JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 15, Number 3, 621–643, 2017

On the symplectic structure over a moduli space of orbifold projective structures

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Let S be a compact connected oriented smooth orbifold surface. We show that using Bers simultaneous uniformization, the moduli space of projective structures on S can be mapped biholomorphically onto the total space of the holomorphic cotangent bundle of the Teichmüller space for S. The total space of the holomorphic cotangent bundle of the Teichmüller space is equipped with the Liouville holomorphic symplectic form, and the moduli space of projective structures also has a natural holomorphic symplectic form. The above identification between the moduli space of projective structures on S and the holomorphic cotangent bundle of the Teichmüller space for S is proved to be compatible with these symplectic structures. Similar results are obtained for biholomorphisms constructed using uniformizations provided by Schottky groups and Earle's version of simultaneous uniformization.

1. Introduction

The holomorphic automorphisms of the complex projective line $\mathbb{CP}^1$ are of the form $z \longmapsto (az+b)/(cz+d)$, where a, b, c, d are complex numbers with $ad-bc=1$; these are known as Möbius transformations. A projective structure on a C^∞ compact oriented surface R is defined by a covering of R by coordinate charts, compatible with the orientation, so that all the transition functions are Möbius transformations. Two projective structures on R are considered isomorphic if they differ by a diffeomorphism of R homotopic to the identity map of R. Let $\mathcal{P}(R)$ denote the space of all isomorphism classes of projective structures on R.

Consider the space of all complex structures on R compatible with the orientation. Two of them are called isomorphic if they differ by a diffeomorphism of R homotopic to the identity map of R. Let $\mathcal{T}(R)$ denote the Teichmüller space of R that parametrizes the isomorphism classes of complex structures on R. Clearly, a projective structure on R induces a complex structure on R compatible with the orientation. So there is a natural map

$$\varphi : \mathcal{P}(R) \longrightarrow \mathcal{T}(R).$$

Both $\mathcal{P}(R)$ and $\mathcal{T}(R)$ are equipped with complex structures, and the map φ is holomorphic.

It is well-known that the space of all projective structures on a fixed Riemann surface can be identified with the space of quadratic differentials on that surface (see [12, p. 292]). This means that any C^∞ section

$$f : \mathcal{T}(R) \longrightarrow \mathcal{P}(R)$$

of the above projection φ produces a diffeomorphism

$$T_f : T^*\mathcal{T}(R) \longrightarrow \mathcal{P}(R),$$

where $T^*\mathcal{T}(R)$ is the holomorphic cotangent bundle of the Teichmüller space (its fibers are identified with the space of quadratic differentials). The above diffeomorphism T_f is holomorphic if and only if f is holomorphic.

Both $T^*\mathcal{T}(R)$ and $\mathcal{P}(R)$ have natural holomorphic symplectic structures. Since $T^*\mathcal{T}(R)$ is a cotangent bundle, it has the Liouville symplectic form

$$\Omega_{\mathcal{T}} := d\sigma,$$

where σ is the tautological holomorphic one-form on $T^*\mathcal{T}(R)$. On the other hand, any projective structure on R produces a flat principal $\mathrm{PSL}(2,\mathbb{C})$–bundle on R (recall that the transition functions for a projective structure lie in $\mathrm{PSL}(2,\mathbb{C})$). Now taking monodromy of flat connections, the space $\mathcal{P}(R)$ is mapped to an open subset of the smooth part of the representation space

$$\mathrm{Hom}(\pi_1(R), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C}).$$

This map is a local biholomorphism. The smooth part of

$$\mathrm{Hom}(\pi_1(R), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$$

is equipped with a holomorphic symplectic form [2], [8]. Pulling back this $\mathcal{P}(R)$ we get a holomorphic symplectic form $\Omega_{\mathcal{P}}$ on $\mathcal{P}(R)$. In [10], Kawai showed that if f is Bers' section B, then one has

$$T_B^*\Omega_{\mathcal{P}} = \pi \cdot \Omega_{\mathcal{T}} .$$

This result was extended to the Schottky's and Earle's sections in [4] and [1] respectively.

An orbifold surface is a surfaces with weighted marked points. Our aim here is to address the question whether the above set-up generalizes to orbifolds, and whether similar results hold for orbifolds. We answer these questions affirmatively. More concretely, we begin by recalling the definition of an orbifold surface S, and explaining what a projective structure on an orbifold means. This leads us to the definitions of Teichmüller space $\mathcal{T}(S)$ and the space of projective structures $\mathcal{P}(S)$ for S. As in the surface case, there is a natural holomorphic projection

$$\widetilde{f}_S : \mathcal{P}(S) \longrightarrow \mathcal{T}(S)$$

that sends a projective structure to its underlying complex structure. There is a natural holomorphic symplectic structure on $\mathcal{P}(S)$, which we will denote by $\Omega^S_{\mathcal{P}}$.

By a Galois covering of a surface we will mean a covering map of it which is possibly ramified (locally isomorphic to $z \longmapsto z^n$ for some positive integer n) such that the group of deck transformations acts transitively on every fiber of the covering map.

To define a section of the above projection $\widetilde{f}_S$, we consider Bers' section B of an appropriate finite Galois cover of S. Then we average B over the Galois group Γ. This construction gives us a biholomorphism

$$T_{S,B} : T^*\mathcal{T}(S) \longrightarrow \mathcal{P}(S) .$$

Let $\Omega^S_{\mathcal{T}}$ denote the Liouville symplectic form on $T^*\mathcal{T}(S)$. In Theorem 4.2 we show that this mapping $T_{S,B}$ preserves the symplectic structures of the spaces, up to a constant:

$$T^*_{S,B}\Omega^S_{\mathcal{P}} = \pi \cdot \Omega^S_{\mathcal{T}} .$$

Finally, we generalize this result to the biholomorphisms $T^*\mathcal{T}(S) \longrightarrow \mathcal{P}(S)$ corresponding to Schottky's and Earle's sections.

2. Orbifold Riemann surface and projective structure

2.1. Definition of orbifold projective structure

Let $\mathbb{N}^{>1}$ be the set of integers bigger than one. An *orbifold surface* is a triple $(X, \mathbb{D}, \varpi)$, where X is a compact connected oriented C^∞ surface,

$$\mathbb{D} := \{x_1, \dots, x_n\} \subset X$$

is a finite collection of distinct ordered points, and

$$\varpi : \mathbb{D} \longrightarrow \mathbb{N}^{>1} \tag{2.1}$$

is a function. Since the elements of $\mathbb{D}$ are ordered, ϖ can be considered as a function on $\{1, \dots, n\}$.

A *coordinate function* on $(X, \mathbb{D}, \varpi)$ is a pair of the form (V, ϕ), where $V \subset \mathbb{CP}^1$ is a connected open subset, and

$$\phi : V \longrightarrow X \tag{2.2}$$

is an orientation preserving C^∞ open map, such that $\#\phi(V) \bigcap \mathbb{D} \leq 1$, and

1) if $\phi(V) \bigcap \mathbb{D} = \emptyset$, then ϕ is an embedding, and
2) if $\phi(V) \bigcap \mathbb{D} = x_i$, then ϕ is a ramified Galois covering of $\phi(V)$ with Galois group $\mathbb{Z}/\varpi(x_i)\mathbb{Z}$, and it is totally ramified over x_i but unramified over the complement $\phi(V) \setminus \{x_i\}$.

The second condition implies that $\phi(V)$ can contain at most one point of $\mathbb{D}$.

The group $\mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{CP}^1$; the action of any

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

sends any $z \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ to $(az+b)/(cz+d) \in \mathbb{CP}^1$. This action of $\mathrm{SL}(2, \mathbb{C})$ factors through the quotient group $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\pm I$. This way, $\mathrm{PGL}(2, \mathbb{C})$ gets identified with the group of all holomorphic automorphisms of $\mathbb{CP}^1$. The holomorphic automorphisms of $\mathbb{CP}^1$ are also called Möbius transformations.

A *projective atlas* on $(X, \mathbb{D}, \varpi)$ is a collection of coordinate functions $\{(V_j, \phi_j)\}_{j\in J}$ such that

1) $\bigcup_{j\in J} \phi_j(V_j) = X$,

2) for every $j \in J$, each deck transformation of the Galois covering

$$\phi_j : V_j \longrightarrow \phi_j(V_j)$$

is the restriction of some Möbius transformation (if ϕ_j is an embedding, then this condition is automatically satisfied because the Galois group is trivial), and

3) for every $j, k \in J$, and for every connected and simply connected open subset

$$V \subset \phi_k^{-1}\left((U_j \bigcap U_k) \setminus \mathbb{D}\right),$$

each branch of the function $\phi_j^{-1} \circ \phi_k$ over V is the restriction of some Möbius transformation.

By a branch of $\phi_j^{-1} \circ \phi_k$ over V we mean a holomorphic map

$$f : V \longrightarrow \mathbb{CP}^1$$

such that $\phi_k = \phi_j \circ f$. Note that if $f : V \longrightarrow \mathbb{CP}^1$ is continuous and

$$\phi_k = \phi_j \circ f,$$

then f is holomorphic.

In view of the second condition in the above definition of projective structure, if some branch of the function $\phi_j^{-1} \circ \phi_k$ over V is the restriction of some Möbius transformation, then each branch of the function $\phi_j^{-1} \circ \phi_k$ over V is the restriction of some Möbius transformation.

Two projective atlases $\{(V_j, \phi_j)\}_{j \in J}$ and $\{(V_i, \phi_i)\}_{i \in I}$ will be called *equivalent* if their union $\{(V_j, \phi_j)\}_{j \in J \cup I}$ is also a projective atlas.

Definition 2.1. A *projective structure* on $(X, \mathbb{D}, \varpi)$ is an equivalence class of projective atlases.

Given a projective structure P on $(X, \mathbb{D}, \varpi)$, a coordinate function (V, ϕ) is called *compatible with* P if (V, ϕ) lies in some projective atlas in the equivalence class defined by P.

When the orbifold structure $(\mathbb{D}, \varpi)$ on X is clear from the context, a projective structure on $(X, \mathbb{D}, \varpi)$ will also be called an *orbifold projective structure* on X.

A projective structure on $(X, \mathbb{D}, \varpi)$ produces a complex structure on X. Indeed, this is an immediate consequence of the following fact: if

$$\mathbb{CP}^1 \supset V \xrightarrow{\phi} X$$

is a coordinate map (as in (2.2)) such that each deck transformation of ϕ is the restriction of some Möbius transformation, then there is a unique complex structure on $\phi(V)$ such that ϕ is a holomorphic map.

An *orbifold Riemann surface* is an orbifold surface $(X, \mathbb{D}, \varpi)$ such that X is equipped with a complex structure compatible with the orientation of X.

From the above observation that a projective structure produces a complex structure it follows that a projective structure on an orbifold surface produces an orbifold Riemann surface.

Given an orbifold Riemann surface $(X, \mathbb{D}, \varpi)$, a projective structure P on the orbifold surface $(X, \mathbb{D}, \varpi)$ will be called *compatible* with the complex structure if the complex structure on X given by P coincides with the given complex structure on X. A compatible projective structure on the orbifold Riemann surface $(X, \mathbb{D}, \varpi)$ will also be called a *projective structure* on the orbifold Riemann surface $(X, \mathbb{D}, \varpi)$.

When the orbifold structure $(\mathbb{D}, \varpi)$ on the Riemann surface X is clear from the context, a projective structure on the orbifold Riemann surface $(X, \mathbb{D}, \varpi)$ will also be called an *orbifold projective structure on the Riemann surface X*.

We now recall Lemma 3.2 of [5] on the existence of orbifold projective structures.

Lemma 2.2 ([5]). *An orbifold Riemann surface $(X, \mathbb{D}, \varpi)$ admits a compatible projective structure if and only if at least one of the following three conditions are satisfied:*

1) $\text{genus}(X) \geq 1$,
2) $\#\mathbb{D} \notin \{1, 2\}$,
3) *if* $\#\mathbb{D} = 2$, *then* $\varpi(x_1) = \varpi(x_2)$.

Therefore, $(X, \mathbb{D}, \varpi)$ does not admit a compatible projective structure if and only if either $\text{genus}(X) = 0 = n - 1$ *or* $\text{genus}(X) = 0 = n - 2$ *with* $\varpi(x_1) \neq \varpi(x_2)$.

Assumption A: Henceforth, for all orbifold surfaces considered, we assume that at least one of the three conditions in Lemma 2.2 is satisfied.

In view of Assumption A and Lemma 2.2, all orbifold Riemann surfaces considered henceforth admit a projective structure.

2.2. Parameter space for orbifold projective structures

Let S be a compact connected oriented surface of genus g. Fix n ordered points $\mathbb{D} := \{x_1, \ldots, x_n\}$ on S. Let $\mathcal{T}(S)$ be the Teichmüller space corresponding to this n-pointed surface S. We recall a construction of $\mathcal{T}(S)$. The space of all complex structures on the smooth surface S compatible with its orientation will be denoted by $\mathrm{Com}(S)$. Let $\mathrm{Diff}_{\mathbb{D}}(S)$ be the group of all orientation preserving diffeomorphisms of S that fix the subset $\{x_1, \ldots, x_n\}$ pointwise. Let

$$\mathrm{Diff}^0_{\mathbb{D}}(S) \subset \mathrm{Diff}_{\mathbb{D}}(S)$$

be the subgroup consisting of all diffeomorphisms homotopic, fixing $\mathbb{D}$ pointwise, to the identity map of S. This group $\mathrm{Diff}^0_{\mathbb{D}}(S)$ acts on $\mathrm{Com}(S)$; the action of any $f \in \mathrm{Diff}^0(S)$ sends a complex structure to its pullback using f^{-1}. The above Teichmüller space $\mathcal{T}(S)$ is the quotient

$$\mathcal{T}(S) = \mathrm{Com}(S)/\mathrm{Diff}^0_{\mathbb{D}}(S)\,.$$

The space $\mathrm{Com}(S)$ has a natural complex structure which induces a complex structure on $\mathcal{T}(S)$.

Let $\mathrm{Proj}(S)$ denote the space of all projective structures on $(S, \mathbb{D}, \varpi)$. Consider the group of diffeomorphisms $\mathrm{Diff}^0_{\mathbb{D}}(S)$ defined above. This group has a natural action on $\mathrm{Proj}(S)$. The action of any $\tau \in \mathrm{Diff}^0_{\mathbb{D}}(S)$ on $\mathrm{Proj}(S)$ takes a projective structure P to the one uniquely determined by the following property: a coordinate function (V, ϕ) is compatible with this projective structure if and only if the coordinate function $(V, \tau^{-1} \circ \phi)$ is compatible with P. Let

$$\mathcal{P}(S) := \mathrm{Proj}(S)/\mathrm{Diff}^0_{\mathbb{D}}(S) \tag{2.3}$$

be the quotient. There is a natural projection

$$\widetilde{f}_S : \mathcal{P}(S) \longrightarrow \mathcal{T}(S) \tag{2.4}$$

that sends a projective structure to the complex structure underlying it.

There is a natural complex structure on $\mathrm{Proj}(S)$ that induces a complex structure on the quotient space $\mathcal{P}(S)$. An alternative way of describing this complex structure on $\mathrm{Proj}(S)$ is the following. A projective structure on the orbifold S defines a flat principal $\mathrm{PSL}(2,\mathbb{C})$–bundle on the complement $S \setminus \mathbb{D}$. Sending a flat connection to its monodromy representation, the space $\mathcal{P}(S)$ gets identified with a submanifold of the smooth locus of the representation space

$$\mathrm{Hom}(\pi_1(S \setminus \mathbb{D}),\ \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})\,.$$

The smooth locus of $\mathrm{Hom}(\pi_1(S \setminus \mathbb{D}),\ \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$ has a complex structure given by the complex structure on $\mathrm{PSL}(2,\mathbb{C})$, and the submanifold $\mathcal{P}(S)$ is preserved by the underlying almost complex structure. Therefore, $\mathcal{P}(S)$ gets an induced complex structure. The projection $\widetilde{f}_S$ in (2.4) is holomorphic.

Proposition 2.3. *As before, let g denote the genus of S. The dimension of this complex manifold $\mathcal{P}(S)$ is*

- $6g - 6 + 2n$ *if* genus$(S) \geq 2$,
- $2n$ *(respectively,* 2*) if* genus$(S) = 1$ *with* $n > 0$ *(respectively,* $n = 0$*), and*
- $2(n-3)$ *(respectively,* 0*) if* genus$(S) = 0$ *with* $n \geq 4$ *(respectively,* $n \leq 3$*).*

Proof. Since the two cases genus$(S) = 0 = n-1$ and genus$(X) = 0 = n-2$ with $\varpi(x_1) \neq \varpi(x_2)$ are omitted (see Assumption A), a theorem due to Bundgaard–Nielsen and Fox says that there is a finite Galois covering

$$\psi : Y \longrightarrow S \tag{2.5}$$

such that ψ is unramified over $S \setminus \mathbb{D}$, and for each $x_i \in \mathbb{D}$, the order of ramification at every point of $\psi^{-1}(x_i)$ is $\varpi(x_i)$ [13, p. 26, Proposition 1.2.12], where ϖ is the function in (2.1). We call the order of ramification at 0 of the map $z \longmapsto z^m$ to be m. Let $\widetilde{g}$ denote the genus of Y.

Let $\mathrm{Proj}_0(Y)$ denote the space of all projective structures on the compact oriented surface Y (for Y, the subset of orbifold points is empty); the subscript "0" is to emphasize that the orbifold structure on Y is trivial. The space of all complex structures on the smooth surface Y compatible with its orientation will be denoted by $\mathcal{C}(Y)$. There is a natural map

$$f'_Y : \mathrm{Proj}_0(Y) \longrightarrow \mathcal{C}(Y) \tag{2.6}$$

that sends a projective structure on Y to the complex structure on Y defined by it.

Let $\mathrm{Diff}^0(Y)$ denote the group of all diffeomorphisms of Y homotopic to the identity map of Y. The group $\mathrm{Diff}^0(Y)$ acts on both $\mathrm{Proj}(Y)$ and $\mathcal{C}(Y)$. Define

$$\mathcal{P}(Y) := \mathrm{Proj}_0(Y)/\mathrm{Diff}^0(Y) \quad \text{and} \quad \mathcal{T}(Y) := \mathcal{C}(Y)/\mathrm{Diff}^0(Y)\,.$$

The quotient $\mathcal{T}(Y)$ is called the *Teichmüller space* for Y. It is a complex manifold of dimension $3\widetilde{g}-3$ or 1 or 0 depending on whether $\widetilde{g} \geq 2$ or $\widetilde{g} = 1$ or $\widetilde{g} = 0$. Also, $\mathcal{T}(Y)$ is contractible (diffeomorphic to the unit ball). The quotient $\mathcal{P}(Y)$ is a complex manifold of dimension $2 \cdot \dim_{\mathbb{C}} \mathcal{T}(Y)$, and it is also contractible. The map f'_Y in (2.6) descends to a projection

$$f_Y : \mathcal{P}(Y) \longrightarrow \mathcal{T}(Y)\,. \tag{2.7}$$

This map f_Y is a holomorphic submersion. More precisely, f_Y makes $\mathcal{P}(Y)$ a holomorphic fiber bundle over $\mathcal{T}(Y)$. In fact, $\mathcal{P}(Y)$ is a torsor for the holomorphic cotangent bundle $T^*\mathcal{T}(Y)$, which means that for any $Z \in \mathcal{T}(Y)$ the vector space of $T^*_Z\mathcal{T}(Y)$ acts freely and transitively on the fiber of f_Y over the point Z. That $\mathcal{P}(Y)$ is a torsor for $T^*\mathcal{T}(Y)$ follows from the facts that the space of all projective structure on a given compact Riemann surface Z is an affine space for the space of quadratic differentials $H^0(Z,\, T^*Z \otimes T^*Z)$, while the fiber of $T^*\mathcal{T}(Y)$ at any point $Z \in \mathcal{T}(Y)$ is also $H^0(Z,\, T^*Z \otimes T^*Z)$.

Let

$$\Gamma := \mathrm{Gal}(\psi) \tag{2.8}$$

be the Galois group for the covering map ψ in (2.5). We will show that Γ has a natural action on both $\mathcal{P}(Y)$ and $\mathcal{T}(Y)$.

Take any $T \in \Gamma$. Since T is an orientation preserving diffeomorphism of Y, it produces a self–map of $\mathrm{Proj}_0(Y)$ that sends a projective structure P to the one uniquely determined by the following property: a coordinate function $(V\,, \phi)$ is compatible with this projective structure if and only if the coordinate function $(V\,, T^{-1} \circ \phi)$ is compatible with P. This way we get an action of Γ on $\mathrm{Proj}_0(Y)$. Similarly, Γ acts on the space of complex structures $\mathcal{C}(Y)$: the action of any $T \in \Gamma$ sends a complex structure to the pullback of it by T^{-1}. The map f'_Y in (2.6) evidently intertwines these actions of Γ on $\mathrm{Proj}_0(Y)$ and $\mathcal{C}(Y)$.

Next we note that the conjugation action Γ on the group of diffeomorphisms of Y preserves $\mathrm{Diff}^0(Y)$, meaning for any $T \in \Gamma$ and $T' \in \mathrm{Diff}^0(Y)$, we have

$$T^{-1}T'T \in \mathrm{Diff}^0(Y)\,.$$

Therefore, the above actions of Γ on $\mathrm{Proj}_0(Y)$ and $\mathcal{C}(Y)$ descend to actions of Γ on the quotient spaces $\mathcal{P}(Y)$ and $\mathcal{T}(Y)$ respectively. Since f'_Y in (2.6) is Γ–equivariant, the descended map f_Y in (2.7) is also Γ–equivariant.

From the construction of the actions of Γ on $\mathcal{P}(Y)$ and $\mathcal{T}(Y)$ it follows that these actions preserve the complex structures of $\mathcal{P}(Y)$ and $\mathcal{T}(Y)$.

The space $\mathcal{P}(S)$ in (2.3) is the fixed point locus

$$\mathcal{P}(S) = \mathcal{P}(Y)^{\Gamma} \subset \mathcal{P}(Y) \tag{2.9}$$

for the action of Γ on $\mathcal{P}(Y)$. Indeed, the pullback to Y of a projective structure on $(S\,,\mathbb{D}\,,\varpi)$ is a projective structure on Y. This pulled back projective structure on Y is clearly invariant under the action of Γ. Conversely, if we have a Γ–invariant projective structure P on Y, then P defines a projective structure P' on $(S\,,\mathbb{D}\,,\varpi)$. A coordinate function $\phi : V \longrightarrow S$, with V simply connected, is compatible with P' if the lift $V \longrightarrow Y$ of ϕ is compatible with P; note that in view of the definition of a projective structure on $(S\,,\mathbb{D}\,,\varpi)$, the properties of the covering map ψ imply that ϕ lifts to a map to Y.

Since the action of Γ on $\mathcal{P}(Y)$ preserves the complex structure of $\mathcal{P}(Y)$, the fixed point locus $\mathcal{P}(Y)^{\Gamma}$ is a complex submanifold of $\mathcal{P}(Y)$.

To compute the dimension of the fixed point locus $\mathcal{P}(S)$, we first note that the map f_Y being Γ–equivariant restricts to a map

$$F_Y := f_Y|_{\mathcal{P}(Y)^{\Gamma}} : \mathcal{P}(S) = \mathcal{P}(Y)^{\Gamma} \longrightarrow \mathcal{T}(Y)^{\Gamma}\,, \tag{2.10}$$

where $\mathcal{T}(Y)^{\Gamma} \subset \mathcal{T}(Y)$ is the fixed point locus for the action of Γ on $\mathcal{T}(Y)$. We note that $\mathcal{T}(Y)^{\Gamma}$ is a complex submanifold because the action of Γ preserves the complex structure of $\mathcal{T}(Y)$. Any Γ–invariant complex structure on Y produces a complex structure on S. On the other hand, any complex structure on S defines a Γ–invariant complex structure on Y. It is known that $\mathcal{T}(Y)^{\Gamma}$ is identified with the earlier defined Teichmüller space $\mathcal{T}(S)$ for the n-pointed surface S [9]. In particular, $\dim \mathcal{T}(Y)^{\Gamma}$ coincides with the dimension of $\mathcal{T}(S)$. Therefore,

- if $\mathrm{genus}(S) \geq 2$, then $\dim \mathcal{T}(Y)^{\Gamma} = 3g-3+n$,
- if $\mathrm{genus}(S) = 1$ and $n \geq 1$, then $\dim \mathcal{T}(Y)^{\Gamma} = n$,
- if $\mathrm{genus}(S) = 1$ and $n = 0$, then $\dim \mathcal{T}(Y)^{\Gamma} = 1$,

- if genus$(S) = 0$ and $n \geq 4$, then $\dim \mathcal{T}(Y)^{\Gamma} = n - 3$, and
- if genus$(S) = 0$ and $n \leq 3$, then $\dim \mathcal{T}(Y)^{\Gamma} = 0$.

The map F_Y in (2.10) is a holomorphic fiber bundle whose fiber over any Riemann surface $Z \in \mathcal{T}(Y)^{\Gamma}$ is an affine space for the complex vector space consisting of all Γ–invariant holomorphic sections of $H^0(Z, K_Z^{\otimes 2})$ (the space of all Γ–invariant quadratic differentials on Z). Indeed, this follows from the fact that the space of all Γ–invariant projective structures on Z is an affine space for $H^0(Z, K_Z^{\otimes 2})^{\Gamma}$.

Take any point

$$Z \in \mathcal{T}(Y)^{\Gamma}.$$

We consider Z as Y equipped with a Γ–invariant complex structure. We have

$$H^0(Z, K_Z^{\otimes 2})^{\Gamma} = H^0(Z/\Gamma, K_{Z/\Gamma}^{\otimes 2} \otimes \mathcal{O}_{Z/\Gamma}(\mathbb{D})).$$

Using Serre duality, we have

$$H^0(Z/\Gamma, K_{Z/\Gamma}^{\otimes 2} \otimes \mathcal{O}_{Z/\Gamma}(\mathbb{D})) = H^1(Z/\Gamma, T(Z/\Gamma) \otimes \mathcal{O}_{Z/\Gamma}(-\mathbb{D}))^*,$$

where $T(Z/\Gamma)$ is the holomorphic tangent bundle of Z/Γ. But

$$H^1(Z/\Gamma, T(Z/\Gamma) \otimes \mathcal{O}_{Z/\Gamma}(-\mathbb{D}))$$

is the holomorphic tangent space to $\mathcal{T}(Y)^{\Gamma}$ at the point $Z/\Gamma \in \mathcal{T}(Y)^{\Gamma}$. Hence $\dim \mathcal{T}(Y)^{\Gamma} = \dim H^0(Z, K_Z^{\otimes 2})^{\Gamma}$. Since

$$\dim \mathcal{P}(S) = \dim \mathcal{T}(Y)^{\Gamma} + \dim H^0(Z, K_Z^{\otimes 2})^{\Gamma},$$

the proposition follows. □

Remark 2.4. Take any $Z/\Gamma \in \mathcal{T}(S)$ as above. The fiber of the holomorphic cotangent bundle $T^*\mathcal{T}(S)$ over Z/Γ is identified with $H^0(Z, K_Z^{\otimes 2})^{\Gamma}$. Therefore, $\mathcal{P}(S)$ is a holomorphic affine bundle over $\mathcal{T}(S)$ for the holomorphic cotangent bundle $T^*\mathcal{T}(S)$.

3. Bers' Section

Let us continue with the setting of the proof of Proposition 2.3: we have a surface Y and a finite group of diffeomorphisms Y, acting on Y. An element

γ of Γ induces a holomorphic automorphism of $\mathcal{T}(Y)$ as well as of $\mathcal{P}(Y)$, which we denote by

$$\gamma_{\mathcal{T}} : \mathcal{T}(Y) \longrightarrow \mathcal{T}(Y) \quad \text{and} \quad \gamma_{\mathcal{P}} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(Y) \tag{3.1}$$

respectively. The mapping f_Y of (2.7) is Γ-equivariant, which means that

$$f_Y \circ \gamma_{\mathcal{P}} = \gamma_{\mathcal{T}} \circ f_Y . \tag{3.2}$$

The above holomorphic mapping $\gamma_{\mathcal{T}} : \mathcal{T}(Y) \longrightarrow \mathcal{T}(Y)$ induces in a natural way a holomorphic self-map of the cotangent space

$$d^*\gamma_{\mathcal{T}} : T^*\mathcal{T}(Y) \longrightarrow T^*\mathcal{T}(Y) .$$

This space $T^*\mathcal{T}(Y)$, being the total space of the holomorphic cotangent bundle of a complex manifold, has a natural holomorphic symplectic form, which is known as the Liouville form. The Liouville symplectic form on $T^*\mathcal{T}(Y)$ will be denoted by $\Omega_{\mathcal{T}}$. Since $\gamma_{\mathcal{T}}$ is a biholomorphism of $\mathcal{T}(Y)$, it is easy to check that the induced map $d^*\gamma_{\mathcal{T}}$ of $T^*\mathcal{T}(Y)$ preserves the form Liouville $\Omega_{\mathcal{T}}$.

On the other hand, the space $\mathcal{P}(Y)$ is mapped onto an open subset of the smooth locus of the representation space

$$\mathrm{Hom}(\pi_1(Y), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C}) .$$

This map is a local biholomorphism. Hence $\mathcal{P}(Y)$ has a natural holomorphic symplectic structure [2], [8], [1]. We will denote the holomorphic symplectic form on $\mathcal{P}(Y)$ by $\Omega_{\mathcal{P}}$.

Lemma 3.1. *The mapping $\gamma_{\mathcal{P}}$ in* (3.1) *preserves the symplectic form $\Omega_{\mathcal{P}}$.*

Proof. Let Z be a compact connected oriented surface. Fix a base point $z_0 \in Z$. Let G be a semisimple Lie group. Consider $\mathrm{Hom}(\pi_1(Z, z_0), G)/G$ equipped with the natural symplectic form Ω (see [2] and [8] for Ω). Let

$$\alpha : Z \longrightarrow Z$$

be an orientation preserving diffeomorphism. The isomorphism

$$\alpha_* : \pi_1(Z, z_0) \longrightarrow \pi_1(Z, \alpha(z_0))$$

induced by α produces a diffeomorphism

$$\alpha' : \mathrm{Hom}(\pi_1(Z, z_0), G)/G \longrightarrow \mathrm{Hom}(\pi_1(Z, \alpha(z_0)), G)/G .$$

By choosing a path from z_0 to $\alpha(z_0)$, the group $\pi_1(Z, \alpha(z_0))$ is naturally identified with $\pi_1(Z, z_0)$ up to an inner automorphism of $\pi_1(Z, z_0)$. Therefore, α' produces a diffeomorphism

$$\widetilde{\alpha} : \mathrm{Hom}(\pi_1(Z, z_0), G)/G \longrightarrow \mathrm{Hom}(\pi_1(Z, z_0), G)/G .$$

This diffeomorphism $\widetilde{\alpha}$ preserves Ω. Indeed, this follows immediately from the construction of Ω (see [2] and [8]). We mentioned earlier that $\Omega_{\mathcal{P}}$ coincides with the above symplectic form Ω for $G = \mathrm{PGL}(2, \mathbb{C})$. Therefore, we now conclude that $\gamma_{\mathcal{P}}$ in (3.1) preserves $\Omega_{\mathcal{P}}$. □

The projection f_Y in (2.7) has a holomorphic section constructed by Bers using the notion of simultaneous uniformization [3]. We will denote that section by B:

$$B : \mathcal{T}(Y) \longrightarrow \mathcal{P}(Y) . \tag{3.3}$$

Let

$$T_B : T^*\mathcal{T}(Y) \longrightarrow \mathcal{P}(Y)$$

be the holomorphic mapping that sends any $(Z, \theta) \in T^*\mathcal{T}(Y)$ to $B(Z) + \theta$; note that $\theta \in H^0(Z, K_Z^{\otimes 2})$ and the fiber of $\mathcal{P}(Y)$ over $Z \in \mathcal{T}(Y)$ is an affine space for $H^0(Z, K_Z^{\otimes 2})$, so $B(Z) + \theta$ is also an element of the fiber of $\mathcal{P}(Y)$ over Z. This map T_B is clearly a biholomorphism In [10], Kawai proved that T_B preserves the symplectic structures of $\mathcal{T}(Y)$ and $\mathcal{P}(Y)$ in the sense that

$$\frac{1}{\pi} \cdot T_B^*\Omega_{\mathcal{P}} = \Omega_{\mathcal{T}} . \tag{3.4}$$

For an element γ of Γ, we define a holomorphic mapping

$$B_\gamma : \mathcal{T}(Y) \longrightarrow \mathcal{P}(Y), \quad B_\gamma := \gamma_{\mathcal{P}} \circ B \circ \gamma_{\mathcal{T}}^{-1} . \tag{3.5}$$

We note that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{P}(Y) & \xrightarrow{\gamma_{\mathcal{P}}} & \mathcal{P}(Y) \\ \big\downarrow f_Y & & \big\downarrow f_Y \\ \mathcal{T}(Y) & \xrightarrow{\gamma_{\mathcal{T}}} & \mathcal{T}(Y) \end{array}$$

Therefore, B_γ in (3.5) is also a holomorphic section of the projection f_Y.

Let

$$T_{B_\gamma} : T^*\mathcal{T}(Y) \longrightarrow \mathcal{P}(Y) \tag{3.6}$$

be the biholomorphism that sends any $(Z, \theta) \in T^*\mathcal{T}(Y)$ to $B_\gamma(Z) + \theta$.

Lemma 3.2. *For the above map T_{B_γ} the following holds:*

$$T^*_{B_\gamma}\Omega_{\mathcal{P}} = \pi \cdot \Omega_{\mathcal{T}} .$$

Proof. The following diagram of holomorphic maps is commutative

$$\begin{array}{ccc} T^*\mathcal{T}(Y) & \xrightarrow{T_B} & \mathcal{P}(Y) \\ \Big\downarrow d^*\gamma_{\mathcal{T}} & & \Big\downarrow \gamma_{\mathcal{P}} \\ T^*\mathcal{T}(Y) & \xrightarrow{T_{B_\gamma}} & \mathcal{P}(Y) \end{array} \tag{3.7}$$

We noted earlier that $d^*\gamma_{\mathcal{T}}$ preserves the Liouville symplectic form $\Omega_{\mathcal{T}}$ because $d^*\gamma_{\mathcal{T}}$ is induced by a biholomorphism of $\mathcal{T}(Y)$. From Lemma 3.1 we know that $\gamma_{\mathcal{P}}$ preserves $\Omega_{\mathcal{P}}$. Therefore, in view of the commutative diagram in (3.7), from (3.4) we conclude that $T^*_{B_\gamma}\Omega_{\mathcal{P}} = \pi \cdot \Omega_{\mathcal{T}}$. □

Since the fibers of f_Y are affine spaces for the fibers of the holomorphic cotangent bundle of $\mathcal{T}(Y)$, we have

$$B_\gamma - B \in H^0(\mathcal{T}(Y), T^*\mathcal{T}(Y)) ,$$

in other words, $B_\gamma - B$ is a holomorphic one-form on $\mathcal{T}(Y)$.

Lemma 3.3. *The above holomorphic one-form $B_\gamma - B$ on $\mathcal{T}(Y)$ is closed.*

Proof. Let

$$\beta : T^*\mathcal{T}(Y) \longrightarrow T^*\mathcal{T}(Y)$$

be the holomorphic automorphism of the fiber bundle

$$p : T^*\mathcal{T}(Y) \longrightarrow \mathcal{T}(Y) \tag{3.8}$$

defined by $v \longmapsto v + (B_\gamma - B)(p(v))$. Clearly, we have

$$T_B \circ \beta = T_{B_\gamma} .$$

Therefore, from (3.4) and Lemma 3.2 it follows that

$$\pi \cdot \Omega_T \,=\, (T_{B_\gamma})^*\Omega_P \,=\, (T_B \circ \beta)^*\Omega_P \,=\, \beta^*(T_B)^*\Omega_P \,=\, \pi \cdot \beta^*\Omega_T\,.$$

Hence $\beta^*\Omega_{\mathcal{T}} - \Omega_{\mathcal{T}} \,=\, 0$. On the other hand, from the construction of $\Omega_{\mathcal{T}}$ it follows immediately that

$$\beta^*\Omega_{\mathcal{T}} - \Omega_{\mathcal{T}} \,=\, p^*d(B_\gamma - B)\,,$$

where p is the projection in (3.8). Combining these two we conclude that the form $B_\gamma - B$ is closed. □

Since the fibers of f_Y (see (2.7)) are affine spaces for the fibers of $T^*\mathcal{T}(Y)$, and B_γ is a holomorphic section of f_Y, we conclude that

$$B' \,:=\, \frac{1}{\#\Gamma}\sum_{\gamma\in\Gamma} B_\gamma \tag{3.9}$$

is a holomorphic section of f_Y, where $\#\Gamma$ is the order of the group Γ. Let

$$T_{B'} \,:\, T^*\mathcal{T}(Y) \,\longrightarrow\, \mathcal{P}(Y) \tag{3.10}$$

be the holomorphic isomorphism that sends any

$$(Z\,,\theta) \,\in\, T^*\mathcal{T}(Y)$$

to $B'(Z) + \theta$.

Proposition 3.4. *For the above map $T_{B'}$, the following holds:*

$$T^*_{B'}\Omega_{\mathcal{P}} \,=\, \pi \cdot \Omega_{\mathcal{T}}\,.$$

Proof. Let

$$\omega \,:=\, \frac{1}{\#\Gamma}\sum_{\gamma\in\Gamma}(B_\gamma - B) \tag{3.11}$$

be the holomorphic one-form on $\mathcal{T}(Y)$, where $B_\gamma - B$ is the one-form in Lemma 3.3. Let

$$\beta' \,:\, T^*\mathcal{T}(Y) \,\longrightarrow\, T^*\mathcal{T}(Y)$$

be the holomorphic automorphism of the fiber bundle $T^*\mathcal{T}(Y)$ defined by $v \longmapsto v + \omega(p(v))$, where p is the projection in (3.8). Clearly, we have

$$T_B \circ \beta' = T_{B'} . \tag{3.12}$$

As noted earlier, from the construction of $\Omega_{\mathcal{T}}$ it follows immediately that

$$(\beta')^*\Omega_{\mathcal{T}} - \Omega_{\mathcal{T}} = p^* d\omega = dp^*\omega . \tag{3.13}$$

From Lemma 3.3 we have

$$d\omega = \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} d(B_\gamma - B) = 0 . \tag{3.14}$$

Hence $p^*d\omega = 0$. Consequently, from (3.13) we have

$$(\beta')^*\Omega_{\mathcal{T}} = \Omega_{\mathcal{T}} .$$

Therefore, using (3.4),

$$(T_B \circ \beta')^*\Omega_{\mathcal{P}} = (\beta')^* T_B^*\Omega_{\mathcal{P}} = (\beta')^*(\pi \cdot \Omega_{\mathcal{T}}) = \pi \cdot (\beta')^*\Omega_{\mathcal{T}} = \pi \cdot \Omega_{\mathcal{T}} .$$

Now from (3.12) it follows that $T_{B'}^*\Omega_{\mathcal{P}} = \pi \cdot \Omega_{\mathcal{T}}$. □

4. Bers' section in the orbifold set-up

As before,

$$\mathcal{P}(Y)^\Gamma \subset \mathcal{P}(Y) \quad \text{and} \quad \mathcal{T}(Y)^\Gamma \subset \mathcal{T}(Y)$$

are the fixed point loci for the actions of Γ on $\mathcal{P}(Y)$ and $\mathcal{T}(Y)$ respectively. Consider the projection F_Y in (2.10). We will construct a holomorphic section of it.

From the construction of B' in (3.9) it follows immediately that the action of the Galois group Γ on $\mathcal{P}(Y)$ preserves the image $B'(\mathcal{T}(Y))$ (the action leaves the subset invariant, but not pointwise). The action of Γ on the Teichmüller space $\mathcal{T}(Y)$ produces an action of Γ on the cotangent bundle $T^*\mathcal{T}(Y)$. Since the projection f_Y in (2.7) is Γ–equivariant, and $B'(\mathcal{T}(Y))$ is preserved by the action of Γ, it follows that the biholomorphism $T_{B'}$ in (3.10) is Γ–equivariant.

The image $B'(\mathcal{T}(Y))$ being Γ–equivariant restricts to a holomorphic section

$$\widetilde{B} : \mathcal{T}(Y)^{\Gamma} \longrightarrow \mathcal{P}(Y)^{\Gamma} \tag{4.1}$$

of the projection F_Y constructed in (2.10). As noted before, for any

$$Z \in \mathcal{T}(Y)^{\Gamma},$$

the holomorphic cotangent space $T^*_Z(\mathcal{T}(Y)^{\Gamma})$ coincides with the space of invariants

$$H^0(Z, K_Z^{\otimes 2})^{\Gamma} \subset H^0(Z, K_Z^{\otimes 2}).$$

We also recall that F_Y is a holomorphic fiber bundle whose fiber over any Riemann surface $Z \in \mathcal{T}(Y)^{\Gamma}$ is an affine space for the vector space $H^0(Z, K_Z^{\otimes 2})^{\Gamma}$. Therefore, $\widetilde{B}$ in (4.1) produces a biholomorphism

$$T_{\widetilde{B}} : T^*(\mathcal{T}(Y)^{\Gamma}) \longrightarrow \mathcal{P}(Y)^{\Gamma} \tag{4.2}$$

that sends any $(Z, \theta) \in T^*(\mathcal{T}(Y)^{\Gamma})$ to $\widetilde{B}(Z) + \theta \in \mathcal{P}(Y)^{\Gamma}$.

The symplectic form $\Omega_{\mathcal{P}}$ on $\mathcal{P}(Y)$ restricts to a symplectic form on $\mathcal{P}(Y)^{\Gamma}$. This symplectic form on $\mathcal{P}(Y)^{\Gamma}$ will be denoted by $\Omega^{\Gamma}_{\mathcal{P}}$. On the other hand, the cotangent bundle $T^*(\mathcal{T}(Y)^{\Gamma})$ is equipped with the Liouville symplectic form; this Liouville symplectic form will be denoted by $\Omega^{\Gamma}_{\mathcal{T}}$.

Proposition 4.1. *For the biholomorphism $T_{\widetilde{B}}$ in (4.2), the following holds:*

$$T^*_{\widetilde{B}}\Omega^{\Gamma}_{\mathcal{P}} = \pi \cdot \Omega^{\Gamma}_{\mathcal{T}}.$$

Proof. Consider the action of Γ on $T^*\mathcal{T}(Y)$ induced by the action of Γ on $\mathcal{T}(Y)$. It is easy to see that the fixed-point set $(T^*\mathcal{T}(Y))^{\Gamma} \subset T^*\mathcal{T}(Y)$ is identified with $T^*(\mathcal{T}(Y)^{\Gamma})$. In particular, we have

$$T^*(\mathcal{T}(Y)^{\Gamma}) \subset T^*\mathcal{T}(Y).$$

The Liouville symplectic form $\Omega_{\mathcal{T}}$ on $T^*\mathcal{T}(Y)$ restricts to the Liouville symplectic form $\Omega^{\Gamma}_{\mathcal{T}}$ on $T^*(\mathcal{T}(Y)^{\Gamma})$. The form $\Omega^{\Gamma}_{\mathcal{P}}$, by definition, is the restriction of $\Omega_{\mathcal{P}}$. Also, The map $T_{\widetilde{B}}$ coincides with the restriction of $T_{B'}$ (constructed in (3.10)) to the submanifold $T^*(\mathcal{T}(Y)^{\Gamma})$ of $T^*\mathcal{T}(Y)$. Therefore, the proposition follows from Proposition 3.4. □

We recall that $\mathcal{P}(Y)^\Gamma$ and $\mathcal{T}(Y)^\Gamma$ are identified with $\mathcal{P}(S)$ and $\mathcal{T}(S)$ respectively. Using these identifications, the projection F_Y in (2.10) coincides with the projection $\widetilde{f}_S$ in (2.4).

The construction of the symplectic form on $\mathcal{P}(Y)$ extends to $\mathcal{P}(S)$. Indeed, the symplectic form on the representation space of a compact surface group constructed in [8], [2] can be generalized to the representation space of the fundamental group of a punctured surface once we fix the monodromy around the punctures (see [6]). Let $\Omega^S_{\mathcal{P}}$ denote the holomorphic symplectic form on $\mathcal{P}(S)$. This form $\Omega^S_{\mathcal{P}}$ coincides with $\Omega_{\mathcal{P}}$ using the above mentioned identification of $\mathcal{P}(Y)^\Gamma$ with $\mathcal{P}(S)$. The Liouville symplectic form on $T^*\mathcal{T}(S)$ will be denoted by $\Omega^S_{\mathcal{T}}$.

The section $\widetilde{B}$ (see (4.1)) of the projection F_Y produces a holomorphic section of the projection $\widetilde{f}_S$ in (2.4). As done before, this holomorphic section produces a biholomorphism

$$T_{S,B} : T^*\mathcal{T}(S) \longrightarrow \mathcal{P}(S).$$

This map $T_{S,B}$ clearly coincides with $T_{\widetilde{B}}$ in (4.2) after identifying $\mathcal{P}(S)$ and $T^*\mathcal{T}(S)$ with $\mathcal{P}(Y)^\Gamma$ and $T^*(\mathcal{T}(Y)^\Gamma)$ respectively.

Therefore, Proposition 4.1 gives the following:

Theorem 4.2. *For the above biholomorphism $T_{S,B}$, the following holds:*

$$T^*_{S,B}\Omega^S_{\mathcal{P}} = \pi \cdot \Omega^S_{\mathcal{T}}.$$

5. Schottky and Earle uniformizations

The projection f_Y of (2.7) admits a couple of other natural holomorphic sections, apart from the one described in (3.3). One of these sections is given by Earle in [7], which is a modification of the simultaneous uniformization theorem. The other one is given by the uniformization by Schottky groups. It is natural to ask whether the symplectic structures are preserved by the biholomorphisms $T^*\mathcal{T}(S) \longrightarrow \mathcal{P}(S)$ constructed using these sections, i.e., whether the analogue of Theorem 4.2 holds. Our aim in this final section is to address this question.

In [7], Earle constructed a holomorphic section

$$e : \mathcal{T}(Y) \longrightarrow \mathcal{P}(Y).$$

The construction of e, which follows closely the approach of the simultaneous uniformization theorem of Bers, is done using a marking on Y and an involution of the fundamental group of Y induced by an orientation reversing diffeomorphism of Y. The construction of e follows a modification of the simultaneous uniformization theorem. In a sense this section is intrinsic, since it does not require fixing a base point of $\mathcal{T}(Y)$, unlike in the construction of Bers' section. As said above, the section e is holomorphic.

For each element $\gamma \in \Gamma$, we get another section e_γ defined by

$$e_\gamma = \gamma_{\mathcal{P}} \circ e \circ \gamma_{\mathcal{T}}^{-1}$$

(just as done in (3.5)). Let ρ_γ denote the one-form on $\mathcal{T}(Y)$ defined by

$$\rho_\gamma := e_\gamma - B_\gamma \tag{5.1}$$

Lemma 5.1. *The form ρ_γ is closed.*

Proof. Let ϕ be the C^∞ section of the projection the projection f_Y (see (2.7)) given by the Fuchsian uniformization. This section is not holomorphic. We define

$$\alpha_\gamma := e_\gamma - \phi \quad \text{and} \quad \beta_\gamma := B_\gamma - \phi\,.$$

Since e_γ and B_γ are holomorphic sections, Theorem 9.2 of [11, p. 355] applies, and from it we conclude that

$$d\alpha_\gamma = d\beta_\gamma\,.$$

Therefore, $d\rho_\gamma = d(e_\gamma - B_\gamma) = d\alpha_\gamma - d\beta_\gamma = 0.$ □

Let

$$T_E^\gamma : T^*\mathcal{T}(Y) \longrightarrow \mathcal{P}(Y)$$

be the biholomorphism that sends any $(Z\,,\theta) \in T^*\mathcal{T}(Y)$ to $e_\gamma(Z) + \theta \in \mathcal{P}(Y)$. Let

$$A_{\rho_\gamma} : T^*\mathcal{T}(Y) \longrightarrow T^*\mathcal{T}(Y)$$

be the holomorphic automorphism defined by $v \longmapsto v + \rho_\gamma(p(v))$, where p is the projection in (3.8). Clearly,

$$T_E^\gamma = T_{B_\gamma} \circ A_{\rho_\gamma}\,, \tag{5.2}$$

where T_{B_γ} is constructed in (3.6). Now, using (3.4) and Lemma 5.1,

$$(T_{B_\gamma} \circ A_{\rho_\gamma})^*\Omega_{\mathcal{P}} = (A_{\rho_\gamma})^*(T_{B_\gamma})^*\Omega_{\mathcal{P}}$$

$$= \pi \cdot (A_{\rho_\gamma})^* \Omega_{\mathcal{T}} = \pi \cdot (\Omega_{\mathcal{T}} + d\rho_\gamma) = \pi \cdot \Omega_{\mathcal{T}} .$$

Therefore, from (5.2) we have

$$(T^\gamma_E)^* \Omega_{\mathcal{P}} = \pi \cdot \Omega_{\mathcal{T}} . \tag{5.3}$$

We now average these sections, and define

$$e' := \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} e_\gamma , \tag{5.4}$$

which is a holomorphic section of f_Y. Let

$$T_{e'} : T^*\mathcal{T}(Y) \longrightarrow \mathcal{P}(Y)$$

be the biholomorphism that sends any (Z, θ) to $e'(Z) + \theta$ (as in (3.10)).

Proposition 5.2. *For the above map $T_{e'}$, the following holds:*

$$T^*_{e'} \Omega_{\mathcal{P}} = \pi \cdot \Omega_{\mathcal{T}} .$$

Proof. As in Proposition 3.4, we define a holomorphic one-form on $T(Y)$ by

$$\mu := \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} (e_\gamma - e) .$$

In view of (5.3), it suffices to show that μ is closed (see the proof of Proposition 3.4).

Consider ω constructed in (3.11). We observe that

$$\begin{aligned} \mu - \omega &= \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} (e_\gamma - e - B_\gamma + B) \\ &= \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma} ((e_\gamma - B_\gamma) - (e - B)) \\ &= \frac{1}{\#\Gamma} \sum_{\gamma\in\Gamma\setminus e^0} (e_\gamma - B_\gamma) , \end{aligned}$$

where e^0 is the identity element of Γ. Hence from Lemma 5.1 it follows that $d(\mu - \omega) = 0$. Now from (3.14) we conclude that $d\mu = 0$. □

The constructions of Section 4 can be done with the section e' in (5.4) instead of B', to obtain a section

$$\widetilde{e} : \mathcal{T}(Y)^{\Gamma} \longrightarrow \mathcal{P}(Y)^{\Gamma}$$

of the projection F_Y in (2.10). Just as before, $\widetilde{e}$ produces a biholomorphism

$$T_{S,e} : T^*\mathcal{T}(S) \longrightarrow \mathcal{P}(S)\,.$$

Now just as in Theorem 4.2, we have:

Theorem 5.3. *For biholomorphic mapping $T_{S,e}$,*

$$T_{S,e}^* \Omega_{\mathcal{P}}^S = \pi \cdot \Omega_{\mathcal{T}}^S\,.$$

We finish this section with the observation that all the above constructions carry to the case of the Schottky section. This section also satisfies McMullen's theorem (Theorem 9.2 of [11, p. 355]) which says the following: Let $X = \Omega/\Gamma$ be the quotient Riemann surface for a finitely generated Kleinian group Γ, and let μ, $\nu \in M(X)$ be a pair of sufficiently smooth Beltrami differentials. Then

$$\int_X \phi_\mu\, \nu = \int_X \phi_\nu\, \mu\,,$$

where ϕ_μ, $\phi_\nu \in L^1(X, dz^2)$ give the projective distortions of μ and ν.

Therefore, Theorem 5.3 remains valid for the biholomorphism

$$T^*\mathcal{T}(S) \longrightarrow \mathcal{P}(S)$$

given by the Schottky section.

Acknowledgements

We thank the referee for helpful comments. The second-named author is partially supported by a J. C. Bose Fellowship.

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Received December 18, 2013
Accepted July 22, 2016