

Floer theory and topology of $\text{Diff}(S^2)$

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We say that a fixed point of a diffeomorphism is non-degenerate if 1 is not an eigenvalue of the linearization at the fixed point. We use pseudo-holomorphic curves techniques to prove the following: the inclusion map $i : \text{Diff}^1(S^2) \rightarrow \text{Diff}(S^2)$ vanishes on all homotopy groups, where $\text{Diff}^1(S^2) \subset \text{Diff}(S^2)$ denotes the space of orientation preserving diffeomorphisms of S^2 with a prescribed non-degenerate fixed point. This complements the classical results of Smale and Eels and Earle.

1. Introduction

The group of orientation preserving diffeomorphisms of S^2 has been shown to deformation retract to $SO(3)$ in the classical work of Smale [9]. However it seems we are still very far from completely understanding some finer aspects of its topology. Consider for example the following elementary question.

Question 1.1. *Let $\text{Diff}^1(S^2)$ denote the space of orientation preserving diffeomorphisms of S^2 , with one prescribed non-degenerate fixed point (with possibly other fixed points). Here non-degenerate means that 1 is not an eigenvalue of the linearization at the fixed point. Is this space contractible?*

We expect that the answer is yes, since $\text{Diff}^1(S^2) \cap SO(3)$ is just the space $SO^1(3)$ of rotations about a fixed axis, not containing the identity map, and so forms a contractible space. Moreover Earle and Eells [2] prove a somewhat related fact that the space of orientation preserving diffeomorphisms of S^2 fixing three points is contractible.

The intuition behind the argument of Smale or Earle and Eells is indicative that there should be a deformation retraction of $\text{Diff}^1(S^2)$ onto $SO^1(3)$. One tricky point in directly applying Smale's argument (for instance) is that in constructing the deformation retraction one must be careful to keep our

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fixed point fixed and non-degenerate. If we allow the deformation retraction to take place in the ambient space $\text{Diff}(S^2)$, of orientation preserving diffeomorphisms, some version of Smale's argument may (very likely) give a null-homotopy of the inclusion $\text{Diff}^1(S^2) \rightarrow \text{Diff}(S^2)$, although this is not known at the moment. We show using very different techniques the following:

Theorem 1.2. *The inclusion map $i : \text{Diff}^1(S^2) \rightarrow \text{Diff}(S^2)$ vanishes on all homotopy groups.*

Our proof involves elliptic, or more precisely pseudo-holomorphic curve techniques in symplectic geometry, and is morally closely connected to Floer theory. We certainly hope that our techniques can be sharpened to give that $\text{Diff}^1(S^2, \omega)$ is itself contractible.

Part of the interest in developing the main technique of this paper is that some version of it may be applicable in some other contexts, particularly the context of contactomorphism groups of contact 3-folds.

Remark 1.3. *A technically identical argument but dealing with Hamiltonian fibrations over discs, rather than $\mathbb{C}\mathbb{P}^1$, gives an analogue of the above theorem for the space $\text{Lag}(S^2)$ of Lagrangian submanifolds in S^2 Hamiltonian isotopic to equator, i.e. simple closed smooth unparametrized loops partitioning S^2 into a pair of regions with equal area. The basic ingredient for this is that a Hamiltonian loop of Lagrangian submanifolds, that is a loop obtained by a Hamiltonian flow, gives rise to a sub-fibration of $M \times D^2$ over the boundary ∂D^2 with fiber $L_\theta \subset M_\theta$ over θ the element of the loop at θ . We may then analogously study moduli spaces of holomorphic disks with boundary on this Lagrangian sub-fibration, for suitable almost complex structures, see for instance [1] for a similar story, or [4] for the monotone case.*

Claim 1.4. *The inclusion map $i : \text{Lag}^1(S^2) \rightarrow \text{Lag}(S^2)$ vanishes on all homotopy groups. Where the superscript 1 means that we take the subspace of those Lagrangians transversally intersecting the standard equator at a prescribed point, (there may be other intersections).*

2. Proof of Theorem 1.2

Suppose that we are given a smooth map $g : S^m \rightarrow \Omega\text{Diff}^1(S^2)$, where $\Omega\text{Diff}^1(S^2)$ denotes the space of based smooth loops in $\text{Diff}^1(S^2, \omega)$, which are constant near end points. We show that $i_* \circ g$ is null-homotopic as follows. Associated to each $p \in \Omega\text{Diff}^1(S^2)$ there is a structure group $\text{Diff}(S^2)$

S^2 fibration X_p over $\mathbb{C}P^1$, with $\text{Diff}(S^2)$ the group of orientation preserving diffeomorphisms, which is formed via the clutching construction. It will be more illuminating to think of it as a conformal symplectic fibration, that is a fibration whose transition maps are conformal symplectic.

The fact that we are pushing g from $\Omega\text{Diff}^1(S^2)$, specifically the non degeneracy property of the associated diffeomorphisms is used to construct a very special almost complex structure on $X_s = X_{i_*g(s)}$ parametrically with s and using that a natural foliation of X_s by holomorphic sections. This foliation in turn determines a natural smooth trivialization of X_s . The construction of this foliation uses in particular classical positivity of intersections ideas of Gromov-McDuff, particularly following [3]. Although our techniques are based on the theory of closed pseudo-holomorphic curves, the underlying idea is closely related to Floer theory. In particular we have in mind some potential generalizations which use Floer theory more explicitly. Given all this it is a matter of topology to deduce the main result.

We now give the detailed argument. For each s we get a conformal symplectic S^2 fibration

$$\pi_s : X_s \rightarrow \mathbb{C}P^1,$$

as follows

$$(2.1) \quad X_s := S^2 \times D_-^2 \sqcup_{g(s)} S^2 \times D_+^2,$$

D_\pm^2 identified with D^2 , and X_s is the quotient of $S^2 \times D_-^2 \sqcup S^2 \times D_+^2$ by the equivalence relation:

$$(x, (1, \theta)) \in S^2 \times \partial D_-^2$$

is equivalent to

$$(g(s)(\theta)^{-1}(x), (1, \theta)) \in S^2 \times \partial D_+^2$$

using the polar coordinates (r, θ) , $r \in [0, 1]$ on D^2 . The total space of this s -family can be understood as a smooth conformal symplectic S^2 fibration

$$\tilde{\pi} : \tilde{X} \rightarrow S^m \times S^2.$$

For each s we have a smooth section σ_s of X_s corresponding to our distinguished non-degenerate fixed point x_{fix} , in our coordinates over D_\pm^2 this section is just $z \mapsto x_{fix}$. We shall call these *spectral sections*.

Lemma 2.1. *The spectral sections σ_s have vertical Chern number 0.*

Proof. By construction the restriction N_s of the vertical bundle of X_s to σ_s is obtained by the clutching construction

$$N_s \simeq \mathbb{R}^2 \times D_-^2 \sqcup_{g_{s,*}} \mathbb{R}^2 \times D_+^2,$$

where $g_{s,*}$ denotes the linearization of the loop f_s at x_{fix} . The Conley-Zehnder index obviously extends to the linear conformal symplectic group $\text{LinConfSymp}(\mathbb{R}^{2n})$, and $CZ(g_{s,*}) = 0$ where CZ is the Conley-Zehnder index, and $CZ(g_{s,*}) = 0$ directly follows by the assumption that x_{fix} is a non-degenerate fixed point for $g(s)(\theta)$ for each θ , and by the construction of the Conley-Zehnder index in [7], (we have a loop with no crossing points with the Maslov cycle). On the other hand $CZ(g_{s,*})$ is exactly the Chern number of N_s , (for suitable normalizations.) See for example [5], actually it is obvious that this is true up to a non-zero multiple, since both the Chern number and the Conley-Zehnder index determine injections $\pi_1(\text{LinConfSymp}(\mathbb{R}^{2n})) \rightarrow \mathbb{Z}$, which is all that is necessary here. \square

Next we construct a smooth family $\{\mathcal{A}_s\}$ of $\text{Diff}(S^2)$ connections on $\{X_s\}$, as follows. Let $\mathcal{F} \rightarrow S^k$ be a fibration whose fiber \mathcal{F}_s over s is the space of $\text{Diff}(S^2)$ connections on X_s , for which the spectral section σ_s is flat. Each \mathcal{F}_s is a non-empty, affine space and hence contractible. In particular we may choose a smooth section $\{\mathcal{A}_s\}$ and this is our family.

Fix a smooth family $\{j_{z,s}\}$, $z \in \mathbb{C}\mathbb{P}^1$, $s \in S^m$ of $\{\omega_{z,s}\}$ -compatible almost complex structures on the vertical tangent bundles $\{T^{vert}|_{z,s}\tilde{X}\}$. A smooth $\text{Diff}(S^2)$ connection \mathcal{A} on X_s , gives rise to an almost complex structure $J_{\mathcal{A}}$, by asking that horizontal spaces be $J_{\mathcal{A}}$ -invariant, that on $T^{vert}|_z X_s$ $J_{\mathcal{A}}$ coincides with $j_{z,s}$, and so that the projection map π_s is $J_{\mathcal{A}}$ -holomorphic. We let $\{J_s\}$ denote the family of almost complex structures on $\{X_s\}$ corresponding to $\{\mathcal{A}_s\}$.

Lemma 2.2. *The moduli space $\overline{\mathcal{M}}(J_s, A := [\sigma_s])$ of stable J_s -holomorphic sections of X_s , in the class $A := [\sigma_s]$, contains no nodal curves.*

By a *stable holomorphic section* we mean a stable J_s -holomorphic map σ into X_s , in the classical sense, [6] with domain an unmarked nodal Riemann sphere, one of those components is called *principal*. The restriction σ_{princ} of σ , to the principal component is a J_s -holomorphic section, i.e. we have a

commutative diagram:

$$\begin{array}{ccc}
 & & X_s \\
 & \nearrow \sigma_{\text{princ}} & \downarrow \pi_s \\
 \mathbb{C}\mathbb{P}^1 & \xrightarrow{\text{id}} & \mathbb{C}\mathbb{P}.
 \end{array}$$

All the other components of σ are vertical, that is they are J_s -holomorphic maps into the fibers of X_s .

Proof. By construction we have a smooth, embedded \mathcal{A}_s -flat and hence J_s -holomorphic section σ_s of X_s . If there are stable sections with more than one component in our moduli space $\overline{\mathcal{M}}(J_s, A)$ then taking the principal component of the corresponding nodal section, we would have a smooth holomorphic section σ' in class $[\sigma_s] + A$ for some spherical fiber class A with $c_1(A) < 0$.

Consequently since $[\sigma_s] \cap [\sigma_s] = 0$ by Lemma 2.1, σ' would have negative intersection number with σ_s , which is impossible by positivity of intersections as σ_s is embedded, see [6, Section 2.6]. \square

Lemma 2.3. $\overline{\mathcal{M}}(J_s, A)$ is also regular, (in particular has the expected dimension).

Proof. By automatic transversality the top stratum is regular, as since all elements (in the top stratum) have normal bundle with vanishing Chern number, the associated real linear CR operator is automatically surjective, see for example [6, Appendix C]. But we just showed that there are no nodal curves and hence no other strata. \square

Since we have no nodal curves, all elements of $\overline{\mathcal{M}}(J_s, A)$ are in particular embedded, and this moduli space is non-empty as it has the element σ_s , then by positivity of intersections, and by regularity

$$\text{ev}_s : \overline{\mathcal{M}}(J_s, A) \rightarrow \pi_s^{-1}(z_0) \simeq S^2,$$

taking a section to its value over $0 \in \mathbb{C}$ must be a smooth degree 1 map. But then again by positivity of intersections, we get that the sections of $\overline{\mathcal{M}}(J_s, A)$ induce a canonical smooth foliation of X_s , with all leaves diffeomorphic to

$\mathbb{C}\mathbb{P}^1$, and as they are sections we get a canonical smooth trivialization

$$X_s \xrightarrow{tr_s} S^2 \times \mathbb{C}\mathbb{P}^1,$$

by identifying all fibers with the fiber over $0 \in \mathbb{C}\mathbb{P}^1$ using the leaves of the foliation. In particular each X_s is trivial as a smooth fibration. So for the rest of the argument we assume that $m > 0$.

The total space \tilde{X} of the family $\{X_s\}$ can be understood as a bundle $P_g \rightarrow S^m$ with fiber over s : $X_s \simeq S^2 \times \mathbb{C}\mathbb{P}^1$ and with structure group by the discussion above reducible to the group of $\text{Diff}(S^2)$ bundle automorphisms of $S^2 \times \mathbb{C}\mathbb{P}^1$ fixing the fiber over 0, with 0 identified with the origin of D^2_- . In other words this structure group is:

$$\Omega_{\text{id}}^2 \text{Diff}(S^2),$$

with Ω^2 denoting the based spherical mapping space in the class of the constant map. We drop the subscript id from now on.

By construction P_g is equivalent to the pullback by the sequence

$$S^m \xrightarrow{g} \Omega \text{Diff}^1(S^2, \omega) \xrightarrow{i_*} \Omega \text{Diff}(S^2),$$

of the structure group $\Omega^2 \text{Diff}(S^2)$ bundle U over $\Omega \text{Diff}(S^2)$, whose fiber over a loop γ is the fibration X_γ constructed as in (2.1). It follows by the author's [8, Section 7] that U is the universal structure group $\Omega^2 \text{Diff}(S^2)$ bundle. To note, we state results there for (as a particular case) $\text{Ham}(S^2, \omega)$ rather than $\text{Diff}(S^2)$ but the argument is the same for $\text{Diff}(S^2)$. Strictly speaking what follows by [8] is that U is universal for the subgroup of $\Omega^2 \text{Diff}(S^2)$ corresponding to bundle automorphisms (over id) of $S^2 \times \mathbb{C}\mathbb{P}^1$, which are identity bundle maps over contractible neighborhoods of $0, \infty \in \mathbb{C}\mathbb{P}^1$, however this subgroup can be easily shown to be homotopy equivalent to $\Omega^2 \text{Diff}(S^2)$.

So if P is trivial as a structure group $\Omega^2 \text{Diff}(S^2)$ bundle the map $i_* \circ g$ is null-homotopic. □

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