}$ of R_0-algebras.*

For the discussion in Section 5 we need to define F_{HC} on generators of $\mathcal{C}_K$. We view the braid B as horizontal with strands oriented to the right and numbered to be increasing from top to bottom. Consider a flat disk D, to the right of the braid, with n punctures where it intersects $K = \widehat{B}$ (see Figure 1). We assume that the n punctures of D are collinear, on a line that separates D into upper and lower half-disks. Denote by c_{ij}, a cord of $(K, *)$ that is contained in the upper half-disk of D, and has initial endpoint on the i^{th} strand and terminal endpoint on the j^{th} strand. The cord algebra $\mathcal{C}_K$ is generated by the homotopy classes of the set $\{c_{ij}, 1 \leq i \neq j \leq n\}$ and F_{HC} is defined by $F_{HC}(c_{ij}) = \mathbf{A}_{ij}$ (as in (2.1)).

To understand ϕ_B from this perspective, view c_{ij} as a path in D. Considering B as a mapping class of the punctured disk D, let $B \cdot c_{ij}$ denote the isotopy class (fixing endpoints) of the path to which c_{ij} is sent. Viewing D from the left (the side from which the strands of B point towards D), σ_k acts by rotating the k- and $(k+1)$-punctures an angle of π about their midpoint in counter-clockwise fashion.

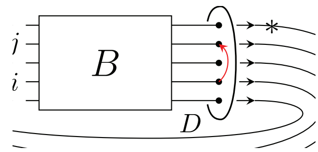

Figure 1: Cord c_{ij} of $K = \widehat{B}$.

Following [18, Section 2], consider the set $P(D)$ of isotopy classes of embedded (oriented) paths in D with endpoints on distinct punctures. There is a unique map $\psi : P(D) \to \mathcal{A}_n$ which satisfies $\psi(c_{ij}) = a_{ij}$ if $i < j$, $\psi(c_{ij}) = -a_{ij}$ if $i > j$, and such that $\psi(B \cdot c_{ij}) = \phi_B(\psi(c_{ij}))$ for any $B \in B_n$. In addition, given representative paths of elements of $P(D)$ which differ only near a puncture as depicted in Figure 2, the relation in Figure 2 is satisfied by the corresponding images under ψ. In Section 5 we use this characterization of ϕ_B to justify some calculations of the matrix $\phi_B(\mathbf{A})$.

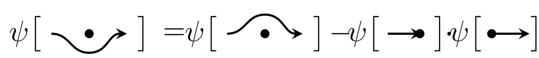

Figure 2: Relation in the image of ψ

To see that F_{HC} produces an isomorphism one must use a framed version of ψ, incorporating μ into the map (see [19, §3.2] for details).

We also require the following results, originally proved in [17]. Following the terminology from that paper, we refer to Theorem 2.5 as the Chain Rule.

Theorem 2.5. *Let B, B' be braids in B_n. Then $\Phi^L_{BB'} = \phi_B(\Phi^L_{B'}) \cdot \Phi^L_B$ and $\Phi^R_{BB'} = \Phi^R_B \cdot \phi_B(\Phi^R_{B'})$.*

Corollary 2.6. *The matrices $\phi_B(\Phi^L_{B^{-1}})$ and $\phi_B(\Phi^R_{B^{-1}})$ are the inverse of Φ^L_B and Φ^R_B, respectively.*

Theorem 2.7. *Let $\mathbf{A}$ be the matrix defined in (2.1). Then for any $B \in B_n$,*

$$\phi_B(\mathbf{A}) = \Phi^L_B \cdot \mathbf{A} \cdot \Phi^R_B.$$

Theorem 2.7 implies that the ideal $\mathcal{I}_B$ used to define $HC_0(K)|_{U=1}$ is generated by entries in $\mathbf{A} - \mathbf{\Lambda} \cdot \Phi^L_B \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \Phi^R_B \cdot \mathbf{\Lambda}^{-1}$ only.

2.3. Augmentations from KCH representations

We now review the algebra $\mathcal{P}_K$, which is defined with elements of π_K, and describe the correspondence in (1.1).

Definition 2.8. Let P_K denote the underlying set of the knot group π_K, where we write $[\gamma] \in P_K$ for $\gamma \in \pi_K$. In π_K let e be the identity, m a choice of meridian, and ℓ the preferred longitude of K. Define $\mathcal{P}_K$ to be the non-commutative unital algebra freely generated over R_0 by P_K modulo the relations:

1) $[e] = 1 - \mu$;
2) $[m\gamma] = \mu[\gamma], [\gamma m] = [\gamma]\mu$ and $[\ell\gamma] = \lambda[\gamma], [\gamma\ell] = [\gamma]\lambda$, for any $\gamma \in \pi_K$;
3) $[\gamma_1\gamma_2] - [\gamma_1 m\gamma_2] = [\gamma_1][\gamma_2]$ for any $\gamma_1, \gamma_2 \in \pi_K$.

Theorem 2.9 ([19]). *$\mathcal{C}_K$ and $\mathcal{P}_K$ are isomorphic as R_0-algebras.*

The isomorphism of the theorem $F_{\mathcal{P}} : \mathcal{P}_K \to \mathcal{C}_K$ may be defined as follows. Suppose the basepoint x for the group π_K is on the boundary torus. Choose a fixed path p from a point on K to x with interior in the tubular neighborhood $n(K)$. Let $\overline{p}$ denote p with reversed orientation. If $g \in \pi_K$ is represented by a loop γ, define $F_{\mathcal{P}}([g])$ to be $\mu^{\mathrm{lk}(\gamma,K)}$ times the cord given by the concatenation $p\gamma\overline{p}$ (here $\mathrm{lk}(\gamma, K)$ is the linking number of γ and K). We use this definition in Section 5.

This identification of $\mathcal{C}_K$ with $\mathcal{P}_K$ uses a basepoint for π_K, as would be expected by the choice of m in the definition of $\mathcal{P}_K$. The oriented boundary of a meridian disk of $n(K)$ that contains x in its boundary is a representative of m.

Remark 2.10. That $\mathcal{P}_K$ is defined to be an algebra over R_0 along with (2) implies the relations $[m\gamma] = [\gamma m]$ and $[\ell\gamma] = [\gamma\ell]$ for any $\gamma \in \pi_K$.

Alternatively, construct the unital algebra $\widetilde{\mathcal{P}}_K$ freely generated over $\mathbb{Z}$ by $P_K \cup \{\lambda^{\pm 1}, \mu^{\pm 1}\}$, modulo the relations (1), (2), (3), and the relation $\lambda\mu = \mu\lambda$. Then $\widetilde{\mathcal{P}}_K$ is isomorphic to the degree zero homology of the *fully noncommutative knot DGA* (see the discussion in [21]).

Working with $\widetilde{\mathcal{P}}_K$, an analogue of Theorem 2.9 has been found in work of K. Cieliebak, T. Ekholm, J. Latschev, and L. Ng.

Theorem 2.11 ([5]). *There is an injective ring homomorphism $\widetilde{\mathcal{P}}_K \hookrightarrow \mathbb{Z}[\pi_K]$ with image generated by the peripheral subgroup $\langle \ell, m\rangle \subset \pi_K$ and elements of the form $\gamma - m\gamma$ where $\gamma \in \pi_K$.*

2.4. Geometric view of augmentations

Here we provide a rough description of a geometric source of augmentations. For more details see [1, Section 6].

Given a Legendrian Λ in a contact manifold Y, and under some conditions on the pair (Y, Λ) (see e.g. [11]), there is an associated Legendrian (or Chekanov-Eliashberg) DGA $(\mathcal{A}(\Lambda), \partial(\Lambda))$ which, with an appropriate notion of equivalence, is invariant under Legendrian isotopy [11].

As alluded to at the start of Section 2 the definition of $HC_0(K)|_{U=1}$ comes from a computation of $(\mathcal{A}(\Lambda_K), \partial(\Lambda_K))$, where Λ_K is the unit conormal lift of K, which is a Legendrian in the standard contact structure on the unit cotangent bundle $ST^*\mathbb{R}^3$. We remark that this does fall into the setting of [11], since $ST^*\mathbb{R}^3$ is contactomorphic to the 1-jet bundle $J^1(S^2)$. In the following, we wish to work with the *fully noncommutative* version of the Legendrian DGA, denoted $\widetilde{\mathcal{A}}(\Lambda_K)$ (cf. [10, Remark 2.2]).

The DGA construction produces a contravariant functor from the category of Legendrians and exact Lagrangian cobordisms to the category of differential graded algebras. In particular, an exact Lagrangian *filling* L – a cobordism from the empty set – of a Legendrian Λ induces a DGA map from $\widetilde{\mathcal{A}}(\Lambda)$ to the ground ring, which is identified with the DGA of the empty set, with zero differential. This induced map is a chain map, hence it is an augmentation.

In the symplectization of $ST^*\mathbb{R}^3$, Λ_K admits an exact Lagrangian filling M_K with the topology of the knot complement. While the augmentation induced from M_K has little information, one can keep track of the homotopy class in $\pi_1(M_K) = \pi_K$ of the boundary of rigid holomorphic disks and obtain a homomorphism $\Phi : \widetilde{\mathcal{A}}(\Lambda_K) \to \mathbb{Z}[\pi_K]$ such that $\Phi \circ \partial_K = 0$. Consideration of 1-parameter families of holomorphic disks shows the image of Φ is the subring of $\mathbb{Z}[\pi_K]$ indicated in Theorem 2.11. It is conjectured that Φ induces an isomorphism on zero-graded homology, which would give a symplecto-geometric source for Theorem 2.11 (this is not the approach taken by Cieliebak, Ekholm, Latschev, and Ng).

Let $\rho : \pi_K \to \mathrm{GL}(V)$ be a KCH representation, μ_0 the eigenvalue of $\rho(m)$ not equal to 1. The longitude ℓ commutes with m, hence the μ_0-eigenspace of $\rho(m)$ is preserved by $\rho(\ell)$. Extending ρ to $\mathbb{Z}[\pi_K]$, the definition of a KCH representation implies that $\rho(m), \rho(\ell)$, and $\rho(\gamma - m\gamma)$ (for any $\gamma \in \pi_K$) are each a linear map preserving the 1-dimensional μ_0-eigenspace, and so each restricted to that eigenspace corresponds to multiplication by an element of $\mathbb{F}$. This lets us assign a scalar to each element in the image of Φ. Identifying that image with $\widetilde{\mathcal{P}}_K$ (via Theorem 2.11), this assignment agrees with the definition of ϵ_ρ presented in Proposition 2.12. Any augmentation induced from a KCH representation thus arises from a flat connection on M_K. Theorem 1.2 says that all augmentations with $\epsilon(\mu) \neq 1$ arise in this way.

2.5. The augmentation induced from a KCH representation

Let S be a ring with 1. Generalize Definition 1.1 by letting V be a right S-module. In this context, we say $\rho : \pi_K \to \mathrm{Aut}_S(V)$ is a KCH representation if there is μ_0 in S such that $1 - \mu_0$ is invertible and there is a generating set $\{e_1, \ldots, e_r\}$ for V such that $\rho(m)e_1 = e_1\mu_0$ and $\rho(m)e_i = e_i$ for $2 \leq i \leq r$. As in the introduction, there is a λ_0 such that $\rho(\ell)e_1 = e_1\lambda_0$ since $\rho(m)$ commutes with $\rho(\ell)$; also μ_0, λ_0 are units as $\rho(m)$ and $\rho(\ell)$ are invertible.

Proposition 2.12. *If $\rho : \pi_K \to Aut_S(V)$ is a KCH representation with $Ann_S(e_1) = \{0\}$, then there is an induced augmentation $\epsilon_\rho : \widetilde{\mathcal{P}}_K \to S$ with $\epsilon_\rho(\mu) = \mu_0$ and $\epsilon_\rho(\lambda) = \lambda_0$.*

Proof. If V is free then the proof of Theorem 2.12 is the same as that which is outlined for $S = \mathbb{C}$ in [21]. In this case, one chooses a basis of V from the set $\{e_1, \ldots, e_r\}$ (which contains e_1 by force). Define a bilinear form so that $\langle e_i, e_j \rangle = \delta_{ij}$ on this basis (where δ_{ij} is the Kronecker-delta). Then ϵ_ρ is defined by setting $\epsilon_\rho(\mu) = \mu_0$, $\epsilon_\rho(\lambda) = \lambda_0$, and $\epsilon_\rho([\gamma]) = (1 - \mu_0)\langle \rho(\gamma)e_1, e_1 \rangle$ for $\gamma \in \pi_K$. The map $\epsilon_\rho : \widetilde{\mathcal{P}}_K \to S$ is then determined. That V is a right module is relevant to ϵ_ρ being well-defined. For example, if $\langle \rho(\gamma)e_1, e_1 \rangle = s$ then $\langle \rho(m\gamma)e_1, e_1 \rangle = \mu_0 s$ uses the right action of S.

The definition of ϵ_ρ above is equally valid when V is not free, despite $\langle v, w \rangle$ not being well-defined for general $w \in V$. For suppose that $\sum_{k=1}^r e_k b_k = v = \sum_{k=1}^r e_k c_k$ for elements $b_k, c_k \in S$, $k = 1, \ldots r$. Then

$$0 = \sum_{k=1}^{r} e_k(b_k - c_k) - \rho(m) \sum_{k=1}^{r} e_k(b_k - c_k) = e_1(1 - \mu_0)(b_1 - c_1).$$

As $\mathrm{Ann}_S(e_1) = \{0\}$ and $1 - \mu_0$ is invertible, $b_1 = c_1$. □

Remark 2.13. Given another KCH representation $\rho' : \pi_K \to \mathrm{Aut}_S(V')$ and a linear isomorphism $\varphi : V' \to V$ such that $\varphi \circ \rho'(g) = \rho(g) \circ \varphi$ for all $g \in \pi_K$, the vectors $e_i' = \varphi^{-1}(e_i)$, $i = 1, \ldots, r$, are eigenvectors of $\rho'(m)$ (with the same eigenvalue as e_i). Noting how the bilinear form we used depends on these eigenvectors, this implies that $\epsilon_\rho = \epsilon_{\rho'}$.

Remark 2.14. There is a different version of Proposition 2.12 for representations of π_K that generalize KCH representations. Suppose that there is a basis of an $\mathbb{F}$-vector space V and $\rho : \pi_K \to \mathrm{GL}(V)$ with which $\rho(m) = \begin{pmatrix} M_0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$ for a $k \times k$ invertible matrix M_0 with $\mathrm{Id}_k - M_0$ also invertible. Let $W \subset V$ be the subspace spanned by the first k basis vectors. Then ρ induces an augmentation $\epsilon : \widetilde{\mathcal{P}}_K \to \mathrm{Mat}_k(\mathbb{F})$ by setting $\epsilon(\mu) = M_0$ and $\epsilon([g]) = (\mathrm{Id}_W - M_0)\mathrm{Proj}_W \rho(g)$.

3. KCH representations and augmentations

In this section, after some inital remarks, we prove Theorem 1.2 by constructing a representation from a certain universal augmentation. The construction is the content of Theorem 3.5 in Section 3.3. In Section 3.4 we then restrict to the case that the target of our augmentation is a field and address the irreducibility of inducing KCH representations. The proof of Theorem 1.2 appears at the end of Section 3.4.

Fix a meridian m of K. Consider a set $\Gamma = \{\gamma_1, \dots, \gamma_r\} \subset \pi_K$, with γ_1 the identity, such that $\mathcal{G} = \left\{g_i \mid g_i = \gamma_i^{-1} m \gamma_i, 1 \le i \le r\right\}$ generates π_K.

Note from Theorem 2.9 that any augmentation $\epsilon : \widetilde{\mathcal{P}}_K \to S$ has values that, for any $g, h \in \pi_K$, satisfy

$$\text{(3.1)} \qquad \begin{aligned} &\epsilon([e]) = 1 - \epsilon(\mu), \quad \epsilon([mg]) = \epsilon(\mu[g]), \epsilon([gm]) = \epsilon([g]\mu), \\ &\qquad \text{and } \epsilon([gmh]) = \epsilon([gh]) - \epsilon([g])\epsilon([h]). \end{aligned}$$

3.1. The universal augmentation of K

Define $E = \{[e]^n \mid n \ge 0\}$ and let $\mathcal{Q}_K$ be the localization $E^{-1}\mathcal{P}_K$. In $\mathcal{P}_K$ the set E satisfies Ore's condition, hence the localization homomorphism is injective and $\mathcal{Q}_K$ is isomorphic to a ring of fractions. We may also define $\widetilde{\mathcal{Q}}_K$, the localization of $\widetilde{\mathcal{P}}_K$ with respect to E, though E does not satisfy Ore's condition in $\widetilde{\mathcal{P}}_K$; for example, see [6]. Though we cannot consider $\widetilde{\mathcal{Q}}_K$ as a ring of fractions containing $\widetilde{\mathcal{P}}_K$, the localization $\iota : \widetilde{\mathcal{P}}_K \to \widetilde{\mathcal{Q}}_K$ has the expected universal property, that if $f : \widetilde{\mathcal{P}}_K \to S$ is an E-inverting homomorphism then there is a unique $g : \widetilde{\mathcal{Q}}_K \to S$ such that $g \circ \iota = f$.

Definition 3.1. The *universal augmentation of* K is the map $\iota : \widetilde{\mathcal{P}}_K \to \widetilde{\mathcal{Q}}_K$, which may not be injective.

The augmentations we consider all factor through the universal augmentation; further, if $\epsilon(\mu), \epsilon(\lambda)$ are central then they define a map on $\mathcal{P}_K$ and factor through $\mathcal{P}_K \hookrightarrow \mathcal{Q}_K$. We will abuse notation, writing $[g]$ for the image $\iota([g])$ in $\widetilde{\mathcal{Q}}_K$.

3.2. Notation and setup

Consider the direct sum $\oplus^r \widetilde{\mathcal{Q}}_K$ as a right $\widetilde{\mathcal{Q}}_K$-module. Write $[\Gamma g]$ for the element $([\gamma_1 g], \ldots, [\gamma_r g])$ in $\oplus^r \widetilde{\mathcal{Q}}_K$. Also, for $1 \le j \le r$, we define $v_j = [\Gamma \gamma_j^{-1}]$ and let V be the right submodule over $\widetilde{\mathcal{Q}}_K$ generated by $\{v_1, v_2, \ldots, v_r\}$.

We need some preparatory lemmas to prove Theorem 3.5, which shows that the universal augmentation is induced from a representation.

Lemma 3.2. *For $h, h' \in \pi_K$, if $g = hh'$ then there are elements $c_1, \ldots, c_r \in \widetilde{\mathcal{P}}_K$ such that $[\Gamma g] = \sum_{k=1}^r [\Gamma h \gamma_k^{-1}] c_k$. Thus $[\Gamma g] \in V$.*

Proof. As h' is a product of elements in $\mathcal{G}$, we write $h' = g_{i_1}^{\varepsilon_1} \ldots g_{i_l}^{\varepsilon_l}$, with $\varepsilon_k = \pm 1$ for $1 \le k \le l$. In $\widetilde{\mathcal{P}}_K$, $[a\gamma_{i_k}^{-1} m^{\varepsilon_k} \gamma_{i_k} b] = [ab] - \varepsilon_k [a\gamma_{i_k}^{-1}][m^{\frac{\varepsilon_k - 1}{2}} \gamma_{i_k} b]$ for any $a, b \in \pi_K$ and $1 \le k \le l$. Hence

$$[\Gamma h h'] = [\Gamma h \gamma_1^{-1}] + \sum_{k=1}^{l} -\varepsilon_k [\Gamma h \gamma_{i_k}^{-1}]([m^{\frac{\varepsilon_k - 1}{2}} \gamma_{i_k} w(k)])$$

where $w(k) = g_{i_{k+1}}^{\varepsilon_{k+1}} \ldots g_{i_l}^{\varepsilon_l}$. Taking $h = e$ and using ι gives that $[\Gamma g] \in V$. □

Remark 3.3. The elements $c_1, \ldots, c_r$ are chosen independently of h.

Lemma 3.4. *Suppose that $c_1, \ldots, c_r \in \widetilde{\mathcal{Q}}_K$ are such that $\sum_{k=1}^r v_k c_k = 0$. If $g \in \pi_K$ then $\sum_{k=1}^r [\Gamma g \gamma_k^{-1}] c_k = 0$.*

Proof. The statement trivially holds if g is the identity since $v_k = [\Gamma \gamma_k^{-1}]$. Let $g = g_i^{\varepsilon} g'$ for some $g_i \in \mathcal{G}$ and $\varepsilon = \pm 1$. Letting $\delta = (\varepsilon - 1)/2$, if we suppose that the statement holds for g' then

$$\sum_{k=1}^{r} [\Gamma g \gamma_k^{-1}] c_k = \sum_{k=1}^{r} [\Gamma g' \gamma_k^{-1}] c_k - \varepsilon [\Gamma \gamma_i^{-1}] \mu^{\delta} \sum_{k=1}^{r} [\gamma_i g' \gamma_k^{-1}] c_k = 0,$$

as $\sum [\gamma_i g' \gamma_k^{-1}] c_k$ is the i coordinate of $\sum [\Gamma g' \gamma_k^{-1}] c_k = 0$. □

3.3. KCH representations from augmentations

In this section we show that the universal augmentation is induced from a KCH representation, from which it will follow that the same is true of any augmentation sending $1-\mu$ to a unit.

Theorem 3.5. *There is a well-defined KCH representation* $\rho_\iota : \pi_K \to Aut_{\widetilde{\mathcal{Q}}_K}(V)$ *that induces the universal augmentation* $\iota : \widetilde{\mathcal{P}}_K \to \widetilde{\mathcal{Q}}_K$.

Remark 3.6. The analogous statement for $\mathcal{P}_K \hookrightarrow \mathcal{Q}_K$ also holds.

Proof. Given $g \in \pi_K$ define $\rho_\iota(g)v_j = [\Gamma g\gamma_j^{-1}]$, for each $1 \le j \le r$, which is an element of V by Lemma 3.2. Extend $\rho_\iota(g)$ to a $\widetilde{\mathcal{Q}}_K$-linear map. By Lemma 3.4, this determines a well-defined map on V.

We show below that $\rho_\iota(gh) = \rho_\iota(g)\rho_\iota(h)$, and it follows that $\rho_\iota(g)$ is invertible and $\rho_\iota : \pi_K \to \mathrm{Aut}_{\widetilde{\mathcal{Q}}_K}(V)$ is a well-defined homomorphism.

Fix $1 \le j \le r$. Choose any $h, h'' \in \pi_K$ and set $h' = h''\gamma_j^{-1}$. Using the calculation from Lemma 3.2 we write $[\Gamma hh']$ as a sum $\sum_{k=1}^r [\Gamma h\gamma_k^{-1}]c_k$. By our definitions this implies

$$\rho_\iota(hh'')v_j = [\Gamma hh'] = \sum_{k=1}^r [\Gamma h\gamma_k^{-1}]c_k = \rho_\iota(h)\left(\sum_{k=1}^r v_k c_k\right).$$

By Remark 3.3 the c_k are independent of h. Hence we may set $h = e$ in the above equation and obtain $\sum_{k=1}^r v_k c_k = [\Gamma h'] = \rho_\iota(h'')v_j$. Hence $\rho_\iota(hh'') = \rho_\iota(h)\rho_\iota(h'')$, showing ρ_ι is a homomorphism.

Recall that $\gamma_1 = e$ which makes $g_1 = m$. To see that ρ_ι is a KCH representation we find a generating set $\{e_1, \ldots, e_r\}$ for V as discussed in Section 2.3.

Set $e_1 = v_1$ and $e_j = v_j - v_1(1-\mu)^{-1}[\gamma_j^{-1}]$ for $j = 2, \ldots, r$. One finds that $\rho_\iota(m)e_1 = e_1\mu$ and $\rho_\iota(m)e_j = e_j$ for $j = 2, \ldots, r$, as $\rho_\iota(m)v_j = [\Gamma g_1\gamma_j^{-1}] = v_j - v_1[\gamma_j^{-1}]$ by (3) in Theorem 2.9.

To determine the induced augmentation (which by Proposition 2.12 exists since the first coordinate of e_1 is invertible), for given $g \in \pi_K$ we choose $c_1, \ldots, c_r \in \mathcal{P}_K$ as in Lemma 3.2, with $h = e$, so that

$$\begin{aligned}\rho_\iota(g)e_1 = [\Gamma g] = \sum_{k=1}^r v_k c_k &= e_1\left(c_1 + (1-\mu)^{-1}\sum_{k=2}^r [\gamma_k^{-1}]c_k\right) + \sum_{k=2}^r e_k c_k \\ &= e_1(1-\mu)^{-1}\sum_{k=1}^r [\gamma_k^{-1}]c_k + \sum_{k=2}^r e_k c_k,\end{aligned}$$

the last equality since $[\gamma_1^{-1}] = [e] = 1 - \mu$. Now, the first coordinate of $[\Gamma g] \in V$ is $[g]$ and the first coordinate of $\sum_{k=1}^r v_k c_k$ is $\sum_{k=1}^r [\gamma_k^{-1}] c_k$. Thus, $\rho_\iota(g) e_1 = e_1(1-\mu)^{-1}[g] + \sum_{k=2}^r e_k c_k$. Hence $(1-\mu)\langle \rho_\iota(g)e_1, e_1\rangle = [g] \in \widetilde{\mathcal{Q}}_K$ and the induced augmentation is $\iota : \widetilde{\mathcal{P}}_K \to \widetilde{\mathcal{Q}}_K$. □

Corollary 3.7. *Let $\epsilon : \widetilde{\mathcal{P}}_K \to S$ (or, alternatively $\epsilon : \mathcal{P}_K \to S$) be an augmentation with $\epsilon(1-\mu)$ invertible in S. Then there is a KCH representation $\rho : \pi_K \to Aut_S(W)$ for some right S-module W, such that $\epsilon_\rho = \epsilon$.*

Proof. There is a unique homomorphism $\epsilon' : \widetilde{\mathcal{Q}}_K \to S$ such that $\epsilon' \circ \iota = \epsilon$. Let V be as in Theorem 3.5 and define W to be generated over S by vectors $w_j = \epsilon'(v_j)$ (here we apply ϵ' to each coordinate). The map $\rho(g)$ is defined by setting $\rho(g)w_j = \epsilon'(\rho_\iota(g)v_j)$. From Theorem 3.5 it follows that ρ is well-defined and $\epsilon_\rho([g]) = \epsilon([g])$. □

Remark 3.8. Note that $w_j = \rho(\gamma_j^{-1})w_1$ for each $j = 1, \ldots, r$.

3.4. KCH representations on vector spaces and the meridian subspace

We restrict our attention to the case $S = \mathbb{F}$ is a field. Let $\rho : \pi_K \to \mathrm{GL}(V)$ be any KCH representation, $\dim V = d$. Take a basis for V of eigenvectors $e_1, \ldots, e_d$ of $\rho(m)$ such that $\rho(m)e_1 = \mu_0 e_1$.

Definition 3.9. Define $w_j = \rho(\gamma_j^{-1})e_1$ for each $1 \le j \le r$. Define $W_\rho(\Gamma) = \mathbb{F}\langle w_1, w_2, \ldots, w_r\rangle$ to be the *meridian subspace* of V.

Lemma 3.10. *For $1 \le i \le r$, the vector w_i satisfies $\rho(g_i)w_i = \mu_0 w_i$ and $W_\rho(\Gamma)$ is an invariant subspace.*

Proof. For each $1 \le i \le r$ we have

$$\rho(g_i)w_i = \rho(\gamma_i^{-1}m)e_1 = \mu_0 w_i.$$

By definition of w_j, we find that $w_j - \rho(g_i)w_j = \rho(\gamma_i^{-1})(\mathrm{Id}_V - \rho(m))\rho(\gamma_i\gamma_j^{-1})e_1$ for each $1 \le j \le r$. In addition, if $w = \sum_{k=1}^d c_k e_k$ then $(\mathrm{Id}_V - \rho(m))w =$

$(1-\mu_0)c_1e_1$. Taking $w = \rho(\gamma_i\gamma_j^{-1})e_1$, this indicates the equality

$$w_j - \rho(g_i)w_j = (1-\mu_0)\langle\rho(\gamma_i\gamma_j^{-1})e_1, e_1\rangle w_i = \epsilon_\rho([\gamma_i\gamma_j^{-1}])w_i. \tag{3.2}$$

where, as in Section 2.3, $\langle\cdot,\cdot\rangle$ is the bilinear form on V given by the Kronecker-delta $\langle e_i, e_j\rangle = \delta_{ij}$. This proves the lemma since $\mathcal{G}$ generates π_K. □

We remark that equation (3.2) will be important in Lemma 3.14 below. The following lemma was shown for $\mathbb{F} = \mathbb{C}$ in [7, §3.2]. The proof given there carries over to our setting.

Lemma 3.11. *If $\rho : \pi_K \to GL(V)$ is a KCH representation and $W \subset V$ a subspace on which the action of ρ is the identity, then the quotient representation $\overline{\rho} : \pi_K \to GL(V/W)$ is a KCH representation and $\epsilon_\rho = \epsilon_{\overline{\rho}}$.*

Lemma 3.12. *Let $W \subseteq V$ be an invariant subspace of $\rho : \pi_K \to GL(V)$. Then either the action of ρ restricted to W is the identity, or $W_\rho(\Gamma) \subset W$.*

Proof. Given $x \in W$ we have that $(1-\mu_0)\langle x, e_1\rangle e_1 = (\mathrm{Id}_V - \rho(m))x \in W$. Thus, as $1-\mu_0$ is a unit, either $e_1 \in W$ or $\langle x, e_1\rangle = 0$ for every $x \in W$.

If $e_1 \in W$, then $w_i = \rho(\gamma_i^{-1})e_1 \in W$ for $i = 1, \ldots, r$. This implies $W_\rho(\Gamma) \subset W$.

Alternatively it must be that $\rho(m)x = x$ for every $x \in W$. But then, for any $g \in \pi_K$ and any $x \in W$, we have $\rho(g^{-1}mg)x = x$ since $\rho(g)x \in W$. As π_K is generated by conjugates of m, the action of ρ on W is the identity. □

Given an augmentation $\epsilon : \mathcal{C}_K \to \mathbb{F}$ that is induced from a KCH representation $\rho : \pi_K \to \mathrm{GL}(V)$, Lemma 3.11 states that if the action on $W \subset V$ is the identity then the quotient representation induces the same augmentation. Taking such a quotient sufficiently many times gives a KCH representation $\rho' : \pi_K \to \mathrm{GL}(V')$ such that V' has no such subspace. Restricting to the meridian subspace $W_{\rho'}(\Gamma)$, and applying Lemmas 3.12 and 3.10, we obtain the following corollary.

Corollary 3.13. *If $\epsilon : \mathcal{C}_K \to \mathbb{F}$ is induced from a KCH representation, it is induced from a KCH irrep on the meridian subspace of some KCH representation.*

Now for an augmentation $\epsilon : \mathcal{C}_K \to \mathbb{F}$, define $\mathcal{E}(\Gamma)$ to be the $r \times r$ matrix over $\mathbb{F}$ having columns $\epsilon([\Gamma\gamma_j^{-1}])$ (note that these are the vectors w_j from the KCH representation defined in Section 3.3).

Lemma 3.14. *Let $\rho : \pi_K \to GL(V)$ be any KCH representation that induces ϵ and suppose V has no proper subspace where the action of ρ is the identity. Then*

1) *the rank of $\mathcal{E}(\Gamma)$ equals* $\dim W_\rho(\Gamma)$*;*
2) *the dimension of any KCH irrep inducing ϵ equals* $\dim W_\rho(\Gamma)$*.*

Proof. As previously we write $w_j = \rho(\gamma_j^{-1})e_1$ for $j = 1, \dots, r$, and the vectors $w_1, \dots, w_r$ generate $W_\rho(\Gamma)$. For $1 \le j \le r$ let $c_j \in \mathbb{F}$ be such that $\sum_j c_j w_j = 0$. From equation (3.2) we see that for each $g_i \in \mathcal{G}$

$$0 = (Id - \rho(g_i)) \sum_{j=1}^{r} c_j w_j = \sum_{j=1}^{r} \epsilon([\gamma_i \gamma_j^{-1}]) c_j w_i.$$

As no w_i is zero, it must be that $\sum_j \epsilon([\gamma_i \gamma_j^{-1}]) c_j = 0$ for each i. This implies that $\dim W_\rho(\Gamma)$ is at least the rank of $\mathcal{E}(\Gamma)$.

In addition, if there are scalars $c_j, 1 \le j \le r$ such that $\sum_j \epsilon([\gamma_i \gamma_j^{-1}]) c_j = 0$ for each i, then the same equality implies that $\rho(g_i) \sum_j c_j w_j = \sum_j c_j w_j$ for each i. Then $\sum_j c_j w_j \in W_\rho(\Gamma)$ is fixed by π_K and so $\sum c_j w_j = 0$ by hypothesis. Hence $\dim W_\rho(\Gamma) \le$ rank of $\mathcal{E}(\Gamma)$, so they are equal.

To see (2) holds, if $\sigma : \pi_K \to \mathrm{GL}(X)$ is any KCH irrep inducing ϵ then $W_\sigma(\Gamma)$, which is nonzero and invariant, is X. As no proper subspace of X is acted upon trivially, $\dim X = \mathrm{rank}(\mathcal{E}(\Gamma)) = \dim W_\rho(\Gamma)$ by (1). □

We can now prove our main theorem.

Theorem 1.2. *Let $\epsilon : \mathcal{C}_K \to \mathbb{F}$ be an augmentation such that $\epsilon(\mu) \ne 1$. Then a KCH irrep $\rho : \pi_K \to GL(V)$ can be constructed explicitly from ϵ with the property that $\epsilon_\rho = \epsilon$. Moreover, any KCH irrep that induces ϵ is isomorphic to (V, ρ).*

Proof of Theorem 1.2. Given $\epsilon : \mathcal{C}_K \to \mathbb{F}$ with $\epsilon(\mu) \ne 1$, there is a KCH representation ρ that induces ϵ, constructed in Corollary 3.7. This representation acts on the vector space generated by the columns of $\mathcal{E}(\Gamma)$, denoted by V. By Corollary 3.13 a subspace of a quotient of V is a KCH irrep that induces ϵ, and by Lemma 3.14 this KCH irrep also has dimension equal to the rank of $\mathcal{E}(\Gamma)$. Thus the original V was irreducible.

If $\rho' : \pi_K \to \mathrm{GL}(V')$ is any other KCH irrep that induces ϵ then the meridian subspace is V'. Extend the assignment $w_j = \rho'(\gamma_j^{-1})e_1 \mapsto \epsilon([\Gamma \gamma_j^{-1}])$, for each $1 \le j \le r$, to a linear map $\psi : V' \to V$. Examining the proof of Lemma 3.14 we see ψ is a well-defined isomorphism of vector spaces. It is also π_K-equivariant since (3.2) and (3.1) together imply that $\psi \circ \rho' = \rho \circ \psi$ on a generating set of π_K. $\square$

4. Constructing augmentations from a matrix

As in the previous section, fix a meridian m of K and a set $\Gamma = \{\gamma_1, \dots, \gamma_r\} \subset \pi_K$, with γ_1 the identity, such that $\mathcal{G} = \{g_i \mid g_i = \gamma_i^{-1} m \gamma_i, 1 \le i \le r\}$ generates π_K. We find criteria sufficient for an $r \times r$ matrix over $\mathbb{F}$ to be $\mathcal{E}(\Gamma)$ (defined in Section 3.4) for some augmentation $\epsilon : \mathcal{C}_K \to \mathbb{F}$. The criteria are used to prove Theorem 1.9. We begin with the following lemma.

Lemma 4.1. *Let $\epsilon, \epsilon' : \mathcal{C}_K \to S$ be two augmentations satisfying $\epsilon([\gamma_i\gamma_j^{-1}]) = \epsilon'([\gamma_i\gamma_j^{-1}])$ for every pair $1 \le i, j \le r$, and such that $\epsilon(1-\mu)$ is invertible. Then $\epsilon' = \epsilon$.*

Proof. Since $\mathcal{P}_K$ is generated by elements $[g] \in \pi_K$ over R_0 we need to check that ϵ, ϵ' agree on μ, λ and any $[g] \in \pi_K$. The equality $[\gamma_1\gamma_1^{-1}] = [e] = 1 - \mu$ implies $\epsilon(\mu) = \epsilon'(\mu)$ by our assumption.

Given $g \in \pi_K$, choose a product equal to g of elements in $\mathcal{G}$ and their inverses. From the form of the elements c_k determined in the proof of Lemma 3.2, iterating the process there determines an expansion of $[g] = [\gamma_1 g]$ solely in terms of powers of μ and elements $[\gamma_i\gamma_j^{-1}]$, $1 \le i, j \le r$. The assumption on the homomorphisms ϵ and ϵ' implies that $\epsilon([g]) = \epsilon'([g])$.

Finally, $\epsilon(\lambda)\epsilon(1-\mu) = \epsilon([\ell]) = \epsilon'([\ell]) = \epsilon'(\lambda)\epsilon(1-\mu)$ and $\epsilon(1-\mu)$ is invertible. The result follows. $\square$

4.1. Matrices that determine an augmentation

Consider now each element of $\mathcal{G}$ (and the inverses) as formal words $g_i^{\pm 1} = \gamma_i^{-1} m^{\pm 1} \gamma_i$, $i = 1, \dots, r$. For $\mathcal{R}$, a set of words in $\mathcal{G}$ and its inverses, denote the set of formal inverses of words in $\mathcal{R}$ by $\mathcal{R}^{-1}$. Given an $r \times r$ matrix E write E_{ij} for the (i, j)-entry in E. Following the proof of Lemma 4.1, use the explicit expansion determined by iterating Lemma 3.2 and the assignments $\epsilon(\gamma_i\gamma_j^{-1}) = E_{ij}$ and $\epsilon(\mu) = 1 - E_{11}$ to assign a value $\epsilon_E(g)$ to a word $g = \gamma_1 g \gamma_1^{-1}$, written in $\{\gamma_i\gamma_j^{-1} | 1 \le i, j \le r\} \cup \{m^{\pm 1}\}$. For $[g] \in \pi_K$, the element represented by the word g, we find conditions on E such that $\epsilon([g]) = \epsilon_E(g)$ determines a well-defined augmentation.

Lemma 4.2. *Let $\langle \mathcal{G} \mid \mathcal{R} \rangle$ be a presentation of π_K with meridian generators and notation as above. Given an $r \times r$ matrix E, define a function $\epsilon : \mathcal{C}_K \to$*

$\mathbb{F}$ by setting $\epsilon(\mu) = 1 - E_{11}$ and $\epsilon([g]) = \epsilon_E(g)$ for $[g] \in \pi_K$. Then ϵ is a well-defined augmentation if the diagonal entries of E are all equal and not 0 or 1, and $\epsilon_E(\gamma_i R\gamma_j^{-1}) = E_{ij}$ for every $R \in \mathcal{R} \cup \mathcal{R}^{-1}$ and each $1 \le i, j \le r$.

Proof. We have defined $\epsilon(\mu) = \mu_0$ so that $1 - \mu_0$ equals any diagonal entry of E.

Let F be the free group generated by $\mathcal{G}$ and its inverses. Since $\pi_K \cong \langle \mathcal{G} \mid \mathcal{R}\rangle$ is the quotient of F by the smallest normal subgroup containing $\mathcal{R}$, if g and h represent the same element in π_K there is a finite sequence of allowed moves on the words g and h, after which the resulting words agree. The allowed moves are the insertion or deletion into a word of either (1) a cancelling pair xx^{-1} or $x^{-1}x$, $x \in \mathcal{G}$, or (2) an element of $\mathcal{R}$ or its inverse in $\mathcal{R}^{-1}$.

We must show that $\epsilon([g]) = \epsilon([h])$ when $[g] = [h]$ in π_K. By the previous paragraph it is sufficient to prove that if h may be obtained from g by just one allowable insertion move then $\epsilon_E(g) = \epsilon_E(h)$. First suppose the insertion is a cancelling pair. Let $\varepsilon = \pm 1$ and $\delta_\pm = (\pm\varepsilon - 1)/2$ and apply (3.1) and $\epsilon_E(\gamma_k\gamma_k^{-1}) = E_{kk} = 1 - \mu_0$ to see that

$$\begin{aligned} &\epsilon_E(\gamma_i\gamma_k^{-1})\epsilon_E(\gamma_k g_k^{-\varepsilon}\gamma_j^{-1}) \\ &= \epsilon_E(\gamma_i\gamma_k^{-1})\left(\epsilon_E(\gamma_k\gamma_j^{-1}) + \varepsilon\mu_0^{\delta_-}\epsilon_E(\gamma_k\gamma_k^{-1})\epsilon_E(\gamma_k\gamma_j^{-1})\right) \\ &= \mu_0^{-\varepsilon}\epsilon_E(\gamma_i\gamma_k^{-1})\epsilon_E(\gamma_k\gamma_j^{-1}). \end{aligned}$$

From this, and the fact that $\delta_+ - \varepsilon = \delta_-$, we calculate

$$\begin{aligned} \epsilon_E(\gamma_i g_k^{\varepsilon} g_k^{-\varepsilon}\gamma_j^{-1}) &= \epsilon_E(\gamma_i g_k^{-\varepsilon}\gamma_j^{-1}) - \varepsilon\mu_0^{\delta_+ - \varepsilon}\epsilon_E(\gamma_i\gamma_k^{-1})\epsilon_E(\gamma_k\gamma_j^{-1}) \\ &= \epsilon_E(\gamma_i\gamma_j^{-1}). \end{aligned}$$

Now to compare $\epsilon_E(g)$ to $\epsilon_E(h)$ we expand both words to be expressed completely in terms of $\left\{\epsilon_E(\gamma_i\gamma_j^{-1}) \mid 1 \le i, j \le r\right\}$ except that for some k, i_0, j_0 the expansion of h involves $\epsilon_E(\gamma_{i_0} g_k^{\pm 1} g_k^{\mp 1}\gamma_{j_0}^{-1})$. But we've shown our procedure assigns this the same value as $\epsilon_E(\gamma_{i_0}\gamma_{j_0}^{-1})$, and thus $\epsilon_E(g) = \epsilon_E(h)$.

If the insertion is of the second type, there is some $R \in \mathcal{R} \cup \mathcal{R}^{-1}$ inserted into h, and this is the only difference between g and h as words. Applying the same argument as in the previous paragraph, and using the assumption that $\epsilon_E(\gamma_{i_0} R\gamma_{j_0}^{-1}) = E_{i_0 j_0} = \epsilon_E(\gamma_{i_0}\gamma_{j_0}^{-1})$, we see that $\epsilon_E(g) = \epsilon_E(h)$.

By the definition of ϵ and the isomorphism $\mathcal{C}_K \cong \mathcal{P}_K$, we only need to check that $\epsilon([\ell g]) = \epsilon([g\ell])$ for any $g \in \pi_K$. But since ℓ commutes with m,

$$\epsilon(\mu^{-1}[\ell])\epsilon([g]) = \epsilon(\mu^{-1}[\ell g]) - \epsilon(\mu^{-1}[\ell m g]) = (\mu_0^{-1} - 1)\epsilon([\ell g]).$$

By considering $\epsilon([g])\epsilon(\mu^{-1}[\ell])$, this also equals $(\mu_0^{-1} - 1)\epsilon([g\ell])$. Thus $\epsilon([\ell g]) = \epsilon([g\ell])$ since $\mu_0 \neq 1$. □

4.2. Connect sums

Recall the definition of the rank of an augmentation, and the augmentation rank of a knot (Definition 1.8). In [19, Prop. 5.8] the augmentation variety $V_{K_1\#K_2}$ was related to V_{K_1} and V_{K_2}. Here we relate the rank of the augmentations in each.

Theorem 1.9. *Let $K_1, K_2 \subset \mathbb{R}^3$ be oriented knots and suppose $1 \neq \mu_0 \in \mathbb{F}^*$ is such that for $n = 1, 2$ there is an augmentation $\epsilon_n : \mathcal{C}_{K_n} \to \mathbb{F}$ with rank d_n and so that $\epsilon_n(\mu) = \mu_0$. Then $K_1\#K_2$ has an augmentation with rank $d_1 + d_2 - 1$. Furthermore, $ar(K_1\#K_2, \mathbb{F}) = ar(K_1, \mathbb{F}) + ar(K_2, \mathbb{F}) - 1$.*

Proof. For $n = 1, 2$ let $\langle \mathcal{G}_n \mid \mathcal{R}_n\rangle$ be a presentation of π_{K_n}, where $\mathcal{G}_n$ is a set of meridians. Define $r_n = |\mathcal{G}_n|$. Let m_n be a meridian such that $[m_n h] = \mu[h]$ (for all $h \in \pi_{K_n}$) in the cord algebra $\mathcal{C}_{K_n}$. We may assume that $m_n \in \mathcal{G}_n$. Define $\mathcal{G} = (\mathcal{G}_1 \setminus \{m_1\}) \cup (\mathcal{G}_2 \setminus \{m_2\}) \cup \{m\}$. Order $\mathcal{G}$ so that $m = g_1$ is first and the last $r_2 - 1$ elements are in $\mathcal{G}_2$. Write $g_i = \gamma_i^{-1} m \gamma_i$ for the i^{th} generator, $i = 1, \ldots, r_1 + r_2 - 1$.

The group $\pi_{K_1\#K_2}$ has a presentation $\langle \mathcal{G} \mid \mathcal{R}\rangle$ where $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ except that in each $\mathcal{R}_n$, m_n is replaced by m. We may take $\mathcal{R}$ to have $r_1 + r_2 - 2$ relators (with $r_n - 1$ from each summand). In fact, we may assume (and do) each relator to have the form $R = g_\ell g_m g_\ell^{-1} g_k^{-1}$, where $g_k, g_\ell, g_m \in \mathcal{G}_n$ if $R \in \mathcal{R}_n$.

Define an $(r_1 + r_2 - 1) \times (r_1 + r_2 - 1)$ matrix E by setting

$$(E)_{ij} = \begin{cases} \epsilon_1([\gamma_i\gamma_j^{-1}]), \text{ if } i, j \le r_1 \\ \epsilon_2([\gamma_i\gamma_j^{-1}]), \text{ if } i, j > r_1 \\ \frac{1}{1-\mu_0}\epsilon_1([\gamma_i])\epsilon_2([\gamma_j^{-1}]), \text{ if } i \le r_1 < j \\ \frac{1}{1-\mu_0}\epsilon_2([\gamma_i])\epsilon_1([\gamma_j^{-1}]), \text{ if } j \le r_1 < i \end{cases}$$

To each word g in the generators $\mathcal{G}$ and their inverses we define $\epsilon_E(g)$ as in Section 4.1. Note that since $\epsilon_n(\mu) = \mu_0$ for both $n = 1, 2$ the definition of

E is such that the diagonal entries all agree. By Lemma 4.2 then, this defines an augmentation $\epsilon : \mathcal{C}_{K_1\#K_2} \to \mathbb{C}$ provided that $\epsilon_E(\gamma_i R\gamma_j^{-1}) = \epsilon_E(\gamma_i\gamma_j^{-1})$ for each $R \in \mathcal{R} \cup \mathcal{R}^{-1}$. We will need to use the following observation, which the reader may check using an argument similar to that in the proof of Lemma 4.2.

(∗) If $\gamma_k^{-1} m\gamma_k = g_k \in \mathcal{G}_n$, and h represents any element in $\pi_{K_{n'}} \subset \pi_{K_1\#K_2}$ where $n' \neq n$, then $\epsilon_E(\gamma_k h) = \frac{1}{1-\mu_0}\epsilon_n([\gamma_k])\epsilon_{n'}([h])$ and $\epsilon_E(h\gamma_k^{-1}) = \frac{1}{1-\mu_0}\epsilon_n([\gamma_k^{-1}])\epsilon_{n'}([h])$.

Let $R \in \mathcal{R}$ and consider $\epsilon_E(\gamma_i R\gamma_j^{-1})$ for some $1 \le i, j \le r_1 + r_2 - 1$. The check for $R \in \mathcal{R}^{-1}$ is essentially the same.

If both $i, j \le r_1$ and $R \in \mathcal{R}_1$ then $\epsilon_E(\gamma_i R\gamma_j^{-1}) = \epsilon_1([\gamma_i R\gamma_j^{-1}])$ and this equals $\epsilon_1([\gamma_i\gamma_j^{-1}]) = \epsilon_E(\gamma_i\gamma_j^{-1})$ since ϵ_1 is well-defined. We have a similar argument when both $i, j > r_1$ and $R \in \mathcal{R}_2$.

If $i \le r_1 < j$, then by (∗) we see that $\epsilon_E(\gamma_i R\gamma_j^{-1})$ is either

$$\frac{1}{1-\mu_0}\epsilon_1([\gamma_i R])\epsilon_2([\gamma_j^{-1}]) \quad \text{or} \quad \frac{1}{1-\mu_0}\epsilon_1([\gamma_i])\epsilon_2([R\gamma_j^{-1}])$$

depending on whether $R \in \mathcal{R}_1$ or $R \in \mathcal{R}_2$. In both cases this equals

$$\frac{1}{1-\mu_0}\epsilon_1([\gamma_i])\epsilon_2([\gamma_j^{-1}]) = \epsilon_E(\gamma_i\gamma_j^{-1}).$$

The case when $j \le r_1 < i$ is similar.

Finally, suppose both $i, j \le r_1$ but $R \in \mathcal{R}_2$. Here we use our assumption on the form of relators: that $R = g_\ell g_m g_\ell^{-1} g_k^{-1}$ for some $g_k, g_\ell, g_m \in \mathcal{G}_2$. Thus using our definition of ϵ and (∗) we find that

$$\begin{aligned}
&\epsilon_E(\gamma_i R\gamma_j^{-1}) \\
&= \epsilon_E(\gamma_i g_k^{-1}\gamma_j^{-1}) - \epsilon_E(\gamma_i g_\ell\gamma_m^{-1})\epsilon_E(\gamma_m g_\ell^{-1} g_k^{-1}\gamma_j^{-1}) \\
&= \epsilon_E(\gamma_i\gamma_j^{-1}) + \mu_0^{-1}\epsilon_E(\gamma_i\gamma_k^{-1})\epsilon_E(\gamma_k\gamma_j^{-1}) - \epsilon_E(\gamma_i g_\ell\gamma_m^{-1})\epsilon_E(\gamma_m g_\ell^{-1} g_k^{-1}\gamma_j^{-1}) \\
&= \epsilon_E(\gamma_i\gamma_j^{-1}) \\
&\quad + \frac{\epsilon_1([\gamma_i])\epsilon_1([\gamma_j^{-1}])}{(1-\mu_0)^2}\left(\mu_0^{-1}\epsilon_2([\gamma_k^{-1}])\epsilon_2([\gamma_k]) - \epsilon_2([g_\ell\gamma_m^{-1}])\epsilon_2([\gamma_m g_\ell^{-1} g_k^{-1}])\right) \\
&= \epsilon_E(\gamma_i\gamma_j^{-1}) \\
&\quad + \frac{\epsilon_1([\gamma_i])\epsilon_1([\gamma_j^{-1}])}{(1-\mu_0)^2}\left(\epsilon_2([g_k^{-1}]) - \epsilon_2([e]) - (\epsilon_2([g_k^{-1}]) - \epsilon_2([g_\ell g_m g_\ell^{-1} g_k^{-1}]))\right) \\
&= \epsilon_E(\gamma_i\gamma_j^{-1}),
\end{aligned}$$

the last equality since ϵ_2 is well-defined. The case $i, j > r_1$ and $R \in \mathcal{R}_1$ is treated similarly. By Lemma 4.2 we obtain a well-defined augmentation.

Since the top-left $r_1 \times r_1$ block of E is the matrix $\mathcal{E}(\Gamma_1)$, we may choose d_1 from among the first r_1 columns of E that are independent (the first being one of them). We may also choose $d_2 - 1$ columns from the last $r_2 - 1$ that (with the first column) also form an independent set. The union is an independent set by a standard argument. This shows that $K_1 \# K_2$ has an augmentation of rank $d_1 + d_2 - 1$.

Suppose that an augmentation $\epsilon : \mathcal{C}_{K_1 \# K_2} \to \mathbb{F}$ with rank d is given. Choose generators and elements of Γ as above, ordered as above. Consider the KCH irrep $\rho : \pi_{K_1 \# K_2} \to \mathrm{GL}(W_\rho(\Gamma))$ corresponding to ϵ as constructed in Section 3, where $W_\rho(\Gamma)$ is the vector space generated by the columns of $\mathcal{E}(\Gamma)$. By definition, $\rho(g)(\epsilon([\Gamma\gamma_j^{-1}])) = \epsilon([\Gamma g \gamma_j^{-1}])$ for any $g \in \pi_{K_1 \# K_2}$. By Lemma 3.2 the restriction of ρ to $\pi_{K_1} \subset \pi_{K_1 \# K_2}$ is a KCH representation on the subspace W_1 of $W_\rho(\Gamma)$ spanned by the first r_1 columns of $\mathcal{E}(\Gamma)$. Let ϵ_1 be the induced augmentation. Similarly restriction of ρ to π_{K_2} gives a KCH representation (inducing ϵ_2 say), on the space W_2 spanned by the first column along with columns $r_1 + 1$ through $r_1 + r_2 - 1$.

Consider the projection $\mathrm{pr}(W_1) \subset \mathbb{F}^{r_1}$ of W_1 onto the first r_1 factors. By the definition of E, letting E_j be column $j \in \{1, \ldots, r_1\}$ of E, any linear relation $\sum c_j \, \mathrm{pr}(E_j) = 0$ will also hold among the $E_j \in \mathbb{F}^{r_1 + r_2 - 1}$. This makes it clear that pr is an isomorphism between W_1 and the KCH irrep that induces ϵ_1. In particular, W_1 is irreducible. An analogous statement holds for W_2. Thus $\dim W_n \le \mathrm{ar}(K_n, \mathbb{F})$ for $n = 1, 2$.

Since $W_\rho(\Gamma) = W_1 + W_2$ and W_1, W_2 have a common 1-dimensional subspace, we see that $\dim W_\rho(\Gamma) \le \mathrm{ar}(K_1, \mathbb{F}) + \mathrm{ar}(K_2, \mathbb{F}) - 1$. By Lemma 3.14, the rank of ϵ is at most $\mathrm{ar}(K_1, \mathbb{F}) + \mathrm{ar}(K_2, \mathbb{F}) - 1$. □

Corollary 4.3. *If* $ar(K_i, \mathbb{F}) = b(K_i)$ *for* $i = 1, 2$ *(in which case* $mr(K_i) = b(K_i)$*), then* $ar(K_1 \# K_2, \mathbb{F}) = mr(K_1 \# K_2) = b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$*.*

Proof. Theorem 1.5 shows that $\mathrm{ar}(K_i, \mathbb{F}) \le \mathrm{mr}(K_i)$ and it is well-known that $\mathrm{mr}(K_i) \le b(K_i)$. Applying Theorem 1.9 and observing that $b(K) - 1$ is additive we obtain the result. □

Some knots that are known to satisfy the hypothesis of Corollary 4.3 include torus knots, two-bridge knots, a family of pretzel knots [7], and others by results in Section 5.

5. Augmentation rank of braid closures

In this section we construct augmentations by considering K as the closure of a braid $B \in B_n$. To do so we pass to the algebra $HC_0(K)|_{U=1}$, discussed in Section 2 above, which is isomorphic as an R_0-algebra to $\mathcal{C}_K$. We begin showing how to understand augmentations with rank n, proving Theorem 1.10. This allows us to find knots with augmentation rank smaller than meridional rank. Afterwards we indicate a method to construct, from a knot that has a rank n augmentation, a new knot with a rank $n+1$ augmentation. As a consequence we prove Theorem 1.13.

5.1. Augmentations with rank equal to braid index

Recall the isomorphisms $F_{\mathcal{P}} : \mathcal{P}_K \to \mathcal{C}_K$ and $F_{HC} : \mathcal{C}_K \to HC_0(K)|_{U=1}$ of Theorems 2.9 and 2.4, respectively. In the next lemma $\mathbf{A}$ is the $n \times n$ matrix used to define $HC_0(K)|_{U=1}$. Recall that $\mathbf{A}_{ij} = a_{ij}$ if $i < j$, $\mathbf{A}_{ij} = -\mu a_{ij}$ if $i > j$, and the diagonal entries are $1 - \mu$.

Lemma 5.1. *Let $\epsilon : \mathcal{P}_K \to \mathbb{F}$ be an augmentation. The rank of $\mathcal{E}(\Gamma)$ equals the rank of $\epsilon(\mathbf{A})$.*

We admit to abusing notation, as ϵ is defined on $\mathcal{P}_K$ not $HC_0(K)|_{U=1}$. By $\epsilon(\mathbf{A})$ we actually mean $\epsilon \circ (F_{HC} \circ F_{\mathcal{P}})^{-1}(\mathbf{A})$.

Proof. Let K be the closure of $B \in B_n$. Taking a basepoint x for π_K, consider the generating set $\mathcal{G} = \{g_1, g_2, \ldots, g_n\}$ where g_i is the meridian of K contained in D, as depicted on the right in Figure 3 (D is the disk from the discussion in Section 2.2). For each $i > 1$ define a loop $\gamma_i^{-1} \in \pi_K$ that follows g_i in D from x until g_i leaves the upper half-disk. Leaving D, γ_i^{-1} then runs parallel to K, framed as in Figure 3, until returning to x. By convention γ_1 is the identity.

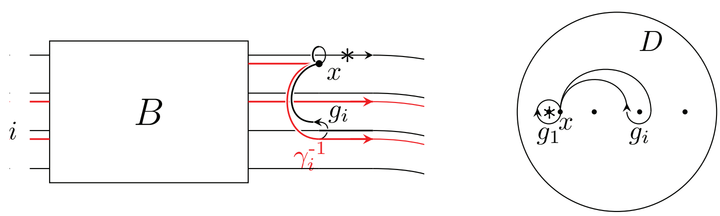

Figure 3: Computing $(F_{HC} \circ F_{\mathcal{P}})^{-1}(a_{ij})$.

Define $\Gamma = \{\gamma_1, \dots, \gamma_r\}$. By construction, g_i and $\gamma_i^{-1} g_1 \gamma_i$ are equal in π_K. Define r_i to be the linking number $\mathrm{lk}(\gamma_i, K)$. After homotopy of cords we see that $F_{\mathcal{P}}([\gamma_i \gamma_j^{-1}]) = \mu^{r_i - r_j} c_{ij}$ and so $F_{HC} \circ F_{\mathcal{P}}([\gamma_i \gamma_j^{-1}]) = \mu^{r_i - r_j} \mathbf{A}_{ij}$.

If Δ is the diagonal matrix with $\Delta_{ii} = \mu^{r_i}$ the discussion above implies $\mathcal{E}(\Gamma)\epsilon(\Delta) = \epsilon(\Delta)\epsilon(\mathbf{A})$. As $\epsilon(\mu) \neq 0$, $\epsilon(\mathbf{A})$ and $\mathcal{E}(\Gamma)$ must have the same rank. □

Theorem 1.10. *Suppose that K is the closure of $B \in B_n$, and that $\epsilon : \mathcal{C}_K \to \mathbb{C}$ is an augmentation of K with rank n and $\epsilon(\mu) = \mu_0$. Then $\epsilon(\lambda) = (-\mu_0)^{-w(B)}$, where $w(B)$ is the writhe (or algebraic length) of B. Furthermore, there is a curve of rank n augmentations in the closure of V_K that corresponds to a factor $\lambda\mu^{w(B)} - (-1)^{w(B)}$ of $Aug_K(\lambda, \mu)$.*

Proof. The proof below uses an argument with determinants for which we work in the commutative algebra $\mathcal{A}_n^c$, defined as $\mathcal{A}_n$ modulo the ideal generated by $\{xy - yx \mid x, y \in \mathcal{A}_n\}$. Write $\mathcal{I}_B^c$ also for the ideal in $\mathcal{A}_n^c$ generated by the quotients of the elements in $\mathcal{I}_B$, the ideal of Definition 2.3. For any augmentation $\epsilon : \mathcal{C}_K \to \mathbb{C}$, $\epsilon \circ F_{HC}^{-1}$ factors through $(\mathcal{A}_n^c \otimes R_0)/\mathcal{I}_B^c$. Without altering notation we suppose during the proof that, for example, the entries of a matrix which are traditionally in $\mathcal{A}_n$, are instead the corresponding class in $\mathcal{A}_n^c$.

Let Λ be the diagonal matrix $\mathrm{diag}[\lambda\mu^{w(B)}, 1, \dots, 1]$ used to define $HC_0(K)|_{U=1}$ (see Section 2.2). That ϵ is well-defined implies

$$\epsilon(\mathbf{A}) - \epsilon(\mathbf{A})\epsilon(\Phi_B^R)\epsilon(\Lambda^{-1}) = 0.$$

By Lemma 5.1 the $n \times n$ matrix $\epsilon(\mathbf{A})$ is invertible, and so $\epsilon(\Phi_B^R) = \epsilon(\Lambda)$. (Note also that $\epsilon(\Phi_B^L) = \epsilon(\Lambda)^{-1}$.)

By Corollary 2.6, Φ_B^R is invertible in the ring of matrices over $\mathbb{Z}[\{a_{ij}\}]$, its inverse being $\phi_B(\Phi_{B^{-1}}^R)$. So $\det \Phi_B^R$ is a unit in $\mathbb{Z}[\{a_{ij}\}]$ and can only be ± 1.

As $\epsilon(\Phi_B^R) = \epsilon(\Lambda)$ and $\det(\epsilon(\Phi_B^R)) = \epsilon(\det(\Phi_B^R)) = \pm 1$, we see that $\epsilon(\lambda\mu^{w(B)}) = \pm 1$. To determine the sign, let σ_k be a standard generator of B_n. Use the definition of ϕ_{σ_k} to check $\det(\Phi_{\sigma_k^{\pm 1}}^R) = -1$. Since ϕ_B is an algebra map we see that $\det(\phi_B(\Phi_{\sigma_k^{\pm 1}}^R)) = -1$ for any $B \in B_n$.

That $\det(\epsilon(\Phi_B^R)) = (-1)^{w(B)}$ now follows from the Chain Rule (Theorem 2.5) and the fact that $w(B)$ has the same parity as the length of B as a word in the generators $\{\sigma_1, \dots, \sigma_{n-1}\}$ of B_n. It follows that $\epsilon(\lambda) = (-\mu_0)^{-w(B)}$.

Let $\Delta(B)$ be the $n \times n$ diagonal matrix $\mathrm{diag}[(-1)^{w(B)}, 1, \ldots, 1]$. We see that ϵ satisfies $\epsilon(\Phi_B^L) = \Delta(B) = \epsilon(\Phi_B^R)$. For any $\mu_1 \in \mathbb{C}^*$, define a map ϵ_1 on $\mathcal{A}_n \otimes R_0$ by setting $\epsilon_1(\mu) = \mu_1$, $\epsilon_1(\lambda) = (-\mu_1)^{-w(B)}$, and $\epsilon_1(a_{ij}) = \epsilon(a_{ij})$ for $1 \le i \ne j \le n$. Then $\epsilon_1(\Phi_B^L) = \Delta(B) = \epsilon_1(\Phi_B^R)$ and $\epsilon_1(\Lambda) = \Delta(B)$, and so ϵ_1 defines an augmentation as

$$\epsilon_1(\mathbf{A}) - \epsilon_1(\Lambda)\epsilon_1(\Phi_B^L)\epsilon_1(\mathbf{A}) = 0 = \epsilon_1(\mathbf{A}) - \epsilon_1(\mathbf{A})\epsilon_1(\Phi_B^R)\epsilon_1(\mathbf{A}).$$

Since $\det(\epsilon_1(\mathbf{A}))$ is a polynomial in $\mathbb{C}[\mu_1]$, and non-zero when $\mu_1 = \mu_0$, it can be zero for only finitely many μ_1. Thus we have a rank n augmentation for all but finitely many $\mu_1 \in \mathbb{C}^*$ and the algebraic closure of V_K contains a 1-dimensional component of rank n augmentations which is the zero locus of $\lambda\mu^{w(B)} - (-1)^{w(B)}$. □

Remark 5.2. The conventions adopted in [20] for infinity transverse homology would make $\epsilon(\lambda) = \mu_0^{w(B)}$, since the translation to our conventions is $\mu \leftrightarrow -\mu^{-1}$.

5.2. Rank gap

Let n be the braid index of K and write $\mathrm{ar}(K)$ for $\mathrm{ar}(K, \mathbb{C})$. From (1.2) the augmentation rank $\mathrm{ar}(K)$ cannot be n if $b(K) < n$, where $b(K)$ is the bridge number. Recall that the meridional rank $\mathrm{mr}(K)$ is the minimal number of meridians that generate π_K. Ideas from the proof of Theorem 1.10 allow us to find knots with a gap between their augmentation rank and meridional rank. That is, $\mathrm{mr}(K) = b(K) = n$, but $\mathrm{ar}(K) < n$.

Theorem 5.3. *Let K be a knot with crossing number at most 10, which has bridge number and minimal braid index equal to three. If K (or its mirror) is one of* $8_{16}, 8_{17}, 10_{91}, 10_{94}$ *then* $2 = ar(K) < mr(K) = 3$. *Otherwise,* $ar(K) = mr(K) = 3$.

Proof. For $B \in B_n$ recall the matrix $\Delta(B)$ defined in the proof of Theorem 1.10. If K is the closure of B and $\mathrm{ar}(K) = n$, there is a map $\epsilon : \mathcal{A}_n \to \mathbb{C}$ with $\epsilon(\Phi_B^L) = \Delta(B)$. As the augmentation rank is an invariant of K, the choice of B (provided it has index n) does not affect the existence of such a map.

Define $\mathcal{A}_n^{ab}$ to be $\mathcal{A}_n^c$ modulo the ideal generated by $\{a_{ij} - a_{ji} \mid 1 \le i \ne j \le n\}$. Given $B \in B_n$, if $\epsilon : \mathcal{A}_n \to \mathbb{C}$ exists with $\epsilon(\Phi_B^L) = \Delta(B)$, and further ϵ descends to $\mathcal{A}_n^{ab}$, then ϵ defines an augmentation by Proposition 2.2 and

Theorem 2.7. Furthermore, for any extension of ϵ to $\mathcal{A}_n \otimes R_0$, a calculation of $\det(\epsilon(\mathbf{A}))$ shows that its degree in $\epsilon(\mu)$ is $n > 0$. So such an ϵ determines a family of rank n augmentations for all but finitely many choices for $\epsilon(\mu)$.

Consulting the database at KnotInfo [4], there are 42 prime knots (up to mirroring) which have bridge number and braid index three and have crossing number at most 10. For each we take a braid representative $B \in B_3$ (e.g. take the representative provided at [4]). Let J_B be the ideal in $\mathcal{A}_3^c$ generated by the polynomials appearing as entries in $\Phi_B^L - \Delta(B)$ (there is no issue working in $\mathcal{A}_3^c$ rather than $\mathcal{A}_3$ as ϵ must factor through $\mathcal{A}_3^c$ in any case). We calculate $\Phi_B^L - \Delta(B)$ and compute a reduced Gröbner basis for J_B.[1]

For the braid representatives of $8_{16}, 8_{17}, 10_{91}, 10_{94}$, the Gröbner basis of J_B is $\{1\}$, so the polynomials have no common zero in $\mathbb{C}$ and there does not exist a map $\epsilon : \mathcal{A}_n \to \mathbb{C}$ with $\epsilon(\Phi_B^L) = \Delta(B)$. For each of the other braid representatives one can see that the computed basis of polynomials has a common zero on which $a_{ij} - a_{ji}$, $1 \le i < j \le 3$, evaluates to zero. By the above discussion we get a rank 3 augmentation.

It was shown in [8] and [2] that every non-trivial knot has a non-abelian $\mathrm{SL}_2\mathbb{C}$ representation, and so $\mathrm{ar}(8_{16}) = \mathrm{ar}(8_{17}) = \mathrm{ar}(10_{91}) = \mathrm{ar}(10_{94}) = 2$. As shown in [3], a consequence of Thurston's orbifold geometrization and the fact that only 2-bridge knots have double branched cover a lens space ([13]) is that 3-bridge knots have meridional rank 3.

For the mirror $m(K)$ of any knot K that has already been checked, [19, Prop. 4.2] implies that the cord algebra of $m(K)$ is isomorphic to the algebra obtained from $HC_0(K)|_{U=1}$ by transforming $\mu \leftrightarrow \mu^{-1}$. Thus $\mathrm{ar}(m(K)) = 3$ if and only if $\mathrm{ar}(K) = 3$. Finally, if K is not prime then since $b(K) = 3$ and $b(K) - 1$ is additive under connect sum, K is the connect sum of two 2-bridge knots, each of which has augmentation rank 2. Then $\mathrm{ar}(K) = 3$ by Theorem 1.9. □

5.3. Increasing augmentation rank with the braid index

Let K be a knot with braid index n and a rank n augmentation $\mathcal{A}_n \to \mathbb{C}$ that descends to $\mathcal{A}_n^{ab}$. We describe a method for constructing knots with

[1]The reader is referred to the Mathematica notebook 3bridge3braids.nb, available at `http://thales.math.uqam.ca/~cornwell/mathematica/`. Our computation of $\Phi_B^L - \Delta(B)$ uses the Mathematica package transverse.m, found at `http://www.math.duke.edu/~ng/math/programs.html` and written by Lenny Ng. Then we use the Mathematica function GroebnerBasis. Note that the transverse.m was written for transverse homology, so one would consider entries in $\Phi_B^L - \mathrm{Id}$ by Remark 5.2.

higher braid index that have a rank $n+1$ augmentation. Let $\sigma_{i,j}$, for $i<j$, denote the braid $\sigma_i \dots \sigma_{j-2}\sigma_{j-1}\sigma_{j-2}^{-1}\dots\sigma_i^{-1}$ (visually, this braid crosses the i and j strands above the intermediate strands). We will prove the following.

Theorem 5.4. *Let u, v be integers with $|u| \geq 2$, $|v| \geq 3$, and let $\delta = \pm 1$. If the closure of $B \in B_n$ has a rank n augmentation that descends to $\mathcal{A}_n^{ab}$ then, for $1 \leq i < n$, the closure of $B\sigma_n^{\delta}\sigma_{i,n}^{u}\sigma_n^{v} \in B_{n+1}$ has a rank $n+1$ augmentation with the same property, provided $u+v-1$ is even or $i=1$ and u is odd.*

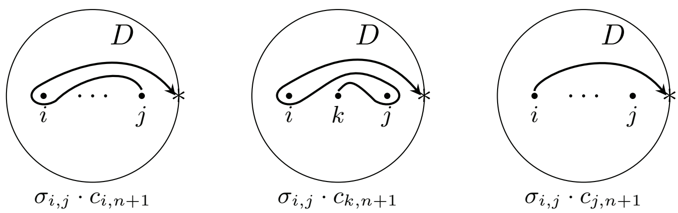

Figure 4: Computing $\Phi_{B'}^{L}$.

For the proof of Theorem 5.4 we first require a pair of lemmas.

Lemma 5.5. *Let $B' = \sigma_{i,j}^{u} \in B_n$ for some $1 \leq i < j \leq n$ and $|u| \geq 2$ and let $X = \{a_{ij}, a_{ji}\}$. If u is odd or $u+v-1$ is even then there exists a map $e : \mathcal{A}_n^{ab} \to \mathcal{A}_n^{ab} \otimes \mathbb{C}$ such that $e(a_{kl}) = a_{kl}$ for $a_{kl} \notin X$, and such that $e(\Phi_{B'}^{L})$ is the $n \times n$ diagonal matrix diag$[Id_{i-1}, (-1)^{u+v-1}, Id_{j-i-1}, (-1)^{v-1}, Id_{n-j}]$.*

Proof. We give a proof for the case $u \geq 2$. Writing a verbatim copy of this proof, but interchanging the roles of i and j throughout (with the one exception that $\sigma_{i,j}$ is simply replaced with $\sigma_{i,j}^{-1}$) one obtains a proof for the case $u \leq -2$.

Include $B_n \subset B_{n+1}$ and consider $\phi_B \in \mathrm{Aut}(\mathcal{A}_{n+1})$. Placing the $n+1$ puncture on D at the boundary we note that the class of $\sigma_{i,j} \cdot c_{i,n+1}$ is represented by the leftmost arc in Figure 4. Using relations from Figure 2, we compute $\phi_{\sigma_{i,j}}(a_{i,n+1}) = a_{j,n+1} - a_{ji}a_{i,n+1}$. The arc on the right of Figure 4 is $\sigma_{i,j} \cdot c_{j,n+1}$ and so $\phi_{\sigma_{i,j}}(a_{j,n+1}) = a_{i,n+1}$. We also find $\phi_{\sigma_{i,j}}(a_{ij}) = -a_{ji}$ and $\phi_{\sigma_{i,j}}(a_{ji}) = -a_{ij}$.

First, $\phi_{\sigma_{i,j}}$ descends to $\mathcal{A}_n$ modulo the ideal generated by $a_{ij} - a_{ji}$. Write $x = a_{ij}$, then $\phi_{\sigma_{i,j}}(x) = -x$. In addition, for any $k \geq 1$, we see $\phi_{\sigma_{i,j}^k}(a_{i,n+1}) =$

$P_k(x)a_{i,n+1} + Q_k(x)a_{j,n+1}$ for some polynomials P_k, Q_k in $\mathbb{Z}[x]$. Using a computation above, we have $P_0(x) = 1$, $Q_0(x) = 0$, $P_1(x) = -x$, and $Q_1(x) = 1$. Now we compute

$$\begin{aligned}\phi_{\sigma_{i,j}^{k+1}}(a_{i,n+1}) &= \phi_{\sigma_{i,j}}(P_k(x)a_{i,n+1} + Q_k(x)a_{j,n+1})\\ &= P_k(-x)(a_{j,n+1} - xa_{i,n+1}) + Q_k(-x)a_{i,n+1},\end{aligned}$$

so $Q_{k+1}(x) = P_k(-x)$, and also

$$P_{k+1}(x) = P_{k-1}(x) - xP_k(-x). \tag{5.1}$$

The recurrence in (5.1) with initial data $P_0(x) = 1$ and $P_1(x) = -x$ then determines $P_k(x)$ (and $Q_k(x) = P_{k-1}(-x)$) for all $k \geq 1$. Note that P_k is an odd or even function, when k is odd or even respectively, that the degree of P_k is k, and that $\phi_{\sigma_{i,j}^k}(a_{j,n+1}) = \phi_{\sigma_{i,j}^{k-1}}(a_{i,n+1}) = P_{k-1}(x)a_{i,n+1} + P_{k-2}(-x)a_{j,n+1}$.

By our computations rows i and j of $\Phi^L_{B'}$ consist of only four non-zero entries (in columns i and j), namely: $P_u(x), P_{u-1}(-x)$ in row i; $P_{u-1}(x)$, $P_{u-2}(-x)$ in row j. For $k < i$ or $k > j$ we have that $\phi_{B'}(a_{k,n+1}) = a_{k,n+1}$. To understand the intermediate rows, let $i < k < j$ and check that the center of Figure 4 gives $\sigma_{i,j} \cdot c_{k,n+1}$. Then

$$\begin{aligned}\phi_{\sigma_{i,j}}(a_{k,n+1}) &= (a_{k,n+1} - a_{ki}a_{i,n+1}) - a_{kj}\phi_{\sigma_{i,j}}(a_{i,n+1})\\ &= (a_{k,n+1} - a_{ki}a_{i,n+1}) + \phi_{\sigma_{i,j}}(a_{ki}a_{i,n+1}),\end{aligned}$$

since $\phi_{\sigma_{i,j}}(a_{ki}) = -a_{kj}$. But then $\phi_{\sigma_{i,j}}(a_{k,n+1} - a_{ki}a_{i,n+1}) = a_{k,n+1} - a_{ki}a_{i,n+1}$, implying $\phi_{\sigma^u_{i,j}}(a_{k,n+1}) = (a_{k,n+1} - a_{ki}a_{i,n+1}) + \phi_{\sigma^u_{i,j}}(a_{ki}a_{i,n+1})$. And so the k^{th} row of $\Phi^L_{B'}$ consists of

$$\phi_{B'}(a_{ki})P_u(x) - a_{ki}, \ 1, \ \text{ and } \ \phi_{B'}(a_{ki})P_{u-1}(-x)$$

in the i, k, j spots, respectively (and 0 elsewhere).

To prove the lemma then, it suffices to show there is a choice for $e(x) = x_0 \in \mathbb{C}$ that makes $P_u(x_0) = (-1)^{u+v-1}, P_{u-1}(x_0) = 0, P_{u-2}(-x_0) = (-1)^{v-1}$ and $\phi_{B'}(a_{ki})|_{x=x_0} = (-1)^{u+v-1}a_{ki}$ for $i < k < j$ (recall that P_{u-1} is either an odd or even function of x).

As $u - 1 > 0$ there is x_0 for which $P_{u-1}(x_0) = 0$. Choose $e(x) = x_0$ and $e(a_{kl}) = a_{kl}$ for $a_{kl} \notin X$. Then only the i^{th} column of $e(\Phi^L_{B'})$ can possibly have non-zero off-diagonal entries. Hence $\det e(\Phi^L_{B'}) = P_u(x_0)P_{u-2}(-x_0)$, as other diagonal entries are 1. In the proof of Theorem 1.10 we saw that this determinant is $(-1)^{w(B')} = (-1)^u$. Moreover, from equation (5.1), $P_u(x_0) =$

$P_{u-2}(x_0) = (-1)^{u-2}P_{u-2}(-x_0)$. Putting this all together requires that $P_{u-2}(-x_0) = \pm 1$.

If u is odd then P_{u-2} is an odd function, so we can guarantee $P_{u-2}(-x_0) = (-1)^{v-1}$ by swapping x_0 for $-x_0$ if needed. As P_{u-1} is even it remains zero, and we have $P_u(x_0) = (-1)^{u-2}P_{u-2}(-x_0) = (-1)^{u+v-1}$. If $u+v-1$ is even we need only consider u even. In this case take $x_0 = 0$ and this makes $P_u(x_0) = 1 = P_{u-2}(-x_0)$ as needed.

Having determined the diagonal, consider $e(\phi_{B'}(a_{ki}))$. Since $k > i$, Theorem 2.7 implies that $-\mu e(\phi_{B'}(a_{ki}))$ equals the (k,i)-entry of $e(\Phi^L_{B'} \cdot \mathbf{A} \cdot \Phi^R_{B'})$. By calculations above, row k of $e(\Phi^L_{B'})$ has at most two non-zero entries and column i of $e(\Phi^R_{B'})$ has one, since row i of $e(\Phi^L_{B'})$ does. So the (k,i)-entry in question is $(-1)^{u+v-1}((e(\phi_{B'}(a_{ki}))P_u(x_0) - a_{ki})(1-\mu) - \mu a_{ki})$. Equating this with $-\mu e(\phi_{B'}(a_{ki}))$ and using that $P_u(x_0) = (-1)^{u+v-1}$ we find that, as required, $e(\phi_{B'}(a_{ki})) = (-1)^{u+v-1}a_{ki}$. □

Lemma 5.6. *Consider $B, B' \in B_n$ included into B_{n+1} so the last strand does not interact and with Φ^L_B and $\Phi^L_{B'}$ as $(n+1)\times(n+1)$ matrices. Let $b = B\sigma_n^{-1}B'\sigma_n^v$ for some $v \geq 3$. Suppose there exists a map $e : \mathcal{A}^{ab}_{n+1} \to \mathbb{C}[X]$ where $e(a_{n,n+1}) = e(a_{n+1,n}) = X$, with the properties*

(a) $e(\Phi^L_B) = \Delta(B)$;

(b) $e(\phi_{B\sigma_n^{-1}}(\Phi^L_{B'})) = diag[(-1)^{w(B')+v-1}, 1\dots, 1, (-1)^{v-1}, 1]$.

Then there is a map $\bar{e} : \mathbb{C}[X] \to \mathbb{C}$ such that $\bar{e} \circ e(\Phi^L_b) = \Delta(b)$.

Proof. We treat the case when v is positive first. The n^{th} row of $e(\Phi^L_B)$ being $(0,\dots,0,1,0)$ implies $e(\phi_B(a_{n,n+1})) = X$. By the Chain Rule we have

$$e\left(\Phi^L_{B\sigma_n^{-1}}\right) = \begin{pmatrix} \mathrm{Id}_{n-1} & & \\ & 0 & 1 \\ & 1 & -X \end{pmatrix} \Delta(B) = \begin{pmatrix} (-1)^{w(B)} & & & \\ & \mathrm{Id}_{n-2} & & \\ & & 0 & 1 \\ & & 1 & -X \end{pmatrix}.$$

By property (b), the Chain Rule, and $w(b) = w(B) + w(B') + v - 1$ we get,

$$e\left(\Phi^L_{B\sigma_n^{-1}B'}\right) = e\left(\phi_{B\sigma_n^{-1}}(\Phi^L_{B'}) \cdot \Phi^L_{B\sigma_n^{-1}}\right) = \begin{pmatrix} (-1)^{w(b)} & & & \\ & \mathrm{Id}_{n-2} & & \\ & & 0 & (-1)^{v-1} \\ & & 1 & -X \end{pmatrix}.$$

If we show that $\bar{e}(X)$ may be chosen so that

$$\bar{e} \circ e\left(\phi_{B\sigma_n^{-1}B'}(\Phi^L_{\sigma_n^v})\right) = \begin{pmatrix} \mathrm{Id}_{n-1} & & \\ & (-1)^{v-1}\bar{e}(X) & 1 \\ & (-1)^{v-1} & 0 \end{pmatrix},$$

then applying the Chain Rule again will prove the claim. First note that $\sigma_{n,n+1} = \sigma_n$, so upon setting $a_{n+1,n} = a_{n,n+1} = x$ the proof of Lemma 5.5 implies that

$$\Phi^L_{\sigma_n^v} = \begin{pmatrix} \mathrm{Id}_{n-1} & & \\ & P_v(x) & P_{v-1}(-x) \\ & P_{v-1}(x) & P_{v-2}(-x) \end{pmatrix}. \tag{5.2}$$

To understand $e(\phi_{B\sigma_n^{-1}B'}(a_{n,n+1}))$ we use Theorem 2.7 to make a calculation similar to the one at the end of the proof of Lemma 5.5. Set $b' = B\sigma_n^{-1}B'$ and consider the equation $\phi_{b'}(\mathbf{A}) = \Phi^L_{b'}\mathbf{A}\Phi^R_{b'}$. With our knowledge of $e(\Phi^L_{b'})$ and Proposition 2.2 we find that

$$e(\phi_{b'}(a_{n,n+1})) = (-1)^{v-1}(-\mu e(a_{n+1,n}) - (1-\mu)e(a_{n,n+1})) = (-1)^v X.$$

Recalling that P_v is odd or even as v is odd or even, this implies that $e\left(\phi_{b'}(\Phi^L_{\sigma_n^v})\right)$ is the matrix in (5.2) after substituting $P_{v-1}(\pm x) \mapsto P_{v-1}(\pm X)$, $P_v(x) \mapsto (-1)^v P_v(X)$, and $P_{v-2}(-x) \mapsto (-1)^v P_{v-2}(-X)$.

As $v-1>0$, there is a choice $\bar{e}(X) = X_0 \in \mathbb{C}$ so that $P_{v-1}(X_0) = (-1)^{v-1}$. Now $\Phi^L_{\sigma_n^{v-1}} = \Delta(\sigma_n^{v-1})$ has a solution when $v \geq 3$, so X_0 may be chosen so that $P_{v-2}(X_0) = 0$. Then

$$(-1)^v P_v(X_0) = (-1)^{v-1}X_0(-1)^{v-1}P_{v-1}(X_0) = (-1)^{v-1}X_0$$

by (5.1).

Finishing the proof, if we extend $\bar{e}$ to $\mathbb{C}[X]$ then

$$\begin{aligned}
\bar{e}\circ e(\Phi^L_b) &= \bar{e}\circ e\left(\phi_{B\sigma_n^{-1}B'}(\Phi^L_{\sigma_n^v})\right)\cdot \bar{e}\circ e\left(\Phi^L_{B\sigma_n^{-1}B'}\right) \\
&= \begin{pmatrix} \mathrm{Id}_{n-1} & & \\ & (-1)^{v-1}\bar{e}(X) & 1 \\ & (-1)^{v-1} & 0 \end{pmatrix} \begin{pmatrix} (-1)^{w(b)} & & & \\ & \mathrm{Id}_{n-2} & & \\ & & 0 & (-1)^{v-1} \\ & & 1 & -\bar{e}(X) \end{pmatrix} \\
&= \Delta(b).
\end{aligned}$$

The only difference when $v \leq -3$ is that the switching of i and j in the proof of Lemma 5.5 means the matrix in (5.2) should instead be

$$\Phi^L_{\sigma_n^v} = \begin{pmatrix} \mathrm{Id}_{n-1} & & \\ & P_{-v+2}(-x) & P_{-v+1}(x) \\ & P_{-v+1}(-x) & P_{-v}(x) \end{pmatrix}. \tag{5.3}$$

As $v \leq -3$ it is possible to choose X_0 so that $P_{-v+1}(-X_0) = (-1)^{v-1}$ and $P_{-v}(X_0) = 0$ as in the positive case. □

Proof of Theorem 5.4. Lemmas 5.5 and 5.6 handle much of the work. Suppose that $\epsilon : \mathcal{A}_n^{ab} \to \mathbb{C}$ determines a rank n augmentation on $HC_0(K)|_{U=1}$, and K is the closure of $B \in B_n$. Define $B' = \sigma_{i,n}^u$. We compare $\epsilon(\phi_{B\sigma_n^{-1}}(\Phi_{B'}^L))$ to $\epsilon(\Phi_{B'}^L)$ with an argument like that used in Lemma 5.6.

By calculations made in the proof of Lemma 5.5, along with the fact that $\phi_{\sigma_{i,n}}(a_{kn}) = a_{ki} - a_{kn}a_{ni}$ (and the analgous identity for $\phi_{\sigma_{i,n}}(a_{nk})$), the matrix $\Phi_{B'}^L$ has entries involving only a_{ik}, a_{ki}, a_{kn}, a_{nk} and a_{in}, a_{ni}, for $i < k < n$.

We note that for any $j < n$,

$$\begin{aligned} &\phi_{\sigma_n^{-1}}(a_{jn}) = a_{j,n+1},\ \phi_{\sigma_n^{-1}}(a_{nj}) = a_{n+1,j} \\ \text{and} \quad &\phi_{\sigma_n^{-1}}(a_{ki}) = a_{ki},\ \phi_{\sigma_n^{-1}}(a_{ik}) = a_{ik}. \end{aligned}$$

Extend ϵ to $\mathcal{A}_{n+1}^{ab}$ so that $\epsilon(a_{j,n+1}) = a_{j,n+1}$ for $1 \leq j \leq n$. The matrices Φ_B^L and Φ_B^R, by definition, record the image under ϕ_B of $a_{j,n+1}$ and $a_{n+1,j}$ respectively. We observed in Theorem 1.10 that $\epsilon(\Phi_B^L) = \Delta(B) = \epsilon(\Phi_B^R)$. Moreover, we note that $\epsilon(\phi_B(a_{ik}))$ is either a_{ik} or $(-1)^{w(B)}a_{ik}$ according to whether $i > 1$ or $i = 1$, since $\epsilon(\phi_B(\mathbf{A})) = \epsilon(\mathbf{\Lambda^{-1}} \cdot \mathbf{A} \cdot \mathbf{\Lambda})$. A similar statement holds for a_{ki}.

If $i > 1$ and $u + k - 1$ is even, then the previous paragraph and Theorem 2.7 imply that $\epsilon(\phi_{B\sigma_n^{-1}}(\Phi_{B'}^L))$ is the matrix $\Phi_{B'}^L$, but with a_{jn} (resp. a_{nj}) replaced by $a_{j,n+1}$ (resp. $a_{n+1,j}$) for $j = i, k$. Now Lemma 5.5 gives a map $e : \mathcal{A}_n^{ab} \to \mathcal{A}_n^{ab} \otimes \mathbb{C}$ such that

$$e(\Phi_{B'}^L) = \mathrm{diag}[\mathrm{Id}_{i-1}, (-1)^{u+v-1}, \mathrm{Id}_{n-i-1}, (-1)^{v-1}] = \mathrm{diag}[(-1)^{u+v-1}, \mathrm{Id}_{n-2}, (-1)^{v-1}]$$

as $u + v - 1$ is even.

Define $\bar{e}$ on $\mathcal{A}_{n+1}^{ab}$ so that $\bar{e}(a_{j,n+1}) = e(a_{j,n})$ for $j = i, k$, and $\bar{e}(a_{n,n+1}) = X$. Now define $\bar{\epsilon} = \bar{e} \circ \epsilon : \mathcal{A}_{n+1}^{ab} \to \mathbb{C}[X]$. Then by Lemma 5.6, since $\bar{\epsilon}$ satisfies properties (a) and (b), it determines a rank $n+1$ augmentation for the closure of $B\sigma_n^{-1}B'\sigma_n^v$.

If $i = 1$ and u is odd then $\epsilon(\phi_{B\sigma_n^{-1}}(\Phi_{B'}^L))$ is $\Phi_{B'}^L$ but with a_{1n} (resp. a_{n1}) replaced by $(-1)^{w(B)}a_{1,n+1}$ (resp. $(-1)^{w(B)}a_{n+1,1}$). The fact that the polynomials P_u, P_{u-2} on the diagonal of $\Phi_{B'}^L$ are odd functions, along with the relation in (5.1), allows us to choose the sign of $P_u(\bar{e}(x))$, with $P_{u-2}(-\bar{e}(x))$ having opposite sign (in similar fashion to the proof of Lemma 5.5). This means that we may find $\bar{\epsilon}$, in similar fashion to the case $i > 1$, satisfying the hypotheses of Lemma 5.6.

This proves the theorem for the case that $\delta = -1$. When $\delta = 1$ the closure of $B\sigma_n^\delta B'\sigma_n^v$ is the mirror of a knot K for which the theorem is already proved. By [19, Prop. 4.2], the mirror of K has a rank $n+1$ augmentation if K has a rank $n+1$ augmentation. This finishes the proof. □

Corollary 1.13. *If $|u|, |w| \geq 2$, $|v| \geq 3$, and $\delta = \pm 1$ and a knot K is the closure of $b = \sigma_1^w \sigma_2^\delta \sigma_1^u \sigma_2^v$, then the closure of V_K contains a curve of rank 3 augmentations.*

Proof. If u is odd then this follows from Theorem 5.4 by taking $n = 2$ and $i = 1$. If u is even then w must be odd as K is a knot. There is either a positive or negative flype (according to the sign of δ) taking b to the braid $\sigma_1^u \sigma_2^\delta \sigma_1^w \sigma_2^v$, which also has closure K. Now apply Theorem 5.4 to this braid. □

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