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Floer cohomologies of non-torus fibers of the Gelfand-Cetlin system

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The Gelfand-Cetlin system has non-torus Lagrangian fibers on some of the boundary strata of the moment polytope. We compute Floer cohomologies of such non-torus Lagrangian fibers in the cases of the 3-dimensional full flag manifold and the Grassmannian of 2-planes in a 4-space.

1. Introduction

Let P be a parabolic subgroup of $\mathrm{GL}(n,\mathbb{C})$ and $F := \mathrm{GL}(n,\mathbb{C})/P$ be the associated flag manifold. The Gelfand-Cetlin system, introduced by Guillemin and Sternberg [GS83], is a completely integrable system

$$\Phi : F \longrightarrow \mathbb{R}^{(\dim_{\mathbb{R}} F)/2},$$

i.e., a set of functionally independent and Poisson commuting functions. The image $\Delta = \Phi(F)$ is a convex polytope called the *Gelfand-Cetlin polytope*, and Φ gives a Lagrangian torus fibration structure over the interior $\operatorname{Int}\Delta$ of Δ. Unlike the case of toric manifolds where the fibers over the relative interior of a d-dimensional face of the moment polytope are d-dimensional isotropic tori, the Gelfand-Cetlin system has non-torus Lagrangian fibers over the relative interiors of some of the faces of Δ.

Let (X, ω) be a compact toric manifold of $\dim_{\mathbb{C}} X = N$, and $\Phi : X \to \mathbb{R}^N$ be the toric moment map with the moment polytope $\Delta = \Phi(X)$. For an interior point $\boldsymbol{u} \in \operatorname{Int} \Delta$, let $L(\boldsymbol{u})$ denote the Lagrangian torus fiber $\Phi^{-1}(\boldsymbol{u})$. Lagrangian intersection Floer theory endows the cohomology group $H^*(L(\boldsymbol{u}); \Lambda_0)$ over the Novikov ring

$$\Lambda_0 := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\ \lambda_i \geq 0,\ \lim_{i \to \infty} \lambda_i = \infty \right\}$$

with a structure $\{\mathfrak{m}_k\}_{k \geq 0}$ of a unital filtered A_∞-algebra [FOOO09]. Let Λ and Λ_+ be the quotient field and the maximal ideal of the local ring Λ_0 respectively. An odd-degree element $b \in H^{\mathrm{odd}}(L(\boldsymbol{u}); \Lambda_0)$ is said to be a *bounding cochain* if it satisfies the *Maurer-Cartan equation*

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) = 0. \tag{1.1}$$

A solution $b \in H^{\mathrm{odd}}(L(\boldsymbol{u}); \Lambda_0)$ to the *weak Maurer-Cartan equation*

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) \equiv 0 \mod \Lambda_0 \, \mathbf{e}_0 \tag{1.2}$$

is called a *weak bounding cochain*, where $\mathbf{e}_0$ is the unit in $H^*(L(\boldsymbol{u}); \Lambda_0)$. The set of weak bounding cochains will be denoted by $\widehat{\mathcal{M}}_{\mathrm{weak}}(L(\boldsymbol{u}))$. The *potential function* is a map $\mathfrak{PO} \colon \widehat{\mathcal{M}}_{\mathrm{weak}}(L(\boldsymbol{u})) \to \Lambda_0$ defined by

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \dots, b) = \mathfrak{PO}(b) \mathbf{e}_0. \tag{1.3}$$

A weak bounding cochain gives a deformed filtered A_∞-algebra whose A_∞-operations are given by

$$\begin{aligned} &\mathfrak{m}_k^b(x_1, \dots, x_k) \\ &\quad = \sum_{m_0=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \mathfrak{m}_{m_0+\cdots+m_k+k}(b^{\otimes m_0} \otimes x_1 \otimes b^{\otimes m_1} \otimes \cdots \otimes x_k \otimes b^{\otimes m_k}). \end{aligned} \tag{1.4}$$

The weak Maurer-Cartan equation implies that $\mathfrak{m}_1^b$ squares to zero, and the *deformed Floer cohomology* is defined by

$$HF((L(\boldsymbol{u}), b), (L(\boldsymbol{u}), b); \Lambda_0) = \mathrm{Ker}(\mathfrak{m}_1^b) \Big/ \mathrm{Im}(\mathfrak{m}_1^b). \tag{1.5}$$

More generally, one can deform the Floer differential $\mathfrak{m}_1$ by

$$\delta_{b_0,b_1}(x) = \sum_{k_0,k_1 \geq 0} \mathfrak{m}_{k_0+k_1+1}(\underbrace{b_0, \dots, b_0}_{k_0}, x, \underbrace{b_1, \dots, b_1}_{k_1}) \tag{1.6}$$

for a pair (b_0, b_1) of weak bounding cochains with $\mathfrak{PO}(b_0) = \mathfrak{PO}(b_1)$. The Floer cohomology of the pair $((L(\boldsymbol{u}), b_0), (L(\boldsymbol{u}), b_1))$ is defined by

$$HF((L(\boldsymbol{u}), b_0), (L(\boldsymbol{u}), b_1); \Lambda_0) = \mathrm{Ker}(\delta_{b_0,b_1}) / \mathrm{Im}(\delta_{b_0,b_1}). \tag{1.7}$$

If the toric manifold X is Fano, then the following hold [FOOO10]:

- $H^1(L(\boldsymbol{u}); \Lambda_0)$ is contained in $\widehat{\mathcal{M}}_{\mathrm{weak}}(L(\boldsymbol{u}))$.
- The potential function $\mathfrak{PO}$ on

$$\bigcup_{\boldsymbol{u} \in \mathrm{Int}\,\Delta} H^1(L(\boldsymbol{u}); \Lambda_0/2\pi\sqrt{-1}\mathbb{Z}) \cong \mathrm{Int}\,\Delta \times (\Lambda_0/2\pi\sqrt{-1}\mathbb{Z})^N \tag{1.8}$$

  can be considered as a Laurent polynomial, which can be identified with the superpotential of the Landau-Ginzburg mirror of X.

- Each critical point of $\mathfrak{PO}$ corresponds to a pair $(\boldsymbol{u}, b)$ such that the deformed Floer cohomology $HF((L(\boldsymbol{u}), b), (L(\boldsymbol{u}), b); \Lambda)$ over the Novikov field Λ is non-trivial.
- If the deformed Floer cohomology group over the Novikov field is non-trivial, then it is isomorphic to the classical cohomology group as a vector space;

$$HF((L(\boldsymbol{u}), b), (L(\boldsymbol{u}), b); \Lambda) \cong H^*(T^N; \Lambda). \tag{1.9}$$

- The quantum cohomology ring $QH(X; \Lambda)$ is isomorphic to the Jacobi ring $\mathrm{Jac}(\mathfrak{PO})$ of the potential function.

In particular, the number of pairs $(L(\boldsymbol{u}), b)$ with nontrivial Floer cohomology coincides with $\mathrm{rank}\, QH(X; \Lambda) = \mathrm{rank}\, H^*(X; \Lambda)$ provided that the potential function is Morse.

Nishinou and the authors [NNU10] introduced the notion of a toric degeneration of an integrable system, and used it to compute the potential function of Lagrangian torus fibers of the Gelfand-Cetlin system. The resulting potential function can be considered as a Laurent polynomial just as in the toric Fano case, which can be identified with the superpotential of the Landau-Ginzburg mirror of the flag manifold given in [Giv97, BCFKvS00]. In contrast to the toric case, the rank of $H^*(F;\Lambda)$ is greater in general than the rank of the Jacobi ring $\operatorname{Jac}(\mathfrak{PO})$, and hence than the number of Lagrangian torus fibers with non-trivial Floer cohomology. In the case of the 3-dimensional flag manifold $\mathrm{Fl}(3)$, the potential function has six critical points, which is equal to the rank of $H^*(\mathrm{Fl}(3);\Lambda)$. Similarly, the potential function for the Grassmannian $\mathrm{Gr}(2,5)$ of 2-planes in $\mathbb{C}^5$ has ten critical points, which is equal to the rank of $H^*(\mathrm{Gr}(2,5);\Lambda)$. On the other hand, the number of critical points of the potential function for the Grassmannian $\mathrm{Gr}(2,4)$ of 2-planes in $\mathbb{C}^4$ is four, which is less than the rank of $H^*(\mathrm{Gr}(2,4);\Lambda)$, which is six.

In this paper, we study non-torus Lagrangian fibers of the Gelfand-Cetlin system over the boundary of the Gelfand-Cetlin polytope in the cases of $\mathrm{Fl}(3)$, $\mathrm{Gr}(2,4)$, and $\mathrm{Gr}(2,5)$. The main results are the following:

Theorem 1.1. *Let $\Phi\colon \mathrm{Fl}(3) \to \mathbb{R}^3$ be the Gelfand-Cetlin system with the Gelfand-Cetlin polytope $\Delta = \Phi(\mathrm{Fl}(3))$.*

1) *There exists a vertex $\boldsymbol{u}_0$ of Δ such that a fiber $L(\boldsymbol{u}) = \Phi^{-1}(\boldsymbol{u})$ over a boundary point $\boldsymbol{u} \in \partial\Delta$ is a Lagrangian submanifold if and only if $\boldsymbol{u} = \boldsymbol{u}_0$.*
2) *The Lagrangian fiber $L(\boldsymbol{u}_0)$ is diffeomorphic to $\mathrm{SU}(2) \cong S^3$.*
3) *The Floer cohomology of $L(\boldsymbol{u}_0)$ is given by*

$$HF(L(\boldsymbol{u}_0), L(\boldsymbol{u}_0); \Lambda_0) \cong \Lambda_0 / T^{\lambda} \Lambda_0, \tag{1.10}$$

where $\lambda > 0$ is a constant depending on the symplectic structure of $\mathrm{Fl}(3)$. *In particular, the Floer cohomology of $L(\boldsymbol{u}_0)$ over the Novikov field Λ is trivial;*

$$HF(L(\boldsymbol{u}_0), L(\boldsymbol{u}_0); \Lambda) = 0. \tag{1.11}$$

Theorem 1.2. *Let $\Phi\colon \mathrm{Gr}(2,4) \to \mathbb{R}^4$ be the Gelfand-Cetlin system with the Gelfand-Cetlin polytope $\Delta = \Phi(\mathrm{Gr}(2,4))$.*

1) *There exists an edge of* Δ *such that a fiber* $L(\boldsymbol{u}) = \Phi^{-1}(\boldsymbol{u})$ *over* $\boldsymbol{u} \in \partial\Delta$ *is a Lagrangian submanifold if and only if* $\boldsymbol{u}$ *is in the relative interior of the edge.*

2) *The Lagrangian fiber* $L(\boldsymbol{u})$ *over any point* $\boldsymbol{u}$ *in the relative interior of the edge is diffeomorphic to* $\mathrm{U}(2) \cong S^1 \times S^3$.

3) $H^1(L(\boldsymbol{u}); \Lambda_0)$ *is contained in* $\widehat{\mathcal{M}}_{\mathrm{weak}}(L(\boldsymbol{u}))$.

4) *The potential function is identically zero on* $H^1(L(\boldsymbol{u}); \Lambda_0)$.

5) *The Floer cohomology* $HF((L(\boldsymbol{u}), b), (L(\boldsymbol{u}), b); \Lambda)$ *of a Lagrangian* $\mathrm{U}(2)$*-fiber* $L(\boldsymbol{u})$ *over the Novikov field* Λ *is non-trivial if and only if* $\boldsymbol{u}$ *is the barycenter* $\boldsymbol{u}_0$ *of the edge and* $b = \pm\pi\sqrt{-1}/2\, \mathbf{e}_1$, *where* $\mathbf{e}_1$ *is a generator of* $H^1(L(\boldsymbol{u}); \mathbb{Z}) \cong \mathbb{Z}$.

6) *If the deformed Floer cohomology group over the Novikov field is non-trivial, then it is isomorphic to the classical cohomology group;*

$$HF((L(\boldsymbol{u}_0), \pm\pi\sqrt{-1}/2\, \mathbf{e}_1), (L(\boldsymbol{u}_0), \pm\pi\sqrt{-1}/2\, \mathbf{e}_1); \Lambda) \cong H^*(S^1 \times S^3; \Lambda). \tag{1.12}$$

7) *The Floer cohomology of the pair* $((L(\boldsymbol{u}_0), \pi\sqrt{-1}/2\, \mathbf{e}_1), (L(\boldsymbol{u}_0), -\pi\sqrt{-1}/2\, \mathbf{e}_1))$ *is trivial;*

$$HF((L(\boldsymbol{u}_0), \pi\sqrt{-1}/2\, \mathbf{e}_1), (L(\boldsymbol{u}_0), -\pi\sqrt{-1}/2\, \mathbf{e}_1); \Lambda) = 0. \tag{1.13}$$

More precise statements, which describe the Floer cohomology groups over the Novikov ring Λ_0, are given in Theorem 4.16, and Theorem 4.20.

Theorem 1.3. *Let* $\Phi\colon \mathrm{Gr}(2,5) \to \mathbb{R}^6$ *be the Gelfand-Cetlin system with the Gelfand-Cetlin polytope* $\Delta = \Phi(\mathrm{Gr}(2,5))$.

1) *There exist two 3-dimensional faces of* Δ *such that a fiber* $L(\boldsymbol{u}) = \Phi^{-1}(\boldsymbol{u})$ *over* $\boldsymbol{u} \in \partial\Delta$ *is a Lagrangian submanifold if and only if* $\boldsymbol{u}$ *is an interior point of one of these faces.*

2) *The Lagrangian fibers over these faces are diffeomorphic to* $S^3 \times T^3$.

3) *Each Lagrangian fiber* $L(\boldsymbol{u})$ *over these faces is displaceable from itself by a Hamiltonian diffeomorphism. In particular, the Floer cohomology over the Novikov field is trivial;*

$$HF((L(\boldsymbol{u}), b), (L(\boldsymbol{u}), b); \Lambda) = 0$$

for any weak bounding cochain b.

Remark 1.4. The preimages of the faces stated in Theorem 1.1, Theorem 1.2, and Theorem 1.3 are the loci where the Gelfand-Cetlin systems fail to be differentiable. Fibers over other boundary faces are lower dimensional isotropic tori, as in the toric case.

A symplectic manifold (X, ω) is *monotone* if the cohomology class $[\omega]$ is positively proportional to the first Chern class;

$$\exists \lambda > 0 \quad [\omega] = \lambda c_1(X). \tag{1.14}$$

The quantum cohomology ring of a monotone symplectic manifold does not have any convergence issue, and hence is defined over $\mathbb{C}$. A Lagrangian submanifold L is *monotone* if the symplectic area of a disk bounded by L is positively proportional to the Maslov index;

$$\exists \lambda > 0 \quad \forall \beta \in \pi_2(M, L) \quad \beta \cap \omega = \lambda \mu(\beta). \tag{1.15}$$

The A_∞-operations on the Lagrangian intersection Floer complex of a monotone Lagrangian submanifold is defined over $\mathbb{C}$. The minimal Maslov number of oriented monotone Lagrangian submanifold is greater than or equal to 2, so that the obstruction class $\mathfrak{m}_0(1)$ can be written as $\mathfrak{m}_0(1) = \mathfrak{m}_0(L)\,\mathbf{e}_0$, where $\mathfrak{m}_0(L) \in \mathbb{C}$ is the count of Maslov index 2 disks bounded by L, weighted by their symplectic areas and holonomies of a flat $U(1)$-bundle on L along the boundaries of the disks. The *monotone Fukaya category* is defined as the direct sum

$$\mathcal{F}(X) := \bigoplus_{\lambda \in \mathbb{C}} \mathcal{F}(X; \lambda), \tag{1.16}$$

where $\mathcal{F}(X; \lambda)$ is an A_∞-category over $\mathbb{C}$ whose objects are monotone Lagrangian submanifolds, equipped with flat $U(1)$-bundles, satisfying $\mathfrak{m}_0(L) = \lambda$. For any monotone Lagrangian submanifold L, there is a natural ring homomorphism

$$QH(X) \to HF(L, L), \tag{1.17}$$

which is known by Auroux [Aur07], Kontsevich, and Seidel to send $c_1(X) \in QH(X)$ to $\mathfrak{m}_0(1) \in HF(L, L)$. It follows that $\mathcal{F}(X; \lambda)$ is trivial unless λ is an eigenvalue of the quantum cup product by $c_1(X)$.

Now consider the case when $X = \mathrm{Gr}(2,4)$, which can be written as a quadric hypersurface

$$X = \left\{ [z_0 : \cdots : z_5] \in \mathbb{P}^5 \;\middle|\; z_0^2 = z_1^2 + \cdots + z_5^2 \right\}. \tag{1.18}$$

The real locus $X_{\mathbb{R}}$ is a monotone Lagrangian sphere, which is the vanishing cycle along a degeneration into a nodal quadric and split-generates the nilpotent summand $D^\pi \mathcal{F}(X;0)$ of the monotone Fukaya category [Smi12, Lemma 4.6]. The Floer cohomology $HF(X_{\mathbb{R}}, X_{\mathbb{R}})$ is semisimple, and carries a formal A_∞-structure [Smi12, Lemma 4.7]. It follows that $D^\pi \mathcal{F}(X;0)$ is equivalent to the direct sum of two copies of the derived category $D^b(\mathbb{C})$ of $\mathbb{C}$-vector spaces. On the other hand, $(L(\boldsymbol{u}_0), \pm\pi\sqrt{-1}/2\,\mathbf{e}_1)$ are also objects of the nilpotent summand $D^\pi \mathcal{F}(X;0)$ of the monotone Fukaya category, which are non-zero by (1.12). Since $(L(\boldsymbol{u}_0), \pm\sqrt{-1}/2\,\mathbf{e}_1)$ is a pair of orthogonal non-zero objects in a triangulated category equivalent to $D^b(\mathbb{C}) \oplus D^b(\mathbb{C})$, they split-generate the whole category:

Corollary 1.5. *The pair* $(L(\boldsymbol{u}_0), \pm\pi\sqrt{-1}/2\,\mathbf{e}_1)$ *split-generate* $D^\pi \mathcal{F}(\mathrm{Gr}(2,4);0)$.

This paper is organized as follows: In Section 2, we recall the construction of the Gelfand-Cetlin system, and study non-torus Lagrangian fibers in the cases of the full flag manifold $Fl(3)$ and the Grassmannians $\mathrm{Gr}(n,2n)$, $\mathrm{Gr}(2,5)$. In Section 3, we discuss critical points of the potential function and eigenvalues of the quantum cup product by the first Chern class. In Section 4 we compute the Floer cohomologies over the Novikov ring of non-torus fibers in Fl(3) and $\mathrm{Gr}(2,4)$. An observation about the displacement energy of a Lagrangian $\mathrm{U}(n)$-fiber in $\mathrm{Gr}(n,2n)$ is also given in this section.

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2. Non-torus fibers of the Gelfand-Cetlin system

2.1. Flag manifolds

For a sequence $0 = n_0 < n_1 < \cdots < n_r < n_{r+1} = n$ of integers, let $F = F(n_1, \dots, n_r, n)$ be the flag manifold consisting of flags

$$0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n, \quad \dim V_i = n_i$$

of $\mathbb{C}^n$. We write the full flag manifold and the Grassmannian as $\mathrm{Fl}(n) = F(1, 2, \dots, n)$ and $\mathrm{Gr}(k, n) = F(k, n)$ respectively. The complex dimension of $F(n_1, \dots, n_r, n)$ is given by

$$N = N(n_1, \dots, n_r, n) := \dim_{\mathbb{C}} F(n_1, \dots, n_r, n) = \sum_{i=1}^{r} (n_i - n_{i-1})(n - n_i).$$

Let $P = P(n_1, \dots, n_r, n) \subset \mathrm{GL}(n, \mathbb{C})$ be the stabilizer subgroup of the standard flag $(V_i = \langle e_1, \dots, e_{n_i} \rangle)_{i=1}^r$, where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{C}^n$. The intersection of P and $\mathrm{U}(n)$ is $\mathrm{U}(k_1) \times \dots \times \mathrm{U}(k_{r+1})$ for $k_i = n_i - n_{i-1}$, and F is written as

$$F = \mathrm{GL}(n, \mathbb{C})/P = \mathrm{U}(n)/(\mathrm{U}(k_1) \times \dots \times \mathrm{U}(k_{r+1})).$$

We take a $\mathrm{U}(n)$-invariant inner product $\langle x, y \rangle = \operatorname{tr} xy^*$ on the Lie algebra $\mathfrak{u}(n)$ of $\mathrm{U}(n)$, and identify the dual vector space $\mathfrak{u}(n)^*$ of $\mathfrak{u}(n)$ with the space $\sqrt{-1}\mathfrak{u}(n)$ of Hermitian matrices. For $\boldsymbol{\lambda} = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \in \sqrt{-1}\mathfrak{u}(n)$ with

$$\underbrace{\lambda_1 = \dots = \lambda_{n_1}}_{k_1} > \underbrace{\lambda_{n_1+1} = \dots = \lambda_{n_2}}_{k_2} > \dots > \underbrace{\lambda_{n_r+1} = \dots = \lambda_n}_{k_{r+1}}, \tag{2.1}$$

the flag manifold F is identified with the adjoint orbit $\mathcal{O}_{\boldsymbol{\lambda}} \subset \sqrt{-1}\mathfrak{u}(n)$ of $\boldsymbol{\lambda}$. Note that $\mathcal{O}_{\boldsymbol{\lambda}}$ consists of Hermitian matrices with fixed eigenvalues $\lambda_1, \dots, \lambda_n$. Let

$$\omega(\mathrm{ad}_\xi(x), \mathrm{ad}_\eta(x)) = \frac{1}{2\pi} \langle x, [\xi, \eta] \rangle, \quad \xi, \eta \in \mathfrak{u}(n)$$

be the (normalized) Kostant-Kirillov form on $\mathcal{O}_{\boldsymbol{\lambda}}$.

For each $i = 1, \dots, r$, we set $\mathbb{P}_i := \mathbb{P}\left(\bigwedge^{n_i} \mathbb{C}^n\right) \cong \mathbb{P}^{\binom{n}{n_i}-1}$. Then the Plücker embedding is given by

$$\iota : F \hookrightarrow \prod_{i=1}^{r} \mathbb{P}_i, \quad (0 \subset V_1 \subset \dots \subset V_r \subset \mathbb{C}^n) \mapsto (\textstyle\bigwedge^{n_1} V_1, \dots, \bigwedge^{n_r} V_r).$$

Let $\omega_{\mathbb{P}_i}$ be the Fubini-Study form on $\mathbb{P}_i$ normalized in such a way that it represents the first Chern class $c_1(\mathcal{O}(1))$ of the hyperplane bundle. Then the

Kostant-Kirillov form ω and the first Chern form $c_1(F)$ of F are given by

$$\omega = \sum_{i=1}^{r} (\lambda_{n_i} - \lambda_{n_{i+1}}) \omega_{\mathbb{P}_i}$$

and

$$c_1(F) = \sum_{i=1}^{r} (n_{i+1} - n_{i-1}) \omega_{\mathbb{P}_i}$$

respectively.

Example 2.1. The 3-dimensional full flag manifold $\mathrm{Fl}(3)$ is embedded into

$$\mathbb{P}_1 \times \mathbb{P}_2 = \mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\textstyle\bigwedge^2 \mathbb{C}^3) \cong \mathbb{P}^2 \times \mathbb{P}^2$$

as a hypersurface. The image of $\mathrm{Fl}(3)$ is given by the Plücker relation

$$Z_1 Z_{23} + Z_2 Z_{31} + Z_3 Z_{12} = 0,$$

where $[Z_1 : Z_2 : Z_3]$ and $[Z_{23} : Z_{31} : Z_{12}]$ are the Plücker coordinates on $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively.

Example 2.2. The Grassmannian $\mathrm{Gr}(2,4)$ of 2-planes in $\mathbb{C}^4$ is embedded into $\mathbb{P}(\bigwedge^2 \mathbb{C}^4) \cong \mathbb{P}^5$ as a hypersurface. The Plücker relation is given by

$$Z_{12} Z_{34} - Z_{13} Z_{24} + Z_{14} Z_{23} = 0,$$

where $[Z_{12} : Z_{13} : Z_{14} : Z_{23} : Z_{24} : Z_{34}]$ is the Plücker coordinates.

2.2. The Gelfand-Cetlin system

For $x \in \mathcal{O}_{\boldsymbol{\lambda}}$ and $k = 1, \dots, n-1$, let $x^{(k)}$ denote the upper-left $k \times k$ submatrix of x. Since $x^{(k)}$ is also a Hermitian matrix, it has real eigenvalues $\lambda_1^{(k)}(x) \geq \lambda_2^{(k)}(x) \geq \cdots \geq \lambda_k^{(k)}(x)$. By taking the eigenvalues for all $k = 1, \dots, n-1$, we obtain a set $(\lambda_i^{(k)})_{1 \leq i \leq k \leq n-1}$ of $n(n-1)/2$ functions, which satisfy the inequalities

$$\begin{array}{ccccccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 & \cdots & \lambda_{n-1} & & & & \lambda_n \\
& \geq & & \geq & & \geq & & \geq & & & & \geq & & \geq & \\
& & \lambda_1^{(n-1)} & & & & \lambda_2^{(n-1)} & & & & & & \lambda_{n-1}^{(n-1)} & & \\
& & & \geq & & \geq & & & & & & \geq & & & \\
& & & & \lambda_1^{(n-2)} & & & & & & \lambda_{n-2}^{(n-2)} & & & & \\
& & & & & \geq & & & & \geq & & & & & \\
& & & & & & \ddots & & \cdot^{\cdot^{\cdot}} & & & & & & \\
& & & & & & & \geq \quad \geq & & & & & & & \\
& & & & & & & \lambda_1^{(1)} & & & & & & &
\end{array} \tag{2.2}$$

It follows that the number of non-constant $\lambda_i^{(k)}$ coincides with $N = \dim_{\mathbb{C}} F$. Let $I = I(n_1, \dots, n_r, n)$ denotes the set of pairs (i, k) such that $\lambda_i^{(k)}$ is non-constant. Then the *Gelfand-Cetlin system* is defined by

$$\Phi = (\lambda_i^{(k)})_{(i,k) \in I} : F(n_1, \dots, n_r, n) \longrightarrow \mathbb{R}^{N(n_1, \dots, n_r, n)}.$$

Proposition 2.3 (Guillemin and Sternberg [GS83]). *The map Φ is a completely integrable system on $(F(n_1, \dots, n_r, n), \omega)$. The functions $\lambda_i^{(k)}$ are action variables, and the image $\Delta = \Phi(F)$ is a convex polytope defined by* (2.2)*. The fiber $L(\boldsymbol{u}) = \Phi^{-1}(\boldsymbol{u})$ over each interior point $\boldsymbol{u} \in \operatorname{Int} \Delta$ is a Lagrangian torus.*

The image $\Delta \subset \mathbb{R}^{N(n_1, \dots, n_r, n)}$ is called the *Gelfand-Cetlin polytope*. The Gelfand-Cetlin system is not smooth on the locus where $\lambda_k^{(i)} = \lambda_k^{(i+1)}$ for some (i, k), or equivalently, where the Gelfand-Cetlin pattern (2.2) contains a set of equalities of the form

$$\begin{array}{ccccc}
& & \lambda_{k+1}^{(i+1)} & & \\
& /\!/ & & \backslash\!\backslash & \\
\lambda_k^{(i)} & & & & \lambda_k^{(i+1)} \\
& \backslash\!\backslash & & /\!/ & \\
& & \lambda_{k-1}^{(i)} & &
\end{array}.$$

The image of such loci are faces of Δ of codimension greater than two where Δ does not satisfy the Delzant condition. Away from such faces, each fiber

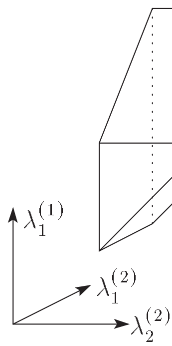

Figure 2.1: The Gelfand-Cetlin polytope for Fl(3).

$\Phi^{-1}(\boldsymbol{u})$ of Φ is an isotropic torus whose dimension is that of the face of Δ containing $\boldsymbol{u}$ in its relative interior.

2.3. The case of Fl(3)

After a translation by a scalar matrix, we may assume that Fl(3) is identified with the adjoint orbit of $\boldsymbol{\lambda} = \operatorname{diag}(\lambda_1, 0, -\lambda_2)$ for $\lambda_1, \lambda_2 > 0$. Then the Gelfand-Cetlin polytope Δ consists of $(u_1, u_2, u_3) \in \mathbb{R}^3$ satisfying

$$\begin{array}{ccccccccc} \lambda_1 & & & & 0 & & & & -\lambda_2 \\ & \geq & & \geq & & \geq & & \geq & \\ & & u_1 & & & & u_2 & & \\ & & & \geq & & \geq & & & \\ & & & & u_3 & & & & \end{array} \tag{2.3}$$

as shown in Figure 2.1. The non-smooth locus of Φ is the fiber $L_0 = \Phi^{-1}(\mathbf{0})$ over the vertex $\mathbf{0} = (0, 0, 0) \in \Delta$ where four edges intersect.

Definition 2.4 (Evans and Lekili [EL, Definition 1.1.1]). Let K be a compact connected Lie group. A Lagrangian submanifold L in a Kähler manifold X is said to be K-*homogeneous* if K acts holomorphically on X in such a way that L is a K-orbit.

Proposition 2.5. *The fiber* $L_0 = \Phi^{-1}(\mathbf{0})$ *is a Lagrangian 3-sphere given by*

$$L_0 = \left\{ \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ \overline{z}_1 & \overline{z}_2 & \lambda_1 - \lambda_2 \end{pmatrix} \in \sqrt{-1}\mathfrak{u}(3) \,\middle|\, |z_1|^2 + |z_2|^2 = \lambda_1 \lambda_2 \right\},$$

which is K*-homogeneous for*

$$K = \left\{ \begin{pmatrix} a_1 & -\overline{a}_2 & 0 \\ a_2 & \overline{a}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, |a_1|^2 + |a_2|^2 = 1 \right\} \cong \mathrm{SU}(2).$$

Proof. Suppose that $x \in L_0$. Then $\lambda_1^{(2)}(x) = \lambda_2^{(2)}(x) = 0$ implies that $x^{(2)} = 0$ and thus x has the form

$$x = \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ \overline{z}_1 & \overline{z}_2 & x_{33} \end{pmatrix}$$

for some $z_1, z_2 \in \mathbb{C}$ and $x_{33} \in \mathbb{R}$. Since

$$\det(\lambda - x) = \lambda \left(\lambda^2 - x_{33}\lambda - (|z_1|^2 + |z_2|^2) \right) = 0$$

has solutions $\lambda = \lambda_1, 0, -\lambda_2$, we have $x_{33} = \lambda_1 - \lambda_2$ and $|z_1|^2 + |z_2|^2 = \lambda_1\lambda_2$. Hence the fiber L_0 is the K-orbit of

$$\begin{pmatrix} 0 & 0 & \sqrt{\lambda_1\lambda_2} \\ 0 & 0 & 0 \\ \sqrt{\lambda_1\lambda_2} & 0 & \lambda_1 - \lambda_2 \end{pmatrix} = \mathrm{Ad}_{g_0} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 \end{pmatrix} \in \mathcal{O}_{\boldsymbol{\lambda}},$$

where

$$g_0 = \begin{pmatrix} \sqrt{\lambda_2/(\lambda_1+\lambda_2)} & 0 & -\sqrt{\lambda_1/(\lambda_1+\lambda_2)} \\ 0 & 1 & 0 \\ \sqrt{\lambda_1/(\lambda_1+\lambda_2)} & 0 & \sqrt{\lambda_2/(\lambda_1+\lambda_2)} \end{pmatrix} \in \mathrm{SU}(3).$$

Next we see that L_0 is Lagrangian. Since K acts transitively on L_0, the tangent space $T_x L_0$ is spanned by infinitesimal actions $\mathrm{ad}_\xi(x)$ of $\xi \in \mathfrak{k}$, where

$$\mathfrak{k} = \left\{ \xi = \begin{pmatrix} \xi^{(2)} & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{u}(3) \,\middle|\, \xi^{(2)} \in \mathfrak{su}(2) \right\} \cong \mathfrak{su}(2)$$

is the Lie algebra of K. Since $x^{(2)} = 0$ for $x \in L_0$, we have

$$\omega(\mathrm{ad}_\xi(x), \mathrm{ad}_\eta(x)) = \frac{\sqrt{-1}}{2\pi} \operatorname{tr}\left(x^{(2)}[\xi^{(2)}, \eta^{(2)}]\right) = 0$$

for any $\xi, \eta \in \mathfrak{k}$. □

Let $\iota : \mathrm{Fl}(3) \to \mathbb{P}_1 \times \mathbb{P}_2 = \mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\bigwedge^2 \mathbb{C}^3)$ be the Plücker embedding and $([Z_1 : Z_2 : Z_3], [Z_{23} : Z_{31} : Z_{12}])$ be the Plücker coordinates. The Kostant-Kirillov form is given by

$$\omega = \lambda_1 \omega_{\mathbb{P}_1} + \lambda_2 \omega_{\mathbb{P}_2}.$$

Since the Lagrangian fiber L_0 as a submanifold in $\mathrm{SU}(3)/T$ consists of

$$\begin{pmatrix} a_1 & -\overline{a}_2 & 0 \\ a_2 & \overline{a}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g_0 = \frac{1}{\sqrt{\lambda_1 + \lambda_2}} \times \begin{pmatrix} \sqrt{\lambda_2} a_1 & -\sqrt{\lambda_1 + \lambda_2}\, \overline{a}_2 & -\sqrt{\lambda_1} a_1 \\ \sqrt{\lambda_2} a_2 & \sqrt{\lambda_1 + \lambda_2}\, \overline{a}_1 & -\sqrt{\lambda_1} a_2 \\ \sqrt{\lambda_1} & 0 & \sqrt{\lambda_2} \end{pmatrix} \mod T$$

with $|a_1|^2 + |a_2|^2 = 1$, the image $\iota(L_0)$ is given by

$$\iota(L_0) = \left\{ \left(\left[a_1 : a_2 : \sqrt{\frac{\lambda_1}{\lambda_2}} \right], \left[\overline{a}_1 : \overline{a}_2 : -\sqrt{\frac{\lambda_2}{\lambda_1}} \right] \right) \;\middle|\; |a_1|^2 + |a_2|^2 = 1 \right\}. \tag{2.4}$$

Define an anti-holomorphic involution τ on $\mathrm{Fl}(3)$ by

$$\begin{aligned} &\tau\left([Z_1 : Z_2 : Z_3], [Z_{23} : Z_{31} : Z_{12}]\right) \\ &= \left(\left[\overline{Z}_{23} : \overline{Z}_{31} : -\frac{\lambda_1}{\lambda_2} \overline{Z}_{12} \right], \left[\overline{Z}_1 : \overline{Z}_2 : -\frac{\lambda_2}{\lambda_1} \overline{Z}_3 \right] \right). \end{aligned} \tag{2.5}$$

Proposition 2.6. *The Lagrangian L_0 is the fixed point set of τ.*

One can easily see that τ is an anti-symplectic involution if and only if $\lambda_1 = \lambda_2$.

2.4. The case of Gr(2, 4)

For $k < n$, let $\widetilde{V}(k,n)$ be the space of $n \times k$ matrices of rank k, and set

$$V(k,n) = \{Z \in \widetilde{V}(k,n) \mid Z^* Z = I_k\}.$$

Then the Grassmannian $\operatorname{Gr}(k,n)$ is given by

$$\operatorname{Gr}(k,n) = \widetilde{V}(k,n) / \operatorname{GL}(k,\mathbb{C}) = V(k,n)/\operatorname{U}(k).$$

We first consider the Gelfand-Cetlin system on $\operatorname{Gr}(n,2n)$ for general n. Fix $\lambda > 0$ and identify $\operatorname{Gr}(n,2n)$ with the adjoint orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ of

$$\boldsymbol{\lambda} = \operatorname{diag}(\underbrace{\lambda, \dots, \lambda}_{n}, \underbrace{-\lambda, \dots, -\lambda}_{n}).$$

The orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ consists of matrices of the form $2\lambda Z Z^* - \lambda I_{2n}$ for $Z \in V(n,2n)$. The Gelfand-Cetlin polytope Δ of $\operatorname{Gr}(n,2n)$ consists of $\boldsymbol{u} = (u_i^{(k)})_{(i,k)\in I} \in \mathbb{R}^{n^2}$ satisfying

$$\begin{array}{ccccccccc}
 & & & & u_n^{(2n-1)} & & & & \\
 & & & \geq & & \geq & & & \\
\lambda & & \iddots & & & & \ddots & & -\lambda \\
 & \geq & & \geq & & & \geq & \geq & \\
 & & u_1^{(n)} & & \cdots & & u_n^{(n)} & & \\
 & & & \geq & & \geq & & & \\
 & & & \ddots & & \iddots & & & \\
 & & & \geq & & \geq & & & \\
 & & & & u_1^{(1)} & & & &
\end{array}.$$

For $-\lambda < t < \lambda$, let $L_t = \Phi^{-1}(t, \dots, t)$ be the fiber over the boundary point $u_1^{(1)} = \dots = u_n^{(2n-1)} = t$ of Δ.

Proposition 2.7. *The fiber L_t is a Lagrangian submanifold given by*

$$L_t = \left\{ \begin{pmatrix} tI_n & \sqrt{\lambda^2 - t^2} A^* \\ \sqrt{\lambda^2 - t^2} A & -tI_n \end{pmatrix} \in \sqrt{-1}\mathfrak{u}(2n) \;\middle|\; A \in \operatorname{U}(n) \right\} \cong \operatorname{U}(n),$$

which is K-homogeneous for

$$K = \left\{ \begin{pmatrix} P & 0 \\ 0 & I_n \end{pmatrix} \in \mathrm{U}(2n) \,\middle|\, P \in \mathrm{U}(n) \right\} \cong \mathrm{U}(n).$$

Proof. We write $x \in \mathcal{O}_{\boldsymbol{\lambda}}$ as

$$x = 2\lambda ZZ^* - \lambda I_{2n} = \lambda \begin{pmatrix} 2Z_1Z_1^* - I_n & 2Z_1Z_2^* \\ 2Z_2Z_1^* & 2Z_2Z_2^* - I_n \end{pmatrix}$$

for $n \times n$ matrices Z_1, Z_2 with

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \in V(n, 2n).$$

Suppose that $x \in L_t$, or equivalently, $\lambda_1^{(n)}(x) = \cdots = \lambda_n^{(n)}(x) = t$. Then the upper-left $n \times n$ block of x satisfies

$$x^{(n)} = 2\lambda Z_1 Z_1^* - \lambda I_n = tI_n,$$

which means that $Z_1 \in \sqrt{(\lambda + t)/2\lambda}\,\mathrm{U}(n)$. After the right $\mathrm{U}(n)$-action on $V(n, 2n)$, we may assume that $Z_1 = \sqrt{(\lambda + t)/2\lambda} I_n$. Then the condition $Z^*Z = I_n$ implies that

$$Z_2^* Z_2 = I_n - \frac{\lambda + t}{2\lambda} I_n = \frac{\lambda - t}{2\lambda} I_n.$$

Hence Z has the form

$$Z = \begin{pmatrix} \sqrt{(\lambda + t)/2\lambda} I_n \\ \sqrt{(\lambda - t)/2\lambda} A \end{pmatrix} \in V(n, 2n) \tag{2.6}$$

for some $A \in \mathrm{U}(n)$, which shows that

$$x = 2\lambda ZZ^* - \lambda I_{2n} = \begin{pmatrix} tI_n & \sqrt{\lambda^2 - t^2} A^* \\ \sqrt{\lambda^2 - t^2} A & -tI_n \end{pmatrix}.$$

The K-homogeneity is obvious from this expression. Since the tangent space $T_x L_t$ is spanned by the infinitesimal action of the Lie algebra $\mathfrak{k}$ of K, and

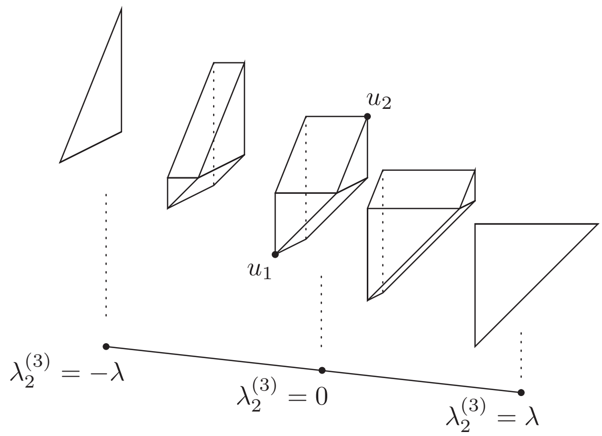

Figure 2.2: The Gelfand-Cetlin polytope for Gr(2, 4).

$x^{(n)} = tI_n$ is a scalar matrix, we have

$$\omega_x(\mathrm{ad}_\xi(x), \mathrm{ad}_\eta(x)) = \frac{1}{2\pi} \operatorname{tr} x^{(n)}[\xi^{(n)}, \eta^{(n)}] = 0$$

for

$$\xi = \begin{pmatrix} \xi^{(n)} & \\ & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta^{(n)} & \\ & 0 \end{pmatrix} \in \mathfrak{k},$$

which shows that L_t is Lagrangian. □

Corollary 2.8. *For $t \neq 0$, the fiber L_t is displaceable, i.e., there exists a Hamiltonian diffeomorphism φ on* Gr$(n, 2n)$ *such that $\varphi(L_t) \cap L_t = \emptyset$.*

Proof. One has $g(L_t) = L_{-t}$ for $g = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \mathrm{U}(2n)$. □

In the rest of this subsection, we restrict ourselves to the case of Gr(2, 4). We write $(u_1, u_2, u_3, u_4) = (u_2^{(3)}, u_1^{(2)}, u_2^{(2)}, u_1^{(1)})$ for simplicity. Figure 2.2 shows the projection

$$\Delta \longrightarrow [-\lambda, \lambda], \quad \boldsymbol{u} = (u_1, u_2, u_3, u_4) \longmapsto u_1.$$

The non-smooth locus of Φ is the inverse image of the edge of Δ defined by $u_1 = \cdots = u_4$. The fiber L_t over $(t, t, t, t) \in \partial\Delta$ is a Lagrangian submanifold consists of $2\lambda ZZ^* - \lambda I_4$ with

$$Z = \frac{1}{\sqrt{2\lambda}} \begin{pmatrix} \sqrt{\lambda + t} I_2 \\ \sqrt{\lambda - t} A \end{pmatrix} \mod \mathrm{U}(2)$$

for $A \in \mathrm{U}(2)$. We identify $\mathrm{U}(2)$ with $\mathrm{U}(1) \times \mathrm{SU}(2) \cong S^1 \times S^3$ by

$$\begin{gathered} \mathrm{U}(1) \times \mathrm{SU}(2) \longrightarrow \mathrm{U}(2), \\ \left(a_0, \begin{pmatrix} a_1 & -\overline{a}_2 \\ a_2 & \overline{a}_1 \end{pmatrix} \right) \longmapsto \begin{pmatrix} a_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & -\overline{a}_2 \\ a_2 & \overline{a}_1 \end{pmatrix}. \end{gathered}$$

Then the image of L_t under the Plücker embedding $\iota : \mathrm{Gr}(2,4) \to \mathbb{P}(\bigwedge^2 \mathbb{C}^4) \cong \mathbb{P}^5$ is given by

$$\begin{aligned} \iota(L_t) = \Bigg\{ & \left[\sqrt{\frac{\lambda + t}{\lambda - t}} : -a_0 \overline{a}_2 : \overline{a}_1 : -a_0 a_1 : -a_2 : \sqrt{\frac{\lambda - t}{\lambda + t}} a_0 \right] \\ & \Bigg| \ |a_0|^2 = |a_1|^2 + |a_2|^2 = 1 \Bigg\}. \end{aligned}$$

This expression implies the following.

Proposition 2.9. *For each $t \in (-\lambda, \lambda)$, we define an anti-holomorphic involution τ_t on $\mathrm{Gr}(2,4)$ defined by*

$$\begin{aligned} & \tau_t([Z_{12} : Z_{13} : Z_{14} : Z_{23} : Z_{24} : Z_{34}]) \\ & = \left[\frac{\lambda + t}{\lambda - t} \overline{Z}_{34} : \overline{Z}_{24} : -\overline{Z}_{23} : -\overline{Z}_{14} : \overline{Z}_{13} : \frac{\lambda - t}{\lambda + t} \overline{Z}_{12} \right] \end{aligned} \tag{2.7}$$

Then L_t is the fixed point set of τ_t.

Remark 2.10. The map τ_0 for $t = 0$ is an anti-symplectic involution as well, and satisfies $\tau_0(L_t) = L_{-t}$ for each $t \in (-\lambda, \lambda)$.

2.5. The case of Gr(2, 5)

We fix $\lambda > 0$ and identify $\mathrm{Gr}(2,5)$ with the adjoint orbit $\mathcal{O}_{\boldsymbol{\lambda}}$ of $\mathrm{diag}(\lambda, \lambda, 0, 0, 0) \in \sqrt{-1}\mathfrak{u}(5)$. The Gelfand-Cetlin polytope Δ is defined by

$$\begin{array}{ccccccccccc}
\lambda & & & & u_1 & & & & & & \\
& \rotatebox[origin=c]{-45}{$\geq$} & & \rotatebox[origin=c]{45}{$\leq$} & & \rotatebox[origin=c]{-45}{$\geq$} & & & & & \\
& & u_2 & & & & u_3 & & & & 0 \\
& & & \rotatebox[origin=c]{-45}{$\geq$} & & \rotatebox[origin=c]{45}{$\leq$} & & \rotatebox[origin=c]{-45}{$\geq$} & & \rotatebox[origin=c]{45}{$\leq$} & \\
& & & & u_4 & & & & u_5 & & \\
& & & & & \rotatebox[origin=c]{-45}{$\geq$} & & \rotatebox[origin=c]{45}{$\leq$} & & & \\
& & & & & & u_6 & & & &
\end{array} \tag{2.8}$$

We first consider the fiber $L_1(s_1, s_2, t)$ over a boundary point given by

$$\begin{array}{ccccccccccc}
\lambda & & & & s_2 & & & & & & \\
& \rotatebox[origin=c]{-45}{$>$} & & \rotatebox[origin=c]{45}{$<$} & & \rotatebox[origin=c]{-45}{$>$} & & & & & \\
& & s_1 & & & & t & & & & 0 \\
& & & \rotatebox[origin=c]{-45}{$>$} & & \rotatebox[origin=c]{45}{$=$} & & \rotatebox[origin=c]{-45}{$=$} & & \rotatebox[origin=c]{45}{$<$} & \\
& & & & t & & & & t & & \\
& & & & & \rotatebox[origin=c]{-45}{$=$} & & \rotatebox[origin=c]{45}{$=$} & & & \\
& & & & & & t & & & &
\end{array}.$$

Proposition 2.11. *The fiber $L_1(s_1, s_2, t)$ is a Lagrangian submanifold diffeomorphic to $\mathrm{U}(2) \times T^2 \cong S^3 \times T^3$. Moreover, $L_1(s_1, s_2, t)$ is K-homogeneous for*

$$\begin{aligned}
K &= \left\{ \begin{pmatrix} P & & & \\ & e^{\sqrt{-1}\theta_1} & & \\ & & e^{\sqrt{-1}\theta_2} & \\ & & & 1 \end{pmatrix} \in \mathrm{U}(5) \;\middle|\; P \in \mathrm{U}(2),\ \theta_1, \theta_2 \in \mathbb{R} \right\} \\
&\cong \mathrm{U}(2) \times T^2.
\end{aligned}$$

Proof. Note that $\mathcal{O}_{\boldsymbol{\lambda}}$ consists of matrices of the form

$$x = \lambda Z Z^* = \lambda (z_i \overline{z}_j + w_i \overline{w}_j)_{1 \leq i,j \leq 5} \tag{2.9}$$

for

$$Z = \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \\ z_3 & w_3 \\ z_4 & w_4 \\ z_5 & w_5 \end{pmatrix} \in V(2,5),$$

i.e.,

$$\sum_{i=1}^{5} |z_i|^2 = \sum_{i=1}^{5} |w_i|^2 = 1, \quad \sum_{i=1}^{5} z_i \overline{w}_i = 0. \tag{2.10}$$

Since the upper-left 2×2 submatrix of $x = \lambda(z_i \overline{z}_j + w_i \overline{w}_j) \in L_1(s_1, s_2, t)$ satisfies

$$x^{(2)} = \lambda \begin{pmatrix} |z_1|^2 + |w_1|^2 & z_1\overline{z}_2 + w_1\overline{w}_2 \\ z_2\overline{z}_1 + w_2\overline{w}_1 & |z_2|^2 + |w_2|^2 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \tag{2.11}$$

we have

$$\sqrt{\frac{\lambda}{t}} \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} \in \mathrm{U}(2), \tag{2.12}$$

and in particular, $|z_1|^2 + |z_2|^2 = |w_1|^2 + |w_2|^2 = t/\lambda$. Then the condition (2.10) implies

$$|z_3|^2 + |z_4|^2 + |z_5|^2 = (\lambda - t)/\lambda, \tag{2.13}$$

$$|w_3|^2 + |w_4|^2 + |w_5|^2 = (\lambda - t)/\lambda, \tag{2.14}$$

$$z_3\overline{w}_3 + z_4\overline{w}_4 + z_5\overline{w}_5 = 0. \tag{2.15}$$

On the other hand, the conditions $\operatorname{tr} x^{(3)} = s_1 + t$, $\operatorname{tr} x^{(4)} = \lambda + s_2$ imply

$$|z_3|^2 + |w_3|^2 = (s_1 - t)/\lambda, \tag{2.16}$$

$$|z_4|^2 + |w_4|^2 = (\lambda - s_1 + s_2 - t)/\lambda, \tag{2.17}$$

$$|z_5|^2 + |w_5|^2 = (\lambda - s_2)/\lambda. \tag{2.18}$$

After the right SU(2)-action on (z, w), we may assume that $(z_5, w_5) = \left(\sqrt{(\lambda - s_2)/\lambda}, 0\right)$. Then (2.13), (2.14), and (2.15) become

$$\begin{aligned} |z_3|^2 + |z_4|^2 &= (s_2 - t)/\lambda, \\ |w_3|^2 + |w_4|^2 &= (\lambda - t)/\lambda, \\ z_3\overline{w}_3 + z_4\overline{w}_4 &= 0, \end{aligned}$$

which mean that the 2×2 submatrix $(z_i, w_i)_{i=3,4}$ has the form

$$\begin{pmatrix} z_3 & w_3 \\ z_4 & w_4 \end{pmatrix} = \begin{pmatrix} \sqrt{(s_2 - t)/\lambda}\, a & -\sqrt{(\lambda - t)/\lambda}\, \bar{b} c \\ \sqrt{(s_2 - t)/\lambda}\, b & \sqrt{(\lambda - t)/\lambda}\, \bar{a} c \end{pmatrix}$$

for some

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathrm{SU}(2), \quad c \in \mathrm{U}(1).$$

Combining this with (2.16) and (2.17) we have

$$|a|^2 = \frac{\lambda - s_1}{\lambda - s_2}, \quad |b|^2 = \frac{s_1 - s_2}{\lambda - s_2},$$

and hence

$$\begin{pmatrix} z_3 & w_3 \\ z_4 & w_4 \end{pmatrix} = \frac{1}{\sqrt{\lambda(\lambda - s_2)}} \times \begin{pmatrix} \sqrt{(s_2 - t)(\lambda - s_1)}\, e^{\sqrt{-1}\theta_1} & -\sqrt{(\lambda - t)(s_1 - s_2)}\, e^{-\sqrt{-1}\theta_2} c \\ \sqrt{(s_2 - t)(s_1 - s_2)}\, e^{\sqrt{-1}\theta_2} & \sqrt{(\lambda - t)(\lambda - s_1)}\, e^{-\sqrt{-1}\theta_1} c \end{pmatrix}$$

for some $\theta_1, \theta_2 \in \mathbb{R}$. After the action of

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{\sqrt{-1}\varphi} \end{pmatrix} \in \mathrm{U}(2) \,\middle|\, \varphi \in \mathbb{R} \right\} \cong \mathrm{U}(1)$$

from the right, we may assume that

$$\begin{pmatrix} z_3 & w_3 \\ z_4 & w_4 \end{pmatrix} = \frac{1}{\sqrt{\lambda(\lambda - s_2)}} \times \begin{pmatrix} \sqrt{(s_2 - t)(\lambda - s_1)}\, e^{\sqrt{-1}\theta_1} & -\sqrt{(\lambda - t)(s_1 - s_2)}\, e^{\sqrt{-1}\theta_1} \\ \sqrt{(s_2 - t)(s_1 - s_2)}\, e^{\sqrt{-1}\theta_2} & \sqrt{(\lambda - t)(\lambda - s_1)}\, e^{\sqrt{-1}\theta_2} \end{pmatrix}.$$

Therefore $Z = (z_i, w_i)_i$ is normalized as

$$\begin{pmatrix} z_1 & w_1 \\ \vdots & \vdots \\ z_5 & w_5 \end{pmatrix} = \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \\ \sqrt{(s_2 - t)(\lambda - s_1)/\lambda(\lambda - s_2)}\, e^{\sqrt{-1}\theta_1} & -\sqrt{(\lambda - t)(s_1 - s_2)/\lambda(\lambda - s_2)}\, e^{\sqrt{-1}\theta_1} \\ \sqrt{(s_2 - t)(s_1 - s_2)/\lambda(\lambda - s_2)}\, e^{\sqrt{-1}\theta_2} & \sqrt{(\lambda - t)(\lambda - s_1)/\lambda(\lambda - s_2)}\, e^{\sqrt{-1}\theta_2} \\ \sqrt{(\lambda - s_2)/\lambda} & 0 \end{pmatrix}$$

with (2.12), which implies that $L_1(s_1, s_2, t)$ is a K-orbit and diffeomorphic to $\mathrm{U}(2) \times T^2$.

The assertion that $L_1(s_1, s_2, t)$ is Lagrangian follows from the K-homogeneity as in the cases of $\mathrm{Fl}(3)$ and $\mathrm{Gr}(n, 2n)$. $\square$

Next we consider the fiber $L_2(s_1, s_2, t)$ over

$$
\begin{array}{ccccccccccc}
\lambda & & & & t & & & & & & \\
 & \searrow & & \diagup\!\!\diagup & & \diagdown\!\!\diagdown & & & & & \\
 & & t & & & & t & & & & 0 \\
 & & & \diagdown\!\!\diagdown & & \diagup\!\!\diagup & & \searrow & & \nearrow & \\
 & & & & t & & & & s_1 & & \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & s_2 & & & &
\end{array}.
$$

Suppose that $x = \lambda(z_i\overline{z}_j + w_i\overline{w}_j)_{1\le i,j\le 5} \in L_2(s_1, s_2, t)$. The condition that $x^{(3)} = \lambda(z_i\overline{z}_j + w_i\overline{w}_j)_{1\le i,j\le 3}$ has eigenvalues $t, t, 0$ is equivalent to

$$
\begin{aligned}
&(2.19) \qquad & |z_1|^2 + |z_2|^2 + |z_3|^2 = t/\lambda, \\
&(2.20) \qquad & |w_1|^2 + |w_2|^2 + |w_3|^2 = t/\lambda, \\
&(2.21) \qquad & z_1\overline{w}_1 + z_2\overline{w}_2 + z_3\overline{w}_3 = 0,
\end{aligned}
$$

and hence

$$
\sqrt{\frac{\lambda}{\lambda - t}} \begin{pmatrix} z_4 & w_4 \\ z_5 & w_5 \end{pmatrix} \in U(2).
$$

On the other hand, the conditions $x^{(1)} = s_2$, $\operatorname{tr} x^{(2)} = t + s_1$, and $\operatorname{tr} x^{(3)} = 2t$ imply

$$
\begin{aligned}
|z_1|^2 + |w_1|^2 &= s_2/\lambda, \\
|z_2|^2 + |w_2|^2 &= (t - s_2 + s_1)/\lambda, \\
|z_3|^2 + |w_3|^2 &= (t - s_1)/\lambda.
\end{aligned}
$$

Then we have the following.

Proposition 2.12. *The fiber $L_2(s_1, s_2, t)$ is a $\mathrm{U}(2) \times T^2$-homogeneous Lagrangian submanifold diffeomorphic to $\mathrm{U}(2) \times T^2 \cong S^3 \times T^3$. Moreover, the* *fibers* $L_1(s_1, s_2, t)$ *and* $L_2(s_1, s_2, t)$ *satisfy*

$$g(L_2(s_1, s_2, t)) = L_1(\lambda - s_1, \lambda - s_2, \lambda - t)$$

for

$$g = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \in \mathrm{U}(5).$$

In particular, $L_1(s_1, s_2, t)$ *and* $L_2(s_1, s_2, t)$ *are displaceable.*

The Hamiltonian isotopy invariance of the Floer cohomology over the Novikov field [FOOO09, Theorem G] implies the following.

Corollary 2.13. *For* $i = 1, 2$, *we have*

$$HF((L_i(s_1, s_2, t), b), (L_i(s_1, s_2, t), b); \Lambda) = 0$$

for any weak bounding cochain b.

Remark 2.14. Other boundary fibers have lower dimensions. For example, the fiber over

$$\begin{array}{ccccccccccc}
\lambda & & & & t & & & & & & \\
& \searrow & & /\!/ & & \backslash\!\backslash & & & & & \\
& & t & & & & t & & & & 0 \\
& & & \backslash\!\backslash & & /\!/ & & \backslash\!\backslash & & \swarrow & \\
& & & & t & & & & t & & \\
& & & & & \backslash\!\backslash & & /\!/ & & & \\
& & & & & & t & & & &
\end{array}$$

consists of

$$\begin{pmatrix} \sqrt{t/\lambda} & 0 \\ 0 & \sqrt{t/\lambda} \\ 0 & 0 \\ z_4 & w_4 \\ z_5 & w_5 \end{pmatrix} \mod \mathrm{U}(2)$$

with

$$\begin{pmatrix} z_4 & w_4 \\ z_5 & w_5 \end{pmatrix} \in \sqrt{(\lambda - t)/\lambda}\, \mathrm{U}(2),$$

which means that the fiber is diffeomorphic to $\mathrm{U}(2)$.

3. Critical points of the potential function

Let $\Phi : F = F(n_1, \dots, n_r, n) \to \Delta$ be the Gelfand-Cetlin system on the flag manifold, and $\{\theta_i^{(k)}\}_{(i,k)\in I}$ be the angle variables dual to the action variables $\{\lambda_i^{(k)}\}_{(i,k)\in I}$. For each $\boldsymbol{u}=(u_k^{(i)})_{(i,k)\in I} \in \operatorname{Int}\Delta$, we identify $H^1(L(\boldsymbol{u});\Lambda_0)$ with Λ_0^N by

$$b = \sum_{(i,k)\in I} x_i^{(k)} d\theta_i^{(k)} \in H^1(L(\boldsymbol{u});\Lambda_0) \longleftrightarrow \boldsymbol{x} = (x_i^{(k)})_{(i,k)\in I} \in \Lambda_0^N,$$

and set

$$\begin{aligned} y_i^{(k)} &= e^{x_i^{(k)}} T^{u_i^{(k)}}, \qquad (i,k) \in I, \\ Q_j &= T^{\lambda_{n_j}}, \qquad j = 1, \dots, r+1. \end{aligned}$$

Theorem 3.1 ([NNU10, Theorem 10.1]). *For any interior point $\boldsymbol{u} \in \operatorname{Int}\Delta$, we have an inclusion $H^1(L(\boldsymbol{u});\Lambda_0) \subset \widehat{\mathcal{M}}_{\mathrm{weak}}(L(\boldsymbol{u}))$. As a function on*

$$\bigcup_{\boldsymbol{u}\in\operatorname{Int}\Delta} H^1(L(\boldsymbol{u});\Lambda_0) \cong \operatorname{Int}\Delta \times \Lambda_0^N,$$

the potential function is given by

$$\mathfrak{PO}(\boldsymbol{u},\boldsymbol{x}) = \sum_{(i,k)\in I} \left(\frac{y_i^{(k+1)}}{y_i^{(k)}} + \frac{y_i^{(k)}}{y_{i+1}^{(k+1)}} \right),$$

where we put $y_i^{(k+1)} = Q_j$ if $\lambda_i^{(k+1)} = \lambda_{n_j}$ is a constant function.

Example 3.2. We identify the 3-dimensional flag manifold Fl(3) with the adjoint orbit of $\boldsymbol{\lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$. The potential function is given by

$$\begin{aligned} \mathfrak{PO} &= e^{-x_1}T^{-u_1+\lambda_1} + e^{x_1}T^{u_1-\lambda_2} + e^{-x_2}T^{-u_2+\lambda_2} \\ &\quad + e^{x_2}T^{u_2-\lambda_3} + e^{x_1-x_3}T^{u_1-u_3} + e^{-x_2+x_3}T^{-u_2+u_3} \\ &= \frac{Q_1}{y_1} + \frac{y_1}{Q_2} + \frac{Q_2}{y_2} + \frac{y_2}{Q_3} + \frac{y_1}{y_3} + \frac{y_3}{y_2}. \end{aligned}$$

The potential function $\mathfrak{PO}$ has six critical points given by

$$\begin{aligned} y_1 &= y_3^2/y_2, \\ y_2 &= \pm\sqrt{Q_3(y_3+Q_2)}, \\ y_3 &= \sqrt[3]{Q_1Q_2Q_3},\ e^{2\pi\sqrt{-1}/3}\sqrt[3]{Q_1Q_2Q_3},\ e^{4\pi\sqrt{-1}/3}\sqrt[3]{Q_1Q_2Q_3}. \end{aligned}$$

It is easy to see that all critical points are nondegenerate and have the same valuation which lies in the interior of the Gelfand-Cetlin polytope. Hence we have as many critical points as $\dim H^*(\mathrm{Fl}(3)) = 6$ in this case. The set of critical values coincides with the set of eigenvalues of the quantum cup product by $c_1(\mathrm{Fl}(3))$. The Floer differential $\mathfrak{m}_1^b$ is trivial for each critical point $(\boldsymbol{u}, \boldsymbol{x})$ of $\mathfrak{PO}$, and the corresponding Floer cohomology is given by

$$HF((L(\boldsymbol{u}), b), (L(\boldsymbol{u}), b); \Lambda_0) \cong H^*(L(\boldsymbol{u}); \Lambda_0) \cong H^*(T^3; \Lambda_0).$$

Example 3.3. We identify $\mathrm{Gr}(2,4)$ with the adjoint orbit of $\mathrm{diag}(2\lambda, 2\lambda, 0, 0)$. Setting $Q = T^{2\lambda}$, the potential function is given by

$$\begin{aligned} \mathfrak{PO} &= e^{-x_2}T^{-u_2+2\lambda} + e^{-x_1+x_2}T^{-u_1+u_2} + e^{x_1-x_3}T^{u_1-u_3} \\ &\quad + e^{x_3}T^{u_3} + e^{x_2-x_4}T^{u_2-u_4} + e^{-x_3+x_4}T^{-u_3+u_4} \\ &= \frac{Q}{y_2} + \frac{y_2}{y_1} + \frac{y_1}{y_3} + y_3 + \frac{y_2}{y_4} + \frac{y_4}{y_3}. \end{aligned} \tag{3.1}$$

This function has four critical points

$$(y_1, y_2, y_3, y_4) = \left((-1)^i \sqrt[4]{Q^2}, \sqrt{-1}^i \sqrt[4]{\frac{Q^3}{4}}, \sqrt{-1}^i \sqrt[4]{4Q}, (-1)^i \sqrt[4]{Q^2} \right)$$

for $i = 0, 1, 2, 3$, and the corresponding critical values are

$$\mathfrak{PO} = 4\sqrt{2}\sqrt{-1}^i \sqrt[4]{Q}. \tag{3.2}$$

Since $\dim H^*(\mathrm{Gr}(2,4)) = 6$, one has less critical point than $\dim H^*(\mathrm{Gr}(2,4))$. These critical points are non-degenerate and have a common valuation

$$\boldsymbol{u}_0 = (\lambda, 3\lambda/2, \lambda/2, \lambda) \in \mathrm{Int}\,\Delta.$$

Hence there exist four weak bounding cochains $b_0, \dots, b_3$ such that

$$HF((L(\boldsymbol{u}_0), b_i), (L(\boldsymbol{u}_0), b_i); \Lambda_0) \cong H^*(L(\boldsymbol{u}_0); \Lambda_0) \cong H^*(T^4; \Lambda_0)$$

for $i = 0, 1, 2, 3$. The set eigenvalues of the quantum cup product by $c_1(\mathrm{Gr}(2,4))$ consists of the four critical values of the potential function and the zero eigenvalue with multiplicity two.

Example 3.4. We identify $\mathrm{Gr}(2,5)$ with the adjoint orbit of $\mathrm{diag}(\lambda, \lambda, 0, 0, 0)$. Since the Gelfand-Cetlin polytope is defined by (2.8), the potential function is given by

$$\mathfrak{PO} = \frac{Q}{y_2} + \frac{y_2}{y_1} + \frac{y_1}{y_3} + \frac{y_2}{y_4} + \frac{y_4}{y_3} + \frac{y_3}{y_5} + y_5 + \frac{y_4}{y_6} + \frac{y_6}{y_5}. \tag{3.3}$$

This function has ten critical points defined by

$$y_6^5 = Q^5, \quad Qy_4 = y_6(y_6^3 - y_4^2),$$

and

$$y_1 = \frac{Q}{y_6}, \quad y_2 = \frac{Q}{y_5}, \quad y_3 = \frac{Q}{y_4}, \quad y_5 = \frac{y_6^2}{y_4}.$$

The set

$$\left\{5(\zeta_5^i + \zeta_5^j)Q^{1/5} \;\middle|\; \zeta_5 = \exp(2\pi\sqrt{-1}/5) \text{ and } 0 \le i < j \le 4\right\} \tag{3.4}$$

of critical values of the potential function coincides with the set of eigenvalues of the quantum cup product by $c_1(\mathrm{Gr}(2,5))$.

4. Floer cohomologies of non-torus fibers

We briefly recall the construction of the A_∞ structure $\{\mathfrak{m}_k\}_{k\ge 0}$, omitting various technical details. Let L be a spin, oriented, and compact Lagrangian submanifold in a symplectic manifold (X,ω). For an almost complex structure J compatible with ω, let $\mathcal{M}_{k+1}(J,\beta)$ be the moduli space of stable J-holomorphic maps $v : (\Sigma, \partial\Sigma) \to (X, L)$ from a bordered Riemann surface Σ in the class $\beta \in \pi_2(X,L)$ of genus zero with $(k+1)$ boundary marked points $z_0, z_1, \ldots, z_k \in \partial\Sigma$. Then $\mathfrak{m}_k = \sum_{\beta\in\pi_2(X,L)} T^{\beta\cap\omega}\mathfrak{m}_{k,\beta} \colon H^*(L;\Lambda_0)^{\otimes k} \to H^*(L;\Lambda_0)$ is defined by

$$\mathfrak{m}_{k,\beta}(x_1, \ldots, x_k) = (\mathrm{ev}_0)_*(\mathrm{ev}_1^* x_1 \cup \cdots \cup \mathrm{ev}_k^* x_k), \tag{4.1}$$

where $\mathrm{ev}_i \colon \mathcal{M}_{k+1}(J,\beta) \to L$, $[v, (z_0, \ldots, z_k)] \mapsto v(z_i)$ is the evaluation map at the ith marked point.

4.1. Holomorphic disks in $(\mathrm{Fl}(3), L_0)$

We identify $\mathrm{Fl}(3)$ with the adjoint orbit of $\mathrm{diag}(\lambda_1, 0, -\lambda_2)$ for $\lambda_1, \lambda_2 > 0$ as in Subsection 2.3. Note that the symplectic form and the first Chern class are given by $\omega = \lambda_1\omega_{\mathbb{P}_1} + \lambda_2\omega_{\mathbb{P}_2}$ and $c_1(\mathrm{Fl}(3)) = 2(\omega_{\mathbb{P}_1} + \omega_{\mathbb{P}_2})$, respectively.

Recall that the homotopy group $\pi_2(\mathrm{Fl}(3)) \cong \mathbb{Z}^2$ is generated by 1-dimensional Schubert varieties X_1 and X_2, which are rational curves of bidegree $(1,0)$ and $(0,1)$ in $\mathbb{P}_1 \times \mathbb{P}_2 \cong \mathbb{P}^2 \times \mathbb{P}^2$, respectively. Since L_0 is diffeomorphic to $\mathrm{SU}(2) \cong S^3$, we have $\pi_1(L_0) = \pi_2(L_0) = 0$. The long exact sequence of homotopy groups yields

$$\pi_2(\mathrm{Fl}(3), L_0) \cong \pi_2(\mathrm{Fl}(3)) \cong \mathbb{Z}^2.$$

Let β_1, β_2 be generators of $\pi_2(\mathrm{Fl}(3), L_0)$ corresponding to X_1 and X_2, respectively. The symplectic area of β_i is given by

$$\beta_i \cap \omega = [X_i] \cap (\lambda_1 \omega_{\mathbb{P}_1} + \lambda_2 \omega_{\mathbb{P}_2}) = \lambda_i.$$

Let τ be the anti-holomorphic involution on $\mathrm{Fl}(3)$ defined in (2.5). For a holomorphic disk $v : (D^2, \partial D^2) \to (\mathrm{Fl}(3), L_0)$, we define a new holomorphic disk $\tau_* v : (D^2, \partial D^2) \to (\mathrm{Fl}(3), L_0)$ by

$$\tau_* v(z) = \tau(v(\overline{z})).$$

Since L_0 is the fixed point set of τ, one can glue v and $\tau_* v$ along the boundary to obtain a holomorphic curve $w = v \# \tau_* v : \mathbb{P}^1 \to \mathrm{Fl}(3)$. The induced involution on $\pi_2(\mathrm{Fl}(3), L_0)$, which is also denoted by τ_*, is given by $\tau_* \beta_1 = \beta_2$. If v represents β_1 or β_2, then $[w] = \beta_1 + \beta_2 = [X_1] + [X_2]$, i.e., w is a rational curve of bidegree $(1,1)$.

Let $\mu_{L_0} : \pi_2(\mathrm{Fl}(3), L_0) \to \mathbb{Z}$ be the Maslov index. If we assume $\lambda_1 = \lambda_2$ so that τ is an anti-symplectic involution, then we have

$$\mu_{L_0}(\beta_i) = \frac{1}{2}(\mu_{L_0}(\beta_i) + \mu_{L_0}(\tau_* \beta_i)) = ([X_1] + [X_2]) \cap c_1(\mathrm{Fl}(3)) = 4$$

for $i = 1, 2$. Since the symplectic form ω and the Lagrangian submanifold L_0 depend continuously on $\lambda_1, \lambda_2 > 0$, the Maslov index $\mu_{L_0}(\beta_1) = \mu_{L_0}(\beta_2) = 4$ is independent of λ_1, λ_2.

To describe holomorphic disks with Lagrangian boundary condition, we identify the unit disk D^2 with the upper half plane $\mathbb{H} = \mathbb{H}_+$.

Proposition 4.1. *Let $w \colon \mathbb{P}^1 \to \mathrm{Fl}(3)$ be a holomorphic curve of bidegree $(1,1)$ such that $w(\mathbb{R} \cup \{\infty\}) \subset L_0$. After the $\mathrm{SU}(2)$-action, we may assume*

$$w(\infty) = ([1 : 0 : \sqrt{\lambda_1/\lambda_2}], [1 : 0 : -\sqrt{\lambda_2/\lambda_1}]). \tag{4.2}$$

We can write

$$w(0) = \left(\left[a_1 : a_2 : \sqrt{\lambda_1/\lambda_2}\right], \left[\overline{a}_1 : \overline{a}_2 : -\sqrt{\lambda_2/\lambda_1}\right]\right) \in L_0 \tag{4.3}$$

for some $(a_1, a_2) \in S^3 \setminus \{(1,0)\}$. *Then* w *is given by*

$$\begin{aligned} w(z) = \Big(&\left[cz + a_1 : a_2 : \sqrt{\lambda_1/\lambda_2}(cz+1)\right], \\ &\left[\overline{c}z + \overline{a}_1 : \overline{a}_2 : -\sqrt{\lambda_2/\lambda_1}(\overline{c}z+1)\right]\Big) \end{aligned}$$

with $c/\overline{c} = -(a_1 - 1)/(\overline{a}_1 - 1)$.

Remark 4.2. After the action of

$$\{g \in \mathrm{PSL}(2,\mathbb{R}) \mid g(0) = 0,\, g(\infty) = \infty\} \cong \mathbb{R}_{>0}$$

on $\mathbb{H}$, we may assume that $|c| = 1$.

Proof. The assumptions (4.2) and (4.3) implies that w has the form

$$\begin{aligned} w(z) = \Big(&\left[c_1 z + a_1 : a_2 : \sqrt{\lambda_1/\lambda_2}(c_1 z+1)\right], \\ &\left[c_2 z + \overline{a}_1 : \overline{a}_2 : -\sqrt{\lambda_2/\lambda_1}(c_2 z+1)\right]\Big) \end{aligned}$$

for some $c_1, c_2 \in \mathbb{C}^*$. The Plücker relation

$$\begin{aligned} 0 &= -(c_1 z + a_1)(c_2 z + \overline{a}_1) - |a_2|^2 + (c_1 z + 1)(c_2 z + 1) \\ &= (c_1 - a_1 c_1 + c_2 - \overline{a}_1 c_2) z \end{aligned}$$

implies $c_1(\overline{a}_1 - 1) + c_2(a - 1) = 0$. On the other hand, the Lagrangian boundary condition $w(\mathbb{R}) \subset L_0$ implies that

$$\frac{c_1 x + a_1}{c_1 x + 1} = \frac{\overline{c_2} x + a_1}{\overline{c_2} x + 1}, \quad \frac{a_2}{c_1 x + 1} = \frac{a_2}{\overline{c_2} x + 1}, \quad x \in \mathbb{R},$$

which means $c_2 = \overline{c_1}$. □

Note that $\arg c$ is determined by a_1 up to sign, and the sign corresponds to whether $v = w|_{\mathbb{H}}$ represents β_1 or β_2. Namely any holomorphic disk in the class β_i satisfying (4.2) and (4.3) is uniquely determined by (a_1, a_2) for $i = 1, 2$.

Example 4.3. Suppose that $(a_1, a_2) = (-1, 0)$. Then $c = \pm\sqrt{-1}$, and the corresponding holomorphic disks are given by

$$v_\pm(z) = \left(\left[z \pm \sqrt{-1} : 0 : \sqrt{\frac{\lambda_1}{\lambda_2}}(z \mp \sqrt{-1}) \right], \right.$$
$$\left. \left[z \mp \sqrt{-1} : 0 : -\sqrt{\frac{\lambda_2}{\lambda_1}}(z \pm \sqrt{-1}) \right] \right).$$

It is easy to see that the image $v_+(\mathbb{H})$ (resp. $v_-(\mathbb{H})$) is the inverse image of the edge of Δ given by $u_1^{(1)} = u_1^{(2)}$ and $u_2^{(2)} = 0$ (resp. $u_1^{(1)} = u_2^{(2)}$ and $u_1^{(2)} = 0$), which is the upper (resp. lower) vertical edge emanating from the vertex $\mathbf{0} = (0, 0, 0)$. Although the disks v_+ and v_- glue to give a holomorphic sphere, its image in the Gelfand-Cetlin polytope is bent because of the failure of the differentiability of Φ. The generators β_1, β_2 of $\pi_2(\mathrm{Fl}(3), L_0)$ are represented by v_+ and v_- respectively.

4.2. Floer cohomology of the SU(2)-fiber in Fl(3)

Let J be the standard complex structure on Fl(3). Since the fiber L_0 is SU(2)-homogeneous, [EL, Proposition 3.2.1] implies the following.

Proposition 4.4. *Any J-holomorphic disk in $(\mathrm{Fl}(3), L_0)$ is Fredholm regular. Hence the moduli space $\mathcal{M}_{k+1}^{\mathrm{reg}}(J, \beta)$ of J-holomorphic disks in the class β with $k+1$ boundary marked points is a smooth manifold of dimension*

$$\begin{aligned} \dim \mathcal{M}_{k+1}^{\mathrm{reg}}(J, \beta) &= \dim L_0 + \mu_{L_0}(\beta) + k + 1 - 3 \\ &= \mu_{L_0}(\beta) + k + 1. \end{aligned}$$

In particular, we have $\dim \mathcal{M}_2(J, \beta_i) = 6$ for $i = 1, 2$. Proposition 4.1 implies the following:

Corollary 4.5. *Let $U = \mathrm{SU}(2) \setminus \{1\} \cong \{(a_1, a_2) \in S^3 \mid a_1 \neq 1\}$. Then $\mathcal{M}_2(J, \beta_i)$ has an open dense subset diffeomorphic to $\mathrm{SU}(2) \times U$ on which the evaluation map is given by*

$$\mathrm{SU}(2) \times U \longrightarrow L_0 \times L_0 \cong \mathrm{SU}(2) \times \mathrm{SU}(2), \quad (g_1, g_2) \longmapsto (g_1, g_1 g_2).$$

In particular, $\mathrm{ev} : \mathcal{M}_2(J, \beta_i) \to L_0 \times L_0$ *is generically one-to-one.*

Since the minimal Maslov number is $\mu_{L_0}(\beta_1) = \mu_{L_0}(\beta_2) = 4$ and

$$\deg \mathfrak{m}_{1,\beta}(x) = \deg x + 1 - \mu_{L_0}(\beta), \quad x \in H^*(L_0; \Lambda_0),$$

the only nontrivial parts of the Floer differential are

$$\mathfrak{m}_{1,\beta_i} : H^3(L_0) \cong H_0(L_0) \longrightarrow H^0(L_0) \cong H_3(L_0)$$

for $i = 1, 2$. Corollary 4.5 implies that for the class $[p] \in H_0(L_0)$ of a point, we have

$$\mathfrak{m}_{1,\beta_i}([p]) = \mathrm{ev}_{0*}[\mathcal{M}_2(J, \beta_i)_{\mathrm{ev}_1} \times \{p\}] = \pm[L_0].$$

To see the sign, we use a result on the orientation of the moduli spaces of pseudo-holomorphic disks by Fukaya, Oh, Ohta, and Ono [FOOO, Theorem 1.5]. The following statement is a slightly weaker version of the result, which is sufficient for our purpose.

Theorem 4.6. *Let (X, ω) be a compact symplectic manifold, and τ an anti-symplectic involution on X whose fixed point set $L = \mathrm{Fix}(\tau)$ is non-empty, compact, connected, and spin. Then $\mathfrak{m}_{k,\beta}$ and $\mathfrak{m}_{k,\tau_*\beta}$ satisfy*

$$\mathfrak{m}_{k,\beta}(P_1, \ldots, P_k) = (-1)^{\epsilon} \mathfrak{m}_{k,\tau_*\beta}(P_k, \ldots, P_1),$$

where

$$\epsilon = \frac{\mu_L(\beta)}{2} + k + 1 + \sum_{1 \le i < j \le k} (\deg P_i - 1)(\deg P_j - 1).$$

Corollary 4.7. *We have $\mathfrak{m}_{1,\beta_1} = \mathfrak{m}_{1,\beta_2}$ for general $\lambda_1, \lambda_2 > 0$.*

Proof. If $\lambda_1 = \lambda_2$, then τ is anti-symplectic, and thus Theorem 4.6 implies

$$\mathfrak{m}_{1,\beta_1} = (-1)^{\mu_{L_0}(\beta_1)/2+2} \mathfrak{m}_{1,\tau_*\beta_1} = \mathfrak{m}_{1,\beta_2}. \tag{4.4}$$

Corollary 4.5 implies that $\mathcal{M}_2(J, \beta_i)$ depends continuously on λ_1, λ_2, and hence its orientation is independent of λ_1, λ_2. Thus (4.4) holds for general λ_1, λ_2. □

Then we have

$$\mathfrak{m}_1([p]) = \sum_{i=1}^{2} \mathfrak{m}_{1,\beta_i}([p]) T^{\omega(\beta_i)} = \pm(T^{\lambda_1} + T^{\lambda_2})[L_0],$$

which implies the following.

Theorem 4.8. *The Floer cohomology of L_0 over the Novikov ring Λ_0 is*

$$HF(L_0, L_0; \Lambda_0) \cong \Lambda_0 / T^{\min\{\lambda_1, \lambda_2\}} \Lambda_0.$$

Theorem 1.1 is an immediate consequence of Theorem 4.8.

4.3. Holomorphic disks in $(\mathrm{Gr}(2,4), L_t)$

We identify $\mathrm{Gr}(2,4)$ with the adjoint orbit of $\mathrm{diag}(\lambda, \lambda, -\lambda, -\lambda)$ for $\lambda > 0$. Note that the Kostant-Kirillov form and the first Chern class are given by

$$\omega = 2\lambda\omega_{\mathrm{FS}}, \quad c_1(\mathrm{Gr}(2,4)) = 4\omega_{\mathrm{FS}},$$

respectively, where ω_{FS} is the Fubini-Study form on $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$.

Recall that $\pi_2(\mathrm{Gr}(2,4)) \cong \mathbb{Z}$ is generated by a 1-dimensional Schubert variety X_1, which is a rational curve of degree one in $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$. Since $\pi_1(\mathrm{Gr}(2,4)) = \pi_2(L_t) = 0$ and $\pi_1(L_t) \cong \mathbb{Z}$, the exact sequence

$$0 \longrightarrow \pi_2(\mathrm{Gr}(2,4)) \longrightarrow \pi_2(\mathrm{Gr}(2,4), L_t) \longrightarrow \pi_1(L_t) \longrightarrow 0$$

implies that $\pi_2(\mathrm{Gr}(2,4), L_t) \cong \mathbb{Z}^2$. Let β_1, β_2 be generators of $\pi_2(\mathrm{Gr}(2,4), L_t)$ such that $\beta_1 + \beta_2 = [X_1] \in \pi_2(\mathrm{Gr}(2,4))$.

Example 4.9. Consider a holomorphic curve $w : \mathbb{P}^1 \to \mathrm{Gr}(2,4)$ of degree one defined by

$$w(z) = \left[\sqrt{\frac{\lambda+t}{\lambda-t}}(z - \sqrt{-1}) : 0 : z - \sqrt{-1} : \right. \\ \left. - z - \sqrt{-1} : 0 : \sqrt{\frac{\lambda-t}{\lambda+t}}(z + \sqrt{-1}) \right]. \tag{4.5}$$

Since w maps $\mathbb{R} \cup \{\infty\}$ to L_t, the restrictions

$$v_+ = w|_{\mathbb{H}_+} : (\mathbb{H}_+, \partial\mathbb{H}_+) \longrightarrow (\mathrm{Gr}(2,4), L_t),$$
$$v_- = w|_{\mathbb{H}_-} : (\mathbb{H}_-, \partial\mathbb{H}_-) \longrightarrow (\mathrm{Gr}(2,4), L_t)$$

to the upper and lower half planes give holomorphic disks representing β_1 and β_2. We define $\beta_1 = [v_+]$ and $\beta_2 = [v_-]$. It is easy to see that the symplectic areas of $v_\pm$ are given by

$$\omega(\beta_1) = \int_{\mathbb{H}_+} v_+^* \omega = \lambda + t, \quad \omega(\beta_2) = \int_{\mathbb{H}_-} v_-^* \omega = \lambda - t.$$

In the case where $t = 0$, the sphere $w(\mathbb{P}^1)$ is mapped by Φ to the slice $\Delta_0 = \Delta \cap \{u_2^{(3)} = 0\}$ of the Gelfand-Cetlin polytope (see Figure 2.2). The image of the disk $v_+(\mathbb{H}_+) \subset w(\mathbb{P}^1)$ is the lower vertical edge emanating from the vertex $\mathbf{0} = (0, 0, 0, 0)$ in Δ_0 where four edges are intersecting, and $v_+(\sqrt{-1}) = [0 : 0 : 0 : -1 : 0 : 1]$ is mapped to the vertex $\boldsymbol{u}_1 \in \Delta_0$ defined by $u_1^{(2)} = u_1^{(1)} = \lambda$ and $u_2^{(2)} = 0$. On the other hand, the remaining part $v_-(\mathbb{H}_-)$ is mapped onto the upper vertical edge of Δ_0 emanating from $\mathbf{0}$. The other vertex $\boldsymbol{u}_2 \in \Delta_0$ of this edge, which is defined by $u_2^{(2)} = u_1^{(1)} = -\lambda$ and $u_1^{(2)} = 0$, is the image of $v_-(-\sqrt{-1}) = [1 : 0 : 1 : 0 : 0 : 0]$.

Let τ_t be the anti-holomorphic involution on $\mathrm{Gr}(2, 4)$ defined in (2.7). Note that $(\tau_t)_*$ is given by $(\tau_t)_* v(z) = \tau_t(v(-\overline{z}))$ for $v : (\mathbb{H}, \partial\mathbb{H}) \to (\mathrm{Gr}(2, 4), L_t)$. Since $(\tau_t)_* v_+ = v_-$, the induced involution on $\pi_2(\mathrm{Gr}(2, 4), L_t)$ is given by $(\tau_t)_*\beta_1 = \beta_2$. Then the Maslov index of β_i is given by

$$\mu_{L_t}(\beta_i) = \frac{1}{2}\left(\mu_{L_t}(\beta_i) + \mu_{L_t}((\tau_t)_*\beta_i)\right) = [X_1] \cap c_1(\mathrm{Gr}(2, 4)) = 4$$

for $i = 1, 2$.

Since any holomorphic disk $v : (\mathbb{H}, \partial\mathbb{H}) \to (\mathrm{Gr}(2, 4), L_t)$ of Maslov index four yields a holomorphic sphere $w = v\#(\tau_t)_* v$ of degree one, we need to describe holomorphic curves $w \colon \mathbb{P}^1 \to \mathrm{Gr}(2, 4)$ of degree one such that $w(\mathbb{R} \cup \{\infty\})$ is contained in the Lagrangian fiber L_t. Proposition 4.10 below is taken from [Sot01, Theorem 2.1], which is well-known in control theory (cf. e.g. [Ros70]).

Proposition 4.10. *Suppose that a holomorphic curve $w \colon \mathbb{P}^1 \to \mathrm{Gr}(k, n) = \widetilde{V}(k, n)/\operatorname{GL}(k, \mathbb{C})$ of degree d is given by*

$$w \colon z \longmapsto \begin{pmatrix} I_k \\ F(z) \end{pmatrix} \mod \operatorname{GL}(k, \mathbb{C})$$

for a rational function $F(z)$ with values in $(n - k) \times k$ matrices. Then there exist matrix valued polynomials $P(z)$, $Q(z)$ of size $(n - k) \times k$ and $k \times k$ respectively such that

1) *$F(z) = P(z)Q(z)^{-1}$, i.e., the curve w is given by*

$$w \colon z \longmapsto \begin{pmatrix} Q(z) \\ P(z) \end{pmatrix} \mod \operatorname{GL}(k, \mathbb{C}),$$

2) *$P(z)$ and $Q(z)$ are coprime in the sense there exist matrix valued polynomials $X(z)$, $Y(z)$ such that $X(z)Q(z) + Y(z)P(z) = I_k$, and*

3) $\deg(\det Q(z)) = d$.

Such $P(z)$ *and* $Q(z)$ *are unique up to multiplication of elements in* $\mathrm{GL}(k, \mathbb{C}[z])$.

Note that (2.6) implies that the $\mathrm{U}(n)$-fiber $L_t \subset \mathrm{Gr}(n, 2n) = \widetilde{V}(n, 2n)/\mathrm{GL}(n, \mathbb{C})$ consists of

$$\begin{pmatrix} I_n \\ \sqrt{(\lambda - t)/(\lambda + t)}\, A \end{pmatrix} \mod \mathrm{GL}(n, \mathbb{C})$$

for $A \in \mathrm{U}(n)$.

Proposition 4.11. *Let* $w \colon \mathbb{P}^1 \to \mathrm{Gr}(n, 2n)$ *be a holomorphic curve of degree one such that* $w(\mathbb{R} \cup \{\infty\}) \subset L_t$*, and let* $F(z)$ *denote the corresponding rational function with values in* $n \times n$ *matrices. By the* $\mathrm{U}(n)$*-action, we assume that*

$$F(\infty) = \sqrt{\frac{\lambda - t}{\lambda + t}} I_n \in \sqrt{\frac{\lambda - t}{\lambda + t}}\, \mathrm{U}(n), \tag{4.6}$$

and set

$$F(0) = \sqrt{\frac{\lambda - t}{\lambda + t}} A \tag{4.7}$$

for $A \in \mathrm{U}(n)$*. Then there exist*

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in S^{2n-1}/S^1 = \mathbb{P}^{n-1}$$

and $c \in \mathbb{C} \setminus \mathbb{R}$ *such that*

$$A = I_n + \left(\frac{c^2}{|c|^2} - 1 \right) a a^*,$$

and

$$F(z) = \sqrt{\frac{\lambda - t}{\lambda + t}} \frac{1}{z - \bar{c}} (z I_n - \bar{c} A) = \sqrt{\frac{\lambda - t}{\lambda + t}} \left(I_n - \frac{c - \bar{c}}{z - \bar{c}} a a^* \right). \tag{4.8}$$

Proof. Let $F(z)=Q(z)P(z)^{-1}$ be the factorization given in Proposition 4.10. Then the assumptions (4.6), (4.7), and $\deg(\det P(z)) = 1$ imply that $F(z)$ has the form

$$F(z) = \sqrt{\frac{\lambda - t}{\lambda + t}} \frac{1}{z - \overline{c}}(zI_n - \overline{c}A)$$

for some $c \in \mathbb{C}$. The Lagrangian boundary condition $w(\mathbb{R} \cup \{\infty\}) \subset L_t$ implies that

$$\frac{1}{x - \overline{c}}(xI_n - \overline{c}A) \in \mathrm{U}(n)$$

for any $x \in \mathbb{R}$, which means $\overline{c}A + cA^* = (c + \overline{c})I_n$, or equivalently, $\overline{c}A - \mathrm{Re}(c)I_n$ is skew-hermitian. Hence $\overline{c}A - \mathrm{Re}(c)I_n$ has pure imaginary eigenvalues $\sqrt{-1}\alpha_1, \ldots, \sqrt{-1}\alpha_n$, and can be diagonalized by some $g \in \mathrm{U}(n)$;

$$g^*(\overline{c}A - \mathrm{Re}(c)I_n)g = \mathrm{diag}(\sqrt{-1}\alpha_1, \ldots, \sqrt{-1}\alpha_n).$$

Since

$$g^*Ag = \mathrm{diag}\left(\frac{\mathrm{Re}(c) + \sqrt{-1}\alpha_1}{\overline{c}}, \ldots, \frac{\mathrm{Re}(c) + \sqrt{-1}\alpha_n}{\overline{c}}\right) \in \mathrm{U}(n)$$

has eigenvalues of unit norm, we have $\alpha_i = \pm \mathrm{Im}(c)$ for $i = 1, \ldots, n$. After the action of a permutation matrix, we may assume that g^*Ag has the form

$$g^*Ag = \mathrm{diag}(\underbrace{c/\overline{c}, \ldots, c/\overline{c}}_{k}, \underbrace{1, \ldots, 1}_{n-k}) =: C \tag{4.9}$$

for some k. Then $F(z)$ is given by

$$\begin{aligned} F(z) &= \sqrt{\frac{\lambda - t}{\lambda + t}} \frac{1}{z - \overline{c}} g(zI_n - \overline{c}C)g^* \\ &= \sqrt{\frac{\lambda - t}{\lambda + t}} g\, \mathrm{diag}\left(\frac{z - c}{z - \overline{c}}, \ldots, \frac{z - c}{z - \overline{c}}, 1, \ldots, 1\right) g^* \end{aligned}$$

In particular, we have

$$\det F(z) = \left(\frac{\lambda - t}{\lambda + t}\right)^{n/2} \left(\frac{z - c}{z - \overline{c}}\right)^k.$$

The condition $\deg(\det P(z)) = 1$ implies that $k = 1$, i.e.,

$$C = \mathrm{diag}(c/\overline{c}, 1, \ldots, 1) = (c/\overline{c} - 1)E_{11} + I_n,$$

where $E_{11} = \operatorname{diag}(1, 0, \ldots, 0) \in \mathfrak{gl}(n, \mathbb{C})$. Let $a \in S^{2n-1} \subset \mathbb{C}^n$ be the first column of g. Then we have

$$A = g\left(\left(\frac{c^2}{|c|^2} - 1\right) E_{11} + I_n\right) g^* = \left(\frac{c^2}{|c|^2} - 1\right) aa^* + I_n,$$

which proves the proposition. □

Remark 4.12. 1) The equation (4.9) (with $k = 1$) implies that $\det A = c/\bar{c} = c^2/|c|^2$.

2) After the $\mathbb{R}_{>0}$-action on the domain, we may assume that $|c| = 1$.

We now assume that $n = 2$. The sign of $\operatorname{Im}(c) = \operatorname{Im}\sqrt{\det A}$ corresponds to the homotopy class of the holomorphic disk $v = w|_{\mathbb{H}}$. The curve w corresponding to $a = [1 : 0]$ and $c = -\sqrt{-1}$ coincides with (4.5), and hence $w|_{\mathbb{H}} = v_+$ represents β_1. Thus $v = w|_{\mathbb{H}}$ represents β_1 (resp. β_2) when $\operatorname{Im}(c) = \operatorname{Im}\sqrt{\det A} < 0$ (resp. $\operatorname{Im}(c) > 0$).

4.4. Floer cohomologies of the U(2)-fibers in Gr(2, 4)

Since the minimal Maslov number of the $U(2)$-fiber L_t is $\mu_{L_t}(\beta_i) = 4$, we have the following by degree reason.

Lemma 4.13. *The potential function $\mathfrak{PO}\colon H^1(L_t; \Lambda_0) \to \Lambda_0$ for L_t is trivial:*

$$\mathfrak{PO} \equiv 0.$$

The cohomology of $L_t \cong S^1 \times S^3$ is given by

$$H^*(L_t) \cong H^*(S^1) \otimes H^*(S^3).$$

Let $\mathbf{e}_1 \in H^1(L_t; \mathbb{Z}) \cong H^1(S^1; \mathbb{Z})$ and $\mathbf{e}_3 \in H^3(L_t; \mathbb{Z}) \cong H^3(S^3; \mathbb{Z})$ be the generators, and write $b = x\mathbf{e}_1 \in H^1(L_t; \Lambda_0)$. Since $\deg \mathfrak{m}^b_{1,\beta} = 1 - \mu_{L_t}(\beta)$ and the minimal Maslov number is four, the only nontrivial parts of the Floer differential $\mathfrak{m}^b_1$ are

$$\mathfrak{m}^b_{1,\beta_i} : H^4(L_t) \cong H^1(S^1) \otimes H^3(S^3) \longrightarrow H^1(L_t) \cong H^1(S^1),$$
$$\mathfrak{m}^b_{1,\beta_i} : H^3(L_t) \cong H^3(S^3) \longrightarrow H^0(L_t) \cong \Lambda_0$$

for $i = 1, 2$.

Since $(\mathrm{Gr}(2,4), L_t)$ is $\mathrm{U}(2)$-homogeneous, any J-holomorphic disk is Fredholm regular for the standard complex structure J by [EL, Proposition 3.2.1]. Hence one has $\dim \mathcal{M}_2(J,\beta_i) = 7$ for $i = 1, 2$. In what follows we identify $L_t \cong \sqrt{(\lambda - t)/(\lambda + t)}\,\mathrm{U}(2)$ with $\mathrm{U}(2)$ by rescaling. Now Proposition 4.11 implies the following:

Corollary 4.14. *Define* $f\colon (0,2\pi)\times\mathbb{P}^1 \to \mathrm{U}(2)$ *by* $f(\theta,a) = (e^{\sqrt{-1}\theta} - 1)aa^* + I_2$. *For* $i = 1, 2$, *the moduli space* $\mathcal{M}_2(J,\beta_i)$ *has an open dense subset diffeomorphic to* $\mathrm{U}(2) \times (0,2\pi) \times \mathbb{P}^1$ *such that the evaluation map is given by*

$$\mathrm{U}(2)\times(0,2\pi)\times\mathbb{P}^1 \longrightarrow L_t\times L_t \cong \mathrm{U}(2)\times\mathrm{U}(2), \quad (g,\theta,a) \longmapsto (g, g\cdot f(\theta,a)).$$

Note that $e^{\sqrt{-1}\theta} = \det f(\theta,a)$ is related to $c \in S^1$ in Proposition 4.11 by $c = \exp(\sqrt{-1}(\theta/2 + \pi))$ or $c = \exp(\sqrt{-1}\theta/2)$ corresponding to $i = 1, 2$.

Next we consider $\mathcal{M}_{k+l+2}(J,\beta_i)$. For a rational curve $w\colon \mathbb{P}^1 \to \mathrm{Gr}(2,4)$ given by (4.8), the composition $\det\circ w|_{\partial\mathbb{H}}\colon \partial\mathbb{H} = \mathbb{R} \to L_t \cong \mathrm{U}(2) \to S^1$ is given by

$$x \longmapsto \frac{x - c}{x - \overline{c}}.$$

Hence each boundary point $x \in \partial\mathbb{H}$ is determined by the argument of $\det w(x) = (x - c)/(x - \overline{c})$. Fixing the 0-th and $(k+1)$-st boundary marked points, we have the following.

Corollary 4.15. *The moduli space* $\mathcal{M}_{k+l+2}(J,\beta_i)$ *has an open dense subset diffeomorphic to*

$$\left\{ (g,\theta,a,(t_i),(s_j)) \in \mathrm{U}(2)\times(0,2\pi)\times\mathbb{P}^1\times\mathbb{R}^k\times\mathbb{R}^l \;\middle|\; \begin{array}{l} 0 < t_1 < \cdots < t_k < \theta, \\ \theta < s_1 < \cdots < s_l < 2\pi \end{array} \right\}$$

on which the evaluation maps $\mathrm{ev}\colon \mathcal{M}_{k+l+2}(J,\beta_i) \to L_t \cong \mathrm{U}(2)$ *satisfy*

$$(\mathrm{ev}_0, \mathrm{ev}_{k+1})\colon (g,\theta,a,(t_i),(s_j)) \longmapsto (g, g\cdot f(\theta,a))$$

and

$$\det \mathrm{ev}_i(g,\theta,a,(t_i),(s_j)) = \begin{cases} e^{\sqrt{-1}t_i}\det g, & i = 1,\ldots,k, \\ e^{\sqrt{-1}\theta}\det g, & i = k+1, \\ e^{\sqrt{-1}s_{i-k-1}}\det g, & i = k+2,\ldots,k+l+2. \end{cases}$$

Theorem 4.16. *For* $b = x\mathbf{e}_1 \in H^1(L_0; \Lambda_0/2\pi\sqrt{-1}\mathbb{Z}) \cong \Lambda_0/2\pi\sqrt{-1}\mathbb{Z}$, *the deformed Floer differential* $\mathfrak{m}_1^b$ *is given by*

$$\mathfrak{m}_1^b(\mathbf{e}_3) = e^x T^{\lambda+t} + e^{-x} T^{\lambda-t}, \tag{4.10}$$

$$\mathfrak{m}_1^b(\mathbf{e}_1 \wedge \mathbf{e}_3) = (e^x T^{\lambda+t} + e^{-x} T^{\lambda-t})\mathbf{e}_1. \tag{4.11}$$

Hence the Floer cohomology of (L_t, b) *is*

$$\begin{aligned} &HF((L_t, b), (L_t, b); \Lambda_0) \\ &\cong \begin{cases} H^*(L_0; \Lambda_0) & \text{if } t = 0 \text{ and } x = \pm\pi\sqrt{-1}/2, \\ (\Lambda_0/T^{\min\{\lambda-t,\lambda+t\}}\Lambda_0)^2 & \text{otherwise.} \end{cases} \end{aligned}$$

The Floer cohomology over the Novikov field is given by

$$HF((L_t, b), (L_t, b); \Lambda) \cong \begin{cases} H^*(L_0; \Lambda) & \text{if } t = 0 \text{ and } x = \pm\pi\sqrt{-1}/2, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathbf{e}_1, \mathbf{e}_3 \in H^*(\mathrm{U}(2))$ are given by

$$\mathbf{e}_1 = \frac{1}{2\pi\sqrt{-1}} \operatorname{tr}(g^{-1}dg) = \frac{1}{2\pi\sqrt{-1}} d\log(\det g), \quad \mathbf{e}_3 = \frac{1}{24\pi^2} \operatorname{tr}\left[(g^{-1}dg)^3\right],$$

where $g^{-1}dg$ is the left-invariant Maurer-Cartan form on $\mathrm{U}(2)$.

Lemma 4.17. *For* $f(\theta, a) = (e^{\sqrt{-1}\theta} - 1)aa^* + I_2$, *we have*

$$f^*\mathbf{e}_1 = \frac{1}{2\pi} \operatorname{tr}(f^{-1}df) = \frac{d\theta}{2\pi}, \tag{4.12}$$

$$f^*\mathbf{e}_3 = \frac{1}{24\pi^2} \operatorname{tr}(f^{-1}df)^3 = (1 - \cos\theta)\frac{d\theta}{2\pi} \wedge \omega_{\mathbb{P}^1}, \tag{4.13}$$

where $\omega_{\mathbb{P}^1}$ *is the Fubini-Study form on* $\mathbb{P}^1$ *normalized in such a way that*

$$\int_{\mathbb{P}^1} \omega_{\mathbb{P}^1} = 1.$$

Proof. The first assertion (4.12) follows from $\det f = e^{\sqrt{-1}\theta}$. Since f is $\mathrm{SU}(2)$-equivariant with respect to the natural action on $\mathbb{P}^1$ and the adjoint action on $\mathrm{U}(2)$, it suffices to show (4.13) at $a = [1 : 0] \in \mathbb{P}^1$. A direct calculation gives

$$f^{-1}df = \begin{pmatrix} \sqrt{-1}d\theta & -(e^{-\sqrt{-1}\theta}-1)d\bar{a}_2 \\ (e^{\sqrt{-1}\theta}-1)da_2 & 0 \end{pmatrix},$$

so that

$$\operatorname{tr}(f^{-1}df)^3 = 3(2 - e^{\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta})\sqrt{-1}d\theta \wedge da_2 \wedge d\bar{a}_2$$

at $a = [1:0]$. On the other hand, the Fubini-Study form on $\mathbb{P}^1$ is given by

$$\omega_{\mathbb{P}^1} = \frac{\sqrt{-1}}{2\pi} da_2 \wedge d\bar{a}_2$$

at $a = [1:0]$, which proves (4.13). □

Proof of Theorem 4.16. Note that for $m : \mathrm{U}(2) \times \mathrm{U}(2) \to \mathrm{U}(2)$, $(g_1, g_2) \mapsto g_1 g_2$, we have $m^*\mathbf{e}_i = \pi_1^*\mathbf{e}_i + \pi_2^*\mathbf{e}_i$ for $i = 1, 3$, where $\pi_1, \pi_2 \colon \mathrm{U}(2) \times \mathrm{U}(2) \to \mathrm{U}(2)$ are the projections to the first and the second factors. Then $\mathrm{ev}_j^*\,\mathbf{e}_i$ are given by

$$\begin{aligned} \mathrm{ev}_i^*\,\mathbf{e}_1 &= \frac{1}{2\pi}dt_i + g^*\mathbf{e}_1, \quad i = 1, \dots, k, \\ \mathrm{ev}_{k+1+i}^*\,\mathbf{e}_1 &= \frac{1}{2\pi}dt_i + g^*\mathbf{e}_1, \quad i = 1, \dots, l, \\ \mathrm{ev}_{k+1}^*\,\mathbf{e}_3 &= f^*\mathbf{e}_3 + g^*\mathbf{e}_3 = (1-\cos\theta)\frac{d\theta}{2\pi} \wedge \omega_{\mathbb{P}^1} + g^*\mathbf{e}_3, \end{aligned}$$

where $g^*\mathbf{e}_i$ is the pull-back of $\mathbf{e}_i$ by the projection

$$\mathrm{U}(2) \times (0, 2\pi) \times \mathbb{P}^1 \longrightarrow \mathrm{U}(2), \quad (g, \theta, a) \longmapsto g$$

to the first factor. For $\theta \in (0, 2\pi)$, set

$$\begin{aligned} D_1(\theta) &= \{(t_1, \dots, t_k) \in \mathbb{R}^k \mid 0 < t_1 < \cdots < t_k < \theta\}, \\ D_2(\theta) &= \{(s_1, \dots, s_l) \in \mathbb{R}^l \mid \theta < s_1 < \cdots < s_l < 2\pi\}. \end{aligned}$$

Taking a suitable orientation on $\mathcal{M}_{k+l+2}(\beta_1, J)$, we have from (4.1) and Corollary 4.15 that

$$
\begin{aligned}
(4.14)\qquad & \mathfrak{m}_{k+l+1,\beta_1}(\underbrace{b,\dots,b}_{k},\mathbf{e}_3,\underbrace{b,\dots,b}_{l}) \\
&= \int_{(0,2\pi)\times\mathbb{P}^1}\left(\int_{D_1(\theta)}\left(\frac{x}{2\pi}\right)^k dt_1\wedge\cdots\wedge dt_k\right) \\
&\qquad\times\left(\int_{D_2(\theta)}\left(\frac{x}{2\pi}\right)^l ds_1\wedge\cdots\wedge ds_l\right)(1-\cos\theta)\frac{d\theta}{2\pi}\wedge\omega_{\mathbb{P}^1} \\
&= \int_{(0,2\pi)}\frac{1}{k!}\left(\frac{\theta}{2\pi}\cdot x\right)^k\frac{1}{l!}\left(\left(1-\frac{\theta}{2\pi}\right)x\right)^l(1-\cos\theta)\frac{d\theta}{2\pi}.
\end{aligned}
$$

Note that the terms $g^*\mathbf{e}_j$ in $\operatorname{ev}_i^*\mathbf{e}_j$ don't contribute to the integral for degree reason. We also note that the factor $1/k!$ comes from the fact that $k!$ copies of the simplex $D_1(\theta)$ tile the k-dimensional cube $[0,\theta]^k$. Hence

$$
\begin{aligned}
\mathfrak{m}^b_{1,\beta_1}(\mathbf{e}_3) &= \int_0^{2\pi}\sum_{k,l\ge 0}\frac{1}{k!}\left(\frac{\theta}{2\pi}\cdot x\right)^k\frac{1}{l!}\left(\left(1-\frac{\theta}{2\pi}\right)x\right)^l(1-\cos\theta)\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} e^{(\theta/2\pi)x}e^{(1-\theta/2\pi)x}(1-\cos\theta)\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} e^{x}(1-\cos\theta)\frac{d\theta}{2\pi} \\
&= e^x.
\end{aligned}
$$

The same argument as the proof of Corollary 4.7 gives

$$
\begin{aligned}
\mathfrak{m}_{k+l+1,\beta_2}(\underbrace{b,\dots,b}_{k},\mathbf{e}_3,\underbrace{b,\dots,b}_{l}) &= (-1)^{k+l}\mathfrak{m}_{k+l+1,\beta_1}(\underbrace{b,\dots,b}_{l},\mathbf{e}_3,\underbrace{b,\dots,b}_{k}) \\
&= \mathfrak{m}_{k+l+1,\beta_1}(\underbrace{-b,\dots,-b}_{l},\mathbf{e}_3,\underbrace{-b,\dots,-b}_{k}),
\end{aligned}
$$

so that

$$\mathfrak{m}^b_{1,\beta_2}(\mathbf{e}_3) = e^{-x}.$$

Hence we have

$$\mathfrak{m}^b_1(\mathbf{e}_3) = \sum_{i=1}^{2}\mathfrak{m}^b_{1,\beta_i}(\mathbf{e}_3)T^{\beta_i\cap\omega} = e^x T^{\lambda+t} + e^{-x}T^{\lambda-t}.$$

Next we compute $\mathfrak{m}_1^b(\mathbf{e}_1 \wedge \mathbf{e}_3) \in H^1(L_0)$. Note that

$$\mathrm{ev}_{k+1}^*(\mathbf{e}_1 \wedge \mathbf{e}_3) = (g^*\mathbf{e}_1 + f^*\mathbf{e}_1) \wedge (g^*\mathbf{e}_3 + f^*\mathbf{e}_3) = g^*\mathbf{e}_1 \wedge f^*\mathbf{e}_3 + \cdots .$$

Since only the term $g^*\mathbf{e}_1 \wedge f^*\mathbf{e}_3$ contribute to $\mathfrak{m}_{k+l+1,\beta_i}(b, \ldots, b, \mathbf{e}_1 \wedge \mathbf{e}_3, b, \ldots, b)$ by degree reason, we have

$$\mathfrak{m}_{k+l+1,\beta_i}(\underbrace{b, \ldots, b}_{k}, \mathbf{e}_1 \wedge \mathbf{e}_3, \underbrace{b, \ldots, b}_{l}) = \mathfrak{m}_{k+l+1,\beta_i}(\underbrace{b, \ldots, b}_{k}, \mathbf{e}_1, \underbrace{b, \ldots, b}_{l}) g^*\mathbf{e}_1.$$

Hence we obtain

$$\begin{aligned} \mathfrak{m}_1^b(\mathbf{e}_1 \wedge \mathbf{e}_3) &= \sum_{i=1}^{2} \mathfrak{m}_{1,\beta_i}^b(\mathbf{e}_1 \wedge \mathbf{e}_3) T^{\beta_i \cap \omega} \\ &= \sum_{i=1}^{2} \mathfrak{m}_{1,\beta_i}^b(\mathbf{e}_1) T^{\beta_i \cap \omega} \mathbf{e}_1 \\ &= (e^x T^{\lambda+t} + e^{-x} T^{\lambda-t}) \mathbf{e}_1. \end{aligned}$$

□

Remark 4.18. Oh [Oh95, Theorem B] computed the Floer cohomology $HF(L, L; \mathbb{Z}/2\mathbb{Z})$ of a real form in a compact Hermitian symmetric space, i.e., a fixed point set $L = \mathrm{Fix}(\tau)$ of an anti-holomorphic and anti-symplectic involution τ. In particular, the Floer cohomology of the U(2)-fiber $L_0 = \mathrm{Fix}(\tau_0)$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is given by

$$HF(L_0, L_0; \mathbb{Z}/2\mathbb{Z}) \cong H^*(L_0; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^4.$$

On the other hand, (4.10) and (4.11) implies that

$$HF(L_0, L_0; \Lambda_0^{\mathbb{Z}}) \cong (\Lambda_0^{\mathbb{Z}} / 2T^{\lambda} \Lambda_0^{\mathbb{Z}})^2,$$

where

$$\Lambda_0^{\mathbb{Z}} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{Z},\ \lambda_i \geq 0,\ \lim_{i \to \infty} \lambda_i = \infty \right\}$$

is the Novikov ring over $\mathbb{Z}$.

Remark 4.19. Here we consider a Lagrangian U(n)-fiber L_t in Gr$(n, 2n)$ for general n. The one-parameter subgroup $g_\theta = \exp(\theta\xi)$ of U$(2n)$ given by

$$\xi = \begin{pmatrix} 0 & -E_{11} \\ E_{11} & 0 \end{pmatrix} \in \mathfrak{u}(2n)$$

sends

$$x = \left(\begin{array}{ccc|ccc} t & & & \overline{x}_1^1 & \cdots & \overline{x}_1^n \\ & \ddots & & \vdots & & \vdots \\ & & t & \overline{x}_n^1 & \cdots & \overline{x}_n^n \\ \hline x_1^1 & \cdots & x_n^1 & -t & & \\ \vdots & & \vdots & & \ddots & \\ x_1^n & \cdots & x_n^n & & & -t \end{array}\right) \in L_t$$

to $\mathrm{Ad}_{g_\theta}(x) \in \mathcal{O}_{\boldsymbol{\lambda}}$ whose upper-left $n \times n$ block is given by

$$(\mathrm{Ad}_{g_\theta}(x))^{(n)} = \begin{pmatrix} t(1 - 2\sin^2\theta) - (x_1^1 + \overline{x}_1^1)\sin\theta\cos\theta & -x_2^1 \sin\theta & \cdots & -x_n^1 \sin\theta \\ -\overline{x}_n^1 \sin\theta & t & & \\ \vdots & & \ddots & \\ -\overline{x}_n^1 \sin\theta & & & t \end{pmatrix}.$$

If $\mathrm{Ad}_{g_\theta}(x)$ is still in L_t, i.e., $(g_\theta x g_\theta^*)^{(n)} = tI_n$, then we have $x_2^1 = \cdots = x_n^1 = 0$ and $\mathrm{Re}\, x_1^1 = -t\tan\theta$. Since $|\mathrm{Re}\, x_1^1| \le \sqrt{\lambda^2 - t^2}$, one has $g_\theta(L_t) \cap L_t = \emptyset$ if

$$|\theta| > \arctan\sqrt{\frac{\lambda^2 - t^2}{t^2}}.$$

Note that the moment map $\mu : \mathcal{O}_{\boldsymbol{\lambda}} \to \mathfrak{u}(2n)$ of the U$(2n)$-action is given by $\mu(x) = (\sqrt{-1}/2\pi)x$ in our setting. Hence the Hamiltonian of g_θ is given by

$$H(x) = \frac{\sqrt{-1}}{2\pi}\langle x, \xi\rangle.$$

Since $\max_{\mathcal{O}_\lambda} H = \lambda/\pi$ and $\min_{\mathcal{O}_\lambda} H = -\lambda/\pi$, the norm of g_θ is given by

$$\int_0^\theta \left(\max_{\mathcal{O}_\lambda} H - \min_{\mathcal{O}_\lambda} H\right) d\theta = \frac{2\lambda}{\pi}\theta.$$

Hence the displacement energy of L_t is bounded from above by

$$h(t) = \frac{2\lambda}{\pi}\arctan\sqrt{\frac{\lambda^2 - t^2}{t^2}}.$$

Note that $h(t)$ is a concave function on $[-\lambda, \lambda]$ such that $h(\pm\lambda) = 0$, $h(0) = \lambda$, and $h(t) > \min\{\lambda - t, \lambda + t\}$ for $t \neq 0, \pm\lambda$.

Theorem 4.20. *The Floer cohomology of the pair* $(L_0, \pi\sqrt{-1}/2\mathbf{e}_1)$, $(L_0, -\pi\sqrt{-1}/2\mathbf{e}_1)$ *is given by*

$$HF((L_0, \pm\pi\sqrt{-1}/2\mathbf{e}_1), (L_0, \mp\pi\sqrt{-1}/2\mathbf{e}_1); \Lambda_0) \cong (\Lambda_0/T^\lambda\Lambda_0)^2.$$

In particular, the Floer cohomology over the Novikov field is trivial;

$$HF((L_0, \pm\pi\sqrt{-1}/2\mathbf{e}_1), (L_0, \mp\pi\sqrt{-1}/2\mathbf{e}_1); \Lambda) = 0.$$

Proof. For $b = \sqrt{-1}\pi/2\mathbf{e}_1 \in H^1(L_0; \Lambda_0)$, we have from (4.1) and (4.14) that

$$\begin{aligned}&\mathfrak{m}_{k+l+1,\beta_i}(\underbrace{b, \dots, b}_{k}, \mathbf{e}_3, \underbrace{-b, \dots, -b}_{l})\\ &= \int_{(0,2\pi)} \frac{1}{k!}\left(\frac{\sqrt{-1}}{4}\theta\right)^k \frac{1}{l!}\left(\frac{\sqrt{-1}}{4}\theta - \frac{\pi\sqrt{-1}}{2}\right)^l (1 - \cos\theta)\frac{d\theta}{2\pi}.\end{aligned}$$

Hence the Floer differential is given by

$$\begin{aligned}\delta_{b,-b}(\mathbf{e}_3) &= \sum_{i=1,2}\sum_{k,l\geq 0} \mathfrak{m}_{k+l+1,\beta_i}(\underbrace{b, \dots, b}_{k}, \mathbf{e}_3, \underbrace{-b, \dots, -b}_{l})T^{\beta_i \cap \omega}\\ &= 2T^\lambda \int_0^{2\pi} \sum_{k,l\geq 0} \frac{1}{k!}\left(\frac{\sqrt{-1}}{4}\theta\right)^k \frac{1}{l!}\left(\sqrt{-1}\left(\frac{\theta}{4} - \frac{\pi}{2}\right)\right)^l (1 - \cos\theta)\frac{d\theta}{2\pi}\\ &= 2T^\lambda \int_0^{2\pi} e^{\sqrt{-1}(\theta/2 - \pi/2)}(1 - \cos\theta)\frac{d\theta}{2\pi}\\ &= \frac{16}{3\pi}T^\lambda.\end{aligned}$$

Similarly we have

$$\delta_{b,-b}(\mathbf{e}_1 \wedge \mathbf{e}_3) = \frac{32}{3\pi}T^\lambda \mathbf{e}_1,$$

and consequently,

$$HF((L_0, \pi\sqrt{-1}/2\mathbf{e}_1), (L_0, -\pi\sqrt{-1}/2\mathbf{e}_1); \Lambda_0) \cong (\Lambda_0/T^\lambda\Lambda_0)^2.$$

The computation of $HF((L_0, -\pi\sqrt{-1}/2\mathbf{e}_1), (L_0, \pi\sqrt{-1}/2\mathbf{e}_1); \Lambda_0)$ is completely parallel. □

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