Categories of (co)isotropic linear relations

ALAN WEINSTEIN

In categories of linear relations between finite dimensional vector spaces, composition is well-behaved only at pairs of relations satisfying transversality and monicity conditions. A construction of Wehrheim and Woodward makes it possible to impose these conditions while retaining the structure of a category. We analyze the resulting category in the case of all linear relations, as well as for (co)isotropic relations between symplectic vector spaces. In each case, the Wehrheim-Woodward category is a central extension of the original category of relations by the endomorphisms of the unit object, which form a free submonoid with two generators in the additive monoid of pairs of nonnegative integers.

1. Introduction

Wehrheim and Woodward [9] have introduced a category of symplectic manifolds allowing compositions between canonical relations (lagrangian submanifolds of products), such that the composition of relations f and g remains as the usual set-theoretic composition when the pair (f, g) satisfies local transversality and global embedding conditions. For linear canonical relations between symplectic vector spaces, these two conditions are equivalent, and we have shown in [5] that the morphisms in the "linear WW category" may be described as pairs consisting of a linear relation and a nonnegative integer which measures non-transversality and may be identified with an endomorphism of the zero-dimensional vector space.

In Poisson geometry, the most important relations (such as the graphs of Poisson maps) are the coisotropic ones, so it is interesting to study the WW category generated by such relations. Here, the conditions of transversality and embedding are no longer equivalent, even in the linear case. We will show that the description of a WW morphism in the linear coisotropic case requires *two* nonnegative integers. For a pair (A, B) of coisotropic subspaces representing an endomorphism of a point, these integers are the dimension of $A \cap B$ and the codimension of $A + B$. (The numbers are equal in the lagrangian case.)

We will first study relations and then WW categories in the context of pure linear algebra, paying special attention to duality. When we pass to symplectic linear algebra, as was the case for the classification of coisotropic pairs in [6], it will be easier to deal directly with the isomorphic (via symplectic orthogonality) category of *isotropic* relations.

We will end with some remarks on the question of extending our results from symplectic to presymplectic and Poisson vector spaces, as was done for the classification of subspace pairs in [6].

In the original WW category, the endomorphisms of a point are equivalence classes of pairs of lagrangian *submanifolds* in symplectic *manifolds*; in that case, the equivalence relation remains quite mysterious, even for endomorphisms of the one-point manifold represented by transversal lagrangian intersections in two points. We hope that our analysis of the linear case will be a useful step toward dealing with the nonlinear coisotropic case.

I would like to thank David Li-Bland and Jonathan Lorand for many discussions of linear relations, and for comments on this manuscript.

2. Selective categories

We begin by recalling some basic notions from [5] and refer to that paper for more details and further examples.

Definition 2.1. A **selective category** is a category with a distinguished class of morphisms, called **suave**, and a class of composable pairs of suave morphisms called **congenial pairs**, such that:

1) Any identity morphism is suave.

2) If f and g are suave, (f, g) is composable, and f or g is an identity morphism, then (f, g) is congenial.

3) If (f, g) is congenial, then fg is suave.

4) If f is a suave isomorphism, its inverse f^{-1} is suave as well, and the pairs (f, f^{-1}) and (f^{-1}, f) are both congenial.

5) If (f, g, h) is a composable triple, then (f, g) and (fg, h) are congenial if and only if (g, h) and (f, gh) are. When these conditions hold, we call (f, g, h) a **congenial triple**.

It follows easily from the definition that any identity morphism is suave, and that notion of congeniality may be extended from pairs and triples to sequences of arbitrary length.

Example 2.2. We denote by **REL** the category whose objects are sets and whose morphisms are relations; i.e., the morphisms in $\mathbf{REL}(X, Y)$ are the subsets of $X \times Y$.

A useful selective structure in **REL** is the one in which all morphisms are suave, but only **monic** pairs are congenial. These are diagrams[1]

$$X \xleftarrow{f} Y \xleftarrow{g} Z$$

for which, whenever (x, y) and (x, y') belong to f, and (y, z) and (y', z) belong to g, then $y = y'$. In other words, (x, z) can belong to fg via at most one element of the intermediate space Y. The only condition in Definition 2.1 which takes a bit of work to check is the last one. For this, it suffices to observe that congenial triples can be defined directly: (f, g, h) is congenial if and only if, whenever (x, w) belongs to fgh, there is exactly one pair (y, z) for which $(x, y) \in f$, $(y, z) \in g$, and $(z, w) \in h$.

Definition 2.3. A **selective functor** between selective categories is one which takes congenial pairs to congenial pairs.

Composition with identity morphisms shows that a selective functor takes suave morphisms to suave morphisms.

The Wehrheim-Woodward construction provides an embedding ι of the the suave morphisms in $\mathbf{C}$ to the morphisms in a category $WW(\mathbf{C})$ in which the composition of any *congenial* pair remains the same as the composition in $\mathbf{C}$. Briefly stated (see [5] for details), the morphisms of $WW(\mathbf{C})$ are equivalence classes of composable sequences in $\mathbf{C}$, where two sequences are equivalent if one can be obtained from the other by moves in which a composable pair of adjacent entries in the sequence is replaced by its composition, or vice versa. In particular, identity morphisms may be freely inserted or removed. We denote the equivalence class of $(f_1, \ldots, f_r)$ by $[f_1, \ldots, f_r]$.

The map ι from the suave morphisms in $\mathbf{C}$ to $WW(\mathbf{C})$ takes each suave morphism f to $[f]$, the equivalence class of the sequence having f as its single entry. $WW(\mathbf{C})$ is then characterized by the following universal property.

Proposition 2.4. *Composition of the entries in composable sequences gives a well-defined functor $c([f_1, \ldots, f_r]) = f_1 \cdots f_r$ to $\mathbf{C}$ from $WW(\mathbf{C})$. The category $WW(\mathbf{C})$ has the universal property that any map to a category $\mathbf{B}$ from*

---

[1] We will usually denote morphisms by arrows pointing to the left, and we will therefore write $\mathrm{Hom}(X, Y)$ to denote the morphisms $X \leftarrow Y$ *to X from Y.*

the suave morphisms of **C** *which takes units to units and which takes congenial compositions to compositions in* **B** *is of the form* $b\iota$ *for a unique functor* **B** $\xleftarrow{b}$ $WW(\mathbf{C})$.

We will refer to c as the **shadow** functor. Composing it with the inclusion ι of the suave morphisms in **C** gives the identity, so ι is injective, i.e., distinct suave morphisms cannot become equal when considered as WW-morphisms via ι.

Remark 2.5. Any selective functor between selective categories induces a functor between their Wehrheim-Woodward categories. In particular, if a selective category **C** admits a transpose operation, i.e. an involutive contravariant endofunctor $f \mapsto f^t$ which fixes objects and takes congenial pairs to congenial pairs, then this operation extends to $WW(\mathbf{C})$, taking a sequence to the same sequence with its order reversed and each entry replaced by its transpose. For instance, the operation which reverses the order of ordered pairs is a transpose operation on **REL**.

Remark 2.6. If the suave morphisms in **C** form a subcategory **C**$'$ and we declare every composable pair to be congenial, then **C**$'$ $\xleftarrow{c}$ $WW(\mathbf{C})$ is an isomorphism of categories.

An important simplification result concerning selective categories requires the following finer structure.

Definition 2.7. A **highly selective category** is a selective category provided with two subcategories of suave morphisms called **reductions** and **coreductions** such that:

1) Any suave isomorphism is a coreduction and a reduction.

2) If (f,g) is a composable pair of suave morphisms, and if f is a coreduction or g is a reduction, then (f,g) is congenial.

3) Any suave morphism f may be factored as gh, where g is a reduction, h is a coreduction, and (g,h) is congenial.

It follows from the injectivity of ι that the subcategories of reductions and coreductions in **C** are mapped isomorphically to subcategories of $WW(\mathbf{C})$, which we will again refer to as reductions and coreductions. Since all identity morphisms are suave, these subcategories are wide; i.e., they contain all the objects.

We will indicate that a morphism is special by decorating its arrow: $X \leftarrow\!\!\!\prec Q$ for a reduction, and $Q \prec\!\!\!\leftarrow Y$ for a coreduction. A composition $X \leftarrow Q \leftarrow Y$ of suave morphisms is then congenial if at least one arrow is decorated at Q.

Example 2.8. In the selective category of relations in Example 2.2, with the congenial pairs the monic ones, we obtain a highly selective structure by defining the reductions to be the single-valued (also called **coinjective**) surjective relations and the coreductions the everywhere-defined (also called **cosurjective**) injective ones.

The following Theorem was proven in [5], and in a special case already in [10]. It shows that arbitrary WW morphisms can be represented by sequences with just two entries.

Theorem 2.9. *Let* **C** *be a highly selective category, and let* $[f_1, \ldots, f_r]$ *be a morphism in* $WW(\mathbf{C})$. *Then there exist an object* Q *and morphisms* $A \in \mathrm{Hom}(X_0, Q)$ *and* $B \in \mathrm{Hom}(Q, X_r)$ *in* **C** *such that* A *is a reduction,* B *is a coreduction, and* $[f_1, \ldots, f_r] = [A, B]$.

A convenient way to analyze WW morphisms is via their graphs. The notion of graph applies when a category **C** has a rigid monoidal structure compatible with its highly selective structure. We recall that a monoidal structure on **C** consists of a "tensor product" bifunctor (often, but not always, denoted $\otimes$) from $\mathbf{C} \times \mathbf{C}$ to **C** and an object **1** which is a unit for this product.[2] The monoidal structure is rigid when each object X has a dual $\overline{X}$ satisfying some identities which, in particular, give a bijective correspondence between $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(X \otimes \overline{Y}, \mathbf{1})$. The morphism $X \otimes \overline{Y} \leftarrow \mathbf{1}$ corresponding to $X \xleftarrow{f} Y$ in $\mathrm{Hom}(X, Y)$ is known as the **graph** of f. Compatibility with a selective structure means that the suave morphisms are closed under the operations and that a certain identity in the definition of rigidity involves only congenial compositions. A key consequence of this compatibility is that a morphism $X \leftarrow Y$ is suave if and only if its graph $X \times \overline{Y} \leftarrow \mathbf{1}$ is.

3. Linear relations

We begin with finite-dimensional vector spaces with no additional structure.

---

[2]For simplicity, we will assume that the associativity and unit axioms are satisfied strictly.

Definition 3.1. The objects of the category **LREL** are the finite-dimensional vector spaces over a fixed ground field (which will usually remain unspecified), and the morphisms **LREL**(X, Y) are the linear subspaces of $X \times Y$. Composition of morphisms is composition of relations.

The category **LREL** was studied in detail by Towber [8], who gave an explicit classification of the endomorphisms up to conjugation by automorphisms, as well as arbitrary morphisms up to "left-right equivalence". The endomorphism classification is also derivable from the classification of subspace quadruples in [4] since, as the authors there observe, a relation $R \subseteq X \times X$ may be recovered from the quadruple of subspaces in $X \times X$ consisting of R, $X \times 0_X$, $0_X \times X$, and the diagonal.

Since a given set may have more than one vector spa9ce structure, **LREL** is not a subcategory of **REL**, but the forgetful functor **REL** $\leftarrow$ **LREL** is faithful. Via this functor, the special subcategories of **REL** (injectives, surjectives, reductions, and coreductions) and the monic pairs have their counterparts in **LREL**. Furthermore, we have the criteria that a linear relation $X \xleftarrow{f} Y$ is injective if and only if its **kernel**, the subspace $\operatorname{Ker} f = f^t(0_X)$, is the zero subspace, and that f is coinjective if and only if the kernel $f(0_Y)$ of f^t is zero. We call this latter kernel the **indeterminacy** of f and denote it by $\operatorname{Indet} f$.

Extending the well known fact that any linear map $X \xleftarrow{f} Y$ may be factored as $X \leftarrow \operatorname{Im} f \leftarrow Y/\operatorname{Ker} f \leftarrow Y$, we have the natural factorization of any linear relation $X \xleftarrow{f} Y$ as the strongly transversal composition

$$(1) \qquad\qquad X \leftarrowtail \operatorname{Im} f/\operatorname{Indet} f \twoheadleftarrow \operatorname{Dom} f/\operatorname{Ker} f \twoheadleftarrow Y.$$

This factorization leads to a proof of the fact stated at the end of Section 1 of [8] that a linear relation is classified up to isomorphism by the dimensions of the six spaces occurring in (1). (Since the middle map is an iaomorphism, only five of these dimensions are independent.)

3.1. Monicity

A composable pair (f, g) of linear relations is monic if and only $\operatorname{Ker} f \cap \operatorname{Indet} g = 0$, in particular if f is injective or g is coinjective. Another criterion for monicity of the pair (f, g) when $f \in \textbf{LREL}(X, Y)$ and $g \in \textbf{LREL}(Y, Z)$ is that the subspaces $f \times g$ and $0_X \times \Delta_Y \times 0_Z$ of $X \times Y \times Y \times Z$ intersect in the zero subspace. In general, the dimension of this intersection will be called the **defect** of the pair.

We may extend the definition and quantitative measure of monicity to composable sequences of arbitrary length.

Definition 3.2. The dimension of

$$(f_1 \times f_2 \times \cdots \times f_r) \cap (0_{X_0} \times \Delta_{X_1} \times \cdots \times \Delta_{X_{r-1}} \times 0_{X_r})$$

will be called the **excess** of the sequence and denoted by $\mathcal{E}(f_1, f_2, \ldots, f_r)$.

The excess $\mathcal{E}(f_1, f_2, \ldots, f_r)$ may be interpreted as the dimension, for any (x_0, x_r) in the composed relation $f_1 \cdots f_r \subseteq (X_0 \times X_r)$, of the affine space of r-tuples

$$((x_0, x_1), (x_1, x_2), \ldots, (x_{r-2}, x_{r-1}), (x_{r-1}, x_r))$$

with each $(x_{i-1}, x_i) \in f_i)$. We will refer to such r-tuples as **trajectories** to x_0 from x_r. We will use the term "monic" by extension from the case $r = 2$ for any composable sequence with zero excess, i.e. with at most one trajectory between any two endpoints. Notice that any 1-entry sequence has zero excess.

Proposition 3.3. *Let $(f_1, f_2, \ldots, f_r)$ be a composable r-tuple of linear relations, and let $1 \leq j \leq r - 1$. Then*

$$(2) \qquad \mathcal{E}(f_1, f_2, \ldots, f_r) = \mathcal{E}(f_1, f_2, \ldots, f_j) + \mathcal{E}(f_1 \cdots f_j, f_{j+1} \cdots f_r)$$
$$+ \mathcal{E}(f_{j+1}, f_{j+2}, \ldots, f_r).$$

Proof. We consider $\mathcal{E}(f_1, f_2, \ldots, f_r)$ as the dimension of the (linear) space of trajectories to 0_{X_0} from 0_{X_r}. Each such trajectory can be decomposed into a pair of trajectories, τ' to 0_{X_0} from some $x_j \in X_j$, and τ'' to this x_j from 0_{X_r}. The set S_j of x_j which occur in this way may be identified with the space of trajectories to 0_{X_0} from 0_{X_r} for the 2-member sequence $(f_1 \cdots f_j, f_{j+1} \cdots f_r)$; it therefore has dimension $\mathcal{E}(f_1 \cdots f_j, f_{j+1} \cdots f_r)$. For each $x_j \in S_j$, the affine spaces τ' and τ'' of trajectories which connect it to 0_{X_0} and 0_{X_1} have dimensions $\mathcal{E}(f_1, f_2, \ldots, f_j)$ and $\mathcal{E}(f_{j+1}, f_{j+2}, \ldots, f_r)$ respectively. Adding the dimensions of these three affine spaces gives the result. $\square$

This leads immediately to:

Corollary 3.4. *Let $(f_1, f_2, \ldots, f_r)$ be a composable r-tuple of linear relations, and consider some implementation of the r-fold composition $f_1 f_2 \cdots f_r$ by a sequence of pairwise compositions, i.e. a "parenthesization." Then the*

excess of the composable r-tuple may be obtained by summing the excesses of the pairwise compositions and is therefore independent of the parenthesization. In particular, the r-tuple is monic if and only if all of the pairwise compositions in the sequence are monic.

3.2. Duality

The contravariant transposition functor on **LREL** which exchanges the entries in each ordered pair is compatible with the one on **REL** via the forgetful functor between these two categories, but there is another contravariant functor on **LREL** which does not correspond to anything in **REL**. It extends the usual duality for linear maps.

Definition 3.5. For $f \in \mathbf{LREL}(X, Y)$, the **dual** $f^* \in \mathbf{LREL}(Y^*, X^*)$ is the relation $\{(\eta, \xi) \in Y^* \times X^* \mid \eta(y) = \xi(x) \text{ for all } (x, y) \in f\}$.

Remark 3.6. The dual f^* may also be seen as the annihilator of f with respect to the bilinear pairing

$$\langle (x, y), (\eta, \xi) \rangle = \xi(x) - \eta(y)$$

between $X \times Y$ and $Y^* \times X^*$. When X and Y are reflexive, this pairing is nondegenerate, from which it follows easily that $f^{**} = f$ when we identify X^{**} with X and Y^{**} with Y.

Proofs of the following result may be found in [1] and [8]; the latter applies in the infinite-dimensional case as well.

Proposition 3.7. *Duality is functorial; i.e., for linear relations*

$$X \xleftarrow{f} Y \xleftarrow{g} Z,$$

$(fg)^* = g^* f^*$.

That duality is a (contravariant) functor is not completely trivial to prove. In fact, it depends on the fact that any linear functional on a subspace of a vector space can be extended to the whole space; functoriality fails for modules over arbitrary rings exactly when there are nonextendible functionals, as in the following example.

Example 3.8. Let R be a ring, and let X be a submodule of the R-module Y for which there is an element ξ_0 of X^* which does not extend to Y. (For

instance, with $R = \mathbb{Z}$, let $X = 2\mathbb{Z}$, $Y = \mathbb{Z}$, and $\xi_0(n) = n/2$.) Let Z be the zero module. Then, if $f \in \mathbf{LREL}(X, Y)$ is the transpose of the inclusion of X in Y, and $g \in \mathbf{LREL}(X, Y)$ is the graph of the zero map, $(fg)^* \neq g^* f^*$.

In fact, fg is the graph of the zero map $X \leftarrow Z$, and hence $(fg)^*$ is the graph of the zero map $Z^* \leftarrow X^*$. On the other hand, g^* is the graph of the zero map $Z^* \leftarrow Y^*$, while f^* is the transpose of the restriction map $X^* \leftarrow Y^*$. The composition $g^* f^*$ thus consists of those pairs $(0_{Z^*}, \xi) \in Z^* \times X^*$ for which there is an $\eta \in Y^*$ whose restriction to X is ξ. Then $(0_{Z^*}, \xi_0)$ belongs to $(fg)^*$ but not to $g^* f^*$.

The duality functor has the following properties (some true for modules as well as vector spaces).

Proposition 3.9. *For any linear relation $X \xleftarrow{f} Y$ between vector spaces:*

1) $(f^t)^* = (f^*)^t$;

2) $\operatorname{Ker} f^* = (\operatorname{Im} f)^\circ$;

3) $\operatorname{Im} f^* = (\operatorname{Ker} f)^\circ$;

4) $\operatorname{Indet} f^* = (\operatorname{Dom} f)^\circ$;

5) $\operatorname{Dom} f^* = (\operatorname{Indet} f)^\circ$,

where $^\circ$ denotes the annihilator of a subspace in the dual of the ambient space.

Consequently, the duality functor exchanges injectives with cosurjectives, coinjectives with surjectives, and reductions with coreductions.

Proof. The first two properties are immediate from the definitions. For 3, when η annihilates the kernel of f, we can use the equation $\xi(x) = \eta(y)$ for $(x, y) \in f$ to define a functional ξ on the image of f, and then extend ξ to all of X. Transposing statements 2 and 3 yields statements 4 and 5.   □

We also have:

Proposition 3.10. *Let (f, g) be a composable pair of linear relations. Then:*

- $(\operatorname{Ker} f \cap \operatorname{Indet} g)^\circ = \operatorname{Dom} g^* + \operatorname{Im} f^*$, *and*
- $(\operatorname{Dom} f + \operatorname{Im} g)^\circ = \operatorname{Ker} g^* \cap \operatorname{Indet} f^*$

Remark 3.11. *All of the results in this section remain valid if V^* is replaced by any space equipped with a nondegenerate pairing with V, such as V itself when it carries a symplectic structure.*

3.3. Transversality

To define selective structures on categories of linear relations, we can of course use the condition of monicity, carried over from **REL**, but compatibility with duality and other considerations (such as continuity) suggest that we impose another condition as well, which does not have a counterpart in **REL**.

Definition 3.12. A pair (f, g) of linear relations $X \xleftarrow{f} Y$ and $Y \xleftarrow{g} Z$ is **transversal** if the sum $f \times g + X \times \Delta_Y \times Z$ is the entire space $X \times Y \times Y \times Z$. A pair which is both transversal and monic is **strongly transversal**.

The pair (f, g) of linear relations is transversal if and only if $\mathrm{Dom}\, f + \mathrm{Im}\, g$ is the entire intermediate space Y. In particular, (f, g) is transversal whenever f is cosurjective or g is surjective.

Recall that the composition of any sequence $(f_1, f_2, \ldots, f_r)$ of linear relations, with $f_j \in \mathbf{LREL}(X_{j-1}, X_j)$ may be obtained by taking the intersection of $f_1 \times f_2 \times \cdots \times f_r$ with $X_0 \times \Delta_{X_1} \times \cdots \times \Delta_{X_{r-1}} \times X_r$ in $X_0 \times X_1 \times \cdots X_r$ and then projecting this along its intersection with the kernel $0_{X_0} \times \Delta_{X_1} \times \cdots \times \Delta_{X_{r-1}} \times 0_{X_r}$ of the projection to $X_0 \times X_r$.

The codimension of

$$(f_1 \times f_2 \times \cdots \times f_r) + (X_0 \times \Delta_{X_1} \times \cdots \times \Delta_{X_{r-1}} \times X_r)$$

in $X_0 \times X_1 \times \cdots X_r$ will be called the **defect** of the sequence and denoted by $\mathcal{D}(f_1, f_2, \ldots, f_r)$. We will call the entire sequence transversal if its defect is zero; this extends the definition for 2-entry sequences.

Using properties of duality, we can now transfer statements about excess and monicity to defect and transversality.

Proposition 3.13. *Let $(f_1, f_2, \ldots, f_r)$ be a composable r-tuple of linear relations. Then*

$$\mathcal{E}(f_1, f_2, \ldots, f_r) = \mathcal{D}(f_r^*, \ldots, f_1^*),$$

and

$$\mathcal{D}(f_1, f_2, \ldots, f_r) = \mathcal{E}(f_r^*, \ldots, f_1^*).$$

In particular, a composable pair (f, g) is monic if and only if (g^*, f^*) is transversal, and vice versa. Thus, the duality functor preserves strong transversality.

Proof. The excess $\mathcal{E}(f_1, f_2, \ldots, f_r)$ is the dimension of the intersection

$$(f_1 \times f_2 \times \cdots \times f_r) \cap (0_{X_0} \times \Delta_{X_1} \times \cdots \times \Delta_{X_{r-1}} \times 0_{X_r}).$$

We now pair the spaces

$$X_0 \times X_1 \times X_1 \times X_2 \cdots \times X_{r-1} \times X_{r-1} \times X_r$$

and

$$X_r^* \times X_{r-1}^* \times X_{r-1}^* \times X_{r-2}^* \times \cdots \times X_2^* \times X_1^* \times X_1^* \times X_0^*$$

by the rule

$$\langle (x_0, x_1', x_1, x_2', \ldots, x_{r-1}', x_{r-1}, x_r'), (\xi_r, \xi_{r-1}', \xi_{r-1}, \xi_{r-2}', \ldots, \xi_1', \xi_1, \xi_0') \rangle$$
$$= \xi_r(x_r') - \xi_{r-1}'(x_{r-1}) + \xi_{r-1}(x_{r-1}') - \xi_{r-2}'(x_{r-2}) + \cdots + \xi_1(x_1') - \xi_0'(x_0)$$

The annihilators of the spaces being intersected are now

$$f_r^* \times f_{r-1}^* \times \cdots \times f_1^* \quad \text{and} \quad X_r^* \times \Delta_{X_{r-1}} \times \cdots \times \Delta_{X_1} \times X_0^*$$

respectively, the codimension of whose sum is the defect $\mathcal{D}(f_r^*, \ldots, f_1^*)$. Since, for a nondegenerate pairing, the dimension of an intersection is the codimension of the sum of the annihilators, our equation is proven.

The second equation in the statement of the proposition follows immediately from the first by reflexivity. $\square$

Corollary 3.14. *Let $(f_1, f_2, \ldots, f_r)$ be a composable r-tuple of linear relations, and let $1 \leq j \leq r - 1$. Then*

$$(3) \qquad \mathcal{D}(f_1, f_2, \ldots, f_r) = \mathcal{D}(f_1, f_2, \ldots, f_j) + \mathcal{D}(f_1 \cdots f_j, f_{j+1} \cdots f_r)$$
$$+ \mathcal{D}(f_{j+1}, f_{j+2}, \ldots, f_r).$$

Proof. Combine Propositions 3.3 and 3.13. $\square$

4. Isotropic and coisotropic relations

We now define two enlargements of the category **SLREL** of symplectic vector spaces and canonical relations. In the categories **ILREL** and **CLREL**,

the objects are again the symplectic vector spaces, and the morphisms $X \leftarrow Y$ are the subspaces of $X \times \overline{Y}$ which are isotropic in the case of **ILREL** and coisotropic in the case of **CLREL**. It is not hard to see that these collections of subspaces are closed under composition and contain the diagonals. The category **SLREL** then sits naturally as a subcategory of **ILREL** and **CLREL** and is the intersection of those two categories.

If I is isotropic in X, then its annihilator with respect to the symplectic pairing is coisotropic, and vice versa. Since the duality functor may be defined in terms of annihilators, we have a bijective functor between **ILREL** and **CLREL** which is the identity on objects. It is convenient to define this functor in contravariant form in such a way that it becomes the identity on objects, as well as on their intersection **SLREL**.

Since a lagrangian relation is isomorphic to its dual, it follows from Proposition 3.10 that the defect and excess are equal in **SLREL**. More generally, we have:

Proposition 4.1. *The excess of a composable sequence $(f_1, f_2, ..., f_r)$ in* **ILREL** *is less than or equal to its defect. In* **CLREL***, the defect is less than or equal to the excess. These are the only constraints on the defect and excess.*

Proof. The case of **CLREL** follows from that of **ILREL** by Proposition 3.10. The defect $\mathcal{D}(f_1, f_2, ..., f_r)$ is the codimension of

$$(f_1 \times f_2 \times \cdots \times f_r) + (X_0 \times \Delta_{X_1} \times \cdots \times \Delta_{X_{r-1}} \times X_r)$$

in $X_0 \times X_1 \times \cdots X_r$. The codimension of this space is equal to the dimension of its symplectic orthogonal

$$(f_1^\omega \times f_2^\omega \times \cdots \times f_r^\omega) \cap (0_{X_0} \times \Delta_{X_1} \times \cdots \times \Delta_{X_{r-1}} \times 0_{X_r}).$$

When the f_j are isotropic, this space contains

$$(f_1 \times f_2 \times \cdots \times f_r) \cap (0_{X_0} \times \Delta_{X_1} \times \cdots \times \Delta_{X_{r-1}} \times 0_{X_r}),$$

whose dimension is the excess of $(f_1, f_2, ..., f_r)$. Hence

$$\mathcal{E}(f_1, f_2, \ldots, f_r) \leq \mathcal{D}(f_1, f_2, \ldots, f_r).$$

Simple examples show that there are no further constraints on $\mathcal{D}$ and $\mathcal{E}$. $\square$

5. Selective structures and WW morphisms

We showed in [5] that the morphisms in $WW(\mathbf{SLREL})$ may be identified with **indexed canonical relations**, which are pairs (L, k) such that L is a canonical relation and k is a nonnegative integer. The composition $(L, k)(L', k')$ is defined, whenever (L, L') is a composable pair, as $(LL', k + k' + \mathcal{E}(L, L'))$. This makes $WW(\mathbf{SLREL})$ a central extension[3] of $\mathbf{SLREL}$ by the endomorphisms of the unit object, which may identified with the additive monoid $\mathbb{Z}_+$ of nonnegative integers and which are composed with other morphisms by the tensor product. The excess $\mathcal{E}$, equal here to the defect $\mathcal{D}$, is the cocycle determining the extension.

When we replace $\mathbf{SLREL}$ by one of the categories $\mathbf{LREL}$, $\mathbf{ILREL}$, or $\mathbf{CLREL}$, the excess and defect are no longer equal. We will therefore use the highly selective structures on $\mathbf{LREL}$, $\mathbf{ILREL}$ and $\mathbf{SLREL}$ in which all morphisms are suave, while the congenial pairs are only those which are strongly transversal, i.e. both monic and transversal. The reductions are again the surjective and single-valued relations, and the coreductions are the everywhere-defined and injective relations.

Remark 5.1. The cotangent functor T^* from $\mathbf{LREL}$ to $\mathbf{SLREL}$ takes a vector space V to the symplectic space $V \oplus V^*$ with the usual structure and a linear relation $V \leftarrow W$ to the canonical relation which is its conormal bundle with the components in W^* multiplied by -1.[4] For a composable pair (f, g) of linear relations, the defect (equal to the excess) of the pair (T^*f, T^*g) is the sum of the defect and excess of (f, g).

Since vanishing of the defect and excess are the defining conditions for congeniality, it is perhaps not surprising that it suffices to keep track of just these two numbers to parametrize the WW morphisms.

Theorem 5.2. *If $\mathbf{C}$ is the category $\mathbf{LREL}$, $\mathbf{ILREL}$, or $\mathbf{CLREL}$, any morphism in $WW(\mathbf{C})$ is determined by its shadow, its defect, and its excess. These are independent, subject only to the inequalities in Proposition 4.1. The category $WW(\mathbf{C})$ is a central extension of $\mathbf{C}$ by the endomorphisms of the unit object. In the case if $\mathbf{LREL}$, the map $(\mathcal{D}, \mathcal{E})$ identifies these endomorphisms with the additive monoid $\mathbb{Z}_+ \times \mathbb{Z}_+$. For $\mathbf{ILREL}$ or $\mathbf{CLREL}$, we*

---

[3]Central extensions of categories are defined in [7].

[4]This is a very special case of the symplectic groupoid construction taking a coisotropic relation between Poisson manifolds to a canonical groupoid relation between their symplectic groupoids.

get the submonoid defined by the inequalities $\mathcal{E} \leq \mathcal{D}$ or $\mathcal{D} \leq \mathcal{E}$. The cocycle determining the central extension is $(\mathcal{D}, \mathcal{E})$.

Proof. We begin with the case of **LREL**. Any morphism in $WW(\mathbf{LREL})$ is represented by a diagram

$$X \xleftarrow{f} Y \xleftarrow{g} Z$$

of linear relations. The graph of this composition may be constructed from the graphs γ_f and γ_g and is represented by the diagram of linear relations

$$X \times Z \xleftarrow{X \times \Delta_Y \times Z} X \times Y \times Y \times Z \xleftarrow{(\gamma_f \times \gamma_g)\delta_{\mathbf{1}}} \mathbf{1},$$

where $X \times \Delta_Y \times Z$ denotes the reduction whose domain is this subspace and whose kernel is $0_X \times \Delta_Y \times 0_Z$, and $\delta_{\mathbf{1}}$ is the diagonal isomorphism $\mathbf{1} \times \mathbf{1} \leftarrow \mathbf{1}$.

Analyzing this composition comes down to analyzing the triple $(X \times \Delta_Y \times Z, 0_X \times \Delta_Y \times 0_Z, f \times g)$ of subspaces in $X \times Y \times Y \times Z$, using the natural isomorphism between $X \times Z$ and the reduced space $(X \times \Delta_Y \times Z)/(0_X \times \Delta_Y \times 0_Z)$. Such triples have been classified in detail, for instance in [2] and [3]. In this classification, any quadruple (V, A, B, C) consisting of a vector space V and its subspaces A, B, and C is isomorphic to a direct sum of copies of "elementary" components (V_i, A_i, B_i, C_i) of the following nine types. V is 1-dimensional in the first eight types and is 2-dimensional in the ninth.

τ_1**::** $A = 0$, $B = 0$, $C = 0$

τ_2**::** $A = 0$, $B = 0$, $C = V$

τ_3**::** $A = 0$, $B = V$, $C = 0$

τ_4**::** $A = V$, $B = 0$, $C = 0$

τ_5**::** $A = 0$, $B = V$, $C = V$

τ_6**::** $A = V$, $B = 0$, $C = V$

τ_7**::** $A = V$, $B = V$, $C = 0$

τ_8**::** $A = V$, $B = V$, $C = V$

τ_9**::** A, B, and C are pairwise independent subspaces, any two of which sum to V.

Applying this decomposition to the case $V = X \times Y \times Y \times Z$, $A = X \times \Delta_Y \times Z$, $B = 0_X \times \Delta_Y \times 0_Z$, $C = f \times g$ and using the compatible decompositions of A and B to obtain a decomposition of $X \times Z = A/B$ we obtain a decomposition of our original WW morphism as a direct sum of nine "elementary morphisms", each of which may contribute to the shadow, defect, and excess of the morphism. Since B is contained in A, the types τ_3, τ_5, and τ_9 cannot appear in the decomposition.

The contributions of the remaining six types are as follows, with the dimension of V_i denoted by n_i:

type	dim(A/B)	dim(shadow)	defect	excess
τ_1	0	0	n_1	0
τ_2	0	0	0	0
τ_4	n_4	0	0	0
τ_6	n_6	n_6	0	0
τ_7	0	0	0	0
τ_8	0	0	0	n_8

The components of type τ_1 and τ_8 are the graphs of endomorphisms of the unit object $\mathbf{1}$ which are determined by their defect and excess. The components of type τ_2 and τ_7 may be ignored, since they give the trivial endomorphism of $\mathbf{1}$ as a morphism in $WW(\mathbf{LREL})$. Finally, the components of types τ_4 and τ_6 are determined by the shadow of (f, g), with dimension n_6, and the space $X \times Z$, with dimension $n_4 + n_6$. It follows that the graph of our WW morphism, and thus the morphism itself, is determined by the shadow, the defect, and the excess. Examples showing that all possibilities for these three items occur are easy to construct, and the composition law based on the cocycle $(\mathcal{D}, \mathcal{E})$ follows from Proposition 3.3 and Corollary 3.14.

We now go on to the case of $WW(\mathbf{ILREL})$; the case of $WW(\mathbf{CLREL})$ is reduced to this one by taking symplectic orthogonals. We will modify the argument above, taking the extra symplectic data into account. Since the kernel of a reduction is determined by the domain, being its symplectic orthogonal, we may work with pairs of subspaces instead of triples.

Each morphism in $WW(\mathbf{ILREL})$ is represented by a diagram

$$X \xleftarrow{f} Y \xrightarrow{g} Z$$

of isotropic relations. As above, the graph of this composition may be constructed from the graphs γ_f and γ_g and is represented by the diagram of

isotropic relations

$$X \times \overline{Z} \overset{X \times \Delta_Y \times \overline{Z}}{\leftarrow} X \times \overline{Y} \times Y \times \overline{Z} \overset{(\gamma_f \times \gamma_g)\delta_{\mathbf{1}}}{\hookleftarrow} \mathbf{1},$$

where $X \times \Delta_Y \times \overline{Z}$ denotes the (lagrangian, hence isotropic) reduction by this coisotropic subspace, and $\delta_{\mathbf{1}}$ is the diagonal isomorphism $\mathbf{1} \times \mathbf{1} \leftarrow \mathbf{1}$.

Analyzing this composition comes down to decomposing the pair $(X \times \Delta_Y \times \overline{Z}, f \times g)$ of subspaces in $X \times \overline{Y} \times Y \times \overline{Z}$, using the natural isomorphism between $X \times \overline{Z}$ and $(X \times \Delta_Y \times \overline{Z})/(0_X \times \Delta_Y \times 0_{\overline{Z}})$. In fact, it is more convenient to pass to the equivalent problem of decomposing the isotropic pair $(0_X \times \Delta_Y \times 0_{\overline{Z}}, f \times g)$, since pairs of isotropic subspaces have already been classified explicitly in [6]. In this classification, any triple (V, A, B) consisting of a symplectic space V and its isotropic subspaces A and B is isomorphic to a monoidal product of "elementary" components (V_i, A_i, B_i) of the following five types:

τ_1:: A and B are lagrangian subspaces, and $A = B$

τ_2:: $B = 0$ and A is a lagrangian subspace

τ_3:: $A = 0$ and B is a lagrangian subspace

τ_4:: A and B are lagrangian subspaces, and $A \cap B = 0$

τ_5:: $A = B = 0$

Applying this decomposition to the case $V = X \times \overline{Y} \times Y \times \overline{Z}$, $A = 0_X \times \Delta_Y \times 0_{\overline{Z}}$, $B = f \times g$ and using the compatible decompositions of A and A^ω to obtain a decomposition of $X \times \overline{Z} = A^\omega/A$ we obtain a decomposition of our original WW morphism as a monoidal product of five "elementary morphisms", each of which may contribute to the shadow, defect, and excess of the morphism. The contributions are now as follows, with *half* the dimension of V_i denoted by n_i.

type	$\dim(A^\omega/A)$	$\dim(\text{shadow})$	defect	excess
τ_1	0	0	n_1	n_1
τ_2	0	0	n_2	0
τ_3	$2n_3$	n_3	0	0
τ_4	0	0	0	0
τ_5	$2n_5$	0	0	0

The components of type τ_1 and τ_2 are the graphs of endomorphisms of the unit object $\mathbf{1}$ which are determined by their defect and excess. The component of type τ_4 may be ignored, since it gives the trivial endomorphism

of 1 as a morphism in $WW(\mathbf{ILREL})$. Finally, the components of types τ_3 and τ_5 are determined by the shadow of (f, g) with dimension n_3, and the space $X \times \overline{Z}$ with dimension $2(n_3 + n_5)$. It follows that the graph of our WW morphism, and thus the morphism itself, is determined by the shadow, the defect, and the excess.

The constraints on the pair $(\mathcal{D}, \mathcal{E})$ are just those from Proposition 4.1, while the composition law involving the stated cocycle follows again from Proposition 3.3 and Corollary 3.14. □

Remark 5.3. Each of the endomorphism monoids above is free with two generators. The generators are $(0, 1)$ and $(1, 0)$ in the case of $\mathbf{LREL}$, $(1, 1)$ and $(1, 0)$ in the case of $\mathbf{ILREL}$, and $(0, 1)$ and $(1, 1)$ in the case of $\mathbf{CLREL}$,

Remark 5.4. We recall that a (constant) presymplectic structure on X is a possibly degenerate skew-symmetric bilinear form, while a (constant) Poisson structure on X is a presymplectic structure on X^*. The latter may be thought of as a bracket operation on the linear functions on X with values in the constants, leading to a (Heisenberg) Lie algebra structure on the affine functions. The notions of isotropic and coisotropic subspace are natural in presymplectic and Poisson vector spaces respectively, giving rise to versions of $\mathbf{ILREL}$ and $\mathbf{SLREL}$ involving these more general objects, and these larger categories are isomorphic via duality. (The annihilator in X^* of an isotropic subspace in presymplectic X is coisotropic in Poisson X^*, and vice versa.)

The classification of (co)isotropic pairs in [6] should make it possible to extend the results here to these larger categories, but we have not yet attempted to carry out this extension. These enlarged categories contain as monoidal subcategories both $\mathbf{LREL}$ and either $\mathbf{ILREL}$ or $\mathbf{SLREL}$ (which generate all objects by monoidal product); the main difficulty is to understand how the morphisms interlace the two subcategories.

References

[1] S. Benenti and W. Tulczyjew, *Relazioni lineari simplettiche*, Memorie dell'Accademia delle Scienze di Torino **5** (1981), 71–140.

[2] S. Brenner, *Endomorphism algebras of vector spaces with distinguished sets of subspaces*, Journal of Algebra **6** (1967), 100–114.

[3] P. Etingof et al., *Introduction to representation theory*, Lecture notes www-math.mit.edu/~etingof/replect.pdf.

[4] I. M. Gelfand and V. A. Ponomarev, *Problems of linear algebra and classification of quadruples of subspaces of a finite-dimensional vector space*, Colloquia Mathematica Societatis Janos Bolyai **5**, Tihany (1970), 163–237.

[5] D. Li-Bland and A. Weinstein, *Selective categories and linear canonical relations*, SIGMA **10** (2014), 100, 31 pages.

[6] J. Lorand and A. Weinstein, *(Co)isotropic pairs in Poisson and presymplectic vector spaces*, SIGMA **11** (2015), 072, 10 pages.

[7] Yu. A. Neretin, *Lectures on Gaussian integral operators and classical groups*, European Mathematical Society, Zürich, 2011.

[8] J. Towber, *Linear relations*, Journal of Algebra **19** (1971), 1–20.

[9] K. Wehrheim and C. Woodward, *Functoriality for Lagrangian correspondences in Floer theory*, Quantum Topology **1** (2010), 129–170.

[10] A. Weinstein, *A note on the Wehrheim-Woodward category*, Journal of Geometric Mechanics **3** (2011), 507–515.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720-3840 USA
E-mail address: alanw@math.berkeley.edu